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Linear stability of Einstein metrics and Perelman's lambda-functional for manifolds with conical singularities

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Linear stability of Einstein metrics and Perelman's lambda-functional for manifolds with conical singularities

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by

Changliang Wang

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To my family.

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Changliang Wang, Stability of Riemannian manifolds with Killing spinors, preprint.

Abstract

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In this thesis, we study linear stability of Einstein metrics and develop the theory of Perelman's λ-functional on compact manifolds with isolated conical singularities. The thesis consists of two parts. In the first part, inspired by works in [DWW05], [GHP03], and [Wan91], by using a Bochner type argument, we prove that complete Riemannian manifolds with non-zero imaginary Killing spinors are stable, and provide a stability condition for Riemannian manifolds with non-zero real Killing spinors in terms of a twisted Dirac operator. Regular Sasaki-Einstein manifolds are essentially principal circle bundles over Kähler-Einstein manifolds. We prove that if the base space of a regular Sasaki-Einstein manifold is a product of at least two Kähler-Einstein manifolds, then the regular Sasaki-Einstein manifold is unstable. More generally, we show that Einstein metrics on principal torus bundles constructed in [WZ90] are unstable, if the base spaces are products of at least two Kähler-Einstein manifolds.

In the second part, we prove that the spectrum of $-4\Delta + R$ consists of discrete eigenvalues with finite multiplicities on a compact Riemannian manifold of dimension n with a single conical singularity, if the scalar curvature of cross section of conical neighborhood is greater than $n-2$. Moreover, we obtain an asymptotic behavior for eigenfunctions near the singularity. As a consequence of these spectrum properties, we extend the theory of Perelman's λ-functional on smooth compact manifolds to compact manifolds with isolated conical singularities.

Contents

Chapter 1

Introduction

A Riemannian manifold (M^n, g) is Einstein if the Ricci curvature Ric_g is constant, i.e.

$$Ric_g = kg, \tag{1.0.1}$$

for some constant k, and k is called the Einstein constant of g. Einstein metrics on a compact manifold M^n of dimension $n \geq 3$ appear in the several variation problems as critical points of some natural Riemannian functionals. For example, Einstein metrics on a compact manifold are critical point of the normalized total scalar curvature functional (see (2.2.1) for definition). Then, it is natural and important to study how the second variation of the normalized total scalar curvature functional behaves at an Einstein metric. This leads to the stability problem of Einstein metrics.

Einstein operator $\Delta_E = \nabla^*\nabla - 2\mathring{R}$ acts on symmetric 2-tensors, where $(\mathring{R}h)_{ij} = R_{ikjl}h_{kl}$ for $h \in C^\infty(S^2(M))$. The second variation of the normalized total scalar curvature functional at an Einstein metric g is given by $-\frac{1}{2V(M)^{\frac{n-2}{n}}}\langle \Delta_E h, h\rangle_{L^2(M)}$, when restricted in traceless transverse directions, i.e. $h \in C^\infty(S^2(M))$ satisfying $tr_g h = 0$ and $\delta_g h = 0$, where $\delta_g h$ is the divergence of h. An Einstein manifold is said stable if $\langle \Delta_E h, h\rangle_{L^2(M)} \geq 0$ for all traceless transverse symmetric 2-tensors h, unstable otherwise, and strictly stable if $\langle \Delta_E h, h\rangle_{L^2(M)} \geq c\langle h, h\rangle_{L^2(M)}$ for some constant $c > 0$. If the manifold is non-compact, we only consider compactly supported symmetric 2-tensors h. In Chapter 2, we will present more detailed background materials for stability of Einstein metrics.

In Chapter 3, we study the stability of a special class of Einstein manifolds, which are Riemannian (spin) manifolds with non-zero Killing spinors. Complete Riemannian manifolds with non-zero Killing spinors have been classified in [Bär93], [Bau89a], [Bau89b], [FK89], and [FK90] (also see [BFGK91]).

Let (M^n, g) be a Riemannian manifold with a non-zero Killing spinor σ with the Killing constant $\mu \neq 0$, i.e.

$$\nabla^S_X \sigma = \mu X \cdot \sigma, \tag{1.0.2}$$

for any vector field X on M^n, where ∇^S denotes the canonical connection on the spinor bundle induced by the Levi-Civita connection on the tangent bundle TM, and " $\cdot$ " denotes the Clifford multiplication. Then the Riemannian manifold (M^n, g) is an Einstein manifold with the scalar curvature $R = 4n(n-1)\mu^2$ (see, e.g. [Fri00]). Because the scalar curvature is real, μ can only be real or purely imaginary. A non-zero Killing spinor is said to be imaginary (resp. real) if its Killing number is imaginary (resp. real). We refer to [Fri00] and [LM89] for spin geometry.

If we set $\mu = 0$ in (3.1.1), i.e. $\nabla^S_X \sigma = 0$ for any vector field X, then σ is called a parallel spinor. Riemannian manifolds with non-zero parallel spinors are Ricci-flat, i.e. the Ricci curvature is zero. X. Dai, X. Wang, and G. Wei proved that manifolds with non-zero parallel spinors are stable in [DWW05] by deriving a Bochner type formula, and rediscovering a result in [Wan91], also see [GHP03] for the formula.

Moreover, an imaginary Killing spinor is of type I if there exists a vector field X

such that $X \cdot \sigma = \sqrt{-1}\sigma$, and otherwise, σ is of type II. H. Baum proved that n-dimensional complete Riemannian manifolds with imaginary Killing spniors of type II with Killing constant $\sqrt{-1}\nu$ are isometric to the n-dimensional hyperbolic space $H^n_{-4\nu^2}$ with constant sectional curvature $-4\nu^2$. N. Koiso proved that Einstein manifolds with negative sectional curvature, in particular, hyperbolic spaces, are stable in [Koi79] (also see [Bes87]). Indeed, by the first inequality in 12.70 in [Bes87], one can see that $\langle \nabla^*\nabla h - 2\mathring{R}h, h\rangle_{L^2} \geq 4(n-2)\nu^2\langle h, h\rangle_{L^2}$ for all compactly supported traceless transverse 2-tensors h on the hyperbolic space $H^n_{-4\nu^2}$.

Therefore, we focus on Riemannian manifolds with imaginary Killing spinors of type I and ones with real Killing spinors. Recently, in [Krö15], K. Kröncke proved that complete Riemannian manifolds with non-zero imaginary Killing spinors are stable by using a warped product structure of these manifolds and a result in [DWW05]. We obtain an estimate for Einstein operator on complete Riemannian manifolds with imaginary Killing spinors of type I by using a Bochner type formula in [DWW05] and [Wan91], and meanwhile, provide a shorter proof for this stability result.

Theorem 1.0.1 *Let (M^n, g) be a complete Riemannian manifold with a non-zero imaginary Killing spinor of type I with Killing constant μ. We have*

$$\int_M \langle \Delta_E h, h\rangle dvol_g \geq -[2(n-2)-4]\mu^2 \int_M \langle h, h\rangle dvol_g. \tag{1.0.3}$$

for all compactly supported traceless transverse symmetric 2-tensor h.

Corollary 1.0.2 *Complete Riemannian manifolds with non-zero imaginary Killing spinors are strictly stable.*

On the other hand, by a similar Bochner type argument, we obtain a stability condition for Riemannian manifolds with non-zero real Killing spinors.

Theorem 1.0.3 *The Riemannian manifold with non-zero real Killing spinor σ with Killing constant μ is stable if the twisted Dirac operator D satisfies*

$$(D-\mu)^2 \geq (n-1)^2\mu^2,$$

on $\{\Phi(h) : h \in C^\infty(S^2(M)), trh = 0, \delta h = 0\}$, where $D : C^\infty(\mathcal{S} \otimes T^*M) \to C^\infty(\mathcal{S} \otimes T^*M)$ with the spinor bundle $\mathcal{S}$, and $\Phi : C^\infty(S^2(M)) \to C^\infty(\mathcal{S} \otimes T^*M)$ is defined as $\Phi(h) = h_{ij}e_i \cdot \sigma \otimes e^j$.

Unlike the case of imaginary Killing spinors, we cannot conclude a general stability result for manifolds with non-zero real Killing spinors. Indeed, standard spheres are well-known stable manifolds with real Killing spinors. On the other hand, Jensen's sphere is an unstable Riemannian manifold with one non-zero real Killing spinor (see, e.g. [ADP83], [Bär93], [Bes87], [Jen73], and [Spa11] for this interesting example). Thus, the real Killing spinors case is more interesting for us. Another reason why Riemnnian manifolds with real Killing spinors, especially whose stability, are interesting and important is that these manifolds play an important role in supergravity theory. By the classification results of Th. Friedrich and I. Kath, O. Hijazi, and C. Bär, even dimensional, except 6 dimensional, Riemannian manifolds with real Killing spinors are standard spheres, which then are strictly stable.

Existence of real Killing spinors on odd dimensional manifolds is closely related to Sasaki-Einstein structures (see, [Bär93], [FK89], and [FK90]). Regular Sasaki-Einstein manifolds are essentially total spaces of principal circle bundles over Kähler-Einstein manifolds (see, e.g. [Bla10]). It is well-known that a product of two Einstein manifolds with the same Einstein constant is an unstable Einstein manifold with a typical unstable direction. By relating the Einstein operator on the total space of a principal circle bundle to the Einstein operator on the base space, we show that if the base space of a regular Sasaki-Einstein manifold is a product of at least two Kähler-Einstein manifolds then the lift of the typical unstable direction on the base is an unstable direction on the regular Sasaki-Einstein manifold. In particular, we obtain the following instability result.

Theorem 1.0.4 *If the base space of a regular Sasaki-Einstein manifold is a product of at least two Kähler-Einstein manifolds, then the regular Sasaki-Einstein manifold is unstable.*

In Chapter 4, we study instability of Einstein metrics on principal torus bundles. Besides regular Sasaki-Einstein manifolds, many other interesting Einstein metrics constructed on the total spaces of principal circle bundles and more generally principal torus bundles. For example, in [WZ90], M. Wang and W. Ziller constructed some Einstein metrics on the total spaces of principal torus bundles over products of positive curved Kähler-Einstein manifolds. Some of their examples are regular Sasaki-Einstein. In most of their examples, the base spaces, which are products of Kähler-Einstein manifolds, however, are not Einstein. As we study the instability of regular Sasaki-Einstein manifolds, we relate the Einstein operator on the total spaces of principal torus bundles to the Einstein operator on the base spaces. As a consequence, we obtain the following instability result for Wang and Ziller's Einstein metrics on principal torus bundles.

Theorem 1.0.5 *Let $\pi : P \to B = M_1 \times \cdots \times M_m$ be a principal torus bundle, and g be the Einstein metric on P constructed by M. Wang and W. Ziller in [WZ90]. If $m \geq 2$, then the Einstein manifold (P, g) is unstable.*

In Chapter 5, we develop the theory of Perelman's λ-functional on compact manifolds with isolated conical singularities. The recent proof of the Yau-Tian-Donaldson conjecture has demonstrated that metrics with conical singularities are not only important in themselves but also providing a powerful tool for studying smooth metrics, see, [CDS15] and [Tian15]. Riemannian manifolds with conical singularities also appear as Gromov-Hausdorff limits of smooth manifolds, and as singularities of Ricci flow. These motivate us to study Einstein manifolds with conical singularities and Ricci flow on manifolds with conical singularities.

The Perelman's λ-functional (see (2.5.4) for definition) on a compact manifold enables us to view Ricci flow as a gradient flow, and Ricci-flat metrics come out as critical points of the λ-functional. Thus, as the first step toward our goal for studying Einstein metrics and Ricci flow on manifolds with conical singularities, we have extended the theory of Perelman's λ-functional to compact Riemannian manifolds with isolated conical singularities defined as the following.

Definition 1.0.6 *We say $(M^n, d, g, p_1, \cdots, p_k)$ is a compact Riemannian manifold with isolated conical singularities at $p_1, \cdots, p_k$, if*

- *(M, d) is a compact metric space,*
- *$(M_0, g|_{M_0})$ is an n-dimensional smooth Riemannian manifold, and the Riemannian metric g induces the given metric d on M_0, where $M_0 = M \setminus \{p_1, \cdots, p_k\}$,*
- *for each singularity p_i, $1 \leq i \leq k$, $\exists$ a neighborhood $U_{p_i} \subset M$ of p_i such that $U_{p_i} \cap \{p_1, \cdots, p_k\} = \{p_i\}$, $(U_{p_i} \setminus \{p_i\}, g|_{U_{p_i} \setminus \{p_i\}})$ is isometric to $((0, \varepsilon_i) \times N_i, dr^2 + r^2 h_r)$ for some $\varepsilon_i > 0$ and compact smooth manifold N_i, where r is coordinate on $(0, \varepsilon_i)$ and h_r is a smooth family of Riemannian metrics on N_i satisfying $h_r = h_0 + o(r^{\alpha_i})$ as $r \to 0$, where $\alpha_i > 0$ and h_0 is a smooth Riemannian metric on N_i.*

Moreover, we say a singularity p is a cone-like singularity, if the metric g on a *neighborhood of p is isometric to* $dr^2 + r^2 h_0$ *for some fixed metric* h_0 *on cross section* N.

In the rest of the thesis, we will only work on manifolds with a single conical singularity because there is no essential difference between one single singularity case and multiple isolated singularities case.

Because Perelman's λ-functional $\lambda(g)$ is essentially the smallest eigenvalue of $-4\Delta_g + R_g$, we first study the spectrum of $-4\Delta_g + R_g$, and we obtain the following spectrum result.

Theorem 1.0.7 (Dai,–) *Let* (M^n, d, g, p) *be a compact Riemannian manifold with a conical singularity at p. If the scalar curvature* $R_{h_0} > (n-2)$ *on* N, *then the operator* $-4\Delta_g + R_g$ *with domain* $C_0^\infty(M \setminus \{p\})$ *is semibounded, and the the spectrum of its Friedrichs extension consists of discrete eigenvalues with finite multiplicity* $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$, *and* $\lambda_k \to +\infty$, *as* $k \to +\infty$.

Theorem 1.0.8 (Dai, –) *Let* (M^n, g, p) *be a compact Riemannian manifold with a single conical singularity p with* $R_{h_0} > (n-2)$ *and satisfying*

$$\begin{gathered} r^i |\nabla^{i+1}(h_r - h_0)| \leq C_i < +\infty, \\ \textit{for some constant } \; C_i, \quad \textit{and each } \; 0 \leq i \leq \frac{n}{2} + 2, \end{gathered} \tag{1.0.4}$$

near p. Then eigenfunctions of $-4\Delta_g + R_g$ *on satisfy*

$$u = o(r^{-\frac{n-2}{2}}), \quad \textit{as } \; r \to 0. \tag{1.0.5}$$

Consequently, the first eigenvalue is simple.

Moreover, if the singularity is cone-like, eigenfunctions have asymptotic expansion *at the conical singularity p as*

$$u \sim \sum_{j=1}^{+\infty} \sum_{l=0}^{+\infty} \sum_{p=0}^{p_j} r^{s_j+l} (\ln r)^p u_{j,l,p}, \tag{1.0.6}$$

where $u_{j,l,p} \in C^\infty(N^{n-1})$, $p_j = 0$ *or* 1, *and* $s_j = -\frac{n-2}{2} \pm \frac{\sqrt{\mu_j - (n-2)}}{2}$, *where* μ_j *are eigenvalues of* $-\Delta_{h_0} + R_{h_0}$ *on* N^{n-1}.

Consequently, we can define the λ-functional on a compact Riemannian manifold with a single conical singularity as the smallest eigenvalue of $-4\Delta_g + R_g$. Then $\lambda(g)$ smoothly depends on g. Let $g(t)$ for $t \in (-\tau, \tau)$ be a smooth family of Riemannian metrics with a single conical singularity at p satisfying $R_{h_0(t)} > (n-2)$ and the asymptotic condition (1.0.4) near p, with $g(0) = g$ and $\frac{d}{dt}|_{t=0} g(t) = h$. We obtain the following first variation formula.

Proposition 1.0.9 (Dai, –)

$$\frac{d}{dt} \lambda(g(t))|_{t=0} = \int_M \langle -Ric_g - Hess_g f, h \rangle_g e^{-f} dvol_g. \tag{1.0.7}$$

From the first variation formula (1.0.7), we can conclude that critical points of the λ-functional are Ricci-flat metrics with a single conical singularity at p, and the λ-functional is non-decreasing along Ricci flow with a single conical singularity at p. The derivations of the first variation formula and this consequence are similar to that on compact manifolds. The main difference and difficulty is that some boundary terms appear when we use the Stoke's theorem on manifolds with conical singularities. It turns out that the asymptotic behavior of eigenfunctions of $-4\Delta_g + R_g$ obtained in Theorem 1.0.8 is exactly what we need such that the boundary terms vanish while approaching to singularity p.

Then, if the initial metric g is a critical point of the λ-functional, i.e. a Ricci-flat metric with a conical singularity at p, we have the following second variation formula.

Proposition 1.0.10 (Dai, –)

$$\frac{d^2}{dt^2}\lambda(g(t))|_{t=0} = \int_M \langle -\frac{1}{2}\Delta_L^g h + \delta_g^* \delta_g h + \frac{1}{2}Hess_g(\nu_h), h\rangle e^{-f} dvol_g, \tag{1.0.8}$$

where $\Delta_g \nu_h = -\delta_g(\delta_g h)$.

Chapter 2

Background materials

In this chapter, we fix some notations and conventions, present some background materials for linear stability problems of Einstein metrics, and briefly review previous works on dynamic stability of Einstein metrics.

2.1 Notation and conventions

In this section, we fix some notations and conventions that we need in this thesis.

Let M be a smooth manifold and $E \to M$ be a smooth vector bundle.

$C^\infty(M) = \{\text{smooth functions } f : M \to \mathbb{R}\}$.

$C^\infty(E)$ denotes the space of smooth sections of a vector bundle E.

$\Omega^k(M) = \{\text{differential } k\text{-forms on } M\}$.

$\mathcal{T}^k M = \otimes^k(T^*M)$ is the bundle of k-tensors, and k-tensors are section of $\mathcal{T}^k M$.

$S^k(M) = \odot^k(T^*M)$ is the k times symmetric tensor product of the cotangent bundle T^*M.

Let (M^n, g) be a n-dimensional Riemannian manifold with the Levi-Civita connection ∇, which naturally extends to tensors. In general, $\{e_1, \cdots, e_n\}$ denotes a local orthonormal frame of TM around the point in the problem. The Riemannian curvature tensor is defined as

$$R(X,Y,Z,W) = g(-\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z, W),$$

and $R_{ijkl} = R(e_i, e_j, e_k, e_l)$. The Ricci curvature tensor is $(Ric_g)_{ik} = \sum_{j=1}^n R_{ijkj}$, and the scalar curvature is $R_g = \sum_{i,j=1}^n R_{ijij}$.

Let $f \in C^\infty(M)$ be a smooth function on M^n. The Hessian $Hess_g f$ of f is a symmetric 2-tensor defined as $Hess_g f(X,Y) = X(Y(f)) - (\nabla_X Y)f$, for any pair of vector fields X and Y on M^n. $\Delta f = tr_g(Hess_g f)$ is the negative Laplacian of f.

δ_g denotes the divergence defined as

$$\begin{aligned} \delta_g : C^\infty(\mathcal{T}^k M) &\to C^\infty(\mathcal{T}^{k-1}M) \\ \alpha &\mapsto (\delta_g \alpha)(X_1, \cdots, X_{k-1}) = -\sum_{i=1}^n (\nabla_{e_i}\alpha)(e_i, X_1, \cdots, X_{k-1}). \end{aligned}$$

δ_g^* denotes the formal adjoint of the $C^\infty(S^k(M))$ restriction of δ with respect to the natural L^2 inner product on tensors induced by the Riemannian metric g.

A Laplacian operator $\nabla^*\nabla$ acting on tensors is defined by

$$\begin{aligned} \nabla^*\nabla : C^\infty(\mathcal{T}^k M) &\to C^\infty(\mathcal{T}^k M) \\ \alpha &\mapsto (\nabla^*\nabla\alpha)(X_1, \cdots, X_k) = -\sum_{i=1}^n (\nabla\nabla\alpha)(e_i, e_i, X_1, \cdots, X_k). \end{aligned}$$

The natural curvature contraction operator $\mathring{R}$ acting on symmetric 2-tensors is defined by

$$\begin{aligned} \mathring{R} : C^\infty(S^2(M)) &\to C^\infty(S^2(M)) \\ h &\mapsto (\mathring{R}h)_{ij} = \sum_{k,l=1}^{n} R_{ikjl} h_{kl}. \end{aligned}$$

Then Einstein operator acting on symmetric 2-tensors is defined as $\Delta_E = \nabla^*\nabla - 2\mathring{R}$.

We also use ∇^g, Δ_g, $(\nabla^g)^*\nabla^g$, $\mathring{R}^g$, and Δ_E^g to denote the Levi-Civita connection, Laplacian on functions, Laplacian on tensors, curvature contraction operator on 2-tensors, and Einstein operator with respect to the Riemannian metric g, respectively, if it is necessary to emphasize the corresponding metric in order to avoid possible ambiguity.

2.2 The normalized total scalar curvature

In this section, we recall variational formulae of the normalized total scalar curvature and discuss a variational characterization of Einstein metrics on a compact manifold. For more detailed information, we refer to [Bes87], [Krö13], and [Via13].

As we have seen in Introduction, (M^n, g) is Einstein if $Ric_g = kg$ for a constant k. In this definition, we require the proportional factor k to be constant. Actually, by using the second Bianchi identity, we have

Proposition 2.2.1 (see, e.g. Corollary 4.19 in [Bes87]) *Let (M^n, g) be a Riemannian manifold of dimension $n \geq 3$. If there is a function f such that $Ric_g = fg$, then f is a constant and (M^n, g) is Einstein.*

Einstein metrics on a compact manifold M^n of dimension $n \geq 3$ naturally appear in the variational problem as critical points of the normalized total scalar curvature functional defined on the space $\mathcal{M}$ of all Riemanian metrics on M^n. The normalized total scalar curvature for $g \in \mathcal{M}$ is defined as

$$\widetilde{\mathbf{S}}(g) = \frac{1}{V(g)^{\frac{n-2}{n}}} \int_M R_g dvol_g, \tag{2.2.1}$$

where $dvol_g$ is the volume form of g, $V(g) = \int_M dvol_g$, and R_g denotes the scalar curvature of g. We note that the functional $\widetilde{\mathbf{S}} : \mathcal{M} \to \mathbb{R}$ is diffeomorphism invariant and scale-invariant.

Let $g(t)$ for $t \in (-\tau, \tau)$ be a smooth family of metrics on M^n with $g(0) = g$ and $\frac{d}{dt} g(t)|_{t=0} = h \in C^\infty(S^2(M))$. We have the following first and second variation formulae of the normalized total scalar curvature functional, see e.g. [Bes87] and [Via13].

$$\widetilde{\mathbf{S}}'_g \cdot h \equiv \frac{d}{dt}\widetilde{\mathbf{S}}(g(t))|_{t=0} = \frac{1}{V(g)^{\frac{n-2}{n}}} \int_M \langle -Ric_g + (\frac{R_g}{2} + \frac{2-n}{2n}\overline{R}_g)g, h\rangle dvol_g, \tag{2.2.2}$$

where $\overline{R} = \frac{1}{V(g)} \int_M R_g dvol_g$ is the average scaler curvature. We can see that the metric g is a critical point of $\widetilde{\mathbf{S}}$, i.e. $\widetilde{\mathbf{S}}'_g \cdot h$ vanishes for an arbitrary variation direction h if and only if

$$Ric_g = (\frac{R_g}{2} + \frac{2-n}{2n}\overline{R}_g)g. \tag{2.2.3}$$

By Proposition 2.2.1, then (M^n, g) is Einstein. Therefore, *a metric on a compact manifold of dimension $n \geq 3$ is a critical point of the total scalar curvature functional $\widetilde{\boldsymbol{S}}$ if and only if it is an Einstein metric.* If $g(0) = g$ is an Einstein metric, then the second variation formula is given by

$$\widetilde{\mathbf{S}}''_g(h, h) \equiv \frac{d^2}{dt^2}\mathbf{S}(g(t))|_{t=0} = \frac{1}{V(g)^{\frac{n-2}{n}}} \int_M \langle P_g h, h\rangle dvol_g, \tag{2.2.4}$$

where

$$\begin{aligned} P_g h = & -\frac{1}{2}\nabla^*\nabla h + \mathring{R}h + \delta_g^*(\delta_g h) + \frac{1}{2}Hess_g(tr_g h) \\ & + [-\frac{1}{2}(\Delta_g(tr_g h)) + \frac{1}{2}\delta_g(\delta_g h) - \frac{R_g}{2n}(tr_g h)]g \\ & - \frac{(2-n)R_g}{2n^2}\overline{(tr_g h)}g, \end{aligned} \tag{2.2.5}$$

with $\overline{(tr_g h)} = \frac{1}{V(g)}\int_M (tr_g h) dvol_g$, and $(\mathring{R}h)_{ij} = R_{ikjl}h_{kl}$. Then we define a symmetric quadratic as

$$\widetilde{\mathbf{S}}''_g(h, \tilde{h}) = \frac{1}{V(g)^{\frac{n-2}{n}}}\int_M \langle P_g h, \tilde{h}\rangle dvol_g. \tag{2.2.6}$$

2.3 Stability of Einstein metrics

To understand the complicated stability operator P_g in (2.2.5), we recall a decomposition of symmetric 2-tensors, and we examine the operator P_g on each factor in the decomposition.

The natural L^2 inner product on $C^\infty(S^2(M))$ is given by

$$(h, \tilde{h}) = \int_M \langle h, \tilde{h}\rangle_g dvol_g, \tag{2.3.1}$$

for $h, \tilde{h} \in C^\infty(S^2(M))$, where $\langle h, \tilde{h}\rangle_g$ is the pointwise inner product on tensors induced by the Riemannian metric g. And let

$$\delta_g^{-1}(0) = \{h \in C^\infty(S^2(M)) | \ \ \delta_g h = 0\},$$

$$tr_g^{-1}(0) = \{h \in C^\infty(S^2(M)) | \ \ tr_g(h) = 0\},$$

$$Im\delta_g^* = \{\delta_g^*\alpha | \ \ \alpha \in \Omega^1(M)\} \subset C^\infty(S^2(M)).$$

Lemma 2.3.1 ([Koi79], Lemma 4.57 in [Bes87]) *For any compact Riemannian manifold (M^n, g), we have the orthogonal decomposition*

$$C^\infty(S^2(M)) = (Im\delta_g^* + C^\infty(M) \cdot g) \oplus (\delta_g^{-1}(0) \cap tr_g^{-1}(0)). \tag{2.3.2}$$

Both factors are infinite dimensional.

