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LINEAR STABILITY OF RIMMING FLOW

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Abstract. We consider the case of rimming flow where a thin film of viscous liquid coats the walls of a cylinder whose axis is horizontal and which is rotating with constant angular velocity. It has been experimentally established that such flows are often unstable and that the liquid often segregates into "rings" along the length of the tube. Using a leading-order lubrication theory, we utilise recently established steady solutions [10], which in some instances contain shocks, to examine the linear stability of the flow when subjected to two-dimensional disturbances. All solutions are shown to be at least neutrally stable. We suggest that further investigations should include higher-order (small) effects and that the origin of the observed instabilities lies in these terms.

1. Introduction. In rimming flow a small quantity of viscous liquid is placed in a cylindrical tube of radius R whose axis is horizontal and which is rotating with constant angular velocity ω (Fig. 1). The liquid is entrained on the cylinder wall and, given suitable wetting properties, a continuous film can be obtained. Flows of this type obviously have practical applications in industry, a typical example being the process of coating the inside of cylindrical fluorescent light bulbs (Fig. 2). Such bulbs consist of a hollow cylindrical glass tube coated with a thin layer of submicron "phosphor" particles which give the light from the tube its characteristic colour. In order to obtain a uniform coating of this type, the particles are suspended in an inert liquid (e.g., water) and the resulting liquid suspension is used to coat the inside of the tube. If a suitably uniform suspension coating is attained, the excess liquid is encouraged to evaporate off (for example, by the strategic location of heating elements), the end result being a (uniform) coating of "phosphor" particles. From a practical point of view, it is well known that viscous flows of this type are often unstable; in particular, an axial stability has often been observed [1] - [4] whereby the liquid segregates into relatively dry and wet cellular areas along the length of the cylinder. Experimentally, a series of "rings" is often observed once the flow has settled down into this apparently secondary stable state. Visually the effect

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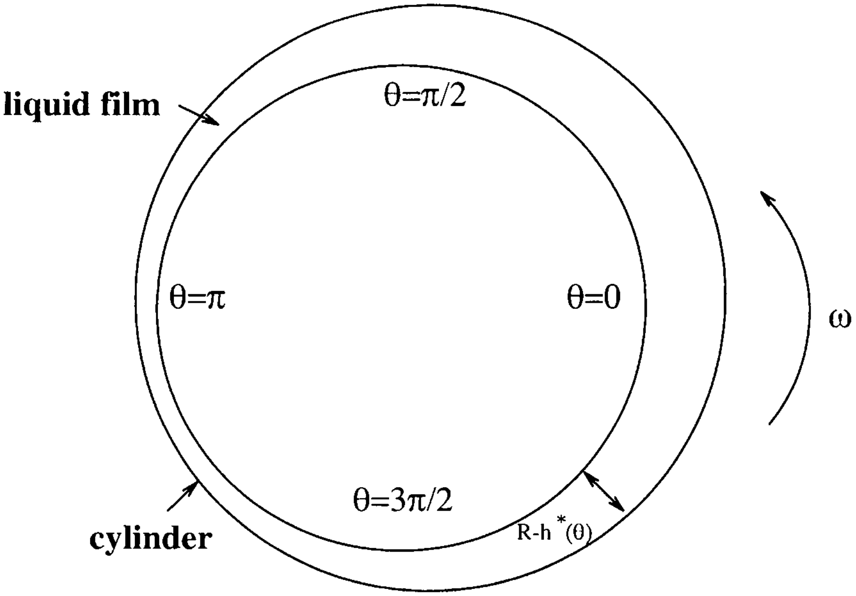

FIG. 1. Schematic (two-dimensional section) of rimming flow

resembles the well-known results obtained by G. I. Taylor for the flow of a viscous liquid in the annular region between two rotating cylinders (see, e.g., [5]). From an industrial point of view, if such structures occur during the coating process, the resulting "phosphor" coating will, practically speaking, be useless. A stability analysis is thus desirable in order to pinpoint the mechanism of stability and, if possible, to deduce the critical values of the physical parameters at which the ring-like structures appear.