Further, if (M^n, g) is Einstein, but not the standard sphere, this decomposition can be refined into

$$C^\infty(S^2(M)) = Im\delta_g^* \oplus C^\infty(M) \cdot g \oplus (\delta_g^{-1}(0) \cap tr_g^{-1}(0)). \tag{2.3.3}$$

Let us consider the second variation formula (2.2.4) restricted on each factor in the decomposition (2.3.3).

1. $h \in Im\delta_g^*$, i.e. $h = \delta_g^*\alpha$ for some $\alpha \in \Omega^1(M)$.

   By the definition of δ_g^* in Notation and Conventions, we have

   $$h = \delta_g^*\alpha = \frac{1}{2}\mathcal{L}_{\alpha^\#} g, \tag{2.3.4}$$

   see e.g. Lemma 1.60 in [Bes87] (note that the sign in the lemma was incorrect). Therefore, h is a variation direction coming from a diffeomorphism action on metrics. Because the functional $\tilde{\mathbf{S}}$ is diffeomorphism invariant, $\widetilde{\mathbf{S}}''_g(h, h) = 0$. In other words, *the second variation of the normalized total scalar curvature functional at an Einstein metric vanishes restricted on $Im\delta_g^*$.*

2. $h \in C^\infty(M) \cdot g$, i.e. $h = fg$ for some $f \in C^\infty(M)$.

By straightforward calculations, we have

$$P_g(fg) = \frac{2-n}{2}(\Delta_g f)g + \frac{2-n}{2n}(R_g f)g + \frac{n-2}{2}Hess_g f - \frac{2-n}{2n}R_g\overline{f}g, \quad (2.3.5)$$

where $\overline{f} = \frac{1}{V(g)}\int_M f dvol_g$. From this, we can easily see that $P_g(cg) = 0$ if f is a constant function $f \equiv c$. Actually, this can be seen from the fact that the functional $\widetilde{\mathbf{S}}$ is scale-invariant, and $h = cg$ is a variation direction coming from a metric rescaling. Therefore, we have

$$(P_g(fg), fg) = (P_g((f-\overline{f})g), (f-\overline{f})g). \quad (2.3.6)$$

So without loose of generality, we can assume $\overline{f} = 0$. Then the second variation in the direction $h = fg$ is given by

$$\begin{aligned}
&\widetilde{\mathbf{S}}''_g(fg, fg) \\
&= (P(fg), fg) \\
&= \frac{1}{V(g)^{\frac{n-2}{n}}}(\frac{2-n}{2}(\Delta_g f)g + \frac{2-n}{2n}(R_g f)g + \frac{n-2}{2}Hess_g f, fg) \\
&= \frac{1}{V(g)^{\frac{n-2}{n}}}\frac{n-2}{2}\int_M (-(n-1)\Delta_g f - R_g f)\cdot f dvol_g \\
&\geq 0.
\end{aligned} \quad (2.3.7)$$

The last inequality follows from the Lichnerowicz eigenvalue estimate for Δ_g (see, e.g. Theorem 4.70 in [GHL]). Thus, *the second variation of the normalized total scalar curvature functional at an Einstein metric is non-negative restricted on conformal variation directions.*

3. $h \in \delta_g^{-1}(0) \cap tr_g^{-1}(0)$.

In this case, we call h a traceless transverse symmetric 2-tensor, or simply a

TT-tensor. The stability operator P_g restricted on TT-tensor will be much simplified, and given by

$$P_g(h) = -\frac{1}{2}\nabla^*\nabla h + \mathring{R}h. \tag{2.3.8}$$

This will be the main operator in the stability problem of Einstein metrics. We make the following definition.

Definition 2.3.2 (Einstein operator) *We call the second order differential operator*

$$\Delta_E = \nabla^*\nabla - 2\mathring{R} : C^\infty(S^2M) \to C^\infty(S^2M) \tag{2.3.9}$$

the Einstein operator.

Remark 2.3.3 *The Einstein operator Δ_E is closely related to the Lichnerowicz Laplacian Δ_L. Indeed, on an Einstein manifold (M^n, g) with Einstein constant k*

$$\Delta_L = \Delta_E + 2k.$$

Then the second variation formula of the normalized total scalar curvature functional $\mathbf{S}$ restricted on TT-tensors is given by

$$\widetilde{\mathbf{S}}''_g(h,h) = -\frac{1}{2V(g)^{\frac{n-2}{n}}}\int_M \langle \Delta_E h, h\rangle dvol_g. \tag{2.3.10}$$

Because the derivative term $\nabla^*\nabla$ of the operator Δ_E is a non-negative term, $\int_M \langle \Delta_E h, h\rangle dvol_g \geq 0$ for most TT-tensors h. However, in general there still are some TT-tensors h so that $\int_M \langle \Delta_E h, h\rangle dvol_g < 0$. More precisely, Δ_E is a self-adjoint elliptic operator and the manifold M is compact. Therefore, its spectrum consists of eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ with finite multiplicities, and $\lambda_i \to +\infty$ as $i \to +\infty$. Thus, *the second variation of the normalized total scalar curvature at an Einstein metric is non-positive in most TT-tensor directions, however, in general may be negative in some TT-tensor directions.* This motivates the following notion of stability of Einstein metrics.

Definition 2.3.4 (Stability of Einstein Manifolds) *Let (M^n, g) be a compact Einstein manifold. (M^n, g) is stable if $\int_M \langle \Delta_E h, h\rangle dvol_g \geq 0$ for all TT-tensors h, and otherwise, (M^n, g) is unstable. (M^n, g) is strictly stable if $\int_M \langle \Delta_E h, h\rangle dvol_g > c\int_M \langle h, h\rangle dvol_g$ for some constant $c > 0$ and all TT-tensors h.*

Remark 2.3.5 *In Definition* 2.3.4*, we only defined stable, unstable, and strictly stable compact Einstein manifolds. Similarly, we can define stable, unstable, and strictly stable non-compact Einstein manifolds by replacing TT-tensors by compactly supported TT-tensors in Definition* 2.3.4*.*

Remark 2.3.6 *The decomposition* (2.3.3) *is orthogonal with respect to the quadratic form $\widetilde{\boldsymbol{S}}''_g(h, \tilde{h})$. Indeed, the first factor is in the null space of $\widetilde{\boldsymbol{S}}''_g(h, \tilde{h})$ because $\boldsymbol{S}$ is diffeomorphism invariant. Thus, it suffices to check that $\widetilde{\boldsymbol{S}}''_g(fg, h) = 0$ if $tr_g h = 0$ and $\delta_g h = 0$. By* (2.3.5)*, we have*

$$
\begin{aligned}
\widetilde{\boldsymbol{S}}''_g(fg, h) &= \frac{1}{V(g)^{\frac{n-2}{n}}} \int_M \langle \frac{n-2}{2} Hess_g f, h\rangle dvol_g \\
&= \frac{1}{V(g)^{\frac{n-2}{n}}} \int_M \langle \frac{n-2}{2} \nabla f, \delta h\rangle dvol_g \\
&= 0.
\end{aligned}
$$

2.4 Some other variational characteristics of Einstein metrics

In addition to the normalized total scalar curvature functional, Einstein metrics on a compact manifold M^n are also critical points of some other Riemannian functionals, for example the total scalar curvature functional $\mathcal{S}(g) = \int_M R_g dvol_g$ restricted to $\mathcal{M}_1$, i.e. $\mathcal{S} \upharpoonright \mathcal{M}_1$. Moreover, Ricci-flat metrics (Einstein metrics with zero Einstein constant) are critical points of Perelman's λ-functional $\lambda(g)$, namely, the first eigenvalue of $-4\Delta_g + R_g$ acting on $C^\infty(M)$. Here

$$\mathcal{M}_1 = \{g \in \mathcal{M} \ \ | \ \ V(g) = 1\},$$

and note that

$$T_g\mathcal{M}_1 = \{h \in C^\infty(S^2M) \ \ | \ \ \int_M \langle g, h\rangle dvol_g = 0\}, \tag{2.4.1}$$

see, e.g. the proof of Theorem 4.21 in [Bes87].

Let us briefly discuss the total scalar curvature functional $\mathcal{S}(g) = \int_M R_g dvol_g$ restricted to $\mathcal{M}_1$. There are very detailed calculations for the first and the second variation formulae of this Riemannian functional in [Bes87].

The first variation formula of the total scalar curvature $\mathbf{S}$ is given by

$$\mathbf{S}'_g \cdot h = \frac{d}{dt}\mathbf{S}(g(t))|_{t=0} = \int_M \langle \frac{R_g}{2} g - Ric_g, h\rangle dvol_g. \tag{2.4.2}$$

Thus, g is a critical point of $\mathbf{S}$ if and only if $\frac{R_g}{2} g - Ric_g = 0$. If we take trace for this, we obtain $\frac{n-2}{2} R_g = 0$. So, if the dimension $n \geq 3$, then $R_g = 0$, and further $Ric_g = 0$. Therefore, *a Riemannian metric g is a critical point of $\boldsymbol{S}$ if and only if it is Ricci-*

flat, i.e. $Ric_g = 0$. On the other hand, by (2.4.1) and (2.4.2), g is a critical point of $\mathbf{S} \restriction \mathcal{M}_1$ if and only if $\frac{R_g}{2} g - Ric_g = cg$ for some constant c. Then by Proposition 2.2.1, g is Einstein. Thus, *a Riemannian metric g is a critical point of $\boldsymbol{S} \restriction \mathcal{M}_1$ if and only if it is Einstein.*

The second variation formula of $\mathbf{S} \restriction \mathcal{M}_1$ at an Einstein metric g is given by

$$\begin{aligned}(\mathbf{S} \restriction \mathcal{M}_1)''_g(h,h) &= \frac{d^2}{dt^2}(\mathbf{S} \restriction \mathcal{M}_1)(g(t))|_{t=0} \\ &= \int_M (\langle h, -\frac{1}{2}\nabla^*\nabla h + \delta^*\delta h + \delta(\delta h) g \\ &\quad - \frac{1}{2}(\Delta(tr_g)h)g - \frac{R_g}{2n}(tr_g h)g + \mathring{R}h\rangle) dvol_g\end{aligned} \tag{2.4.3}$$

For the derivation see 4.53 in [Bes87].

Analyzing the second variation formula (2.4.3) by using the decomposition (2.3.3), we can see that the behavior of the second variation formula (2.4.3) is the same as that of the second variation (2.2.4) of $\widetilde{\mathbf{S}}$. *$(\boldsymbol{S} \restriction \mathcal{M}_1)''_g(h,h)$ vanishes restricted to $Im\delta^*$, and is non-negative restricted on $C^\infty(M)\cdot g$, for the same reason as for* (2.2.4)*. Moreover, $(\boldsymbol{S} \restriction \mathcal{M}_1)''_g(h,h) = -\frac{1}{2}\int_M \langle h, \Delta_E h\rangle dvol_g$, for TT-tensors h. Therefore, an Einstein metric is always a saddle point of $\boldsymbol{S} \restriction \mathcal{M}_1$, and we can make the same notion of stability of Einstein metrics by considering $\boldsymbol{S} \restriction \mathcal{M}_1$.*

Moreover, the stability problem of Einstein metrics was also similarly studied with respect to the variation formulae of the Perelman's ν-entropy, which was introduced in [Per02], for Einstein metrics with positive Ricci curvature, and the variation formulae of ν_+-entropy, which was introduced in [FIN05], for Einstein metrics with negative Ricci curvature. For example, H-D. Cao and C. He studied stability of Einstein metrics with respect to ν-entropy on symmetric spaces of compact type in [CH13]. We refer to [CZ12], [CHI04], and [Zhu11] for the variation formulae of the ν-functional and the ν_+-functional and their detailed derivation.

In the rest of section, we briefly describe a variational characteristic of Ricci-flat metrics by considering the Perelman's λ-functional (see (2.5.4) for definition) introduced by G. Perelman in [Per02]. We refer to [Has12] for detailed calculations of variation formulae of the λ-functional. In [DWW05], they derive the variation formulae of the first eigenvalue of the conformal Laplacian $-\Delta_g + \frac{n-2}{4(n-1)}R_g$. The derivation of the variation formulae of Perelman's λ-functional is very similar. So we also refer to [DWW05]. The λ-functional plays important roles in studying Ricci flow, and in next section, we will discuss this more later.

The first variation formula of the λ-functional is given by

$$\lambda'_g \cdot h = \frac{d}{dt}\lambda(g(t))|_{t=0} = \int_M \langle -Ric_g - Hess_g f, h\rangle e^{-f} dvol_g. \tag{2.4.4}$$

A Riemannian metric g is a critical point of λ if and only if $-Ric_g - Hess_g f = 0$ for some function f. By using the second Bianchi identity and maximal principle, then f has to be a constant function and $Ric_g = 0$, see, e.g. Proposition 1.1.1 in [CZ06]. Thus, *a Riemannian metric is critical point of* λ *if and only if it is Ricci-flat.*

The second variation formula of the λ-functional at a Ricci-flat metric g is given by

$$\lambda''_g(h,h) = \frac{d^2}{dt^2}\lambda(g(t))|_{t=0} = \int_M \langle -\frac{1}{2}\Delta_E h + \delta_g^*\delta_g h + \frac{1}{2}Hess_g \nu_h, h\rangle e^{-f} dovl_g, \tag{2.4.5}$$

where ν_h is a solution of $\Delta_g \nu_h = \delta_g\delta_g h$, and f is constant obtain from the first variation formula. Note that on Ricci-flat manifolds, $\Delta_E = \Delta_L$.

In order to understand the formula (2.4.5) better, we still use the decomposition (2.3.3). The same as two Riemannian functional we discussed before, $Im\delta_g^*$ is in the null space of the quadratic form induced by the second variation λ''_g. In particular,

λ''_g *vanishes on* $Im\delta^*_g$. Then because for $u \in C^\infty(M)$

$$\delta_g(\delta_g(ug)) = \Delta_g u,$$

and

$$\delta^*_g(\delta_g(ug)) = -Hess_g u,$$

when $h = ug$, we have

$$-\frac{1}{2}\Delta_E h + \delta^*_g \delta_g h + \frac{1}{2} Hess_g \nu_h = \frac{1}{2}(\Delta_g u)g - \frac{1}{2} Hess_g u. \tag{2.4.6}$$

Thus,

$$\lambda''_g(ug, ug) = \frac{n-1}{2}\int_M (\Delta_g u) u e^{-f} dvol_g = -\frac{n-1}{2}\int_M |\nabla u|^2 e^{-f} dvol_g \leq 0, \tag{2.4.7}$$

where f is a constant function. In other words, *the second variation* λ''_g *of the* λ*-functional at a Ricci-flat metric* g *is non-positive on the conformal variation directions.*

Another consequence of (2.4.6) is that the second and the third factors in the decomposition (2.3.3) are orthogonal with respect to the quadratic form induced by λ''_g. Indeed, if $h \in \delta_g^{-1}(0) \cap tr_g^{-1}(0)$,

$$\lambda''_g(ug, h) = -\frac{1}{2}\int_M \langle Hess_g u, h\rangle e^{-f} dvol_g = -\frac{1}{2}\int_M \langle \nabla u, \delta h\rangle e^{-f} dvol_g = 0, \tag{2.4.8}$$

where f is a constant function. Therefore, *the decomposition* (2.3.3) *is orthogonal with respect to the quadratic form induced by* λ''_g *at a Ricci-flat metric* g.

When we restrict on TT-tensors, the third factor in the decomposition (2.3.3), $\lambda''_g(h, h) = -\frac{1}{2}\int_M \langle \Delta_E h, h\rangle e^{-f} dvol_g$, where f is a constant function. Thus, by using the λ-functional, we can make the same notion of stability of Ricci-flat metrics (special Einstein metrics) as Definition 2.3.4. However, the second variation formulae of the (normalized) total scalar curvature and that of the λ-functional at a Ricci-flat metric restricted to conformal variation direction have opposite sign. As we have seen, an Einstein metric, in particular, a Ricci-flat metric, is always a saddle point of the total scalar curvature functional and of the normalized total scalar curvature functional because of (2.3.7). But we could expect a Ricci-flat metric to be local maximum point of the λ-functional because of (2.4.7).

2.5 Ricci flow and dynamic stability of Einstein metrics

Besides the stability discussed in the previous sections, which is usually referred as linear stability, we can also discuss a dynamic stability for Ricci-flat metrics via Ricci flow because they are stationary points of Ricci flow. By considering certain normalized Ricci flow whose stationary points are general Einstein metrics, similarly we can discuss a notion of dynamic stability of Einstein metrics. Dynamic stability and the relationship between the linear and dynamic stability of Einstein metrics have been studied in [GIK02], [Has12], [HM14], [Krö15], [Ses06], and [Ye93]. In this section, we briefly review previous main results in this topic.

Let M^n be a manifold of the dimension $n \geq 2$. A family $g(t)$ of Riemannian metrics on M^n is called Ricci flow if it is a solution of the initial value problem

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2Ric_{g(t)}, \\ g(0) = g_0. \end{cases} \tag{2.5.1}$$

R. Hamilton introduced the concept of Ricci flow and proved the short time existence in [Ham82]. We refer to [CK04] and [Top06] for introductions to Ricci flow. Ricci flow has been utilized most notably by G. Perelman in his celebrated proof of the Poincaré Conjecture in [Per02], [Per03a], and [Per03b]. More details of Perelman's work can be found in [CZ06], [KL08], and [MT07]. One of Perelman's breakthrough contributions is the introduction of certain Riemannian functionals for studying Ricci flow in [Per02].

Now let us discuss Perelman's $\mathcal{F}$-functional and λ-functional. Let (M, g) be a compact Riemannian manifold. The $\mathcal{F}$-functional is defined by

$$\mathcal{F}(g, f) = \int_M (R_g + |\nabla f|^2) e^{-f} dV_g, \tag{2.5.2}$$

for $f \in C^\infty(M)$. *Ricci flow can be viewed as the gradient flow of the $\mathcal{F}$-functional.* Let $u = e^{-\frac{f}{2}}$, then $\mathcal{F}$-functional becomes

$$\mathcal{F}(g, u) = \int_M (4|\nabla u|^2 + R_g u^2) dV_g. \tag{2.5.3}$$

Perelman's λ-functional is defined by

$$\lambda(g) = \inf\{\mathcal{F}(g, u) \ \mid \ \int_M u^2 dV_g = 1\}. \tag{2.5.4}$$

Then the λ-functional has the following two properties.

1. $\lambda(g)$ is the smallest eigenvalue of the operator $-4\Delta_g + R_g$, by (2.5.4) and (2.5.3).
2. λ-functional is increasing along Ricci flow, because Ricci flow is the gradient flow of the $\mathcal{F}$-functional and the definition (2.5.4).

We have used the property 1 as a definition of the λ-functional in Section 1.3.

Ricci-flat metrics are stationary points of Ricci flow. So it is natural to study the behavior of Ricci flow starting at a metric close to a Ricci-flat metric, when we view Ricci flow as a dynamic system. This is the dynamic stability of Ricci-flat metrics.

Definition 2.5.1 (Dynamic stability and instability of Ricci-flat metrics) *Let (M^n, g) be a Ricci-flat metric. We say g is dynamically stable if for any neighborhood $\mathcal{V}$ of g in $\mathcal{M}$ there exists a smaller neighborhood $\mathcal{U} \subset \mathcal{V}$ such that Ricci flow starting in $\mathcal{U}$ exists and stays for all time $t \geq 0$ in $\mathcal{V}$ and converges to a Ricci-flat metric in $\mathcal{V}$.*

We say g is dynamically unstable if there exists ancient Ricci flow emerging from g, i.e. a nontrivial Ricci flow $g(t)$ defined on $(-\infty, g)$, which converges to a Ricci-flat metric as $t \to -\infty$.

Remark 2.5.2 *Similarly, we can make a notion of dynamic stability for general Einstein metrics, by replacing "Ricci-flat" by "Einstein", and replacing "Ricci flow" by "a normalized Ricci flow" in Definition* 2.5.1.

For Ricci-flat metrics, N. Sesum proved that dynamic stability implies linear stability, and she also showed that a linear stability together with an integrability assumption implies dynamic stability in [Ses06]. Then, R. Haslhofer provided a new proof for Sesum's result by proving a Łojasiewicz-Simon inequality for Perelman's λ-functional, and he also proved that if a Ricci-flat metric is not linearly stable, then it is dynamically unstable in [Has12]. And further, in [HM14], R. Haslhofer and R. Müller got rid of the integrability assumption in Sesum's result. Therefore, for Ricci-flat metrics, two kinds of stability notions, linear stability and dynamic stability, are equivalent.

For general Einstein metrics, R. Ye proved that strictly linear stability implies dynamic stability in [Ye93]. Recently, by generalizing R. Haslhofer and R. Müller's work, K. Kröncke proved a dynamic stability result under weaker assumptions, and he also proved a dynamic instability result in [Krö13] as follows.

Theorem 2.5.3 (K. Kröncke) *Let (M, g) be a compact Einstein manifold, other than a standard sphere, with Einstein constant μ. Suppose that (M, g) is a local maximizer of the Yamabe functional and if the smallest non-zero eigenvalue λ of the Laplacian satisfies $\lambda > 2\mu$. Then (M, g) is dynamically stable.*

Suppose that (M, g) is not local maximizer of the Yamabe functional or the smallest non-zero eigenvalue λ of the Laplacian satisfies $\lambda < 2\mu$. Then (M, g) is dynamically unstable.

Remark 2.5.4 *The condition, a local maximizer of the Yamabe functional, is in between strictly linear stability and linear stability. More precisely, strictly linear stability implies a local maximizer of the Yamabe functional, and which implies linear stability. Conversely, linear stability together with an integrability assumption implies a local maximizer of the Yamabe functional.*

Chapter 3

Stability of Riemannian manifolds with Killing spinors

In this chapter, we study stability of Riemannian manifolds with non-zero Killing spinors, which then are Einstein manifolds. We prove that all complete Riemannian manifolds with imaginary Killing spinors are strictly stable by using a Bochner type formula in [DWW05], [GHP03], and [Wan91]. This stability result was also proved by Klaus Kröncke recently in a different way. A similar argument for real Kiling spinors gives a stability condition for Riemannian manifold with real Killing spinors in term of a twisted Dirac operator. Existence of real Killing spinors is closely related to the Sasaki-Einstein structure. A regular Sasaki-Einstein manifold is essentially the total space of a certain principal circle bundle over a Kähler-Einstein manifold. We prove that if the base space is a product of at least two Kähler-Einstein manifolds then the regular Sasaki-Einstein manifold is unstable.

3.1 Overview and main results

Let (M^n, g) be a Riemannian manifold with a non-zero Killing spinor σ with the Killing constant μ, i.e.

$$\nabla^S_X \sigma = \mu X \cdot \sigma, \tag{3.1.1}$$

for any vector field X on M^n, where ∇^S denotes the canonical connection on the spinor bundle induced by the Levi-Civita connection on the tangent bundle TM, and " $\cdot$ " denotes the Clifford multiplication. Then the Riemannian manifold (M^n, g) is an Einstein manifold with scalar curvature $R = 4n(n-1)\mu^2$ (see, e.g. [Fri00]). Because the scalar curvature is real, μ can only be real or purely imaginary. A non-trivial Killing spinor is said to be imaginary (resp. real) if its Killing constant is imaginary (resp. real). We refer to [Fri00] and [LM89] for spin geometry.

X. Dai, X. Wang, and G. Wei proved that manifolds with non-zero parallel spinors (which can be viewed as Killing spinors with Kiling constant zero) are stable in [DWW05] by deriving a Bochner type formula, and rediscovering a result in [Wan91], also see [GHP03] for the formula. Inspired by their work, we study the stability of Riemannian manifolds with non-zero Killing spinors, which then are Einstein manifolds. Th. Friedrich initiated the mathematical investigation of Killing spinors in [Fri80]. And then complete Riemannian manifolds with Killing spinors were classified in [Bär93], [Bau89a], [Bau89b], [FK89], and [FK90]. We also refer to the book [BFGK91]. Riemannian manifolds with real and imaginary Killing spinors have several very distinct properties. For example, if the Killing constant is real, then M is compact. On the other hand, if the Killing constant is imaginary, then M is non-compact (see [CGLS86] and [Bau89b]). So we study these two kinds of manifolds separately.

The main ingredient in this Bochner type argument is a Bochner type formula in

[DWW05] and [Wan91]. By combining the Bochner type formula and Baum's classification results for complete Riemannian manifolds with imaginary Killing spinors in [Bau89b], we obtain the following estimate for the Einstein operator on complete Riemannian manifolds with non-zero imaginary Killing spinors of type I. Recall that an imaginary Killing spinor σ is of type I, if there exists a vector field X such that $X \cdot \xi = \sqrt{-1}\xi$, and otherwise we say that it is of type II. Complete Riemannian manifolds with non-zero imaginary Killing spinors of type II are hyperbolic spaces (see, [Bau89b]), and therefore they are strictly stable (See, [Bes87] and [Koi79]).

Theorem 3.1.1 *Let (M^n, g) be a complete Riemannian manifold with a non-zero imaginary Killing spinor of type I with Killing constant μ. We have*

$$\int_M \langle \nabla^*\nabla h - 2\mathring{R}h, h\rangle dvol_g \geq -[2(n-2)-4]\mu^2 \int_M \langle h, h\rangle dvol_g. \tag{3.1.2}$$

for all compactly supported traceless transverse symmetric 2-tensor h.

Consequently, we show that complete Riemannian manifolds with non-zero imaginary Killing spinors are strictly stable.

In the case of real Killing spinors, we have the following estimate.

Theorem 3.1.2 *Let (M^n, g) be a Riemannian manifold with non-zero real Killing spinor with Killing constant μ, then for all traceless transverse $h \in C^\infty(S^2(M))$,*

$$\begin{aligned} \int_M \langle \Delta_E h, h\rangle dvol_g = & \int_M \langle D\Phi(h), D\Phi(h)\rangle dvol_g - 2\mu \int_M \langle D\Phi(h), \Phi(h)\rangle dvol_g \\ & - n(n-2)\mu^2 \int_M \langle h, h\rangle dvol_g. \end{aligned} \tag{3.1.3}$$

Unlike the case of imaginary Killing spinors, from this estimate we cannot conclude a general stability result. Actually, we have both stable and unstable examples: standard spheres are stable Riemannian manifolds with real Killing spinors; the Jensen's sphere is an unstable Riemannian manifold with a real Killing spinor. We obtain a stability condition for manifolds with non-zero real Killing spinors from Theorem 3.1.2.

Corollary 3.1.3 *The Riemannian manifold with non-trivial real Killing spinor with Killing constant μ is stable if the twisted Dirac operator D satisfies*

$$(D-\mu)^2 \geq (n-1)^2\mu^2,$$

on $\{\Phi(h) : h \in C^\infty(S^2(M)), trh = 0, \delta h = 0\}$.

Most Riemannian manifolds with non-zero real Killing spinors are either standard spheres in even dimensions, or Sasaki-Einstein in odd dimensions. And all regular Sasaki-Einstein manifolds are the total spaces of principal S^1-bundles over Kähler-Einstein manifolds. Let $\pi : (M^{2p+1}, g) \to (B^{2p}, G, J)$ be a principal S^1-bundle with a connection η, where (M^{2p+1}, g) is regularly Sasaki-Einstein, (B^{2p}, G, J) is Kähler-Einstein, and π is a Riemannian submersion. Here G is the Kähler metric on B^{2p}, and J is the almost complex structure on B^{2p}. In the following, $\tilde{h} = \pi^* h$, for all $h \in C^\infty(S^2(B))$.

Proposition 3.1.4

$$\langle(\Delta_E^g \tilde{h}, \tilde{h}\rangle = (\langle \Delta_E^G h, h\rangle + 4\langle h, h\rangle + 4\langle h\circ J, h\rangle)\circ\pi, \tag{3.1.4}$$

and therefore,

$$\int_M \langle \Delta_E^g \tilde{h}, \tilde{h}\rangle dvol_g = \int_B (\langle \Delta_E^G h, h\rangle + 4\langle h, h\rangle + 4\langle h\circ J, h\rangle) dvol_G. \tag{3.1.5}$$

where $h\circ J \in C^\infty(S^2(B))$ *with* $h\circ J(X,Y) = h(JX, JY)$.

Corollary 3.1.5 *If there exists a traceless transverse 2-tensor* $h \in C^\infty(S^2(B))$ *such that* $\int_B (\langle (\nabla^G)^* \nabla^G h - 2\mathring{R}^g h, h\rangle dvol_G < -8 \int_B \langle h, h\rangle dvol_G$, *then* (M^{2p+1}, g) *is unstable.*

Corollary 3.1.6 *If the base space* (B^{2p}, g) *is a product of two Kähler-Einstein manifolds, then* (M^{2p+1}, g) *is unstable.*

3.2 Riemannian manifolds with imaginary Killing spinors

In this section, we review classification results of Riemannian manifolds with Killing spinors and some properties of Killing spinors. We will mainly focus on complete Riemannian manifolds with imaginary Killing spinors studied in [Bau89a] and [Bau89b], because Baum's results about the structure of complete Riemannian manifolds with imaginary Killing spinors play a very important role in our estimate of the Einstein operator on these manifolds.

Let us first recall two differences between manifolds with real Killing spinors and manifolds with imaginary Killing spinors pointed out in [Bau89b] (also see [CGLS86]):

1. Let (M^n, g) be a complete Riamnnian manifold with a Killing spinor σ. If σ is real with non-zero real Killing constant, then M^n is compact. If σ is imaginary, then M^n is non-compact.

2. Let $f(x) := \langle \sigma(x), \sigma(x)\rangle_{\mathcal{S}_x}$ denote the length function of a non-zero Killing spnior σ. If σ is real, then f is constant. If σ is imaginary, then f is a non-constant and nowhere vanishing function.

As pointed out by Klaus Kröncke in [Krö15], the fact that the length function f of an imaginary Killing spinor is not constant will cause some issues when we use the Bochner type argument in [DWW05] to estimate the Einstein operator on a Riemannian manifold with imaginary Killing spinors. In order to deal with the issues, we investigate the length function f more carefully, and we recall some properties of the length function f proved in [Bau89b]. Let (M^n, g) be a complete Riemannian manifold with an imaginary Kiling spinor σ with Killing constant $\mu = \sqrt{-1}\nu$.

Lemma 3.2.1 ([Bau89b])

1. *The function*
$$q_\sigma(x) := f^2(x) - \frac{1}{4\nu^2}|\nabla f(x)|^2 \tag{3.2.1}$$
is constant on M^n.

2. *Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame of TM around x. The we have*
$$Re\langle e_i \cdot \sigma(x), e_j \cdot \sigma(x)\rangle = \delta_{ij} f(x), \tag{3.2.2}$$
where Re means taking the real part.

3. *Let dist denote the distance in $\mathcal{S}_x$ with respect to the real scalar product $Re\langle\ ,\ \rangle_{\mathcal{S}_x}$. Then*
$$q_\sigma = f(x) \cdot dist^2(V_\sigma, \sqrt{-1}\sigma(x)) \geq 0, \tag{3.2.3}$$
where $V_\sigma(x) = \{X \cdot \sigma(x)|\ \ X \in T_xM\} \subset \mathcal{S}_x$.

As in [Bau89b], a Killing spinor σ is of type I if $q_\sigma = 0$ and a Killing spinor is of type II if $q_\sigma > 0$. By (3.2.3), this is essentially the same as the simple characteristic of Killing spinors of type I and II mentioned in Introduction. H. Baum has the following classification results for complete Riemannian manifold with imaginary Killing spinors.

Theorem 3.2.2 ([Bau89b]) *Let (M^n, g) be a complete connected Riemannian manifold with an imaginary Killing spinor of type II with the Killing constant $\sqrt{-1}\nu$. Then (M^n, g) is isometric to the hyperbolic space $H^n_{-4\nu^2}$ with the constant sectional curvature $-4\nu^2$.*

Theorem 3.2.3 ([Bau89a][Bau89b]) *Let (M^n, g) be a complete connected Riemannian manifold with an imaginary Killing spinor of type I with the Killing constant $\sqrt{-1}\nu$. Then (M^n, g) is isometric to a warped product $(F^{n-1} \times \mathbb{R}, e^{-4\nu t}h + dt^2)$, where (F^{n-1}, h) is a complete Riemannian manifold with a non-zero parallel spinor.*

Conversely, let (F^{n-1}, h) be a complete Riemannian manifold with non-zero parallel spinors, then the warped product $(M^n, g) := (F^{n-1} \times \mathbb{R}, e^{-4\nu t}h + dt^2)$ is a complete Riemannian manifold with imaginary Killing spinors of type I.

Recall how to construct a Killing spinor of type I on $(F^{n-1} \times \mathbb{R}, e^{-4\nu t}h + dt^2)$ from a parallel spinor on (F^{n-1}, h). When $n-1$ is even, the spinor bundle over the warped product $(F^{n-1} \times \mathbb{R}, e^{-4\nu t}h + dt^2)$ is isometric to the tensor product of the spinor bundle over (F^{n-1}, h) and the spinor bundle over $(\mathbb{R}, dt^2)$. When $n-1$ is odd, the spinor bundle over $(F^{n-1} \times \mathbb{R}, e^{-4\nu t}h + dt^2)$ is isometric to the direct sum of two copies of the tensor product of the spinor bundle over (F^{n-1}, h) and the spinor bundle over $(\mathbb{R}, dt^2)$. The spinor bundle over $(\mathbb{R}, dt^2)$ is a trivial 1-dimensional complex vector bundle. We will use the same notation to denote two isometric spinors.

- If $n-1$ is even, and parallel spinor on F^{n-1} is $\psi = (\psi^+, \psi^-)$, where the decomposition is the $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces decomposition for the action of the complex volume $\omega_{\mathbb{C}} = (\sqrt{-1})^{\frac{n}{2}} e_1 \cdots e_{n-1}$ on the spinor bundle on F^{n-1}, then we can take

$$\sigma = e^{-\nu t}\psi^+ \otimes 1 \tag{3.2.4}$$

as an imaginary Killing spinor of type I on the warped product manifold.

- If $n-1$ is odd, and parallel spinor on F^{n-1} is ψ, then we can take

$$\sigma = e^{-\nu t}(\psi \otimes 1, \hat{\psi} \otimes 1) \tag{3.2.5}$$

as a Killing spinor of type I on the warped product manifold, where " ˆ " denotes the isomorphism between two spin representations coming from projections to the first and the second components of $Cl(n-1) \otimes \mathbb{C} = End(\mathbb{C}^{\frac{n-2}{2}}) \oplus End(\mathbb{C}^{\frac{n-2}{2}})$.

Because the length of a parallel spinor is constant, we can always normalize the parallel spinor ψ on F so that for the Killing spinor σ in (3.2.4) and (3.2.5) we have

$$\langle \sigma, \sigma \rangle = e^{-2\nu t}.$$

Thus for the Killing spinor obtained above we have the length function

$$f = e^{-2\nu t} \tag{3.2.6}$$

only depending on the t variable on $\mathbb{R}$ factor. We can also see that $q_\sigma = 0$. Moreover, we can see that the action of the vector field $\frac{\partial}{\partial t}$ on the Killing spinor σ is given by

$$(\frac{\partial}{\partial t}) \cdot \sigma = \sqrt{-1}\sigma. \tag{3.2.7}$$

3.3 Bochner type formula

In this section, we recall a Bochner type formula coming from Killing spinors in [DWW05] and [Wan91], and present a proof.

Let (M^n, g) be a Riemannian spin manifold with spinor bundle $\mathcal{S} \to M$. The curvature of a connection ∇ on a vector bundle $E \to M$ is defined as

$$R_{XY}\sigma = -\nabla_X \nabla_Y \sigma + \nabla_Y \nabla_X \sigma + \nabla_{[X,Y]}\sigma, \tag{3.3.1}$$

for a section $\sigma \in C^\infty(E)$ and vector field $X, Y \in C^\infty(TM)$. Let R^S be the curvature of ∇^S on the spinor bundle. Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame of the tangent bundle and $\{e^1, \cdots, e^n\}$ be its dual frame. We have

$$R^S_{XY}\sigma = \frac{1}{4}R(X, Y, e_i, e_j)e_i e_j \cdot \sigma, \tag{3.3.2}$$

for any spinor σ. If there exists a Killing spinor σ with Killing constant μ, the Ricci curvature tensor satisfies

$$R_{ij} = 4\mu^2(n-1)g_{ij}, \tag{3.3.3}$$

(see, e.g. [Fri00]). As in [DWW05], we define a linear map $\Phi : S^2(M) \to \mathcal{S} \otimes T^*M$ as

$$\Phi(h) = h_{ij} e_i \cdot \sigma \otimes e^j. \tag{3.3.4}$$

Proposition 3.3.1 ([Wan91]) *Let D be the twisted Dirac operator acting on the twisted spinor bundle $\mathcal{S} \otimes T^*M$, and h be a symmetric 2-tensor on M. Then*

$$\begin{aligned} D^*D\Phi(h) =& \Phi(\Delta_E h) + n(n-2)\mu^2\Phi(h) + 2\mu D\Phi(h) \\ &+ 4\mu^2(trh)e_j \cdot \sigma \otimes e^j - 4\mu(\delta h)_j \cdot \sigma \otimes e^j. \end{aligned} \tag{3.3.5}$$

Proof: Fix a point $x \in M$, choose a local orthonormal frame $\{e_1, \cdots, e_n\}$ around

x such that $\nabla e_i = 0$ at x. Then, at x,

$$\begin{aligned}
D^*D\Phi(h) &= \nabla_{e_k}\nabla_{e_l}h_{ij}e_ke_le_i\cdot\sigma\otimes e^j + \nabla_{e_l}h_{ij}e_ke_le_i\cdot\nabla^S_{e_k}\sigma\otimes e^j \\
&\quad + \nabla_{e_k}h_{ij}e_ke_le_i\cdot\nabla^S_{e_l}\sigma\otimes e^j + h_{ij}e_ke_le_i\cdot\nabla^S_{e_k}\nabla^S_{e_l}\sigma\otimes e^j \\
&= \nabla_{e_k}\nabla_{e_l}h_{ij}e_ke_le_i\cdot\sigma\otimes e^j + \nabla_{e_l}h_{ij}(e_ke_l+e_le_k)e_i\cdot\nabla^S_{e_k}\sigma\otimes e^j \\
&\quad + h_{ij}e_ke_le_i\cdot\nabla^S_{e_k}\nabla^S_{e_l}\sigma\otimes e^j \\
&= \nabla_{e_k}\nabla_{e_l}h_{ij}e_ke_le_i\cdot\sigma\otimes e^j - 2\mu\nabla_{e_k}h_{ij}e_ie_k\cdot\sigma\otimes e^j \\
&\quad + \mu^2 h_{ij}e_ke_le_ie_le_k\cdot\sigma\otimes e^j \\
&= -\nabla_{e_k}\nabla_{e_k}h_{ij}e_i\cdot\sigma\otimes e^j - \frac{1}{2}R_{e_ke_l}h_{ij}e_ke_le_i\cdot\sigma\otimes e^j \\
&\quad - 2\mu\nabla_{e_k}h_{ij}e_ie_k\cdot\sigma\otimes e^j + (n-2)^2\mu^2h_{ij}e_i\cdot\sigma\otimes e^j \\
&= \Phi(\nabla^*\nabla h) + \frac{1}{2}R_{kljp}h_{ip}e_ke_le_i\cdot\sigma\otimes e^j + \frac{1}{2}R_{klip}h_{pj}e_ke_le_i\cdot\sigma\otimes e^j \\
&\quad - 2\mu\nabla_{e_k}h_{ij}e_ie_k\cdot\sigma\otimes e^j + (n-2)^2\mu^2\Phi(h).
\end{aligned} \tag{3.3.6}$$

In the third equality, we use the Clifford relation $e_ke_l+e_le_k = -2\delta_{kl}$, and $\nabla^S_X\sigma = \mu X\cdot\sigma$ for any vector field X. In the fourth equality, we use twice the fact

$$e_le_ie_l\cdot\phi = (n-2)e_i\cdot\phi$$

for any spinor ϕ, which can easily be obtained by using the Clifford relation.

By using the Clifford relation, (3.3.2), and (3.3.3), we have

$$\frac{1}{2}R_{kljp}h_{ip}e_ke_le_i\cdot\sigma\otimes e^j = \Phi(-2\mathring{R}h) - 4\mu^2\Phi(h) + 4\mu^2 trhe_j\cdot\sigma\otimes e^j, \tag{3.3.7}$$

$$\frac{1}{2}R_{klip}h_{pj}e_ke_le_i\cdot\sigma\otimes e^j = 4(n-1)\mu^2\Phi(h), \tag{3.3.8}$$

$$-2\mu\nabla_{e_k}h_{ij}e_ie_k\cdot\sigma\otimes e^j = -4\mu(\delta h)_j\sigma\otimes e^j + 2\mu e_k\cdot\Phi(\nabla_{e_k}h), \tag{3.3.9}$$

$$e_k\cdot\Phi(\nabla_{e_k}h) = D\Phi(h) - (n-2)\mu\Phi(h). \tag{3.3.10}$$

By plugging (3.3.7), (3.3.8), (3.3.9) and (3.3.10) into (3.3.6), we get (3.3.5). ■

3.4 Stability of Riemannian manifolds with imaginary Killing spinors

In this section, we obtain an estimate for the Einstein operator on complete Riemannian manifolds with imaginary Killing spinors of type I. As a consequence of the estimate and Baum's classification results, we prove that all complete Riemannian manifolds with imaginary Killing spinors are strictly stable.

Let (M^n, g) be a Riemannian manifold with an imaginary Killing spinor σ of type I with the Killing constant $\mu = \sqrt{-1}\nu$. We have the following property for the map Φ defined in (3.3.4).

Lemma 3.4.1 *For all $h, \tilde{h} \in C^\infty(S^2(M))$, we have*

$$Re\langle\Phi(h), \Phi(\tilde{h})\rangle = \langle h, \tilde{h}\rangle f, \tag{3.4.1}$$

where $f = \langle\sigma,\sigma\rangle$ is the length function.

Proof:

$$\begin{aligned}
Re\langle\Phi(h),\Phi(\tilde{h})\rangle &= Re(h_{ij}\tilde{h}_{kl}\langle e_i\cdot\sigma\otimes e^j, e_k\cdot\sigma\otimes e^l\rangle)\\
&= Re(h_{ij}\tilde{h}_{kj}\langle e_i\cdot\sigma, e_k\cdot\sigma\rangle)\\
&= h_{ij}\tilde{h}_{kj}Re\langle e_i\cdot\sigma, e_k\cdot\sigma\rangle)\\
&= h_{ij}\tilde{h}_{ij}f.
\end{aligned}$$

In the last step, we use (3.2.2). ■

Lemma 3.4.2 *If σ is a Killing spinor of type I as in* (3.2.4) *or* (3.2.5), *then we have*

$$\|(\frac{\partial}{\partial t})\cdot\Phi(h)\| = \|\Phi(h)\|. \tag{3.4.2}$$

Proof: Choose a local orthonormal frame of TM as $\{e_1 = \frac{\partial}{\partial r}, e_2, \cdots, e_n\}$. Then by (3.2.7), *we have*

$$\begin{aligned}
(\frac{\partial}{\partial t})\cdot\Phi(h) &= (\frac{\partial}{\partial t})\cdot(h_{1j}(\frac{\partial}{\partial t})\cdot\sigma\otimes e^j + \sum_{i\geq 2}h_{ij}e_i\cdot\sigma\otimes e^j)\\
&= \sqrt{-1}h_{1j}(\frac{\partial}{\partial t})\cdot\sigma\otimes e^j - \sqrt{-1}\sum_{i\geq 2}h_{ij}e_i\cdot\sigma\otimes e^j.
\end{aligned}$$

Then by (3.2.2)*, we have*

$$\begin{aligned}\|(\frac{\partial}{\partial t})\cdot\Phi(h)\|^2 &= Re\langle(\frac{\partial}{\partial t})\cdot\Phi(h),(\frac{\partial}{\partial t})\cdot\Phi(h)\rangle\\ &= Re\langle\sqrt{-1}h_{1j}(\frac{\partial}{\partial t})\cdot\sigma\otimes e^j-\sqrt{-1}\sum_{i\geq 2}h_{ij}e_i\cdot\sigma\otimes e^j,\\ &\quad \sqrt{-1}h_{1l}(\frac{\partial}{\partial t})\cdot\sigma\otimes e^l-\sqrt{-1}\sum_{k\geq 2}h_{kl}e_k\cdot\sigma\otimes e^l\rangle\\ &= h_{ij}h_{ij}f\\ &= \|\Phi(h)\|^2.\end{aligned}$$

■

Theorem 3.4.3 *Let (M^n,g) be a complete Riemannian manifold with an imaginary Killing spinor σ of type I with Killing constant $\mu=\sqrt{-1}\nu$. Then we have*

$$\int_M\langle\Delta_E h,h\rangle dvol_g\geq[n(n-2)-4]\nu^2\int_M\langle h,h\rangle dvol_g, \qquad (3.4.3)$$

for all compactly supported traceless transverse $h\in C_0^\infty(S^2(M))$.

Proof: By Proposition 3.3.1, for all traceless transverse symmetric 2-tensor h,

$$\Phi(\Delta_E h)=D^*D\Phi(h)-n(n-2)\mu^2\Phi(h)-2\mu D\Phi(h). \qquad (3.4.4)$$

By Theorem 3.2.3, we can take a Killing spinor as in (3.2.4) or (3.2.5) depending on dimension n of the manifold. Then we know the length function is given by

$$f=e^{-2\nu t}. \qquad (3.4.5)$$

By (3.4.4), and Lemma 3.4.1, for any compactly supported traceless transverse $h\in$

$C_0^\infty(S^2(M))$, we have

$$\begin{aligned}\int_M \langle \Delta_E h, h\rangle dvol_g &= \int_M \frac{Re\langle \Phi(\Delta_E h), \Phi(h)\rangle}{f} dvol_g \\ &= \int_M \frac{Re\langle D^* D\Phi(h), \Phi(h)\rangle}{f} dvol_g \\ &\quad - n(n-2)\mu^2 \int_M \frac{\langle \Phi(h), \Phi(h)\rangle}{f} dvol_g \\ &\quad + \int_M \frac{Re\langle -2\mu D\Phi(h), \Phi(h)\rangle}{f} dvol_g\end{aligned} \tag{3.4.6}$$

By using (3.4.5) and doing an integration by parts, we obtain

$$\begin{aligned}\int_M \frac{Re\langle D^* D\Phi(h), \Phi(h)\rangle}{f} dvol_g &= \int_M \frac{\|D\Phi(h)\|^2}{f} dvol_g \\ &\quad + \int_M \frac{Re\langle D\Phi(h), 2\nu(\frac{\partial}{\partial t})\cdot\Phi(h)\rangle}{f} dvol_g.\end{aligned}$$

By Cauchy inequality, we have

$$\begin{aligned}Re\langle D\Phi(h), 2\nu(\frac{\partial}{\partial t})\cdot\Phi(h)\rangle &\geq -\|D\Phi(h)\|\cdot\|2\nu(\frac{\partial}{\partial t})\cdot\Phi(h)\| \\ &\geq -\frac{\|D\Phi(h)\|^2 + 4\nu^2\|(\frac{\partial}{\partial t})\cdot\Phi(h)\|^2}{2} \\ &= -\frac{\|D\Phi(h)\|^2 + 4\nu^2\|\Phi(h)\|^2}{2}\end{aligned}$$

Thus we have

$$\begin{aligned}\int_M \frac{Re\langle D^* D\Phi(h), \Phi(h)\rangle}{f} dvol_g &\geq \frac{1}{2}\int_M \frac{\|D\Phi(h)\|^2}{f} dvol_g \\ &\quad - 2\nu^2 \int_M \langle h, h\rangle dvol_g\end{aligned} \tag{3.4.7}$$

Similarly, by Cauchy inequality, we have

$$\int_M \frac{Re\langle -2\mu D\Phi(h), \Phi(h)\rangle}{f} dvol_g \geq -\frac{1}{2}\int_M \frac{\|D\Phi(h)\|^2}{f} dvol_g - 2\nu^2 \int_M \langle h, h\rangle dvol_g \tag{3.4.8}$$

Plugging (3.4.7) and (3.4.8) into (3.4.6), we complete the proof. ■

Then Theorem 3.4.3 enables us to prove the following stability result recently obtained in [Krö15] in a differential way.

Corollary 3.4.4 *Complete Riemannian manifolds with non-zero imaginary Killing spinors are strictly stable.*

Proof: By Theorem 3.2.2, complete Riemannian manifolds with Killing spinors of type II are isometric to hyperbolic spaces, and therefore are strictly stable (see [Koi79], and the proof of Theorem 12.67 in [Bes87]). Let (M^n, g) be a Riemannian manifold with Killing spinors of type I. If $n \geq 4$, then by Theorem 3.4.3, (M^n, g) is strictly stable. If $n \leq 3$, we know it has negative constant sectional curvature, and therefore it is also strictly stable. ■

3.5 Stability of Riemannian manifolds with real Killing spinors

In this section, we give a stability condition for manifolds with real Killing spinors in terms of a twisted Dirac operator. Because the length function of a real Killing spinor is constant, an estimate for the Einstein operator can be obtained easier than the case of imaginary Killing spinors. However, unlike imaginary Killing spinor case, from the estimate we cannot conclude a general stability result for manifolds with real Killing spniors.

Let (M^n, g) be a Riemannian manifold with a real Killing spinor σ with Killing constant μ. Without loss of generality, we can choose σ to be of unit length.

Lemma 3.5.1 *For all $h, \tilde{h} \in C^\infty(S^2(M))$, we have*

$$Re\langle \Phi(h), \Phi(\tilde{h})\rangle = \langle h, \tilde{h}\rangle.$$

Then by Proposition 3.3.1, Lemma 3.5.1, and the fact that $\mu \int_M \langle D\Phi(h), \Phi(h)\rangle dvol_g$ is real, we obtain the following estimate for the Einstein operator $\nabla^*\nabla - 2\mathring{R}$.

Theorem 3.5.2 ([GHP03], [Wan91]) *If the Killing constant μ is real, then, for all traceless transverse $h \in C^\infty(S^2(M))$,*

$$\begin{aligned}\int_M \langle \Delta_E h, h\rangle dvol_g = &\int_M \langle D\Phi(h), D\Phi(h)\rangle dvol_g \\ &- 2\mu \int_M \langle D\Phi(h), \Phi(h)\rangle dvol_g \\ &- n(n-2)\mu^2 \int_M \langle h, h\rangle dvol_g.\end{aligned} \tag{3.5.1}$$

Remark 3.5.3 As mentioned in [Die13] and [Krö15], Theorem 3.5.2 has been used to obtain a lower bound on the eigenvalues of the Einstein operator in [GHP03]. The lower bound is $-(n-1)^2\mu^2$, as we can also see in the following Corollary 3.5.4.

Corollary 3.5.4 *A Riemannian manifold with a non-zero real Killing spinor with the Killing constant μ is stable if the twisted Dirac operator D satisfies*

$$(D-\mu)^2 \geq (n-1)^2\mu^2,$$

on $\{\Phi(h) : h \in C^\infty(S^2(M)), trh = 0, \delta h = 0\}$.

Proof: By Theorem 3.5.2, for traceless transverse symmetric 2-tensor h, we have

$$\begin{aligned}\int_M \langle \Delta_E h, h\rangle dvol_g = &\int_M \langle (D-\mu)^2\Phi(h), \Phi(h)\rangle dvol_g \\ &-(n-1)^2\mu^2\int_M \langle h, h\rangle dvol_g.\end{aligned} \tag{3.5.2}$$

This implies the stability condition. ■

3.6 Some unstable regular Sasaki-Einstein manifolds

In this section, we study instability of regular Sasaki-Einstein manifolds, which are essentially total spaces of principal circle bundles over Kähler-Einstein manifolds with positive first Chern classes. A product of two Einstein manifolds (B^{n_1}, g_1) and (B^{n_2}, g_2) with the same positive Einstein constant is an unstable Einstein manifold. Indeed, $h = \frac{g_1}{n_1} - \frac{g_2}{n_2}$ is an unstable traceless transverse direction. We show that if the base manifold of a regular Sasaki-Einstein manifold is a product of two Kähler-Einstein manifolds then we obtain an unstable direction on the Sasaki-Einstein manifold by lifting this unstable direction on the base Kähler-Einstein manifold to the total space.