To this end, we will consider the linear stability of the flow of a viscous liquid with kinematic viscosity ν, (dynamic viscosity μ), density ρ_l, in a long cylindrical tube of radius R, rotating at constant angular velocity ω. The liquid volume fraction (assumed small) will be denoted by V and the acceleration due to gravity by g. In order to test the stability of the flow, we first need a base steady state. A considerable amount of pioneering work on the steady flow problem has been done by Moffatt [6] and Johnson [7] and the steady state behaviour of the problem has been more or less suggested by these authors using the lubrication approximation. (Moffatt concentrated on the exterior problem where the film of liquid occurs on the outside of a rotating cylinder and experimentally he noted the occurrence of ring-like instabilities which rotate around the cylinder.) However, in order to test the stability of the steady flows, we [10] recently obtained closed form expressions for the steady state solutions, while neglecting higher-order smoothing terms [11], and gave an explicit criterion for the existence of shocks in the solution. Such closed form expressions are invaluable when one is carrying out a stability analysis. In the present paper, we will investigate the stability of this axially-independent steady state film thickness $h^* = h^*(\theta)$ to small two-dimensional disturbances using a leading-order lubrication approximation to the equations of motion.

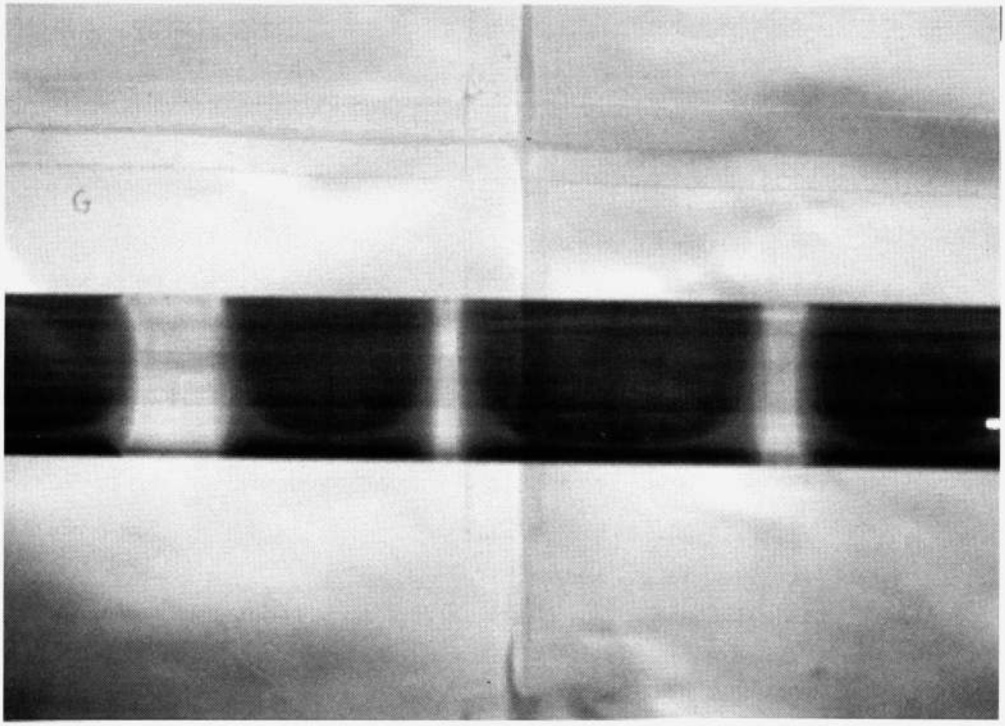

FIG. 2. Photograph of experimental demonstration of the appearance of "rings" during rimming flow