Let us first recall some basic facts about Sasaki manifolds. For details, we refer to [Bla10] and [FOW09]. A quick definition of Sasaki manifolds is given as the following, see, e.g. [FOW09].

Definition 3.6.1 (Definition 1 of Sasaki manifolds) (M^n, g) *is said to be a Sasaki manifold if the cone* $(\mathbb{R}_+ \times M, dr^2 + r^2 g)$ *is Kähler, where* $\mathbb{R}_+ = (0, +\infty)$*, and* r *is coordinate on* $\mathbb{R}_+$*.*

Remark 3.6.2 *From Definition* 3.6.1*, we note that a Sasaki manifold has to be of odd dimension.*

There are several equivalent definitions of Sasaki manifolds. The one given in the following looks more complicated and tells us more about structure on Sasaki manifolds themselves.

Definition 3.6.3 (Definition 2 of Sasaki manifolds) *Let* $(M^{2p+1}, g, \phi, \eta, \xi)$ *be a Riemannian manifold of odd dimension* $2p+1$ *with a* $(1,1)$*-tensor* ϕ*, 1-form* η*, and a vector field* ξ*. It is a Sasaki manifold, if*

(1) $\eta \wedge (d\eta)^p \neq 0,$

(2) $\eta(\xi) = 1,$

(3) $\phi^2 = -id + \eta \otimes \xi,$

(4) $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$

(5) $g(X, \phi Y) = d\eta(X, Y),$

(6) the almost complex structure on $M^{2p+1} \times \mathbb{R}$ *defined by*

$$J(X, f\frac{d}{dr}) = (\phi X - f\xi, \eta(X)\frac{d}{dr})$$

is integrable,

for all vector fields X *and* Y *on* M^{2p+1}*. The vector* ξ *is called the Reeb vector field. And this is a regular Sasaki manifold if the Reeb vector field* ξ *is a regular vector field. If, in addition,* g *is an Einstein metric, then this is a Sasaki-Einstein manifold.*

Remark 3.6.4 *As consequences of Definition* 3.6.3*, we have* $\phi\xi = 0$*,* $\eta \circ \phi = 0$*, and* $\nabla_X \xi = -\phi X$*, in particular,* $\nabla_\xi \xi = 0$*. Moreover,* ξ *is a Killing vector field. For details, see, e.g. [Bla10].*

Remark 3.6.5 *Let us recall one more definition of Sasaki manifold.* (M^n, g) *is a Sasaki manifold if there exists a Killing vector filed* ξ *of unit length on* M^n *so that the Riemann curvature satisfies the condition*

$$R_{X\xi}Y = -g(\xi, Y)X + g(X, Y)\xi, \tag{3.6.1}$$

for any pair of vector fields X *and* Y *on* M^n*. Then from* (3.6.1)*, we can easily see that on a Sasaki-Einstein manifold* (M^n, g) *of dimension* n*,* $Ric_g = (n-1)g$*.*

The relationship between real Killing spinors and the Sasaki-Einstein structures has been observed by T. Friedrich and I. Kath in [FK89] and [FK90], and then was further studied by C. Bär in [Bär93]. We briefly summarize their results as the following.

Theorem 3.6.6 (T. Friedrich and I. Kath, and C. Bär) *A complete simply-connected Sasaki-Einstein manifold of dimension* n *with Einstein constant* $n-1$ *carries at least 2 linearly independent real Killing spinors with distinct Killing constants equal* $\frac{1}{2}$ *and* $-\frac{1}{2}$ *for* $n \equiv 3(mod4)$*, and to the same Killing number equals* $\frac{1}{2}$ *for* $n \equiv 1(mod4)$*, respectively.*

Conversely, a complete Riemannian spin manifold with such spinors in these dimensions is Sasaki-Einstein.

Remark 3.6.7 *T. Friedrich also proved that a complete 4-dimensional manifold with a real Killing spinor is isometric to the standard sphere in [Fri81]. And O. Hijazi* *proved the analogous result in dimension 8 in [Hij86]. More generally, C. Bär proved that all complete manifolds of even dimension* n*,* $n \neq 6$*, with a real Killing spinor are isometric to standard spheres in [Bär93]. Thus, complete manifolds of even dimension* n*,* $n \neq 6$*, with a real Killing spinor are strictly stable.*

Remark 3.6.8 *In the first part of Theorem* 3.6.6*, we need at least two linearly independent real Killing spinors in order to have a Sasaki-Einstein structure. Actually, on a complete Riemannian spin manifold of odd dimension, except 7, existence of one Killing spinor automatically implies the existence of the second one that we need in Theorem* 3.6.6*. The 7-dimensional manifolds with a single linearly independent Killing spinor have been studied in [Kat90] and in more details in [FK97]. We also refer to the book [BFGK91]. The Jensen's sphere is a 7-dimensional complete manifold with a single linearly independent Killing spinor, and it is unstable as mentioned in Introduction. We refer to [ADP83], [Bär93], [Bes87], [Jen73], and [Spa11] for this interesting example.*

Now let us recall the construction of a typical regular Sasaki manifold in [Bla10]. Let (B^{2p}, G, J) be a Kähler manifold of real dimension $2p$, with the Kähler form $\Omega = G(\cdot, J\cdot)$, where G is a Riemannian metric and J is an almost complex structure. Then let $\pi : M^{2p+1} \to B^{2p}$ be a principal S^1-bundle with a connection η with the curvature form $d\eta = 2\pi^*\Omega$. Let ξ be a vertical vector field on M^{2p+1}, generated by S^1-action, such that $\eta(\xi) = 1$, and $\widetilde{X}$ denotes the horizontal lift of X with respect to the connection η for a vector field X on B^{2p}. We set

$$\phi X = \widetilde{J\pi_* X}, \tag{3.6.2}$$

and

$$g(X, Y) = G(\pi_* X, \pi_* Y) + \eta(X)\eta(Y), \tag{3.6.3}$$

for vector fields X and Y on M^{2p+1}. Then $(M^{2p+1}, g, \phi, \eta, \xi)$ is a regular Sasaki manifold.

Conversely, any regular Sasaki manifold can be obtained in this way, see, e.g. Theorem 3.9 and Example 6.7.2 in [Bla10]. Moreover, if (M^{2p+1}, g) is Sasaki-Einstein with Einstein constant $2p$, then (B^{2p}, G, J) is Kähler-Einstein with Einstein constant $2p + 2$.

We fix some notations before carrying on calculations. ∇^g and ∇^G denote the Levi-Civita connections on (M^{2p+1}, g) and on (B^{2p}, G), respectively. R^g and Ric^g, and R^G and Ric^G denote Riemann and Ricci curvatures on (M^{2p+1}, g) and on (B^{2p}, G), respectively. In the rest of this section, we use $X, Y, Z, W, \cdots$ to denote vector fields on B^{2p}, and we use $\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{W}, \cdots$ to denote their horizontal lift to M^{2p+1} with respect to the connection η. And we choose and fix a local orthrnormal frame $\{X_1, X_2, \cdots, X_{2p}\}$ of TB. Then $\{\widetilde{X_1}, \widetilde{X_2}, \cdots, \widetilde{X_{2p}}, \xi\}$ is a local orthonormal frame of TM. We use ∇^g_i to denote $\nabla^g_{\widetilde{X_i}}$, and ∇^G_i to denote $\nabla^G_{X_i}$.

Lemma 3.6.9 *On a regular Sasaki manifold $(M^{2p+1}, g, \phi, \eta, \xi)$ constructed above. We have*

$$\begin{aligned}
[\xi, \widetilde{X}] &= \mathcal{L}_\xi \widetilde{X} = 0, \\
\nabla^g_{\widetilde{X}} \widetilde{Y} &= \widetilde{\nabla^G_X Y} - \Omega(X, Y)\xi, \\
\nabla^g_\xi \widetilde{X} &= \nabla^g_{\widetilde{X}} \xi = -\phi \widetilde{X}, \\
\nabla^g_\xi \xi &= 0.
\end{aligned}$$

Proof: The first equation follows from the fact that the horizontal distribution is S^1 invariant and ξ is generated by the S^1-action. Then the rest properties for covariant derivatives follow from properties in Remark 3.6.4, the first equation, and the fundamental equations of a submersion in [ONe66] (also see [Bes87] for the equations). ■

Let $h \in C^\infty(S^2(B))$, and then $\tilde{h} = \pi^* h \in C^\infty(S^2(M))$. Then by Lemma 3.6.9 and straightforward calculations, we obtain a relationship between $(\nabla^g)^*\nabla^g\widetilde{h}$ and $(\nabla^G)^*\nabla^G h$.

Lemma 3.6.10

$$\begin{aligned} (\nabla^g_k\nabla^g_k\tilde{h})_{ij} &= (\pi^*(\nabla^G_k\nabla^G_k h))_{ij} - 2\tilde{h}_{ij}, \\ (\nabla^g_{\nabla^g_k\widetilde{X_k}}\tilde{h})_{ij} &= (\pi^*(\nabla^G_{\nabla^G_k X_k} h))_{ij}, \\ (\nabla^g_\xi\nabla^g_\xi\tilde{h})_{ij} &= -2\tilde{h}_{ij} + 2\tilde{h}(\phi\widetilde{X_i}, \phi\widetilde{X_j}), \end{aligned}$$

and therefore,

$$((\nabla^g)^*\nabla^g\widetilde{h})_{ij} = (\pi^*((\nabla^G)^*\nabla^G h))_{ij} + 4\tilde{h}_{ij} - 2\tilde{h}(\phi\widetilde{X_i}, \phi\widetilde{X_j}), \tag{3.6.4}$$

for all $1 \leq i, j \leq 2p$, where we take summation for the repeated index k through 1 to $2p$.

Because $\pi : M^{2p+1} \to B^{2p}$ is a Riemannian submersion, by the fundamental equation in [ONe66] and also in Theorem 9.26 in [Bes87], we have the following relationship between curvature tensors on M^{2p+1} and ones on B^{2p}.

Lemma 3.6.11

$$\begin{aligned} R^g(\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{W}) = (\pi^* R^G)(X, Y, Z, W) & \\ - 2(\pi^*\Omega)(\widetilde{X}, \widetilde{Y})(\pi^*\Omega)(\widetilde{Z}, \widetilde{W}) & \\ - (\pi^*\Omega)(\widetilde{X}, \widetilde{Z})(\pi^*\Omega)(\widetilde{Y}, \widetilde{W}) & \\ + (\pi^*\Omega)(\widetilde{X}, \widetilde{W})(\pi^*\Omega)(\widetilde{Y}, \widetilde{Z}), & \end{aligned} \tag{3.6.5}$$

$$R^g(\widetilde{X}, \xi, \widetilde{Y}, \xi) = g(\widetilde{X}, \widetilde{Y}), \tag{3.6.6}$$

and therefore,

$$Ric^g(\widetilde{X}, \widetilde{Y}) = (\pi^* Ric^G)(\widetilde{X}, \widetilde{Y}) - 2g(\widetilde{X}, \widetilde{Y}). \tag{3.6.7}$$

From (3.6.7), we can see that if g is Einstein with Einstein constant k then G is also Einstein with Einstein constant $k+2$. Moreover, the above relations between curvatures directly imply a relation between $\mathring{R}^g \tilde{h}$ and $\mathring{R}^G h$.

Lemma 3.6.12

$$\begin{aligned}(\mathring{R}^g \tilde{h})_{ij} = (\pi^*(\mathring{R}^G h))_{ij} - 3\tilde{h}(\phi\widetilde{X_i}, \phi\widetilde{X_j}) \\ - (\pi^*\Omega)(\widetilde{X_i}, \widetilde{X_j}) \sum_{k=1}^{2p} \tilde{h}(\widetilde{X_k}, \phi\widetilde{X_k}),\end{aligned} \tag{3.6.8}$$

for all $1 \leq i, j \leq 2p$.

Proposition 3.6.13

$$\begin{aligned}\langle(\nabla^g)^*\nabla^g \tilde{h} - 2\mathring{R}^g \tilde{h}, \tilde{h}\rangle = (\langle(\nabla^G)^*\nabla^G h - 2\mathring{R}^g h, h\rangle \\ + 4\langle h, h\rangle + 4\langle h \circ J, h\rangle) \circ \pi.\end{aligned} \tag{3.6.9}$$

Therefore,

$$\begin{aligned}\int_M \langle(\nabla^g)^*\nabla^g \tilde{h} - 2\mathring{R}^g \tilde{h}, \tilde{h}\rangle dvol_g = \int_B (\langle(\nabla^G)^*\nabla^G h - 2\mathring{R}^G h, h\rangle \\ + 4\langle h, h\rangle + 4\langle h \circ J, h\rangle) dvol_G.\end{aligned} \tag{3.6.10}$$

Proof: By Lemma 3.6.10 and Lemma 3.6.12, we directly have

$$\begin{aligned}&\langle(\nabla^g)^*\nabla^g\tilde{h} - 2\mathring{R}^g\tilde{h}, \tilde{h}\rangle \\ &= (\langle(\nabla^G)^*\nabla^G h - 2\mathring{R}^g h, h\rangle + 4\langle h, h\rangle \\ &\quad + 4\langle h(J\cdot, J\cdot), h\rangle + 2(tr_G(h(J\cdot,\cdot)))^2)\circ\pi.\end{aligned} \tag{3.6.11}$$

Then if suffices to show that $tr_G(h(J\cdot,\cdot) = 0$. Because (B^{2p}, G, J) is Kähler, in particular complex, we can choose a local orthonormal frame of TB in the form of

$$\{X_1, \cdots, X_p, JX_1, \cdots, JX_p\}.$$

Then

$$tr_G(h(J\cdot,\cdot)) = \sum_{i=1}^{p} h(JX_i, X_i) + \sum_{j=1}^{p} h(J^2X_j, JX_j) = 0,$$

by using $J^2 = -id$ and the symmetry of h. ■

We choose a local orthonormal frame

$$\{X_1, \cdots, X_p, JX_1, \cdots, JX_p\}$$

of TB as in the proof of Proposition 3.6.13, and set

$$\begin{aligned} h(X_i, X_j) &= h_{ij}, \\ h(X_i, JX_j) &= h_{i\bar{j}}, \\ h(JX_i, X_j) &= h_{\bar{i}j}, \\ h(JX_i, JX_j) &= h_{\bar{i}\bar{j}}, \end{aligned}$$

for all $1 \leq i, j \leq p$.

Then we have

$$\langle h,h\rangle=\sum_{i,j=1}^{p}(h_{ij}h_{ij}+h_{i\bar{j}}h_{i\bar{j}}+h_{\bar{i}j}h_{\bar{i}j}+h_{\bar{i}\bar{j}}h_{\bar{i}\bar{j}}), \tag{3.6.12}$$

$$\langle h\circ J,h\rangle=\sum_{i,j=1}^{p}2(h_{ij}h_{\bar{i}\bar{j}}-h_{\bar{i}j}h_{i\bar{j}})\leq\langle h,h\rangle. \tag{3.6.13}$$

For any $h\in C^\infty(S^2(B))$, by doing directly calculations, we have that $tr_g\tilde{h}=tr_Gh$, $(\delta_g\tilde{h})(\widetilde{X})=(\delta_Gh)(X)$, and $(\delta_g\tilde{h})(\xi)=-tr_G(h(J\cdot,\cdot))=0$. Consequently, if h is traceless and transverse, then so is $\tilde{h}$.

Corollary 3.6.14 *If there exists a traceless transverse 2-tensor $h\in C^\infty(S^2(B))$ such that $\int_B\langle(\nabla^G)^*\nabla^Gh-2\mathring{R}^gh,h\rangle dvol_G\leq-8\int_B\langle h,h\rangle dvol_G$, then (M^{2p+1},g) is unstable.*

Proof: Proposition 3.6.13 and the inequality (3.6.13) directly imply the conclusion. ∎

Corollary 3.6.15 *If the base space (B^{2p},G) of a regular Sasaki-Einstein manifold (M^{2p+1},g) is the Riemannian product of Kähler-Einstein manifolds $(B_1^{2p_1},G_1)$ and $(B_2^{2p_2},G_2)$, where $p_1+p_2=p$, then (M^{2p+1},g) is unstable.*

Proof: Set $h=\frac{G_1}{2p_1}-\frac{G_2}{2p_2}$. h is a traceless transverse symmetric 2-tensor and is an unstable direction of $(B^{2p},G)=(B_1^{2p_1},G_1)\times(B_2^{2p_2},G_2)$. Let us recall

$$Ric_g=(2p_1+2p_2)g, \tag{3.6.14}$$

$$Ric_G=(2p_1+2p_2+2)G. \tag{3.6.15}$$

Then we have

$$\begin{aligned}\langle (\nabla^G)^*\nabla^G h - 2\mathring{R}^G h, h\rangle &= -2\frac{R_{G_1}}{4p_1^2} - 2\frac{R_{G_2}}{4p_2^2} \\ &= -2(p_1+p_2+1)(\frac{1}{p_1}+\frac{1}{p_2}).\end{aligned} \tag{3.6.16}$$

Moreover,

$$\langle h, h\rangle = \langle h\circ J, h\rangle = \frac{1}{2p_1} + \frac{1}{2p_2}. \tag{3.6.17}$$

Thus, by Proposition 3.6.13, we have

$$\langle (\nabla^g)^*\nabla^g \tilde{h} - 2\mathring{R}^g \tilde{h}, \tilde{h}\rangle = -2(p_1+p_2-1)(\frac{1}{p_1}+\frac{1}{p_2}) < 0, \tag{3.6.18}$$

if both $p_1 \geq 1$ and $p_2 \geq 1$. ∎

Chapter 4

Instability of Einstein metrics on principal torus bundles

In this chapter, we generalize the instability result for regular Sasaki-Einstein metrics in Chapter 3 to Einstein metrics on principal torus bundles. In particular, we prove that the most of Einstein metrics on principal torus bundles constructed by M. Wang and W. Ziller in [WZ90] are unstable.

4.1 Overview and main results

In addition to regular Sasaki-Einstein manifolds, S. Kobayashi proved the existence of an Einstein metric on the unit circle bundle of the canonical line bundle over a Kähler-Einstein manifold with positive first Chern class in [Kob63]. Then, more generally, M. Wang and W. Ziller constructed Einstein metrics on principle torus bundles over Riemannian products of Kähler-Einstein manifolds with positive first Chern classes in [WZ90].

Theorem 4.1.1 (M. Wang and W. Ziller) *Let (M_i, g_i), $i = 1, \cdots, m$, be Kähler-Einstein manifolds with the first Chern classes $c_1(M_i) > 0$, and $\pi : P \to B = M_1 \times \cdots \times M_m$ be a principal circle bundle whose Euler class is $e(P) = \sum b_i \pi_i^* \alpha_i$, where $b_i \in \mathbb{Z}$, $\pi_i : B \to M_i$ is the projection onto the ith factor, and $\alpha_i \in H^2(M_i, \mathbb{Z})$ is indivisible. Then if $e(P) \neq 0$, P carries an Einstein metric with positive scalar curvature uniquely characterized up to homothety by the requirements that π is a Riemannian submersion with totally geodesic fibers and that the metric on B is of the form $x_1\pi_1^* g_1 + \cdots + x_m \pi_m^* g_m$ for some choice of $x_1, \cdots, x_m$.*

Theorem 4.1.2 (M. Wang and W. Ziller) *Let (M_i, g_i), $1 \leq i \leq m$, be Kähler manifolds with $c_1(M_i) > 0$, and $\pi : P \to B = M_1 \times \cdot \times M_m$ be the principal T^r bundles, $r \leq m$, with characteristic classes $\beta_i = \sum_{j=1}^m b_{ij} \pi_j^* \alpha_j$, $i = 1, \cdots, r$, where $b_{ij} \in \mathbb{Z}$ and $\alpha_i \in H^2(M_i, \mathbb{Z})$ is indivisible. Then if the matrix $(b_{ij})_{r\times m}$ has maximal rank, there exists an Einstein metric on P with positive scalar curvature such that π is a Riemannian submersion with totally geodesic flat fibers and such that the metric on the base B is a product of the Kähler-Einstein metrics.*

Let $\pi : P \to B$ be a principal G-bundle with a principal connection θ where G is a connected Lie group acting on P on the right. Let $\check{g}$ be a Riemannian metric on B, and let $\hat{g}$ be a left-invariant metric on G. Define a metric g on P as

$$g(X, Y) = \check{g}(\pi_* X, \pi_* Y) + \hat{g}(\theta(X), \theta(Y)), \tag{4.1.1}$$

for any pair of vector fields X and Y on P. Then $\pi : (P, g) \to (B, \check{g})$ is a Riemannian submersion with totally geodesic fibers isometric to $(G, \hat{g})$.

Recall some notations and facts about Riemannian submersions in [Bes87]. $\mathcal{H}$ and $\mathcal{V}$ denote the horizontal and vertical distributions, respectively. A vector field E on P

is *projectable* if there exists a vector field $\check{E}$ on B such that $\pi_*(E) = \check{E}$, and then we say that E and $\check{E}$ are π-related. A vector filed E on P is *basic* if it is projectable and horizontal. In general, X, Y, Z will denote horizontal vector fileds on P and U, V, W vertical vector fields. In [ONe66], B. O'Neil defined the tensor T and the tensor A for Riemannian submersions. The fibers of the principal bundles that we are discussing about are totally geodesic. Thus, the O'Neil's tensor T vanishes on them. Let ω be the curvature form of the principal connection θ. The O'Neil's tensor A is related to ω by

$$\theta(A_X Y) = -\frac{1}{2}\omega(X, Y). \tag{4.1.2}$$

Proposition 4.1.3 *Take G to be S^1 in the above construction of principal bundle, and choose $\hat{g}$ such that the length of S^1 is 2π. Let θ be a principal connection with curvature form $d\theta = \pi^*\eta$, where η is a closed 2-form on B. Then we have*

$$\begin{aligned} \langle (\nabla^g)^*\nabla^g h - 2\mathring{R}^g h, h\rangle = {} & \langle (\nabla^{\check{g}})^*\nabla^{\check{g}}\check{h} - 2\mathring{\check{R}}^{\check{g}}\check{h}, \check{h}\rangle \circ \pi \\ & + (\eta_{ki}\eta_{kl}\check{h}_{lj}\check{h}_{ij} + \eta_{ik}\eta_{jl}\check{h}_{kl}\check{h}_{ij}) \circ \pi, \end{aligned} \tag{4.1.3}$$

for all symmetric 2-tensors $\check{h} \in C^\infty(S^2(B))$, where $h = \pi^*\check{h}$, $h_{ij} = h(X_i, X_j)$, $\eta_{ki} = \eta(\check{X}_k, \check{X}_i)$, and $\check{h}_{ij} = \check{h}(\check{X}_i, \check{X}_j)$, with $\{\check{X}_i\}$ a local orthonormal frame of TB, and X_i the basic vector fields π-related to $\check{X}_i$.

Corollary 4.1.4 *The Einstein metrics constructed in Theorem* 4.1.1 *are unstable if $m \geq 2$.*

More generally, take G to be a torus T^r, which splits into $T^r = S^1 \times \cdots \times S^1$. Let $\{e_1, \cdots, e_r\}$ be a basis of the Lie algebra of T^r coming from this decomposition of T^r. Let $\pi : P \to B$ be a principal T^r-bundle with a principal connection θ with the curvature form $d\theta = \pi^*\eta$ with $\eta = \sum_{\alpha=1}^r \eta_\alpha e_\alpha$. We use U_α to denote the vertical vector fields generated by e_α through T^r-action on P, for each $1 \leq \alpha \leq r$. Let $\hat{g}_{\alpha\beta} = \hat{g}(U_\alpha, U_\beta)$. The we have the following relationship between the Einstein operators on the total space and the one on the base space.

Proposition 4.1.5

$$\begin{aligned}\langle(\nabla^g)^*\nabla^g h - 2\mathring{R}^g h, h\rangle = &\langle(\nabla^{\check{g}})^*\nabla^{\check{g}}\check{h} - 2\mathring{\check{R}}^{\check{g}}\check{h}, \check{h}\rangle \circ \pi \\ &+ [\hat{g}_{\alpha\beta}(\eta_\alpha)_{ki}(\eta_\beta)_{kl}\check{h}_{lj}\check{h}_{ij} \\ &+ \hat{g}_{\alpha\beta}(\eta_\alpha)_{ik}(\eta_\beta)_{jl}\check{h}_{kl}\check{h}_{ij}] \circ \pi,\end{aligned} \tag{4.1.4}$$

for all symmetric 2-tensors $\check{h} \in C^\infty(S^2(B))$*, where* $h = \pi^*\check{h}$*,* $h_{ij} = h(X_i, X_j)$*,* $(\eta_\alpha)_{ki} = (\eta_\alpha)(\check{X}_k, \check{X}_i)$*, and* $\check{h}_{ij} = \check{h}(\check{X}_i, \check{X}_j)$*, with* $\{\check{X}_i\}$ *a local orthonormal frame of* TB*, and* X_i *basic vector fields* π*-related to* $\check{X}_i$*.*

Corollary 4.1.6 *The Einstein metrics constructed in Theorem* 4.1.2 *are unstable if* $m \geq 2$*.*

4.2 Einstein operator on principal circle bundles

In this section, we prove Proposition 4.1.3.