2. Formulation of the flow problem. Referring to Fig. 1, we use polar coordinates, fixed in space. The film thickness is denoted by $h^*(\theta, t^*)$. The relevant equations are the Stokes equations:

$$0 = -\nabla p^* + \mu \nabla^2 \mathbf{u}^* + \rho_l \mathbf{g}, \tag{1}$$

where $p^*(r^*, \theta, t^*)$ is the pressure, $\mathbf{u}^*(r^*, \theta, t^*)$ is the liquid velocity vector and $\mathbf{g}$ is the gravity vector. For incompressible flow, the velocity vector satisfies $\nabla \cdot \mathbf{u}^* = 0$. The boundary conditions for the flow are:

$$\mathbf{u}^* = (u^*, v^*) = (0, \omega R) \quad \text{on } r^* = R \tag{2}$$

while on the liquid free surface $r^* = h^*(\theta, t^*)$, we assume that the air exerts zero tangential and normal stress on the liquid (i.e., we neglect surface tension in this leading-order approximation). In addition, we apply a kinematic condition, i.e.,

$$\frac{D}{Dt}(r^* - h^*) = 0 \quad \text{on } r^* = h^*. \tag{3}$$

We nondimensionalise as follows:

$$u^* = \varepsilon U u; \qquad v^* = \omega R + Uv; \qquad h^* = R - hH;$$
$$r^* = R - \rho H; \qquad p^* = \frac{\mu U R}{H^2} p; \qquad t^* = tT \tag{4}$$

and use the following reference quantities:

$$U = \frac{\rho g H^2}{\mu}; \qquad T = \frac{\mu R}{\rho g H^2}, \tag{5}$$

where H is a typical film thickness, yet to be precisely defined. We define $\varepsilon = \frac{H_0^*}{R}$ and we temporarily take $H = H_0^*$, where the average film thickness H_0^* is

$$H_0^* = \frac{1}{2\pi} \int_0^{2\pi} (R - h^*) d\theta, \tag{6}$$

and we assume that $H_0{}^* << R$. The average dimensionless film thickness is given by

$$H_0 = \frac{1}{2\pi} \int_0^{2\pi} h(\theta) d\theta = \varepsilon \Omega^{-\frac{1}{2}}, \tag{7}$$

where $\Omega = \frac{\omega\nu}{gR}$ is a dimensionless rotation rate and H_0 is an $O(1)$ dimensionless parameter. We also define a related parameter $H_1 = \frac{H}{R}\Omega^{-\frac{1}{2}}$.

Noting that the volume fraction

$$V = \frac{2RH_0^* - H_0^{*2}}{R^2} = 2\varepsilon - \varepsilon^2,$$

we observe that the assumption of a small volume fraction $V << 1$ is approximately equivalent to the assumption that $\varepsilon << 1$ and we use ε as a small parameter. To leading order in ε, the problem reduces (in the lubrication approximation) to:

$$\frac{\partial p}{\partial \rho} = 0, \tag{8}$$
$$\frac{\partial^2 v}{\partial \rho^2} = H_1^2 \cos\theta, \tag{9}$$
$$-\frac{\partial u}{\partial \rho} + \frac{\partial v}{\partial \theta} = 0, \tag{10}$$

with solutions (to leading order):

$$p = 0, \tag{11}$$
$$v = H_1^2 \cos\theta \left(\frac{\rho^2}{2} - h\rho\right), \tag{12}$$
$$u = H_1^2 \left(-\sin\theta \left(\frac{\rho^3}{6} - \frac{h\rho^2}{2}\right) - \frac{\rho^2}{2}\cos\theta h_\theta\right). \tag{13}$$

At this point it is simplest to rescale and let $H \equiv (\frac{\omega R \nu}{g})^{\frac{1}{2}}$, which means that the time scale becomes $T = 1/\omega$ and $H_1 = 1$. The dimensional liquid flux is

$$q^* = -Ch + C\cos\theta \frac{h^3}{3}, \tag{14}$$

where $C \equiv (\frac{\nu\omega^3 R^3}{g})^{1/2}$. In dimensional form the kinematic boundary condition (3) is equivalent to

$$\frac{\partial h^*}{\partial t^*} = \frac{1}{R}\frac{\partial q^*}{\partial \theta}. \tag{15}$$

With the above rescaling, this reduces at leading order to the dimensionless evolution equation:

$$h_t + (h - \frac{1}{3}h^3 \cos\theta)_\theta = 0, \tag{16}$$

where the flux q is given by $q = h - \frac{1}{3}h^3\cos\theta$. We first need to consider steady solutions of (16).