Let $\pi : P^{n+1} \to B^n$ be a principal circle bundle with a principal connection θ with the curvature form $\omega = d\theta = \pi^*\eta$, where η is a closed 2-form on B. As in (4.1.1), let $\check{g}$ be a Riemannian metric on B, and then set $g = \pi^*\check{g} + \theta \otimes \theta$. Then $\pi : (P^{n+1}, g) \to (B^n, \check{g})$ is a Riemannian submersion with totally geodesic fibers. Let U be a vertical vector field generated by S^1-action on P^{n+1} with $\theta(U) = 1$. For any pair of horizontal vector fields X and Y on P, by (4.1.2), we have $A_X Y = -\frac{1}{2}\omega(X, Y)U$. And further, if X and Y are basic, then $A_X Y = -\frac{1}{2}\eta(\check{X}, \check{Y})U$. Throughout the rest of this section, we choose and fix a local orthonormal frame $\{X_1, \cdots, X_n, U\}$ of TP around the point in the problem, where $X_1, \cdots, X_n$ are basic

Lemma 4.2.1 *Let X and Y be basic vector fields. We have*

$$\begin{aligned} [U, X] &= \mathcal{L}_U X = 0, \\ \nabla^g_X Y &= \nabla^{\check{g}}_{\check{X}} \check{Y} - \frac{1}{2}\eta(\check{X}, \check{Y})U, \\ \nabla^g_U X = \nabla^g_X U &= \frac{1}{2}\eta(\check{X}, \check{X}_i)X_i, \\ \nabla^g_U U &= 0. \end{aligned}$$

In the second equality, actually $\nabla^{\check{g}}_{\check{X}} \check{Y}$ is a vector field on the base B. But here, we use it to denote its horizontal lift to P.

Proof: The first equation follows from facts that X is horizontal, U is generated by S^1-action, and the horizontal distribution is S^1-invariant. Then the rest of equations follow from O'Neil's fundamental equations for Riemannian submersions, and facts that tensor T vanishes and $A_X Y = -\frac{1}{2}\eta(\check{X}, \check{Y})U$. ■

Let $\check{h} \in C^\infty(S^2(B^n))$ be a symmetric 2-tensor on B^n, and then $h = \pi^* \check{h}$ be a symmetric 2-tensor on P^{n+1}.

Lemma 4.2.2

$$\begin{aligned} ((\nabla^g)^* \nabla^g h)_{ij} &= (\nabla^{\check{g}})^* \nabla^{\check{g}} \check{h})_{ij} \circ \pi \\ &\quad + \sum_{k,l=1}^{n} (\frac{1}{2}\eta_{ki}\eta_{kl}h_{lj} + \frac{1}{2}\eta_{kj}\eta_{kl}h_{li} - \frac{1}{2}\eta_{ik}\eta_{jl}h_{kl}) \circ \pi, \end{aligned} \tag{4.2.1}$$

where, i and j run through 1 to n.

Proof: By definition, we have

$$
\begin{aligned}
&((\nabla^g)^*\nabla^g h)_{ij} \\
= &-(\nabla^g_k\nabla^g_k h)(X_i,X_j)+(\nabla^g_{\nabla^g_k X_k}h)(X_i,X_j)-(\nabla^g_U\nabla^g_U h)(X_i,X_j),
\end{aligned}
\tag{4.2.2}
$$

where, and throughout this proof, ∇^g_k means $\nabla^g_{X_k}$, $\nabla^{\check{g}}_k$ means $\nabla^{\check{g}}_{\check{X}_k}$, and take sum for repeated indices k,l through 1 to n. Now we compute each of these three terms.

$$
\begin{aligned}
(\nabla^g_k\nabla^g_k h)(X_i,X_j) &= X_kX_k(h(X_i,X_j)) - X_k(h(\nabla^g_k X_i,X_j)) - X_k(h(X_i,\nabla^g_k X_j)) \\
&\quad - X_k(h(\nabla^g_k X_i,X_j)) + h(\nabla^g_k\nabla^g_k X_i,X_j) + h(\nabla^g_k X_i,\nabla^g_k X_j) \\
&\quad - X_k(h(X_i,\nabla^g_k X_j)) + h(\nabla^g_k X_i,\nabla^g_k X_j) + h(X_i,\nabla^g_k\nabla^g_k X_j) \\
&= [\check{X}_k\check{X}_k(\check{h}(\check{X}_i,\check{X}_j)) - \check{X}_k(\check{h}(\nabla^{\check{g}}_k\check{X}_i,\check{X}_j)) - \check{X}_k(\check{h}(\check{X}_i,\nabla^{\check{g}}_k\check{X}_j)) \\
&\quad - \check{X}_k(\check{h}(\nabla^{\check{g}}_k\check{X}_i,\check{X}_j)) + \check{h}(\nabla^{\check{g}}_k\nabla^{\check{g}}_k\check{X}_i,\check{X}_j) \\
&\quad - \frac{1}{4}\eta_{ki}\eta_{kl}\check{h}_{lj} + \check{h}(\nabla^{\check{g}}_k\check{X}_i,\nabla^{\check{g}}_k\check{X}_j) \\
&\quad - \check{X}_k(\check{h}(\check{X}_i,\nabla^{\check{g}}_k\check{X}_j)) + \check{h}(\nabla^{\check{g}}_k\check{X}_i,\nabla^{\check{g}}_k\check{X}_j) \\
&\quad + \check{h}(\check{X}_i,\nabla^{\check{g}}_k\nabla^{\check{g}}_k\check{X}_j) - \frac{1}{4}\eta_{kj}\eta_{kl}\check{h}_{il}]\circ\pi \\
&= [(\nabla^{\check{g}}_k\nabla^{\check{g}}_k\check{h})_{ij} - \frac{1}{4}\eta_{ki}\eta_{kl}\check{h}_{lj} - \frac{1}{4}\eta_{kj}\eta_{kl}\check{h}_{il}]\circ\pi.
\end{aligned}
\tag{4.2.3}
$$

In the second equality, we use $\pi_*(\nabla^g_k\nabla^*_k X_i) = \nabla^{\check{g}}_k\nabla^{\check{g}}_k\check{X}_i - \frac{1}{4}\eta_{ki}\eta_{kl}X_l$.

Then, because $\nabla^g_k X_k = \nabla^{\check{g}}_k\check{X}_k - \frac{1}{2}\eta_{kk}U = \nabla^{\check{g}}_k\check{X}_k$, we have

$$
(\nabla^g_{\nabla^g_k X_k}h)(X_i,X_j) = (\nabla^{\check{g}}_{\nabla^{\check{g}}_k X_k}h)_{ij}\circ\pi. \tag{4.2.4}
$$

For the third term, we have

$$
\begin{aligned}
(\nabla^g_U\nabla^g_U h)_{ij} &= UU(h_{ij}) - 2U(h(\nabla^g_U X_i, X_j)) - 2U(h(X_i, \nabla^g_U X_j)) \\
&\quad + h(\nabla^g_U\nabla^g_U X_i, X_j) + 2h(\nabla^g_U X_i, \nabla^g_U X_j) + h(X_i, \nabla^g_U\nabla^g_U X_j) \qquad (4.2.5)\\
&= [\frac{1}{4}\eta_{ik}\eta_{kl}h_{lj} + \frac{1}{2}\eta_{ik}\eta_{jl}h_{kl} + \frac{1}{4}\eta_{jk}\eta_{kl}\eta_{il}] \circ \pi.
\end{aligned}
$$

Here, we used facts that h_{ij}, $h(\nabla^g_U X_i, X_j)$, and $h(X_i, \nabla^g_U X_j)$ are constant along fibers, since h is the pull-back of a 2-tensor on the base. We also used $\pi_*(\nabla^g_U\nabla^g_U X_i) = \frac{1}{4}\eta_{ik}\eta_{kl}\check{X}_l$.

Plugging (4.2.3), (4.2.4), and (4.2.5) into (4.2.2), we complete the proof of the lemma. ■

By using the fundamental equations for Riemannian curvature tensor for Riemannian submersions (see, Theorem in [ONe66], or equation (9.28f) in [Bes87]) and that $A_X Y = -\frac{1}{2}\eta(\check{X}, \check{Y})U$ for basic vector fields X and Y, we have the following relation between Riemannian curvature tensor on the total space and that on the base.

Lemma 4.2.3

$$R_{ijkl} = \check{R}_{ijkl} \circ \pi + (-\frac{1}{2}\eta_{ij}\eta_{kl} + \frac{1}{4}\eta_{jk}\eta_{il} - \frac{1}{4}\eta_{ik}\eta_{jl}) \circ \pi, \qquad (4.2.6)$$

and therefore,

$$(\mathring{R}h)_{ij} = (\mathring{\check{R}}\check{h})_{ij} \circ \pi + (-\frac{1}{2}\sum_{k,l=1}^{n}\eta_{ik}\eta_{jl}\check{h}_{kl} + \frac{1}{4}\sum_{k,l=1}^{n}\eta_{kj}\eta_{il}\check{h}_{kl}) \circ \pi, \qquad (4.2.7)$$

where i, j, k, and l run through 1 to n.

Proof of Proposition 4.1.3: By using Lemma 4.2.2 and Lemma 4.2.3, we have

$$\begin{aligned}
\langle (\nabla^g)^*\nabla^g h - 2\mathring{R}^g h, h\rangle &= \sum_{i,j=1}^{n} ((\nabla^g)^*\nabla^g h - 2\mathring{R}^g h)_{ij} h_{ij} \\
&= (\sum_{i,j=1}^{n} ((\nabla^{\check{g}})^*\nabla^{\check{g}} \check{h} - 2\mathring{\check{R}}^{\check{g}} \check{h})_{ij} \check{h}_{ij}) \circ \pi \\
&\quad + \sum_{i,j,k,l=1}^{n} [\frac{1}{2}\eta_{ki}\eta_{kl}\check{h}_{lj}\check{h}_{ij} + \frac{1}{2}\eta_{kj}\eta_{kl}\check{h}_{li}\check{h}_{ij} \\
&\quad - \frac{1}{2}\eta_{ik}\eta_{jl}\check{h}_{kl}\check{h}_{ij} + \frac{1}{2}\eta_{ik}\eta_{jl}\check{h}_{kl}\check{h}_{ij} - \frac{1}{2}\eta_{kj}\eta_{il}\check{h}_{kl}\check{h}_{ij}] \circ \pi \\
&= \langle (\nabla^{\check{g}})^*\nabla^{\check{g}} \check{h} - 2\mathring{\check{R}}^{\check{g}} \check{h}, \check{h}\rangle \circ \pi \\
&\quad + \sum_{i,j,k,l=1}^{n} (\eta_{ki}\eta_{kl}\check{h}_{lj}\check{h}_{ij} + \eta_{ik}\eta_{jl}\check{h}_{kl}\check{h}_{ij}) \circ \pi.
\end{aligned}$$

In the last step, we use facts that η_{ij} is anti-symmetric and $\check{h}_{ij}$ is symmetric about indices i and j.

4.3 Instability of Einstein metrics on principal circle bundles

In this section, we prove Corollary 4.1.4.

A necessary and sufficient condition for g defined in (4.1.1) to be Einstein is given in [Bes87] and [WZ90]. We recall the condition for torus bundles given in [WZ90]. As mentioned in [WZ90], any left-invariant metric on a torus is bi-invariant and flat, so $Ric_{\hat{g}} = 0$, and the curvature $\omega = d\theta = \pi^*\eta$ on a principal torus bundle is the pull-back of a closed 2-form η on B. Then, (P, g) is Einstein with Einstein constant k iff

$$\eta \ \ \textit{is a harmonic form on} \ \ (B, \hat{g}), \tag{4.3.1}$$

$$\frac{1}{4}\sum_{i,j}\hat{g}(\eta(\check{X}_i,\check{X}_j),U)\hat{g}(\eta(\check{X}_i,\check{X}_j),V)=k\hat{g}(U,V), \tag{4.3.2}$$

$$Ric_{\check{g}}(\check{X},\check{Y})-\frac{1}{2}\sum_{i}\hat{g}(\eta(\check{X},\check{X}_i),\eta(\check{Y},\check{X}_i))=k\check{g}(\check{X},\check{Y}), \tag{4.3.3}$$

for any pair of vector fields $\check{X}$ and $\check{Y}$ on B, where $\{\check{X}_i\}$ is a local orthonormal frame of TB.

We recall the construction of Einstein metrics on principal circle bundles over products of Kähler-Einstein manifolds in [WZ90]. Let (M_i, g_i), $i = 1, \cdots, m$, be Kähler-Einstein manifolds with first Chern classes $c_1(M_i) > 0$ and real dimension n_i. Write $c_1(M_i) = q_i\alpha_i$, where $\alpha_i \in H^2(M_i, \mathbb{Z})$ is indivisible and $q_i \in \mathbb{Z}$. Normalize g_i such that $[\omega_i] = 2\pi\alpha_i$, equivalently, $Ric_{g_i} = q_i g_i$, where ω_i is the Kähler form of g_i. Let $\pi : P \to B = M_1 \times \cdots \times M_m$ be a principal S^1-bundle whose Euler class is $e(p) = \sum b_i\pi_i^*\alpha_i$, where $b_i \in \mathbb{Z}$, and $\pi_i : B \to M_i$ denotes the projection onto the ith factor. Choose a Riemannian metric on B as $\check{g} = x_1\pi_1^*g_1 + \cdots + x_m\pi_m^*g_m$, where $x_1, \cdots, x_m$ are constants to be determined. Let $\eta = \sum b_i\pi_i^*\omega_i$, and θ be a principal connection on P such that $d\theta = \pi^*\eta$. And choose the left-invariant metric on S^1 such that the length of S^1 is 2π. Then Einstein conditions (4.3.2) and (4.3.3) become

$$\sum n_i\frac{b_i^2}{x_i^2}=4k, \tag{4.3.4}$$

$$\frac{q_j}{x_j}-\frac{1}{2}(\frac{b_j}{x_j})^2=k, \quad j=1,\cdots,m. \tag{4.3.5}$$

M. Wang and W. Ziller proved the existence of an unique solution of the system of equations (4.3.4) and (4.3.5) about x_j, provided $e(P) \neq 0$, and therefore, obtained the existence of Einstein metrics on the circle bundle P.

Proof of Corollary 4.1.4: On M. Wang and W. Ziller's Einstein manifolds constructed above, $\eta = \sum_{i=1}^m b_i\pi_i^*\omega_i$. Assume $m \geq 2$. Let $\check{h} = \frac{x_1\pi_1^*g_1}{n_1} - \frac{x_2\pi_2^*g_2}{n_2}$ be a symmetric 2-tensor on B^n, where $n = \sum_i^m n_i$ is the real dimension of the base product manifold. $\check{h}$ is traceless and transverse, i.e. $\delta_{\check{g}}\check{h} = 0$ and $tr_{\check{g}}\check{h} = 0$. These imply that $\delta_g h = 0$ and $tr_g h = 0$, i.e. $h = \pi^*\check{h}$ is a traceless transverse 2-tensor on P.

$$\begin{aligned}
&\langle(\nabla^{\check{g}})^*\nabla^{\check{g}}\check{h} - 2\mathring{R}^{\check{g}}\check{h}, \check{h}\rangle \\
&= -2\sum_{i,j,k,l=1}^{n}\check{R}_{ijkl}(\frac{x_1\pi_1^*g_1}{n_1} - \frac{x_2\pi_2^*g_2}{n_2})_{ik}(\frac{x_1\pi_1^*g_1}{n_1} - \frac{x_2\pi_2^*g_2}{n_2})_{jl} \\
&= -2(\frac{R_{x_1g_1}}{n_1^2} + \frac{R_{x_2g_2}}{n_2^2}) \\
&= -2(\frac{q_1}{x_1n_1} + \frac{q_2}{x_2n_2}) \\
&= -2k(\frac{1}{n_1} + \frac{1}{n_2}) - (\frac{b_1^2}{n_1x_1^2} + \frac{b_2^2}{n_2x_2^2}).
\end{aligned}$$

In the last step, we use the equation (4.3.5).

$$\begin{aligned}
&\sum_{i,j,k,l=1}^{n}\eta_{ki}\eta_{kl}\check{h}_{lj}\check{h}_{ij} \\
&= \sum_{i,j,k,l=1}^{n}(\sum_{s=1}^{m}b_s\pi_s^*\omega_s)_{ki}(\sum_{t=1}^{m}b_t\pi_t^*\omega_t)_{kl}(\frac{x_1\pi_1^*g_1}{n_1} - \frac{x_2\pi_2^*g_2}{n_2})_{lj}(\frac{x_1\pi_1^*g_1}{n_1} - \frac{x_2\pi_2^*g_2}{n_2})_{ij} \\
&= \frac{b_1^2}{n_1^2}\|\omega_1\|^2_{x_1g_1} + \frac{b_2^2}{n_2^2}\|\omega_2\|^2_{x_2g_2} \\
&= \frac{b_1^2}{2n_1x_1^2} + \frac{b_2^2}{2n_2x_2^2},
\end{aligned}$$

and similarly,

$$\sum_{i,j,k,l=1}^{n}\eta_{ik}\eta_{jl}\check{h}_{kl}\check{h}_{ij} = \frac{b_1^2}{2n_1x_1^2} + \frac{b_2^2}{2n_2x_2^2}.$$

Combining these equations and Proposition 4.1.3, we obtain

$$\langle \nabla^* \nabla h - 2\mathring{R}h, h\rangle = -2k(\frac{1}{n_1} + \frac{1}{n_2}) = -\frac{2n_1 n_2(n_1+n_2)}{n_1^2+n_2^2}\langle h, h\rangle.$$

This implies that if $m \geq 2$, then M. Wang and W. Ziller's Einstein metrics on circle bundles are unstable.

4.4 Einstein operator on torus bundles

In this section, we prove Proposition 4.1.5.

Lemma 4.4.1 *Let X and Y be basic vector fields. We have*

$$\begin{aligned}
[U_\alpha, X] &= \mathcal{L}_{U_\alpha} X = 0, \\
\nabla^g_X Y &= \nabla^{\check{g}}_{\check{X}} \check{Y} - \frac{1}{2}\eta_\alpha(\check{X}, \check{Y}) U_\alpha, \\
\nabla^g_{U_\alpha} X = \nabla^g_X U_\alpha &= \frac{1}{2}\hat{g}_{\alpha\beta}\eta_\beta(\check{X}, \check{X}_i) X_i, \\
\nabla^g_{U_\alpha} U_\beta &= 0,
\end{aligned}$$

where α and β run through 1 to r, and take sum for repeated indices.

Proof: This first equation follows from facts that the horizontal vector field X is T^r−invariant and U_α is generated by e_α through T^r-action. The second and third equations follow from the O'Neil's fundamental equations for Riemannian submersions and facts that $A_X Y = -\frac{1}{2}\eta_\alpha(\check{X}, \check{Y})U_\alpha$, and tensor T vanishes on T^r-bundles that we are considering. Let us check the fourth equality. Because the tensor T vanishes, $\nabla^g_{U_\alpha} U_\beta$ is vertical. Actually, $\nabla^g_{U_\alpha} U_\beta = \nabla^{\hat{g}}_{U_\alpha} U_\beta$, when we restrict on each fiber. Then because any left-invariant metric on a torus is bi-invariant, i.e. $\hat{g}$ is bi-invariant, and by the well-known formula for the connection of a bi-variant metric on a Lie group (see, e.g. Corollary 3.19 in [CE]), we have $\nabla^{g}_{U_\alpha}U_\beta = \nabla^{\hat{g}}_{U_\alpha}U_\beta = \frac{1}{2}[U_\alpha, U_\beta] = 0$. In the last equality, we use the fact that a torus is an Abelian group. ∎

Lemma 4.4.2

$$\begin{aligned}((\nabla^g)^*\nabla^g h)_{ij} = ((\nabla^{\check{g}})^*\nabla^{\check{g}}\check{h})_{ij} &+ \frac{1}{4}\hat{g}_{\alpha\beta}(\eta_\alpha)_{ki}(\eta_\beta)_{kl}\check{h}_{lj} + \frac{1}{4}\hat{g}_{\alpha\beta}(\eta_\alpha)_{kj}(\eta_\beta)_{kl}\check{h}_{li} \\ &- \frac{1}{4}\hat{g}_{\alpha\beta}(\eta_\alpha)_{ik}(\eta_\beta)_{kl}\check{h}_{lj} - \frac{1}{4}\hat{g}_{\alpha\beta}(\eta_\alpha)_{ik}(\eta_\beta)_{jl}\check{h}_{kl} \\ &- \frac{1}{4}\hat{g}_{\alpha\beta}(\eta_\alpha)_{ik}(\eta_\beta)_{jl}\check{h}_{kl} - \frac{1}{4}\hat{g}_{\alpha\beta}(\eta_\alpha)_{jk}(\eta_\beta)_{kl}\check{h}_{li}.\end{aligned} \tag{4.4.1}$$

Lemma 4.4.3

$$R_{ikjl} = \check{R}_{ikjl} - \frac{1}{2}\hat{g}_{\alpha\beta}(\eta_\alpha)_{ik}(\eta_\beta)_{jl} + \frac{1}{4}\hat{g}_{\alpha\beta}(\eta_\alpha)_{kj}(\eta_\beta)_{il} - \frac{1}{4}\hat{g}_{\alpha\beta}(\eta_\alpha)_{ij}(\eta_\beta)_{kl} \tag{4.4.2}$$

Proofs of these two lemmas are very similar to the proofs of lemmas in Section 4.2. So we omit their proofs. Then these two lemmas directly imply Proposition 4.1.5.

4.5 Instability of Einstein metrics on principal torus bundles

In this section, we prove Corollary 4.1.6.

We recall Wang and Ziller's construction of Einstein metrics on principal T^r bundles over a product of Kähler-Einstein manifolds. Consider a principal T^r bundle $\pi : P \to B$. Choose and fix a decomposition $T^r = S^1 \times \cdots \times S^1$. Let $\beta_\alpha \in H^2(B, \mathbb{Z})$, $\alpha = 1, \cdots, r$ be the Euler classes of the circle bundles $P/T^{r-1} \to B$ where $T^{r-1} \subset T^r$ is the subtorus with ith S^1 factor deleted. Then the T^r-bundle is classified by characteristic classes β_α.

Let (M^{n_s}, g_s), $s = 1, \cdots, m$, be Käher-Einstein manifolds with positive first Chern classes $c_1(M_s) = q_s\alpha_s$, where $\alpha_s \in H^2(M_s, \mathbb{Z})$ is indivisible and $q_s \in \mathbb{Z}$. Normalize g_s such that $[\omega_s] = 2\pi\alpha_s$, i.e. $Ric_{g_s} = q_s g_s$, where ω_s is the Kähler form of g_s. Let $\pi : P^{n+r} \to B^n = M_1^{n_1} \times \cdots \times M_m^{n_m}$ be a principal T^r-bundle with characteristic classes $\beta_\alpha = \sum_{s=1}^m b_{\alpha s}\pi_s^*\alpha_j$, $\alpha = 1, \cdots, r$. Let θ be a principal connection on the T^r bundle with the curvature form $\omega = d\theta = \pi^*\eta$ with $\eta = \sum_{\alpha=1}^r \eta_\alpha e_\alpha$ and $\eta_\alpha = \sum_{s=1}^m b_{\alpha s}\pi_s^*\omega_s$. Recall that $\{e_1, \cdots, e_r\}$ is a basis of the Lie algebra of T^r coming from the chosen decomposition of T^r, and $\hat{g}_{\alpha\beta} = \hat{g}(e_\alpha, e_\beta)$ is a left invariant metric on T^r.

Then Einstein conditions (4.3.2) and (4.3.3) become

$$\sum_{s=1}^m \frac{b_{\alpha s}b_{\beta s}n_s}{x_s^2} = 4k\hat{g}^{\alpha\beta}, \quad 1 \le \alpha, \beta \le r, \tag{4.5.1}$$

$$\frac{q_s}{x_s} - \frac{1}{2}\sum_{\alpha,\beta=1}^r \frac{\hat{g}_{\alpha\beta}b_{\alpha s}b_{\beta s}}{x_s^2} = k, \quad s = 1, \cdots, m. \tag{4.5.2}$$

By showing existence of solutions of the system of equations (4.5.1) and (4.5.2), M. Wang and W. Ziller obtain Einstein metrics on these principal torus bundles, provided that the matrix $(b_{\alpha s})_{r\times m}$ with $r \le m$ has maximal rank.

Proof of Corollary 4.1.6: Assume $m \ge 2$. The same as in the proof of Corollary 4.1.4, we take $\check{h} = \frac{x_1 g_1}{n_1} - \frac{x_2 g_2}{n_2}$. Let $h = \pi^*\check{h}$. Then $\delta_g h = 0$, $tr_g h = 0$, and

$$\begin{aligned}\langle(\nabla^{\check{g}})^*\nabla^{\check{g}}\check{h} - 2\mathring{R}^{\check{g}}\check{h}, \check{h}\rangle &= -2\left(\frac{q_1}{x_1 n_1} + \frac{q_2}{x_2 n_2}\right)\\ &= -2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)k\\ &\quad - \left(\sum_{\alpha,\beta=1}^r \frac{\hat{g}_{\alpha\beta}b_{\alpha 1}b_{\beta 2}}{n_1 x_1^2} + \sum_{\alpha,\beta=1}^r \frac{\hat{g}_{\alpha\beta}b_{\alpha 2}b_{\beta 2}}{n_2 x_2^2}\right).\end{aligned} \tag{4.5.3}$$

In the last equality, we use equations (4.5.2). And because $\eta_\alpha = \sum_{s=1}^m b_{\alpha s}\omega_s$, and

$\|\omega_s\|^2_{x_s g_s} = \frac{n_s}{2x_s^2}$ for $s = 1, \cdots, m$,

$$\begin{aligned}\hat{g}_{\alpha\beta}(\eta_\alpha)_{ki}(\eta_\beta)_{kl}\check{h}_{lj}\check{h}_{ij} &= \hat{g}_{\alpha\beta}(\eta_\alpha)_{ik}(\eta_\beta)_{jl}\check{h}_{kl}\check{h}_{ij} \\ &= \sum_{\alpha,\beta=1}^{r} \frac{\hat{g}_{\alpha\beta}b_{\alpha 1}b_{\beta 2}}{2n_1 x_1^2} + \sum_{\alpha,\beta=1}^{r} \frac{\hat{g}_{\alpha\beta}b_{\alpha 2}b_{\beta 2}}{2n_2 x_2^2}.\end{aligned} \tag{4.5.4}$$

Then Proposition 4.1.5 implies

$$\langle \nabla^*\nabla h - 2\mathring{R}h, h\rangle = -2k(\frac{1}{n_1} + \frac{1}{n_2}) = -\frac{2n_1 n_2(n_1+n_2)}{n_1^2+n_2^2}\langle h, h\rangle, \tag{4.5.5}$$

and we complete the proof.

Chapter 5

Perelman's λ-functional on manifolds with conical singularities

In this chapter, we prove that on a compact manifold with a single conical singularity the spectrum of the operator $-4\Delta + R$ consists of discrete eigenvalues with finite multiplicities, if the scalar curvature R satisfies a certain condition near the singularity. Moreover, we obtain an asymptotic behavior for eigenfunctions near the singularity. As a consequence of these spectrum properties, we extend the theory of the Perelman's λ-functional on smooth compact manifolds to compact manifolds with a single conical singularity. All these work and results also go through on compact manifolds with isolated conical singularities.

5.1 Overview and main results

As we have seen from (2.5.4), the Perelman's λ-functional on a smooth compact manifold is essentially the smallest eigenvalue of the operator $-4\Delta+R$. Consequently, we can also define the λ-functional on a compact smooth manifold as the smallest eigenvalue of $-4\Delta+R$, which is the definition of the λ-functional that we want to use on compact manifolds with isolated conical singularities. Therefore, we first study the spectrum of $-4\Delta + R$ on a compact Riemannian manifold with isolated conical singularities defined as the following.

Definition 5.1.1 *We say $(M^n, d, g, p_1, \cdots, p_k)$ is a compact Riemannian manifold with isolated conical singularities at $p_1, \cdots, p_k$, if*

- *(M, d) is a compact metric space,*
- *$(M_0, g|_{M_0})$ is an n-dimensional smooth Riemannian manifold, and the Riemannian metric g induces the given metric d on M_0, where $M_0 = M \setminus \{p_1, \cdots, p_k\}$,*
- *for each singularity p_i, $1 \leq i \leq k$, their exists a neighborhood $U_{p_i} \subset M$ of p_i such that $U_{p_i} \cap \{p_1, \cdots, p_k\} = \{p_i\}$, $(U_{p_i} \setminus \{p_i\}, g|_{U_{p_i}\setminus\{p_i\}})$ is isometric to $((0, \varepsilon_i) \times N_i, dr^2 + r^2 h_r)$ for some $\varepsilon_i > 0$ and a compact smooth manifold N_i, where r is a coordinate on $(0, \varepsilon_i)$ and h_r is a smooth family of Riemannian metrics on N_i satisfying $h_r = h_0 + o(r^{\alpha_i})$ as $r \to 0$, where $\alpha_i > 0$ and h_0 is a smooth Riemannian metric on N_i.*

Moreover, we say a singularity p is a cone-like singularity, if the metric g on a neighborhood of p is isometric to $dr^2 + r^2 h_0$ for some fixed metric h_0 on cross section N.

Remark 5.1.2 *We do analysis on $(M^n, d, g, p_1, \cdots, p_k)$ away from singular points $p_1, \cdots, p_k$.*

In the rest of this chapter, we will only work on manifolds with a single conical singularity because there is no essential difference between one single singularity case *and multiple isolated singularities case. And all work and results on manifolds with a single conical singularity go through on manifolds with isolated conical singularities.*

Recall some basic facts about cones over compact smooth manifolds. Let $C(N, h_0) = (\mathbb{R}_+ \times N, g = dr^2 + r^2 h_0)$ be the Riemannian cone over a compact (n-1)-dimensional smooth Riemannian manifold (N^{n-1}, h_0). Then we have:

$$\Delta_g = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_{h_0}, \tag{5.1.1}$$

$$R_g = \frac{1}{r^2}[R_{h_0} - (n-1)(n-2)]. \tag{5.1.2}$$

From (5.1.1) and (5.1.2), we can see that on the cone the operator $-4\Delta + R$ is a Schrödinger operator with singular potential. Actually, the potential function R_g behaves like $O(\frac{1}{r^2})$ as $r \to 0$, i.e. blows up near the tip of the cone. This type of operators have been studied in several literatures, for example in [BS87] and [RS2].

Let us first look at the simplest one-dimensional example of singular Schrödinger operators that is mentioned in [BS87] and also studied in [RS2]. Let $L_a = -\frac{d^2}{dx^2} + \frac{a}{x^2}$ be an unbounded operator in $L^2(\mathbb{R}_+)$ with the domain $D(L_a) = C_0^\infty(\mathbb{R}_+)$, where $a \in \mathbb{R}$ is a constant and $\mathbb{R}_+ = (0, +\infty)$. By Hardy's inequality, if $a \geq -\frac{1}{4}$, the operator L_a is nonnegative on $C_0^\infty(\mathbb{R}_+)$, in particular semi-bounded. Actually, by some simple scaling technique we can see that $a \geq -\frac{1}{4}$ is not only a sufficient condition but also a necessary condition for L_a to be semi-bounded. Thus, the operator L_a is either nonnegative or not semi-bounded.

In [RS2], Michael Read and Barry Simon give the following criterion for essential self-adjointness of a Schrödinger operator with a spherically symmetric potential: $-\Delta + V(r)$ on $\mathbb{R}^n$, where $r = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$.

Theorem 5.1.3 (Theorem X.11 in [RS2]) *Let $V(r)$ be a continuous symmetric po-*

tential on $\mathbb{R}^n \setminus \{0\}$. *If* $V(r)$ *satisfies*

$$V(r) + \frac{(n-1)(n-3)}{4}\frac{1}{r^2} \geq \frac{3}{4r^2},$$

then $-\Delta + V(r)$ *is essentially self-adjoint on* $C_0^\infty(\mathbb{R}^n \setminus \{0\})$. *If* $V(r)$ *satisfies*

$$0 \leq V(r) + \frac{(n-1)(n-3)}{4}\frac{1}{r^2} \leq \frac{c}{r^2}, \quad c < \frac{3}{4},$$

then $-\Delta + V(r)$ *is not essentially self-adjoint on* $C_0^\infty(\mathbb{R}^n \setminus \{0\})$.

From the above one-dimensional and higher dimensional examples we can see that certain conditions on the potential function R_g, actually on R_{h_0}, should be necessary, if we expect that the operator $-4\Delta + R$ is semibounded, and its Friedrichs extension then has nice spectrum. It turns out that $R_{h_0} > (n-2)$ is a sufficient condition, and we have one of main results in this chapter as the following.

Theorem 5.1.4 (Dai, –) *Let* (M^n, d, g, p) *be a compact Riemannian manifold with a conical singularity at* p. *If the scalar curvature* $R_{h_0} > (n-2)$ *on* N, *then the operator* $-4\Delta_g + R_g$ *with domain* $C_0^\infty(M \setminus \{p\})$ *is semibounded, and the spectrum of its Friedrichs extension consists of discrete eigenvalues with finite multiplicity* $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$, *and* $\lambda_k \to +\infty$, *as* $k \to +\infty$.

In [BP03], B. Botvinnik and S. Preston proved that the spectrum of the conformal Laplacian on a compact Riemannian manifold with isolated tame conical singularities consists of discrete eigenvalues with finite multiplicities. The conformal Laplacian $-\Delta + \frac{n-2}{4(n-1)}R$ is also a singular Schrödinger operator. A tame conical singularity is given as a cone over a product of the standard spheres. Therefore, the scalar curvature of the cross section of a tame conical singularity satisfies the condition in Theorem 5.1.4, and our result is more general. Our idea of the proof of Theorem

5.1.4 is similar to the one in [BP03]. We use certain weighted Sobolev spaces that can be compactly embedded in $L^2(M)$. And by doing some estimates, we show that the operator $-4\Delta+R$ is semi-bounded and the domain of its self-adjoint extension is in a weighted Sobolev space. Then we can use the spectrum theorem for self-adjoint compact operators to obtain the property of the spectrum of the operator $-4\Delta+R$.

Theorem 5.1.4 enables us to define the λ-functional as the smallest eigenvalue of $-4\Delta+R$. However, when we derive variational formulae of the the λ-functional, it turns out that certain asymptotic behavior of eigenfunctions near singularities is necessary. We have another main result in this chapter as the following.

Theorem 5.1.5 (Dai, –) *Let (M^n, g, p) be a compact Riemannian manifold with a single conical singularity p with $R_{h_0} > (n-2)$ and satisfying*

$$\begin{gathered} r^i|\nabla^{i+1}(h_r - h_0)| \leq C_i < +\infty, \\ \textit{for some constant } \; C_i, \quad \textit{and each } \; 0 \leq i \leq \frac{n}{2}+2, \end{gathered} \tag{5.1.3}$$

near p. Then eigenfunctions of $-4\Delta_g + R_g$ on satisfy

$$u = o(r^{-\frac{n-2}{2}}), \quad \textit{as} \;\; r \to 0. \tag{5.1.4}$$

Consequently, the first eigenvalue is simple.

Moreover, if the singularity is cone-like, eigenfunctions have asymptotic expansion at the conical singularity p as

$$u \sim \sum_{j=1}^{+\infty}\sum_{l=0}^{+\infty}\sum_{p=0}^{p_j} r^{s_j+l}(\ln r)^p u_{j,l,p}, \tag{5.1.5}$$

where $u_{j,l,p} \in C^\infty(N^{n-1})$, $p_j = 0$ or 1, and $s_j = -\frac{n-2}{2} \pm \frac{\sqrt{\mu_j-(n-2)}}{2}$, where μ_j are *eigenvalues of* $-\Delta_{h_0} + R_{h_0}$ *on* N^{n-1}.

On a manifold with a cone-like singularity, a small neighborhood of the singularity is a finite exact cone over a compact smooth manifold. On this neighborhood, we can separate variable and explicitly solve the eigenfunction equation in term of eigenfunctions on the cross section of the cone and some hypergeometric functions. By using classical elliptic estimates and some estimates for the hypergeometric functions, we then obtain the asymptotic expansion (5.1.5) of eigenfunctions on manifold with a cone-like singularity.

On a manifold with a conical singularity, we cannot do explicit calculations. Therefore, instead, we do some estimates to obtain an asymptotic order near the singularity for eigenfunctions in (5.1.4). We first work on small finite cones, on which we can obtain some weighted Sobelov inequalities and weighted elliptic estimates by using scaling technique. The asymptotic condition (5.1.3) for the asymptotically conical metric implies weighted Sobelov norms and weighted C^k-norms with respect to exactly conical metric $dr^2 + r^2 h_r$ are equivalent to ones with respect to asymptotically conical metric $dr^2 + r^2 h_r$. Then these weighted Sobelov inequality and weighted elliptic estimates still hold on an asymptotic finite cone. This implies the asymptotic order in (5.1.4) by using elliptic bootstrapping. And further, we can obtain variation formulae of λ-functional on compact manifolds with a single conical singularity.

5.2 Weighted Sobolev Spaces

In this section, we introduce weighted Sobolev spaces on compact Riemannian manifolds with conical singularities and establish the compact embedding property for the weighted Sobolev spaces.

Let (M^n, g, p) be a compact Riemannian manifold with a single conical singularity at p, and U_p be a conical neighborhood of p such that $(U_p \setminus \{p\}, g|_{U_p\setminus\{p\}})$ is isometric to $((0,\epsilon)\times N, dr^2 + r^2 h_r)$. For each $k \in \mathbb{N}$ and $\delta \in \mathbb{R}$, we define the weighted Sobolev space $H^k_\delta(C_\epsilon(N))$ to be the completion of $C_0^\infty(M\setminus\{p\})$ with respect to the weighted Sobolev norm

$$\|u\|^2_{H^k_\delta(M)} = \int_M (\sum_{i=0}^k \chi^{2(\delta-i)+n} |\nabla^i u|^2) dvol_g, \tag{5.2.1}$$

where $\nabla^i u$ denotes the ith covariant derivative, and $\chi \in C^\infty(M\setminus\{p\})$ is a positive weight function satisfying

$$\chi(q) = \begin{cases} 1 & \text{if } \ q \in M \setminus U_p, \\ \frac{1}{r} & \text{if } \ r = dist(q,p) < \frac{\epsilon}{4}, \end{cases} \tag{5.2.2}$$

and $0 < (\chi(q))^{-1} \leq 1$ for all $q \in M \setminus \{p\}$.

For the simplicity of notations, we set $H^k(M) \equiv H^k_{k-\frac{n}{2}}(M)$. The we have the following compact embedding of $H^k(M)$ into $L^2(M)$.

Theorem 5.2.1 (Dai, –) *The continuous embedding*

$$i : H^k(M) \hookrightarrow L^2(M) \tag{5.2.3}$$

is compact for each $k \in \mathbb{N}$.

Before proving Theorem 5.2.1, we prove the analogous compact embedding theorem on finite cones. Let $(C_\epsilon(N), g) = ((0,\epsilon)\times N, dr^2 + r^2 h)$ be a finite cone. We define the weighted Sobolev space $H^k_\delta(C_\epsilon(N))$ on the cone $(C_\epsilon(N), g)$ to be the completion of $C_0^\infty(C_\epsilon(N))$ with respect to the weighted Sobolev norm

$$\|u\|^2_{H^k_\delta(C_\epsilon(N))} = \int_{C_\epsilon(N)} (\sum_{i=0}^k \frac{1}{r^{2(\delta-i)+n}} |\nabla^i u|^2) dvol_g. \tag{5.2.4}$$

We also set $H^k(C_\epsilon(N)) \equiv H^k_{k-\frac{n}{2}}(C_\epsilon(N))$. Then we have the following compact embedding on a finite cone.

Lemma 5.2.2 (Dai, –) *The continuous embedding*

$$i : H^k(C_\epsilon(N)) \hookrightarrow L^2(C_\epsilon(N))$$

is compact for each $k \in \mathbb{N}$.

Proof: Because $\|u\|_{H^k_2(C_\epsilon(N))} \geq \|u\|_{H^l_2(C_\epsilon(N))}$, for $k \geq l \in \mathbb{N}$, we have continuous embedding $H^k(C_\epsilon(N)) \hookrightarrow H^l(C_\epsilon(N))$, for $k \geq l \in \mathbb{N}$. Therefore, it suffices to show that the embedding: $i : H^1(C_\epsilon(N)) \hookrightarrow L^2(C_\epsilon(N))$, is compact. Let $(\widetilde{C}_\epsilon(N), \widetilde{g}) = ((0,\epsilon) \times N, dr^2 + h)$ be a finite cylinder, and $W^{1,2}_0(\widetilde{C}_\epsilon(N))$ be the usual Sobolev space on the cylinder $\widetilde{C}_\epsilon(N)$, which is the completion of $C^\infty_0(\widetilde{C}_\epsilon(N))$ with respect to the norm:

$$\|u\|_{W^{1,2}(\widetilde{C}_\epsilon(N))} = \int_{\widetilde{C}_\epsilon(N)} (u^2 + |\widetilde{\nabla} u|^2_{\widetilde{g}}) dvol_{\widetilde{g}},$$

where $\widetilde{\nabla} u$ is the gradient of u with respect to the metric $\widetilde{g}$. It is obvious that the mapping:

$$L^2(C_\epsilon(N), g) \rightarrow L^2(\widetilde{C}_\epsilon(N), \widetilde{g})$$

$$u \mapsto \widetilde{u} = r^{\frac{n-1}{2}} u$$

is unitary, where $n = dim(N) + 1$. We will show that

$$\|u\|_{H^1(C_\epsilon(N))} \geq \frac{3}{4} \min\{1, \frac{1}{\epsilon^2}\} \|\widetilde{u}\|_{W^{1,2}(\widetilde{C}_\epsilon(N))}, \tag{5.2.5}$$

for all $u \in C^\infty_0((0,\epsilon) \times N)$. This then completes the proof, since the embedding

$$W^{1,2}_0(\widetilde{C}_\epsilon(N)) \hookrightarrow L^2(\widetilde{C}_\epsilon(N), \widetilde{g})$$

is compact by the classical Rellich Lemma.

Now we prove the inequality (5.2.5). Let $0 = \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \nearrow +\infty$ be eigenvalues of the positive Laplacian, $-\Delta_N$, on the compact Riemannian manifold (N, h), and $\psi_1, \psi_2, \psi_3, \cdots$ be corresponding eigenfunctions. Let $u \in C_0^\infty((0,\epsilon) \times N)$. We expand the function u and $\widetilde{u}$, respectively, as

$$\begin{aligned} u(r,x) &= \sum_{i=1}^{\infty} u_i(r)\psi_i(x), \\ \widetilde{u}(r,x) &= \sum_{i=1}^{\infty} \widetilde{u}_i(r)\psi_i(x), \end{aligned} \tag{5.2.6}$$

where $u_i(r) = r^{-\frac{n-1}{2}}\widetilde{u}_i(r)$.

$$\begin{aligned}
&\|u\|^2_{H^1(C_\epsilon(N))} \\
&= \int_{C_\epsilon(N)} (\frac{1}{r^2}u^2 + |\nabla u|_g^2) dvol_g \\
&= \int_{C_\epsilon(N)} (\frac{1}{r^2}u^2 + |\partial_r u|^2 + \frac{1}{r^2}|\nabla_N u|_h^2) dvol_g \\
&= \int_0^\epsilon \int_N [\frac{1}{r^2}(\sum_{i=1}^{\infty} u_i(r)\psi_i(x))^2 + (\sum_{i=1}^{\infty} u_i^{'}(r)\psi_i(x))^2 \\
&\quad + \frac{1}{r^2}(\sum_{i=1}^{\infty} u_i(r)\nabla_N\psi_i(x))^2] r^{n-1} dvol_h dr \\
&= \int_0^\epsilon [\frac{1}{r^2}\sum_{i=1}^{\infty}(u_i(r))^2 + \sum_{i=1}^{\infty}(u_i^{'}(r))^2 + \frac{1}{r^2}\sum_{i=1}^{\infty}\mu_i(u_i(r))^2] r^{n-1} dr \\
&= \int_0^\epsilon [\frac{1}{r^2}\sum_{i=1}^{\infty}(\widetilde{u}_i(r))^2 + \sum_{i=1}^{\infty}(-\frac{n-1}{2}\frac{1}{r}\widetilde{u}(r) + \widetilde{u}_i^{'}(r))^2 + \frac{1}{r^2}\sum_{i=1}^{\infty}\mu_i(\widetilde{u}_i(r))^2] dr \\
&= \int_0^\epsilon [\frac{1}{r^2}\sum_{i=1}^{\infty}(1 + \frac{(n-1)(n-3)}{4} + \mu_i)(\widetilde{u}_i(r))^2 + \sum_{i=1}^{\infty}(\widetilde{u}_i^{'})^2] dr \\
&\geq \int_0^\epsilon [\sum_{i=1}^{\infty}(\frac{3}{4} + \mu_i)(\widetilde{u}_i(r))^2 + \sum_{i=1}^{\infty}(\widetilde{u}_i^{'})^2] dr \\
&= \frac{3}{4}\min\{1, \frac{1}{\epsilon^2}\}\|\widetilde{u}\|^2_{W^{1,2}(\widetilde{C}(N))}
\end{aligned}$$

■

Proof of Theorem 5.2.1. As in the proof of Lemma 5.2.2, it suffices to show that $H^1(M) \hookrightarrow L^2(M)$ is compact. Because $(U_p \setminus \{p\}, g|_{U_p\setminus\{p\}})$ is isometric to $((0,\epsilon) \times N, dr^2 + r^2 h_r)$, where $h_r = h_0 + o(r^\alpha)$, for some $\alpha > 0$, if we define $g_0 = dr^2 + r^2 h_0$ on $(0,\epsilon) \times N$, there exists $0 < \epsilon_1 < \frac{\epsilon}{4}$, such that on $(0,\epsilon_1) \times N$,

$$\frac{1}{2} g_0 \le g \le 2 g_0.$$

Then for any $u \in C_0^\infty((0,\epsilon) \times N)$, we have

$$\frac{1}{2^{1+\frac{n}{2}}} \|u\|^2_{H^1(C_{\epsilon_1}(N),g_0)} \le \|u\|^2_{H^1(C_{\epsilon_1}(N),g)} \le 2^{1+\frac{n}{2}} \|u\|^2_{H^1(C_{\epsilon_1}(N),g_0)}, \tag{5.2.7}$$

$$\frac{1}{2^{\frac{n}{2}}} \|u\|^2_{L^2(C_{\epsilon_1}(N),g_0)} \le \|u\|^2_{L^2(C_{\epsilon_1}(N),g)} \le 2^{\frac{n}{2}} \|u\|^2_{L^2(C_{\epsilon_1}(N),g_0)}. \tag{5.2.8}$$

By Lemma 5.2.2, inequalities (5.2.7) and (5.2.8) imply that the embedding

$$H^1(C_{\epsilon_1}(N), g) \hookrightarrow L^2(C_{\epsilon_1}(N), g) \tag{5.2.9}$$

is compact. Set $M_0 = M \setminus (0, \frac{\epsilon_1}{2}) \times N$. The compactness of embedding $W_0^{1,2}(M_0) \hookrightarrow L^2(M_0)$ and the compactness of the embedding (5.2.9) imply the compactness of the embedding $H^1(M) \hookrightarrow L^2(M)$.

5.3 Spectrum of $-4\Delta + R$ on a finite cone

In this section we study the spectrum of the operator $L = -4\Delta + R$ on a small finite cone $(C_\epsilon(N), g) = ((0,\epsilon) \times N, dr^2 + r^2 h_0)$ with Dirichlet boundary condition. By using the compact embedding results obtained in the previous section and establishing a semi-boundedness estimate for the operator L, we show that the spectrum the Friedrichs extension of L on a small finite cone with the Dirichlet boundary condition consists of discrete eigenvalues with finite multiplicities.

Let

$$L = -4\Delta + R : L^2(C_\epsilon(N)) \to L^2(C_\epsilon(N))$$

be a densely defined unbounded operator with the domain $Dom(L) = C_0^\infty(C_\epsilon(N))$.

Theorem 5.3.1 (Dai, –) *If the scalar curvature R_{h_0} on the cross section (N^{n-1}, h_0) satisfies $R_{h_0} > (n-2)$, then*

$$(Lu, u)_{L^2} \geq \delta_0 \|u\|_{H^1(C_\epsilon(N))}$$

for all $u \in C_0^\infty(C_\epsilon(N))$, and some constant $\delta_0 > 0$ that depends on $\min_{x\in N}\{R_{h_0}(x)\}$ and n. In particular, the operator $(L, Dom(L) = C_0^\infty(C_\epsilon(N)))$ is strictly positive.

Proof: Because the manifold (N^{n-1}, h) is compact, and $R_{h_0} > (n-2)$, we have

$$\min_{x\in N}\{R_{h_0}(x)\} > (n-2).$$

And because

$$(n-1)(n-2) - \frac{4-\delta}{4}[(n-1)(n-3)+1] + \delta \to n-2, \quad \text{as} \quad \delta \searrow 0,$$

there exists $\delta_0 > 0$, such that

$$\min_{x\in N}\{R_{h_0}(x)\} > (n-1)(n-2) - \frac{4-\delta_0}{4}[(n-1)(n-3)+1] + \delta_0. \tag{5.3.1}$$

Set

$$L_{\delta_0} = -(4-\delta_0)\Delta + R - \frac{1}{r^2}\delta_0.$$

Then

$$L = L_{\delta_0} - \delta_0\Delta + \frac{1}{r^2}\delta_0,$$

and for any $u \in C_0^\infty(C_\epsilon(N))$,

$$\begin{aligned}
(Lu, u)_{L^2} &= \int_{C_\epsilon(N)} (Lu) u dvol_g \\
&= \int_{C_\epsilon(N)} (L_{\delta_0} u) u dvol_g \\
&\quad + \int_{C_\epsilon(N)} [(-\delta_0 \Delta u) u + \frac{1}{r^2}\delta_0 u^2] dvol_g \\
&= \int_{C_\epsilon(N)} (L_{\delta_0} u) u dvol_g \\
&\quad + \delta_0 \int_{C_\epsilon(N)} (|\nabla u|^2 + \frac{1}{r^2} u^2) dvol_g \\
&= (L_{\delta_0} u, u)_{L^2} + \delta_0 \|u\|_{H^1(C_\epsilon(N))}.
\end{aligned}$$

Thus it suffices to show that $(L_{\delta_0} u, u)_{L^2} \geq 0$.