3. Steady solutions. Steady solutions to (16) have been derived in [10]. We summarise the main results briefly here. In [10] it is shown that if $H_0 > H_c \approx 0.7071$, then the liquid flux q attains its maximum value of $\frac{2}{3}$ and shock-like solutions can occur in the quadrant $(-\frac{\pi}{2}, 0]$. We note that such shock structures have been observed experimentally [8], [9]. We define

$$h_< = \left(-\frac{3q}{2\cos\theta} + \sqrt{\frac{9q^2}{4\cos^2\theta} - \frac{1}{\cos^3\theta}}\right)^{\frac{1}{3}} - \left(\frac{3q}{2\cos\theta} + \sqrt{\frac{9q^2}{4\cos^2\theta} - \frac{1}{\cos^3\theta}}\right)^{\frac{1}{3}} \tag{17}$$

and

$$h_1 = \frac{2}{\sqrt{\cos\theta}}\cos\left(\frac{\phi}{3}\right); \qquad h_3 = \frac{2}{\sqrt{\cos\theta}}\cos\left(\frac{\phi}{3} - \frac{2\pi}{3}\right), \tag{18}$$

where $\phi \equiv \arccos(-\frac{3q}{2}\sqrt{\cos\theta})$. Note that by using the complex definition of the inverse cosine and by using principal values of the square root and log, one reproduces $h_<$ from h_3 when $\cos\theta < 0$. We can thus relabel $h_<$ as h_3 when $\cos\theta < 0$.

If $H_0 < H_c$, then $q < \frac{2}{3}$ and $h(\theta)$ is smooth, even and periodic and given by

$$\begin{aligned} h_s(\theta) &= q \quad \text{if } \cos\theta = 0, \\ &= h_3\,(h_<) \quad \text{if } \cos\theta < 0, \\ &= h_3 \quad \text{if } \cos\theta > 0. \end{aligned} \tag{19}$$

If $H_0 \geq H_c$, then $q = \frac{2}{3}$ and shocks can occur. In this case, referring to Fig. 3, the shock solution starts on the branch h_3 at $\theta = -\pi$, jumps vertically to the branch h_1 at $\theta = \theta_j$ and then moves (smoothly) back onto the branch h_3 at $\theta = 0$. The location of the shock is determined by demanding that (7) be satisfied [10]. Other authors [7] have commented on the possibility of shock solutions occurring in the quadrant $\theta \in [0, \pi/2)$. This is theoretically possible by switching from h_1 to h_3 but, as we shall show later, such solutions are unstable and do not occur in practice.

For the case $H_0 > H_c$, the solutions are discontinuous with a single jump. We now interpret such solutions in the weak sense. Let $w(\theta, t)$ be a C^1 function and let R be any region in the (θ, t) plane in which (16) is valid. Then