Actually, we claim that

$$(L_{\delta_0} u, u)_{L^2} \geq C\|u\|_{L^2}, \tag{5.3.2}$$

for all $u \in C_0^\infty(C_\epsilon(N))$, where

$$C = \min\{\min_{x\in N}\{R_{h_0}(x)\} - [(n-1)(n-2) - \frac{4-\delta_0}{4}((n-1)(n-3)+1) + \delta_0], 1\} > 0.$$

Now we prove the claim (5.3.2). For any $u \in C_0^\infty(C_\epsilon(N))$, we can expand it as the following in terms of eigenfunctions $\psi_i(x)$ of operator $-(4-\delta_0)\Delta_{h_0} + R_{h_0} - \delta_0$ with eigenvalues μ_i,

$$u = \sum_{i=0}^{\infty} u_i(r)\varphi_i(x). \tag{5.3.3}$$

Then by using (5.1.1) and (5.1.2),

$$L_{\delta_0} u = \sum_{i=0}^{\infty}[-(4-\delta)u_i''(r) - (4-\delta)\frac{n-1}{r}u_i'(r) - \frac{1}{r^2}(-\mu_i + (n-1)(n-2))]\psi_i.$$

Let $\tilde{u}_i(r) = r^{\frac{n-1}{2}}u_i(r)$, then we have

$$L_{\delta_0} u = \sum_{i=0}^{\infty}[-(4-\delta_0)\tilde{u}_i'' + \frac{1}{r^2}(\mu_i - (n-1)(n-2) + \frac{4-\delta_0}{4}(n-1)(n-3))\tilde{u}_i(r)]r^{-\frac{n-1}{2}}\psi_i.$$

Because $\mu_i \to +\infty$ as $i \to +\infty$, we can take large enough $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, $\mu_i - (n-1)(n-2) + \frac{4-\delta_0}{4}(n-1)(n-3) > 1$.

$$
\begin{aligned}
&(L_{\delta_0}u,u)_{L^2}\\
&=\int_{C_\epsilon(N)}(L_{\delta_0}u)udvol_g\\
&=\int_0^\epsilon\int_N\{\sum_{i=0}^{\infty}[-(4-\delta_0)\tilde{u}_i''(r)+\frac{1}{r^2}(\mu_i-(n-1)(n-2)\\
&\quad+\frac{4-\delta_0}{4}(n-1)(n-3))\tilde{u}_i(r)]r^{-\frac{n-1}{2}}\psi_i\}\{\sum_{j=0}^{\infty}\tilde{u}_j(r)r^{-\frac{n-1}{2}}\psi_i\}r^{n-1}dvol_{h_0}dr\\
&=\int_0^\epsilon\sum_{i=0}^{\infty}[-(4-\delta_0)]\tilde{u}_i''(r)\tilde{u}_idr\\
&\quad+\int_0^\epsilon\sum_{i=0}^{\infty}\{\frac{1}{r^2}[\mu_i-(n-1)(n-2)+\frac{4-\delta_0}{4}(n-1)(n-3)](\tilde{u}_i(r))^2\}dr\\
&=\int_0^\epsilon\sum_{i=0}^{\infty}(4-\delta_0)(\tilde{u}_i'(r))^2dr\\
&\quad+\int_0^\epsilon\sum_{i=0}^{\infty}\{\frac{1}{r^2}[\mu_i-(n-1)(n-2)+\frac{4-\delta_0}{4}(n-1)(n-3)](\tilde{u}_i(r))^2\}dr\\
&=\int_0^\epsilon\sum_{i=0}^{i_0}(4-\delta_0)(\tilde{u}_i'(r))^2dr\\
&\quad+\int_0^\epsilon\sum_{i=0}^{i_0}\{\frac{1}{r^2}[\mu_i-(n-1)(n-2)+\frac{4-\delta_0}{4}(n-1)(n-3)](\tilde{u}_i(r))^2\}dr\\
&\quad\int_0^\epsilon\sum_{i=i_0+1}^{\infty}(4-\delta_0)(\tilde{u}_i'(r))^2dr\\
&\quad+\int_0^\epsilon\sum_{i=i_0+1}^{\infty}\{\frac{1}{r^2}[\mu_i-(n-1)(n-2)+\frac{4-\delta_0}{4}(n-1)(n-3)](\tilde{u}_i(r))^2\}dr\\
&=I+II.
\end{aligned}
$$

By using Hardy's inequality,

$$\begin{aligned}
I &= \int_0^{\epsilon} \sum_{i=0}^{i_0} (4-\delta_0)(\tilde{u}_i'(r))^2 dr \\
&\quad + \int_0^{\epsilon} \sum_{i=0}^{i_0} \{\frac{1}{r^2}[\mu_i - (n-1)(n-2) + \frac{4-\delta_0}{4}(n-1)(n-3)](\tilde{u}_i(r))^2\} dr \\
&\geq \int_0^{\epsilon} \sum_{i=0}^{i_0} \frac{4-\delta_0}{4}\frac{1}{r^2}(\tilde{u}_i(r))^2 dr \\
&\quad + \int_0^{\epsilon} \sum_{i=0}^{i_0} \{\frac{1}{r^2}[\mu_i - (n-1)(n-2) + \frac{4-\delta_0}{4}(n-1)(n-3)](\tilde{u}_i(r))^2\} dr \\
&\geq \{\min_{x\in N}\{R_{h_0}(x)\} - [(n-1)(n-2) \\
&\quad - \frac{4-\delta_0}{4}((n-1)(n-3)+1) + \delta_0]\} \int_0^{\epsilon} \frac{1}{r^2} \sum_{i=0}^{i_0} (\tilde{u}_i(r))^2 dr \\
&\geq C \int_0^{\epsilon} \sum_{i=0}^{i_0} (\tilde{u}_i(r))^2 dr,
\end{aligned}$$

and since $\mu_i - (n-1)(n-2) + \frac{4-\delta_0}{4}(n-1)(n-3) > 1$ for all $i > i_0$,

$$\begin{aligned}
II &= \int_0^{\epsilon} \sum_{i=i_0+1}^{\infty} (4-\delta_0)(\tilde{u}_i'(r))^2 dr \\
&\quad + \int_0^{\epsilon} \sum_{i=i_0+1}^{\infty} \{\frac{1}{r^2}[\mu_i - (n-1)(n-2) + \frac{4-\delta_0}{4}(n-1)(n-3)](\tilde{u}_i(r))^2\} dr \\
&\geq \int_0^{\epsilon} \sum_{i=i_0}^{\infty} (\tilde{u}_i(r))^2 dr \\
&\geq C \int_0^{\epsilon} \sum_{i=i_0}^{\infty} (\tilde{u}_i(r))^2 dr.
\end{aligned}$$

This proves (5.3.2). So we complete the proof. ∎

Corollary 5.3.2 (Dai, –) *If the scalar curvature R_{h_0} on (N^{n-1}, h_0) satisfies $R_{h_0} > (n-2)$, then the operator $(L, Dom(L) = C_0^{\infty}(C_{\epsilon}(N)))$ has a self-adjoint strictly pos-*

itive Friedrichs extension $(\widetilde{L}, Dom(\widetilde{L}))$*. Moreover,* $Dom(\widetilde{L}) \subset H^1(C_\epsilon(N))$*, and the image* $Ran(\widetilde{L}) = L^2(C_\epsilon(N))$*.*

Proof: The existence of the self-adjoint strictly positive and surjective extension follows from the Neumann Theorem in [EK], because the operator $(L, Dom(L))$ is strictly positive by Theorem 5.3.1. Moreover, from Theorem 5.3.1, we can obtain that the completion of $C_0^\infty(C_\epsilon(N))$ with respect to the norm $\|u\|_L = (Lu, u)_{L^2}$ is a subspace of $H^1(C_\epsilon(N))$. Thus from the construction of the Friedrichs extension in the proof of the Neumann theorem in [EK], we can easily see that $Dom(\widetilde{L}) \subset H^1(C_\epsilon(N))$.

■

Theorem 5.3.3 (Dai, –) *If the scalar curvature of* (N^{n-1}, h_0)*,* $R_{h_0} > (n-2)$*, then the spectrum of the Friedrichs extension of the operator* $-4\Delta + R$ *on* $(C_\epsilon(N), g = dr^2 + r^2 h_0)$ *consists of discrete eigenvalues with finite multiplicities*

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots,$$

and $\lambda_k \to +\infty$ *as* $k \to +\infty$*.*

Moreover, eigenfunctions $\{\varphi_i\}_{i=1}^\infty$ *form a basis of* $L^2(C_\epsilon(N))$*.*

Proof: By the Corollary 5.3.2, the Friedrichs extension $\widetilde{L} : Dom(\widetilde{L}) \to L^2(C_\epsilon(N))$ is one-to-one and onto. And its inverse

$$\widetilde{L}^{-1} : L^2(C_\epsilon(N)) \to Dom(\widetilde{L}) \hookrightarrow H^1(C_\epsilon(N)) \hookrightarrow L^2(C_\epsilon(N))$$

is a self-adjoint compact operator, because the embeddding $H^1(C_\epsilon(N)) \hookrightarrow L^2(C_\epsilon(N))$ is compact. Then the spectrum theorem of self-adjoint compact operators completes the proof. ■

5.4 Spectrum of $-4\Delta + R$ on compact manifolds with a single conical singularity

In this section, we study the spectrum of the operator $-4\Delta + R$ on compact Riemannian manifolds with a single conical singularity. By using the semi-boundedness estimate for the operator $-4\Delta + R$ on a small finite cone, we establishing the same estimate for the operator $-4\Delta + R$ on compact Riemannian manifolds with a single conical singularities. And then, we prove that the spectrum of the operator $-4\Delta + R$ on compact Riemannian manifolds with a single conical singularity consists of discrete eigenvalues with finite multiplicities.

Theorem 5.4.1 (Dai, –) *Let (M^n, g, p) be a compact Riemannian manifold with a single conical singularity at p. If the scalar curvature R_{h_0} on (N^{n-1}, h_0) satisfies $R_{h_0} > (n-2)$, then there exists a large enough constant A, such that the operator $L_A = L + A$ satisfies:*

$$(L_A u, u)_{L^2(M)} \geq C\|u\|_{H^1(M)}$$

for all $u \in C_0^\infty(M \setminus \{p\})$ and some constant $C > 0$. In particular, the operator $(L_A, Dom(L_A) = C_0^\infty(M \setminus \{p\}))$ is strictly positive.

Proof: The conical neighborhood $(U_p \setminus \{p\}, g|_{U_p\setminus\{p\}})$ of conical singularity p is isometric to $((0,\epsilon) \times N, dr^2 + r^2 h_r)$, where $h_r = h_0 + o(r^\alpha)$, for some $\alpha > 0$. Then the scalar curvature on the conical neighborhood is given by

$$\begin{aligned} R_g &= \frac{1}{r^2}(R_{h_r} - (n-1)(n-2) + o(r^\alpha)) \\ &= \frac{1}{r^2}(R_{h_0} - (n-1)(n-2) + o(r^\alpha)). \end{aligned} \tag{5.4.1}$$

Because $R_{h_0} > (n-2)$, there exists $\beta(n) \in (0,1)$ such that

$$R_{h_0} > (n-1)[\frac{1}{\beta(n)^{2n}}(n-2) - (n-3)] - 1.$$

Then there exists $\epsilon(n) > 0$, such that on $(0, \epsilon(n)) \times N$,

$$\beta(n)^2 g_0 \le g \le \frac{1}{\beta(n)^2} g_0,$$

$$\beta(n) R_{h_0} \le r^2 R_g + (n-1)(n-2) \le \frac{1}{\beta(n)} R_{h_0}.$$

For any $u \in C_0^\infty((0, \epsilon(n)) \times N)$, we have

$$\begin{aligned}
(Lu, u)_{L^2(C_{\epsilon(n)}(N))} &= \int_{C_{\epsilon(n)}(N)} (-4\Delta u + Ru) u dvol_g \\
&= \int_{C_{\epsilon(n)}(N)} (4|\nabla u|^2 + Ru^2) dvol_g \\
&\ge \int_{C_{\epsilon(n)}(N)} [4\beta(n)^n |\nabla u|^2_{g_0} + \frac{1}{r^2}\beta(n)^n R_{h_0} u^2 \\
&\quad + \frac{1}{r^2}(-(n-1)(n-2))\frac{1}{\beta(n)^n}] dvol_{g_0} \\
&= \beta(n)^n \int_{C_{\epsilon(n)}(N)} [-4\Delta_{g_0} u \\
&\quad + \frac{1}{r^2}(R_{g_0} - \beta(n)^{-2n}(n-1)(n-2))u] u dvol_{g_0} \\
&\ge \beta(n)^n C_1 \|u\|_{H^1(C_{\epsilon(n)}(N))}
\end{aligned}$$

The last inequality follows the same argument as in theorem 5.3.1, i.e. for any $u \in C_0^\infty((0, \epsilon(n)) \times N)$,

$$(Lu, u)_{L^2(C_{\epsilon(n)}(N))} \ge \beta(n)^n C_1 \|u\|_{H^1(C_{\epsilon(n)}(N))} \tag{5.4.2}$$

We cover the manifold M by the conical neighborhood $(0, \epsilon(n)) \times N$ of singularity p and the interior part $M_0 = M \backslash C_{(0, \frac{1}{8}\epsilon(n))}(N)$. We construct a partition of unity subordinate to this covering as following. Let ρ_1 a function on $C_\epsilon(N)$ satisfying

$$\rho_1(r,x) = \begin{cases} 1, & 0 < r < \frac{\epsilon(n)}{4} \\ 0, & r > \frac{\epsilon(n)}{2}, \end{cases}$$

with $0 \leq \rho_1(r,x) \leq 1$. We extend ρ_1 trivially to the whole M, and we still use ρ_1 to denote the extended function. Let $\rho_2 = 1 - \rho_1$. Then $\{\rho_1, \rho_2\}$ is a partition of unity subordinate to the covering.

For any $u \in C_0^\infty(M)$,

$$\begin{aligned} (L_B u, u) &= \int_M (L_B u_1 + L_B u_2)(u_1 + u_2) dvol_g \\ &= \int_M (L_B u_1) u_1 dvol_g + \int_M (L_B u_1) u_2 dvol_g \\ &\quad + \int_M (L_B u_2) u_1 dvol_g + \int_M (L_B u_2) u_2 dvol_g, \end{aligned}$$

where $u_1 = \rho_1 u$, $u_2 = \rho_2 u$, and $L_B = L + B$ for some $B > 0$.

By (5.4.2), we have

$$\int_M (L_B u_1) u_1 dvol_g \geq \beta(n)^n C_1 \int_M (\chi^2 |u_1|^2 + |\nabla u_1|^2) dvol_g,$$

where C_1 is a positive constant.

Because u_2 is compactly supported in M_0 and R is bounded on $\overline{M_0}$, i.e. there exists $C_2 < 0$ such that $R > C_2$ on M_0, we have

$$\begin{aligned}\int_M (L_B u_2) u_2 dvol_g &= \int_{M_0} (-4\Delta u_2 + (R+B)u_2) u_2 dvol_g \\ &= \int_{M_0} (4|\nabla u_2|^2 + (R+B)|u_2|^2) dvol_g \\ &\geq C_2 \int_{M_0} (|\nabla u_2|^2 + \chi^2 |u_2|^2) dvol_g\end{aligned}$$

By integration by parts,

$$\begin{aligned}\int_M (L_B u_1) u_2 dvol_g &= \int_M (L_B u_2) u_1 dvol_g \\ &= \int_M \langle \nabla u_1, \nabla u_2 \rangle dvol_g + \int_M (R+B) u_1 u_2 dvol_g \\ &= \int_M \langle u \nabla \rho_1 + \rho_1 \nabla u, u \nabla \rho_2 + \rho_2 \nabla u \rangle dvol_g \\ &\quad + \int_M (R+B) u_1 u_2 dvol_g \\ &= \int_M u^2 (\partial_r \rho_1)(\partial_r \rho_2) dvol_g + \int_{C_\epsilon(N)} u \rho_2 (\partial_r \rho_1)(\partial_r u) dvol_g \\ &\quad + \int_{C_\epsilon(N)} u \rho_1 (\partial_r \rho_2)(\partial_r u) dvol_g + \int_{C_\epsilon(N)} \rho_1 \rho_2 |\nabla u|^2 dvol_g \\ &\quad + \int_M (R+B) u_1 u_2 dvol_g.\end{aligned}$$

Then we have

$$\int_M u^2 (\partial_r \rho_1)(\partial_r \rho_2) dvol_g > C_3 \int_M u^2 dvol_g,$$

$$\begin{aligned}
\int_{C_\epsilon(N)} u\rho_2(\partial_r\rho_1)(\partial_r u)dvol_g &= \int_0^\epsilon \int_N u\rho_2(\partial_r\rho_1)(\partial_r u)r^{n-1}dvol_{h_r}dr \\
&= -\frac{1}{2}\int_0^\epsilon \int_N (\partial_r\rho_2)(\partial_r\rho_1)u^2 r^{n-1}dvol_{h_r}dr \\
&\quad -\frac{1}{2}\int_0^\epsilon \int_N \rho_2(\partial_r^2\rho_1)u^2 r^{n-1}dvol_{h_r}dr \\
&\quad -\frac{1}{2}\int_0^\epsilon \int_N u^2\frac{\rho_2(\partial_r\rho_1)}{r}(n-1)r^{n-1}dvol_{h_r}dr \\
&\quad -\frac{1}{2}\int_0^\epsilon \int_N u^2\rho_2(\partial_r\rho_1)tr(h_r^{-1}\frac{\partial}{\partial r}h_r)r^{n-1}dvol_{h_r}dr \\
&> C_4\int_M u^2 dvol_g,
\end{aligned}$$

for some negative constant C_3 and C_4.

Similarly, we have

$$\int_{C_\epsilon(N)} u\rho_1(\partial_r\rho_2)(\partial_r u)dvol_g > C_5\int_M u^2 dvol_g,$$

for some constant C_5.

Thus

$$\begin{aligned}
\int_M (L_B u_1)u_2 dvol_g &> \int_M (\rho_1\rho_2|\nabla u|^2 + (R+B)u_1u_2)dvol_g \\
&\quad + (C_3+C_4+C_5)\int_M u^2 dvol_g \\
&> \int_M (\rho_1\rho_2|\nabla u|^2 + u_1u_2)dvol_g \\
&\quad + C_6\int_M u^2 dvol_g,
\end{aligned}$$

where, $C_6 = C_3 + C_4 + C_5$

Similarly, we can show

$$(u_1, u_2)_{H_2^1(M)} < C_7 \int_M (\rho_1\rho_2|\nabla u|^2 + u_1u_2)dvol + C_8 \int_M u^2 dvol_g,$$

for some $C_7 > 0$, such that $\frac{1}{C_7} < \beta(n)^n C_1, C_2$, and $C_8 > 0$

Thus

$$\int_M (L_B u_1)u_2 dvol_g > \frac{1}{C_7}(u_1, u_2)_{H_2^1(M)} + (C_6 - \frac{C_8}{C_7}) \int_M u^2 dvol_g,$$

and therefore,

$$\int_M (L_B u)u dvol_g > \frac{1}{C_7}(u, u)_{H_2^1(M)} + 2(C_6 - \frac{C_8}{C_7}) \int_M u^2 dvol_g.$$

Let $A = B + 2(\frac{C_8}{C_7} - C_6)$, then we have

$$\int_M (L_A u)u dvol_g > \frac{1}{C_7}(u, u)_{H_2^1(M)},$$

in particular,

$$(L_A u, u)_{L^2} > \frac{1}{C_7}\|u\|_{L^2}^2,$$

i.e. $(L = -4\Delta + R, Dom(L) = C_0^\infty(M))$ is strictly positive. ■

Theorem 5.4.2 (Dai, –) *Let (M, g, p) be a compact Riemannian manifold with a single conical singularity p. If the scalar curvature R_{h_0} on (N^{n-1}, h_0) satisfies $R_{h_0} > (n-2)$, then the spectrum of the Friedrichs extension of the operator $-4\Delta + R$ on (M, g, p) consists of discrete eigenvalues with finite multiplicity*

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots,$$

and $\lambda_k \to +\infty$.

Moreover, eigenfunctions $\{\varphi_i\}_{i=1}^{\infty}$ *form a basis of* $L^2(M)$.

Proof: The proof is the same as the proof of the Theorem 5.3.3 ■

5.5 Asymptotic behavior of eigenfunctions of $-4\Delta + R$ on compact manifolds with a single cone-like singularity

In this section, we obtain an asymptotic expansion for eigenfunctions of $-4\Delta + R$ near a singularity on manifolds with cone-like singularities, on which we can explicitly express eigenfunction in terms of some hpyergeometric functions, and eigenvalues and eigenfunctions on the cross section.

Let (M^n, g, p) be a compact Riemannian manifold with cone-like singularity p, and U_p be a neighborhood of p so that $U_p \backslash \{p\}$ is diffeomorphic to $(0, \varepsilon) \times N$, and on $U_p \backslash \{p\}$, $g = dr^2 + r^2 h_0$. Let $\mu_1 \leq \mu_2 \leq \mu_3 \leq \cdots$ be eigenvalues of the operator $-4\Delta_{h_0} + R_{h_0}$ on the Riemannian manifold (N, h_0), which is the cross section of conical part of (M^n, g, p), and $\psi_1, \psi_2, \psi_3, \cdots$ be corresponding normalized eigenfunctions, i.e. $\|\psi_i\|_{L^2} = 1$.

By using the classical Sobolev embedding theorem and elliptic regularity, let $s =$

$2n$, we have

$$
\begin{aligned}
\|\psi_i\|_{L^\infty} &\leq K_s\|\psi_i\|_{W^{s,2}} \\
&\leq K_s C_s(\|\psi_i\|_{W^{s-2,2}} + \|(-4\Delta_{h_0} + R_{h_0})\psi_i\|_{W^{s-2,2}}) \\
&= K_s C_s(1 + |\mu_i|)\|\psi_i\|_{W^{s-2,2}} \\
&\dots \\
&\leq C(1 + |\mu_i|)^n\|\psi_i\|_{W^{0,2}} \\
&= C(1 + |\mu_i|)^n.
\end{aligned}
$$

Let u be an eigenfunction of the operator $-4\Delta_g + R(g)$ with eigenvalue λ, i.e.

$$-4\Delta_g u + R(g)u = \lambda u. \tag{5.5.1}$$

On the conical neighborhood $U_p\backslash\{p\}$, we do the the following expansion for the eigenfunction u:

$$u = \sum_{i=1}^{+\infty} u_i(r)\psi_i(x), \tag{5.5.2}$$

where x is the coordinate on N.

By plugging (5.1.1), (5.1.2), and (5.5.2) in the equation (5.5.1), we have

$$\begin{aligned}
-4\Delta_g u + R(g)u &= -4(\partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_{g_0})\sum_{i=1}^{+\infty}(u_i\psi_i) \\
&\quad + \frac{1}{r^2}(R(g_0) - (n-1)(n-2))\sum_{i=1}^{+\infty}(u_i\psi_i) \\
&= \sum_{i=1}^{+\infty}(-4u_i^{''}\psi_i - 4\frac{n-1}{r}u_i^{'}\psi_i - 4\frac{1}{r^2}u_i\Delta_{g_0}\psi_i) \\
&\quad + \sum_{i=1}^{+\infty}(\frac{1}{r^2}R(g_0)u_i\psi_i - (n-1)(n-2)\frac{1}{r^2}u_i\psi_i) \\
&= \sum_{i=1}^{+\infty}[-4u_i^{''}\psi_i - 4\frac{n-1}{r}u_i^{'}\psi_i + \frac{1}{r^2}\mu_i\psi_i - (n-1)(n-2)\frac{1}{r^2}u_i\psi_i] \\
&= \sum_{i=1}^{+\infty}[-4u_i^{''} - 4\frac{n-1}{r}u_i^{'} - \frac{1}{r^2}(-\mu_i + (n-1)(n-2))u_i]\psi_i \\
&= \sum_{i=1}^{+\infty}\lambda u_i\varphi_i.
\end{aligned}$$

Thus we obtain the following equations.

$$-4u_i^{''} - 4\frac{n-1}{r}u_i^{'} - \frac{1}{r^2}(-\mu_i + (n-1)(n-2))u_i = \lambda u_i.$$

We rearrange it to get the equation

$$u_i^{''} + \frac{n-1}{r}u_i^{'} + \frac{1}{4}(\mu - \frac{1}{r^2}(\lambda_i - (n-1)(n-2)))u_i = 0. \tag{5.5.3}$$

Now let's solve the equation (5.5.3) in three cases for different signs of λ.

Case 1: $\lambda = 0$. Then equation (5.5.3) becomes the following Euler equation.

$$u_i^{''} + \frac{n-1}{r}u_i^{'} - \frac{1}{4}\frac{1}{r^2}[\mu_i - (n-1)(n-2)]u_i = 0 \tag{5.5.4}$$

By directly solving equation (5.5.4), we obtain

$$u_i(r) = A_i r^{-\frac{n-2}{2}+\frac{\sqrt{\mu_i-(n-2)}}{2}} + B_i r^{-\frac{n-2}{2}-\frac{\sqrt{\mu_i-(n-2)}}{2}}, \tag{5.5.5}$$

where A_i and B_i are some constants. And because $u = \sum_{i=0}^{+\infty} u_i\psi_i(x) \in L^2(M,g)$, obviously for large i, $u_i(r) = A_i r^{-\frac{n-2}{2}+\frac{\sqrt{\mu_i-(n-2)}}{2}}$, i.e. $B_i = 0$ for large i.

Case 2: $\lambda > 0$. In this case, we will make a transformation for our function to get a Bessel equation.

Let

$$h_i(r) = (\frac{\sqrt{\lambda}}{2})^{-\frac{n-2}{2}} r^{\frac{n-2}{2}} u_i(\frac{2r}{\sqrt{\lambda}}).$$

Then

$$u_i(r) = r^{-\frac{n-2}{2}} h_i(\frac{\sqrt{\lambda}}{2}r),$$

$$u_i^{'}(r) = (-\frac{n-2}{2})r^{-\frac{n-2}{2}-1}h_i(\frac{\sqrt{\lambda}r}{2}) + \frac{\sqrt{\lambda}}{2}r^{-\frac{n-2}{2}}h_i^{'}(\frac{\sqrt{\lambda}r}{2}),$$

$$\begin{aligned} u_i^{''}(r) = &(-\frac{n-2}{2})(-\frac{n-2}{2}-1)r^{-\frac{n-2}{2}-2}h_i(\frac{\sqrt{\lambda}r}{2}) \\ &+ \sqrt{\lambda}(-\frac{n-2}{2})r^{-\frac{n-2}{2}-1}h_i^{'}(\frac{\sqrt{\lambda}r}{2}) + \frac{\lambda}{4}r^{-\frac{n-2}{2}}h_i^{''}(\frac{\sqrt{\lambda}r}{2}). \end{aligned}$$

Plugging them in the equation (5.5.3), we obtain the following Bessel equation.

$$h_i^{''}(\frac{\sqrt{\lambda}r}{2}) + \frac{1}{\frac{\sqrt{\lambda}r}{2}}h_i^{'}(\frac{\sqrt{\lambda}r}{2}) + [1 - \frac{1}{\frac{\lambda r^2}{4}}\frac{1}{4}(\mu_i - (n-2))]h_i(\frac{\sqrt{\lambda}r}{2}) = 0.$$

Thus

$$h_i(\frac{\sqrt{\lambda}r}{2}) = A_i J_{\frac{1}{2}\sqrt{\mu_i-(n-2)}}(\frac{\sqrt{\lambda}r}{2}) + B_i Y_{\frac{1}{2}\sqrt{\mu_i-(n-2)}}(\frac{\sqrt{\lambda}r}{2}),$$

where A_i and B_i are some constants, and $J_\nu(z)$ and $Y_\nu(z)$ Bessel functions of first and second kind respectively. Hence we obtain

$$u_i(r) = A_i r^{-\frac{n-2}{2}} J_{\frac{1}{2}\sqrt{\mu_i-(n-2)}}(\frac{\sqrt{\lambda}r}{2}) + B_i r^{-\frac{n-2}{2}} Y_{\frac{1}{2}\sqrt{\mu_i-(n-2)}}(\frac{\sqrt{\lambda}r}{2}). \tag{5.5.6}$$

Bessel functions have the following asymptotic behavior. If $\nu \to +\infty$ through real values, with $z \neq 0$ fixed, then

$$J_\nu(z) \sim \frac{1}{\sqrt{2\pi\nu}}(\frac{ez}{2\nu})^\nu,$$

$$Y_\nu(z) \sim -\sqrt{\frac{2}{\pi\nu}}(\frac{ez}{2\nu})^{-\nu}.$$

Thus as in Case 1, for large i, $u_i(r) = A_i r^{-\frac{n-2}{2}} J_{\frac{1}{2}\sqrt{\mu_i-(n-2)}}(\frac{\sqrt{\lambda}r}{2})$, i.e. $B_i = 0$.