$$\int_R w(h_t + q_\theta)\,d\theta\,dt = 0. \tag{20}$$

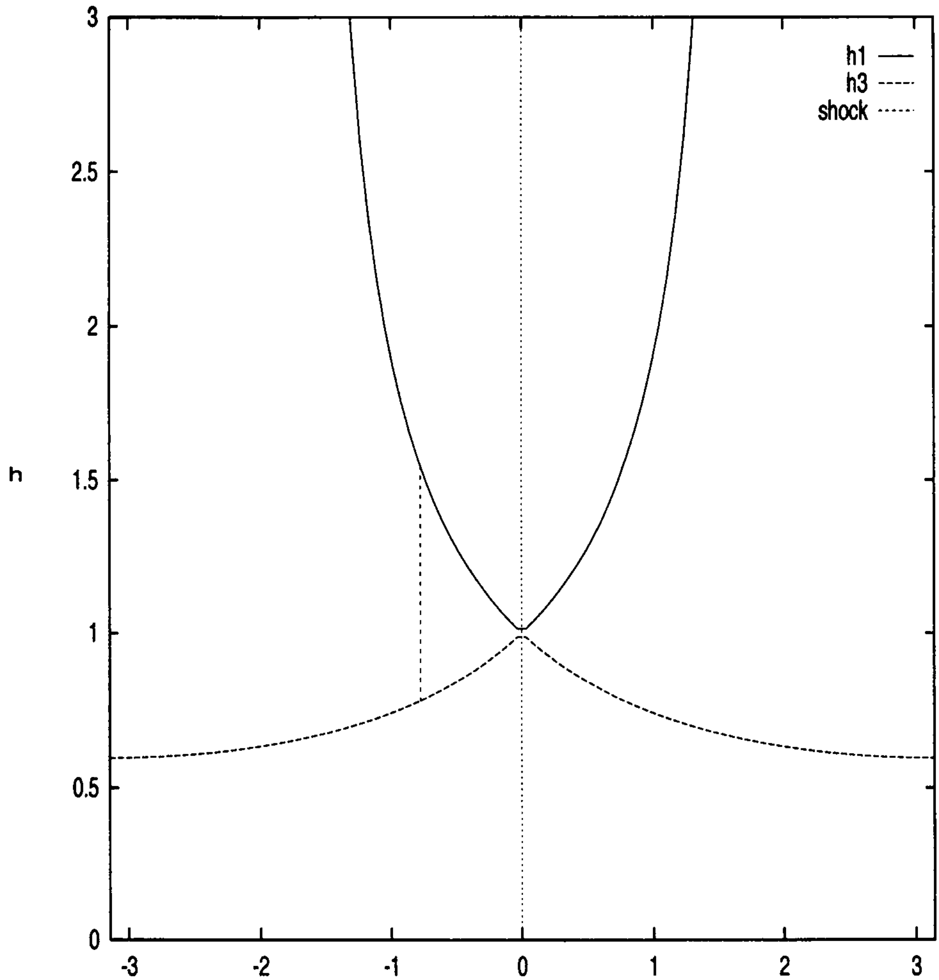

FIG. 3. Plots of steady solutions of Eq. (16) when $H_0 > H_c$ and a shock is always present. Different values of H_0 give rise to different shock positions in $(-\pi/2, 0]$. In this figure, the shock is located at $\theta_j = -0.7757$.

If we impose the further constraint that w vanishes on ∂R, then applying the divergence theorem yields

$$\int_R (hw_t + qw_\theta) d\theta\, dt = 0. \tag{21}$$

We can define a weak solution of (16) to be one that is piecewise continuous and for which (21) is valid. Equation (21) places a restriction on the curves of discontinuity. If we suppose that h has a discontinuity across some curve Γ in (θ, t) space, then we choose R so as to contain a portion of the discontinuity curve. Using the divergence theorem on

(21), we find that along the part of the discontinuity curve contained within R:

$$\int w[(h_- - h_+)d\theta - (q^- - q^+)dt] = 0, \tag{22}$$

where a "+" sub or superscript refers to a quantity evaluated on the right-hand side of the shock and a "-" to a quantity on the left-hand side. Hence we obtain the well-known result [12] for the shock speed c:

$$c = \frac{q^- - q^+}{h_- - h_+}. \tag{23}$$

Applying these results to (16), we note first that the solution with one jump in $(-\pi/2, 0]$ jumps from a value $h_- = h_3(\theta)$ to a value $h_+ = h_1(\theta)$ at $\theta = \theta_j$. Using (17) and (18), we establish the following identity by direct computation:

$$h_+^2 + 2h_+h_- + h_-^2 = \frac{3}{\cos\theta}. \tag{24}$$

In the shock evolving from (16), the shock speed c is given by

$$c = \frac{q^- - q^+}{h_- - h_+} = 1 - \frac{1}{3}\cos\theta\, h_+^2 + 2h_+h_- + h_-^2 = 0 \tag{25}$$

on applying (24) and so the shock is stationary. This verifies the steady of solutions of the last section.

4. Linear stability of smooth solution $h_s(\theta)$ ($q < \frac{2}{3}$). We test the stability of $h_s(\theta)$ as given in (19) as a steady solution of (16). We thus set

$$h = h_s(\theta) + \delta\, \mathrm{Re}[f(\theta)e^{st}], \tag{26}$$

where $\delta << 1$ and $f(\theta)$ can be complex-valued but is periodic with period 2π. Omitting terms of $O(\delta^2)$ or smaller we obtain the following ODE for $f(\theta)$:

$$f_\theta(1 - \cos\theta h_s^2) + f(s - 2h_s\frac{dh_s}{d\theta}\cos\theta + h_s^2\sin\theta) = 0. \tag{27}$$

The solution to (27) can be written explicitly as

$$f(1 - h_s^2\cos\theta) = A\exp\left(-s\int^\theta \frac{1}{(1 - h_s^2\cos\theta)}d\theta\right). \tag{28}$$

We note that s and A are constants. Furthermore, the integrand in the integral is always positive (for $q < \frac{2}{3}$) and is periodic with period 2π as are h_s and $\cos\theta$. Thus the integral term is an increasing function of θ. However, we recall that $f(\theta)$ must be a periodic function, period 2π; so we deduce from (28) that all eigenvalues s must be imaginary. Hence the solution $h_s(\theta)$ is neutrally stable.

Since the integrand in (28) is a strictly positive quantity for any value of $q < 2/3$, it is clear that its integral over the period 2π is a constant quantity $I(q)$. In order for f in (28) to remain periodic, it is necessary that

$$|s|I(q) = |s|\int_0^{2\pi} \frac{1}{1 - h_s^2\cos\theta}\,d\theta = 2N\pi, \tag{29}$$

where N is an integer. Hence the eigenvalues in (28) must satisfy

$$s = \frac{i2N\pi}{\int_0^{2\pi} \frac{1}{1-h_s^2 \cos\theta}\, d\theta}. \tag{30}$$

4.1. *Numerical verification of neutral stability of $h_s(\theta)$.* We write (27) in the form

$$f_\theta D(\theta) + f E(\theta) + sf = 0, \tag{31}$$

where the known coefficients $D(\theta)$ and $E(\theta)$ are periodic functions with period 2π (like $h_s(\theta)$). We write $D(\theta), E(\theta)$ and $f(\theta)$ as Fourier series:

$$D(\theta) = \sum_{j=-\infty}^{\infty} D_j e^{ij\theta}; \quad E(\theta) = \sum_{j=-\infty}^{\infty} E_j e^{ij\theta}; \quad f(\theta) = \sum_{j=-\infty}^{\infty} C_j e^{ij\theta}, \tag{32}$$

where the D_j and E_j are known constants given by

$$D_j = \frac{1}{2\pi}\int_0^{2\pi} D(\theta) e^{-ij\theta} d\theta; \qquad E_j = \frac{1}{2\pi}\int_0^{2\pi} E(\theta) e^{-ij\theta} d\theta \tag{33}$$

with $D_{-j} = D_j^*$ and $E_{-j} = E_j^*$. The constants C_j in (32) are unknown as are the eigenvalues s in (31). Noting that

$$\sum_{j=-\infty}^{\infty} \alpha_j e^{ij\theta} \sum_{l=-\infty}^{\infty} \beta_l e^{il\theta} = \sum_{m=-\infty}^{\infty} e^{im\theta} \sum_{n=-\infty}^{\infty} \alpha_{m-n}\beta_n, \tag{34}$$