Case 3: $\lambda < 0$. We use the results in [Bat53]. If $1 + \sqrt{\mu_i - (n-2)}$ is not an integer, then

$$\begin{aligned} u_i =& A_i r^{-\frac{n-1}{2}} (\sqrt{-\lambda}r)^{\frac{1+\sqrt{\mu_i-(n-2)}}{2}} e^{-\frac{\sqrt{-\lambda}r}{2}} \sum_{k=0}^{+\infty} \frac{(\frac{1+\sqrt{\mu_i-(n-2)}}{2})_k}{(1+\sqrt{\mu_i-(n-2)})_k} \frac{(\sqrt{-\lambda}r)^k}{k!} \\ &+ B_i r^{-\frac{n-1}{2}} (\sqrt{-\lambda}r)^{\frac{1-\sqrt{\mu_i-(n-2)}}{2}} e^{-\frac{\sqrt{-\lambda}r}{2}} \sum_{k=0}^{+\infty} \frac{(\frac{1-\sqrt{\mu_i-(n-2)}}{2})_k}{(1-\sqrt{\mu_i-(n-2)})_k} \frac{(\sqrt{-\lambda}r)^k}{k!}, \end{aligned} \tag{5.5.7}$$

where A_i and B_i are some constants, and $(x)_k = x(x+1)\cdots(x+n-1)$.

If $1+\sqrt{\mu_i-(n-2)}=1+m$ is a positive integer, then

$$\begin{aligned} u_i =& A_i r^{-\frac{n-1}{2}}(\sqrt{-\lambda}r)^{\frac{m+1}{2}} e^{-\frac{\sqrt{-\lambda}r}{2}} \sum_{k=0}^{+\infty} \frac{(\frac{1+m}{2})_k}{(1+m)_k}\frac{(\sqrt{-\lambda}r)^k}{k!} \\ &+ B_i r^{-\frac{n-1}{2}}(\sqrt{-\lambda}r)^{\frac{1+m}{2}} e^{-\frac{\sqrt{-\lambda}r}{2}} \frac{(-1)^{m-1}}{m!\Gamma(\frac{1-m}{2})}\{(\sum_{k=0}^{+\infty}\frac{(\frac{1+m}{2})_k}{(1+m)_k}\frac{(\sqrt{-\lambda}r)^k}{k!})\log(\sqrt{-\lambda}r) \\ &+ \sum_{k=0}^{+\infty}\frac{(\frac{1+m}{2})_k}{(1+m)_k}(\psi(\frac{1+m}{2}+k)-\psi(1+k)-\psi(1+m+k))\frac{(\sqrt{-\lambda}r)^k}{k!} \\ &+ \frac{(m-1)!}{\Gamma(\frac{1+m}{2})}\sum_{k=0}^{m-1}\frac{(\frac{1-m}{2})_k}{(1-m)_k}\frac{(\sqrt{-\lambda}r)^{k-m}}{k!}\}, \end{aligned} \tag{5.5.8}$$

where A_i and B_i are some constants, and $\psi(x)$ is the logarithmic derivative of the Gamma function $\Gamma(x)$. And as in the previous cases, for the large i, $B_i=0$.

Combining the above explicit computations and estimates for eigenfunctions φ_i, we obtain the following asymptotic behavior for eigenfunction u.

Theorem 5.5.1 (Dai, –) *Let (M^n,g,p) be a compact Riemannian manifold with a single cone-like singularity p with $R_{h_0}>(n-2)$, and u be an eigenfunction of the operator $-4\Delta_g+R_g$ on M. Then u has an asymptotic expansion at the conical singularity p as*

$$u\sim\sum_{j=1}^{+\infty}\sum_{l=0}^{+\infty}\sum_{p=0}^{p_j} r^{s_j+l}(lnr)^p u_{j,l,p},$$

where $u_{j,l,p}\in C^\infty(N^{n-1})$, $p_j=0$ or 1, and $s_j=-\frac{n-2}{2}\pm\frac{\sqrt{\mu_j-(n-2)}}{2}$, where μ_j are eigenvalues of $-\Delta_{h_0}+R_{h_0}$ on N^{n-1}.

Proof: On the conical part $U_p\setminus\{p\}$, as above we expand an eigenfunction with eigenvalue λ as

$$u(r,x)=\sum_{i=1}^{\infty}u_i(r)\psi_i(x).$$

In the rest of the proof, we set

$$\nu_i = \sqrt{\mu_i - (n-2)}$$

.

If $\lambda = 0$, by the solution (5.5.5), there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$,

$$u_i(r) = A_i r^{-\frac{n-2}{2}+\frac{\nu_i}{2}}.$$

For a fixed r_0, $u(r_0, x) \in L^2(N)$, and

$$+\infty > \|u(r_0, x)\|_{L^2(N)} = \sum_i^{\infty} |u_i(r_0)|^2 \geq \sum_{i=i_0}^{\infty} |A_i|^2 r_0^{-(n-2)+\nu_i}.$$

Then for all $r < r_0$,

$$\begin{aligned}\sum_{i=i_0}^{\infty} |u_i(r)\psi_i(x)| &\leq C \sum_{i=i_0}^{\infty} |A_i| r^{-\frac{n-2}{2}+\frac{\nu_i}{2}} (1+|\mu_i|)^n \\ &= C \sum_{i=i_0}^{\infty} |A_i| r_0^{-\frac{n-2}{2}+\frac{\nu_i}{2}} (1+|\mu_i|)^n (\frac{r}{r_0})^{-\frac{n-2}{2}+\frac{\nu_i}{2}} \\ &\leq C (\sum_{i=i_0}^{\infty} |A_i|^2 r_0^{-(n-2)+\nu_i})^{\frac{1}{2}} (\sum_{i=i_0}^{\infty} (1+|\mu_i|)^{2n} (\frac{r}{r_0})^{-(n-2)+\nu_i})^{\frac{1}{2}} \\ &< +\infty\end{aligned}$$

If $\lambda > 0$, by the solution (5.5.6), there exists $i_1 \in \mathbb{N}$ such that for all $i \geq i_1$,

$$u_i(r) = A_i r^{-\frac{n-2}{2}} J_{\frac{1}{2}\nu_i}(\frac{\sqrt{\lambda} r}{2}) = A_i r^{-\frac{n-2}{2}} (\frac{\sqrt{\lambda} r}{4})^{\frac{\nu_i}{2}} \sum_{m=0}^{\infty} \frac{(\frac{\sqrt{\lambda} r}{4})^{2m}}{m!\Gamma(\frac{1}{2}\nu_i + m + 1)}.$$

Fix $r_0 > 0$. Then for $r \leq r_0$ and $i > i_1$,

$$|A_i| r^{-\frac{n-2}{2}} \frac{1}{\Gamma(\frac{1}{2}\nu_i + 1)} (\frac{\sqrt{\lambda} r}{4})^{\frac{\nu_i}{2}} < |u_i(r)| < |A_i| r^{-\frac{n-2}{2}} \frac{C(r_0)}{\Gamma(\frac{1}{2}\nu_i + 1)} (\frac{\sqrt{\lambda} r}{4})^{\frac{\nu_i}{2}},$$

where $C(r_0) = e^{\frac{\lambda r_0^2}{16}}$. Then

$$+\infty > \|u(r_0, x)\|_{L^2(N)} = \sum_{i=0}^{\infty} |u_i(r_0)|^2 \geq \sum_{i=i_1}^{\infty} |A_i|^2 r_0^{-(n-2)} \frac{1}{(\Gamma(\frac{1}{2}\nu_i + 1))^2} (\frac{\sqrt{\lambda} r_0}{4})^{\nu_i}.$$

Thus for all $r < r_0$,

$$\sum_{i=i_1}^{\infty} |u_i(r) \varphi_i(x)| \leq C(r_0) C \sum_{i=i_1}^{\infty} |A_i| r^{-\frac{n-2}{2}} \frac{1}{\Gamma(\frac{1}{2}\nu_i + 1)} (\frac{\sqrt{\lambda} r}{4})^{\frac{\nu_i}{2}} (1 + |\mu_i|)^n < +\infty.$$

If $\lambda < 0$, by the solutions (5.5.7) and (5.5.8) there exists $i_2 \in \mathbb{N}$ such that for all $i \geq i_2$

$$u_i = A_i r^{-\frac{n-1}{2}} (\sqrt{-\lambda} r)^{\frac{1+\nu_i}{2}} e^{-\frac{\sqrt{-\lambda} r}{2}} \sum_{k=0}^{\infty} \frac{(\frac{1+\nu_i}{2})_k}{(1+\nu_i)_k} \frac{(\sqrt{-\lambda} r)^k}{k!}.$$

Then for $r < r_0$ and $i > i_2$

$$|A_i| r^{-\frac{n-1}{2}} (\sqrt{-\lambda} r)^{\frac{1+\nu_i}{2}} \leq |u_i(r)| \leq e^{\frac{\sqrt{-\lambda} r_0}{2}} |A_i| r^{-\frac{n-1}{2}} (\sqrt{-\lambda} r)^{\frac{1+\nu_i}{2}}.$$

Thus as above,

$$\sum_{i=i_2}^{\infty} |u_i(r) \psi_i(x)| \leq e^{\frac{\sqrt{-\lambda} r_0}{2}} C \sum_{i=i_2}^{\infty} |A_i| r^{-\frac{n-1}{2}} (\sqrt{-\lambda} r)^{\frac{1+\nu_i}{2}} (1 + |\mu_i|)^n < +\infty.$$

Hence in all three cases, $\sum_{i=1}^{\infty} u_i(r) \psi_i(x)$ absolutely converges to $u(r, x)$ for all $r < r_0$ uniformly about $x \in N$. By plugging (5.5.5), (5.5.6), (5.5.7) or (5.5.8) in $u(r, x) = \sum_{i=1}^{\infty} u_i(r) \psi_i(x)$, we obtain the asymptotic expansion.

Similarly, we can show that derivatives of the expansion series with respect to r variable also absolutely converge. And then we complete the proof. ∎

Corollary 5.5.2 (Dai, –) *Let (M^n, g, p) be a compact Riemannian manifold with a single cone-like singularity p with $R_{h_0} > (n-2)$. The eigenfunctions of $-4\Delta_g + R_g$ on satisfy*

$$u = o(r^{-\frac{n-2}{2}}), \quad as \;\; r \to 0.$$

Consequently, the first eigenvalue is simple.

Proof: By combining the fact that eigenfunctions in $H^1(M)$ and the asymptotic expansion in Theorem 5.5.1, we obtain the asymptotic order in the Corollary. And this asymptotic order enable the proof of Courant's nodal domain theorem in [Cha] work on manifolds with a single cone-like singularity with $R_{h_0} > n-2$. Thus, the first eigenvalue is simple. ∎

5.6 Asymptotic behavior of eigenfunctions of $-4\Delta + R$ on compact manifolds with a single conical singularity

In this section, we obtain an asymptotic order for eigenfunctions near the singularity on manifolds with a single conical singularity. For this purpose, we first establish Sobolev inequality and elliptic estimate for weighted norms on a finite cone analogous to that on $\mathbb{R}^n$ in [Bar86].

We first work on a finite cone $(C_\epsilon(N) = (0,\epsilon) \times N, g = dr^2 + r^2 h_0)$. We define weighted uniform C^k-norms on a finite cone $C_\epsilon(N)$ as

$$\|u\|_{C^k_\delta} = \sup_{C_\epsilon(N)} (\sum_{i=0}^{k} r^{i-\delta}|\nabla^i u|), \tag{5.6.1}$$

for $k \in \mathbb{N}$ and $\delta \in \mathbb{R}$. When $k = 0$, we use C_δ to denote C^0_δ. Then similar to (iv) of Theorem 1.2 in [Bar86], we use scaling technique to obtain the following weighted Sobolev inequality.

Lemma 5.6.1 (Dai, –) *If $u \in H^k_\delta(C_\epsilon(N))$, and $k > \frac{n}{2} + l$, then*

$$\|u\|_{C^l_\delta(C_\epsilon(N))} \leq C\|u\|_{H^k_\delta(C_\epsilon(N))}, \tag{5.6.2}$$

for some constant $C = C(n, k, \delta, \epsilon)$.

Moreover,

$$|\nabla^l u(r, x)| = o(r^{-l+\delta}) \quad as \quad r \to 0.$$

Proof: Let $u(r, x)$ be a function on the finite cone $C_\epsilon(N)$, where x is a coordinate on N, and set

$$u_a(r, x) = u(ar, x), \tag{5.6.3}$$

for a positive constant a. And let $C_{r_1,r_2} = (r_1, r_2) \times N$ be a annulus on the finite cone $C_\epsilon(N)$, for $r_1 < r_2 \leq \epsilon$. Then by a simple change of variables, we have

$$\|u\|_{H^k_\delta(C_{ar_1,ar_2})} = a^{-\delta}\|u_a\|_{H^k_\delta(C_{r_1,r_2})}, \tag{5.6.4}$$

and

$$\|u\|_{C^l_\delta(C_{ar_1,ar_2})} = a^{-\delta}\|u_a\|_{C^l_\delta(C_{r_1,r_2})}. \tag{5.6.5}$$

Let $C_j = ((\frac{1}{2})^{j+1}\epsilon, (\frac{1}{2})^j\epsilon) \times N$ be an annulus on the cone $C_\epsilon(N)$. For any fixed

$j \in \mathbb{N}$, by choosing $a = (\frac{1}{2})^j$, $r_1 = (\frac{1}{2})\epsilon$, and $r_2 = \epsilon$ in (5.6.4) and (5.6.5), and using the usual Sobolev inequality, we have

$$\begin{aligned}
\|u\|_{C^l_\delta(C_j)} &= (\frac{1}{2})^{-j\delta}\|u_{(\frac{1}{2})^j}\|_{C^l_\delta(C_0)} \\
&\leq (\frac{1}{2})^{-j\delta}C\|u_{(\frac{1}{2})^j}\|_{H^k_\delta(C_0)} \\
&= C\|u\|_{H^k_\delta(C_j)} \\
&\leq C\|u\|_{H^k_\delta(C_\epsilon(N))},
\end{aligned}$$

where the constant C is independent of j. Therefore, we obtain the Sobolev inequality

$$\|u\|_{C_\delta(C_\epsilon(N))} \leq C\|u\|_{H^k_\delta(C_\epsilon(N))}.$$

Because $\|u\|_{H^k_\delta(C_\epsilon((N))} < \infty$ we have $\|u\|_{H^k_\delta(C_j)} = o(1)$ as $j \to \infty$. Therefore, we have $|\nabla^l u(r,x)| = o(r^{-l+\delta})$ as $r \to 0$, since $\sup_{(\frac{1}{2})^{j+1}\epsilon < r < (\frac{1}{2})^j\epsilon} r^{l-\delta}|\nabla^l u(r,x)| \leq \|u\|_{C^l_\delta(C_j)} \leq C\|u\|_{H^k_\delta(C_j)}$. ■

Similar to Proposition 1.6 in [Bar86], we also have the following elliptic estimate.

Lemma 5.6.2 (Dai, –) *If* $u \in H^{k-2}_\delta(C_\epsilon(N))$, *and* $Lu \in H^{k-2}_{\delta-2}(C_\epsilon(N))$, *then*

$$\|u\|_{H^k_\delta(C_\epsilon(N))} \leq C(\|Lu\|_{H^{k-2}_{\delta-2}(C_\epsilon(N))} + \|u\|_{H^{k-2}_\delta(C_\epsilon(N))}),$$

for some constant $C = C(n,k,\delta,\epsilon)$.

Proof: The inequality follows from the usual interior elliptic estimates and the scaling technique as in the proof of Lemma 5.6.1. ■

Now we consider finite asymptotic cones. Let $(C_\epsilon(N) = (0,\epsilon) \times N, g = dr^2 + r^2 h_r)$ be a finite asymptotic cone, where h_r is a family of Riemannian metrics on N satisfying

$h_r = h_0 + o(r^\alpha)$ as $r \to 0$ for some $\alpha > 0$ and a Riemannian metric h_0 on N. On the finite asymptotic cone, we can also define weighted Sobolev norms and weighted uniform C^k-norms the same as ones on a finite cone. We use $\|\cdot\|_{\widetilde{H}^k_\delta(C_\epsilon(N))}$ and $\|\cdot\|_{\widetilde{C}^k_\delta(C_\epsilon(N))}$ to denote weighted norms on the finite asymptotic cone.

We make an extra assumption for the asymptotically conical metric $g = dr^2 + r^2 h_r$ as

$$|\nabla^{i+1}(h_r - h_0)| \in C_{-i}(C_\epsilon(N)), \quad for \;\; 0 \le i \le \frac{n}{2} + 2, \tag{5.6.6}$$

where the covariant derivative ∇ and the norm $|\cdot|$ of tensors are with respect to the exactly conical metric $dr^2 + r^2 h_0$. Then asymptotic condition (5.6.6) of the metric implies that $r^i|\nabla^i \omega|$ is bounded for all $0 \le i \le \frac{n}{2} + 2$, where ω if the difference tensor between the Levi-Civita connection for the asymptotically conical metric and the one for the exactly conical metric. And then as arguments in the proof of Theorem 5.2.1, for sufficiently small ϵ, these weighted norms with respect to the asymptotically conical metric on $C_\epsilon(N)$ are equivalent to corresponding weighted norms with respect to the exact cone metric on $C_\epsilon(N)$. Therefore, by Lemma 5.6.1 and Lemma 5.6.2, we have the following Sobolev inequality and elliptic estimates on a sufficiently small finite asymptotic cone.

Lemma 5.6.3 (Dai, –) *If ϵ is sufficiently small, $u \in \widetilde{H}^k_\delta(C_\epsilon(N))$, and $k > \frac{n}{2} + 1$, then*

$$\|u\|_{\widetilde{C}^l_\delta(C_\epsilon(N))} \le C\|u\|_{\widetilde{H}^k_\delta(C_\epsilon(N))}, \tag{5.6.7}$$

for $l = 0$, and 1, and some constant $C = C(n, k, \delta, \epsilon)$.

Lemma 5.6.4 (Dai, –) *If ϵ is sufficiently small, $u \in \widetilde{H}^{k-2}_\delta(C_\epsilon(N))$, and $Lu \in$*

$\widetilde{H}^{k-2}_{\delta-2}(C_\epsilon(N))$, *then*

$$\|u\|_{\widetilde{H}^k_\delta(C_\epsilon(N))} \le C(\|Lu\|_{\widetilde{H}^{k-2}_{\delta-2}(C_\epsilon(N))} + \|u\|_{\widetilde{H}^{k-2}_\delta(C_\epsilon(N))}),$$

for $2 \le k \le \frac{n}{2}+2$, *and some constant* $C = C(n,\delta,\epsilon)$, *where* L *is also the operator* $-4\Delta + R$ *with respect to the asymptotically conical metric.*

These Sobolev inequality and elliptic estimates imply the following asymptotic order for eigenfunctions of $-4\Delta + R$ near the tip of a finite asymptotic cone.

Theorem 5.6.5 (Dai, –) *Let* u *be an eigenfunction of* $L = -4\Delta + R$ *on a finite asymptotic cone* $(C_\epsilon(N), dr^2 + r^2 h_r)$ *with* $R_{h_0} > (n-2)$ *and* (5.6.6)*. Then*

$$|\nabla^i u| = o(r^{-\frac{n-2}{2}-i}), \quad as \ \ r \to 0,$$

for $i = 0$ *and* 1*.*

Proof: Because we only consider the asymptotic behavior of the eigenfunction near the tip of the cone, without loose of generality, we can assume ϵ is sufficiently small so that Lemma 5.6.3 and Lemma 5.6.4 hold on $C_\epsilon(N)$. In the proof of Theorem 5.3.3, we have obtained that the eigenfunction $u \in \widetilde{H}^1(C_\epsilon) = \widetilde{H}^1_{1-\frac{n}{2}}(C_\epsilon(N))$. Then $Lu \in \widetilde{H}^1_{1-\frac{n}{2}}(C_\epsilon(N)) \subset \widetilde{H}^1_{1-2-\frac{n}{2}}(C_\epsilon(N))$, since Lu is a scale multiple of u. Then by Lemma 5.6.4, $u \in \widetilde{H}^3_{1-\frac{n}{2}}(C_\epsilon(N))$. By applying this elliptic bootstrapping, we obtain that $u \in \widetilde{H}^{[\frac{n}{2}]+2}_{1-\frac{n}{2}}(C_\epsilon(N))$. Therefore, by Lemma 5.6.3, $u = o(r^{-\frac{n-2}{2}})$, and $|\nabla u| = o(r^{-\frac{n-2}{2}-1})$, as $r \to 0$. ■

As a direct consequence of Theorem 5.6.5, eigenfunctions of $-4\Delta+R$ on a manifold with a single conical singularity have an asymptotic behavior near the singularity.

Corollary 5.6.6 (Dai, –) *Let* (M^n, g, p) *be a compact Riemannian manifold with a single conical singularity* p *with* $R_{h_0} > (n-2)$ *and* (5.6.6) *near the singularity* p*. The* *eigenfunctions of* $-4\Delta_g + R_g$ *on satisfy*

$$|\nabla^i u| = o(r^{-\frac{n-2}{2}-i}), \quad as \;\; r \to 0,$$

for $i = 0$ *and* 1. *Consequently, the first eigenvalue is simple.*

5.7 λ-functional on manifolds with a single conical singularity

In this section, we define the Perelman's λ-functional on manifolds with a single conical singularity and obtain its first and second variation formulae as an application of spectrum properties of the operator $-4\Delta + R$ we obtained in previous sections.

Let (M^n, g, p) be a compact Riemannian manifold with a single conical singularity at p with $R_{h_0} > (n-2)$ and (5.6.6) near p. We define the λ-functional as the first eigenvalue of $-4\Delta + R$. Let u be the corresponding normalized positive eigenfunction, i.e. $\int_M u^2 dvol_g = 1$ and

$$-4\Delta u + Ru = \lambda u. \tag{5.7.1}$$

Let $u = e^{-\frac{f}{2}}$, then (5.7.1) becomes

$$\lambda = 2\Delta f - |\nabla f|^2 + R. \tag{5.7.2}$$

Let $g(t)$ for $t \in (-\tau, \tau)$ be a smooth family of metrics on M^n with a single conical singularity at p satisfying $R_{h_0(t)} > (n-2)$, and (5.6.6) near p for all $g(t)$, and $g(0) = g$. Differentiating (5.7.2) in t gives

$$\dot{\lambda} = 2\dot{\Delta} f + 2\Delta \dot{f} + \dot{R} - (\dot{|\nabla f|^2}), \tag{5.7.3}$$

where "upperdot" denotes the derivative with respect to t at $t = 0$. Multiplying the equation (5.7.3) by e^{-f} and then integrating over M^n, we have

$$\dot{\lambda} = \int_M (2\dot{\Delta} f + 2\Delta \dot{f} + \dot{R} - (\dot{|\nabla f|^2}))e^{-f} dvol_g. \tag{5.7.4}$$

Let's look at the second term in the integral in (5.7.4).

$$\begin{aligned}\int_{M \setminus C_\epsilon(N)} 2\Delta \dot{f} e^{-f} dvol_g &= \int_{\partial C_\epsilon(N)} (\partial_r \dot{f}) e^{-f} r^{n-1} dvol_{h_\epsilon} \\ &\quad - \int_{M \setminus C_\epsilon(N)} 2\langle \nabla \dot{f}, \nabla f \rangle e^{-f} dvol_g \\ &= o(\epsilon^{-(n-2)+(n-1)-1}) \\ &\quad - \int_{M \setminus C_\epsilon(N)} 2\langle \nabla \dot{f}, \nabla f \rangle e^{-f} dvol_g \\ &\to - \int_M 2\langle \nabla \dot{f}, \nabla f \rangle e^{-f} dvol_g \quad \text{as} \quad \epsilon \to 0,\end{aligned}$$

where the boundary goes away along the limit because of the asymptotic behavior of the eigenfunction in Theorem 5.6.6.

For other terms, plug the standard variation formulae for the scale curvature R and the Laplacian Δ (see [Bes87] or [DWW05]) into (5.7.4). Then similar to the second term when we do integration by parts all boundary terms go away. Therefore, we obtain the same first variation formula as that on the smooth compact manifolds.

Proposition 5.7.1 (Dai, –)

$$\dot{\lambda} = \int_M \langle -Ric_g - Hess_g f, h \rangle_g e^{-f} dvol_g, \tag{5.7.5}$$

where $h = \dot{g}$.

Corollary 5.7.2 (Dai, –) *The critical points of λ-functional are Ricci-flat metrics with a single conical singularity at p.*

Proof: By Proposition 5.7.1, a critical point is a metric g with a single conical singularity at p satisfying

$$-Ric_g - Hess_g f = 0.$$

$$\begin{aligned}\int_{M\setminus C_\epsilon(N)} \Delta(e^{-f}) dvol_g &= \int_{\partial C_\epsilon(N)} \partial_r(e^{-f}) r^{n-1} dvol_{h_\epsilon} \\ &= o(r^{-(n-2)-1+(n-1)}) \\ &= o(1) \to 0 \quad \text{as} \quad r \to 0,\end{aligned}$$

i.e. $\int_M \Delta(e^{-f}) dvol_g = 0$. Therefore, the proof of Proposition 1.1.1 in [CZ06] applies here and completes the proof. ■

Proposition 5.7.3 (Dai, –) *At a critical point, i.e. a Ricci-flat metric g with a single conical singularity, the second variation formula is given by*

$$\ddot{\lambda} = \int_M \langle -\frac{1}{2}\Delta_{L,g} h + \delta_g^* \delta_g h + \frac{1}{2}\nabla_g^2(\nu_h), h\rangle_g e^{-f} dvol_g, \tag{5.7.6}$$

where $\Delta_g \nu_h = -\delta_g(\delta_g h)$.

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