(31) becomes

$$\sum_{n=-\infty}^{\infty} (D_{m-n}(in) + E_{m-n}) C_n = -sC_m. \tag{35}$$

On defining

$$\mathbf{f} = (\ldots C_{-1}, C_0, C_1 \ldots)^{\mathrm{T}} \tag{36}$$

and

$$M_{mn} = D_{m-n}(in) + E_{m-n} \qquad (-\infty < m, n < \infty), \tag{37}$$

we can write (35) in the form

$$\sum_{n=-\infty}^{\infty} M_{mn} C_n = -sC_m, \text{ i.e., } \mathbf{Mf} = -s\mathbf{f}, \tag{38}$$

where $\mathbf{M}$ is a square matrix of infinite order. Equation (38) is a matrix eigenvalue problem. In practice we truncate at some arbitary integer value N so that $-N \leq m, n \leq N$ and (38) is a $(2N+1) \times (2N+1)$ matrix eigenvalue problem.

During computations, we took $N = 20$ or 30 and, in all cases examined, we found the real part of s to be zero to within the tolerance used, thus verifying the neutral stability of $h_s(\theta)$. In addition, the eigenvalues were always found to satisfy the relationship (30).

5. Stability of shocks. If $H_0 > H_c$, then $q = \frac{2}{3}$ and a shock (or puddle) appears in the quadrant $(-\frac{\pi}{2}, 0]$. The exact location of the shock can be determined as in [10].

We can interpret (16) by using the method of characteristics; whence we have

$$\frac{dh}{dt} = -\frac{1}{3}h^3 \sin\theta, \qquad \frac{d\theta}{dt} = 1 - h^2\cos\theta, \tag{39}$$

and the term $-\frac{1}{3}h^3 \sin\theta$ is associated with damping of any resulting motion (see [12]) while the term $1-h^2\cos\theta$ can be interpreted as the characteristic velocity. Alternatively, we can interpret (16) as representing nonlinear kinematic waves with kinematic wave speed

$$\frac{dq}{dh} = 1 - h^2\cos\theta. \tag{40}$$

Figure 4 graphs the steady state film thickness h and $1 - h^2\cos\theta$ against θ for the case where $H_0 = 0.72$ when the shock is located at $\theta_j \approx -0.4397$ [10]. The dotted line is thus a curve in the $(\frac{d\theta}{dt}, \theta)$-plane, and it is clear that the steady solution must be stable as the kinematic wave-speed on each side of the shock is towards the shock. Although Fig. 3 represents one particular case, if the shock occurs at $\theta_j \in (-\pi/2, 0]$, the graph of $1 - h^2\cos\theta$ will have the same basic shape, i.e., $1 - h^2\cos\theta$ is always positive on $(-\pi/2, \theta_j)$ and it is negative on $(\theta_j, 0]$. More specifically, we recall that a shock solution occurs when $H_0 > H_c$ and we have $q = \frac{2}{3}$ and $h(0) = 1$. If the shock is located at $\theta_j \in (-\frac{\pi}{2}, 0]$, referring to Fig. 4, we find by direct computation for all $\theta_j \in (-\pi/2, 0]$ that $1 - h_-^2\cos\theta = 1 - h_3^2\cos\theta > 0$ while $1 - h_+^2\cos\theta = 1 - h_1^2\cos\theta < 0$. Hence the wave speed in the immediate vicinity of the shock is always toward it and so the situation is stable. We thus see that a shock occurring in the quadrant $(-\frac{\pi}{2}, 0]$ will be stable.

Previous authors [7] have alluded to the possibility of shocks occurring in the quadrant $[0, \pi/2)$, but a similar argument to that used above indicates that such shocks are unstable. If a shock occurs at $\theta_j \in [0, \pi/2)$, then the wave speed at θ_j^- is $1 - h_-^2\cos\theta = 1 - h_1^2\cos\theta < 0$ while at θ_j^+ it is $1 - h_+^2\cos\theta = 1 - h_3^2\cos\theta > 0$. Thus the local wave speed on either side of the shock is in the direction away from the shock and small disturbances would thus not be absorbed. We conclude that such solutions will not occur in practice.

6. Discussion. Using a leading-order lubrication theory, we have found that both the smooth solutions (when $q < \frac{2}{3}$) and the solutions with a shock in $(-\frac{\pi}{2}, 0]$ $(q = \frac{2}{3})$ are stable situations. As pointed out in the introduction, it has been established experimentally that films of this type can definitely become *unstable*. At this stage there are two possibilities. Either the flow is in fact unstable to two-dimensional disturbances, but the approximation used is insufficiently sharp, or the flow is indeed stable to all two-dimensional disturbances but is unstable to three-dimensional disturbances (i.e., disturbances with an axial variation). Our suspicion is that the flow is in fact unstable to two-dimensional disturbances but that a leading-order lubrication theory is an insufficiently sharp approximation to the physical situation. If higher-order effects (including surface tension) are included in a lubrication theory model, we speculate that instability will be found to occur (presumably on a relatively long timescale since the neglected effects are small). We are motivated to speculate in this way by the nature of the evolution

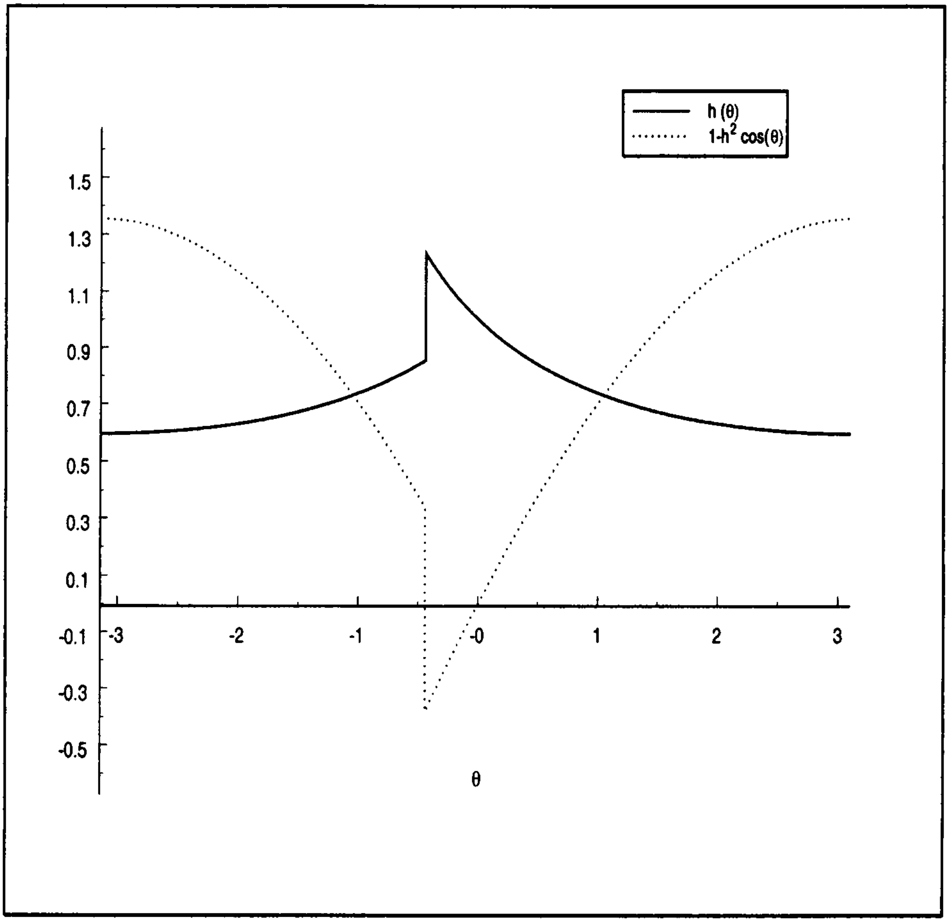

FIG. 4. Plots of $h, 1 - h^2 \cos\theta$ for the case $H_0 = 0.72$

equation when higher-order terms are included. In particular, the higher-order version of (16) will contain terms like $A(\theta)h_{\theta\theta}$ and $B(\theta)h_{\theta\theta\theta\theta}$ where the coefficient $A(\theta)$ varies in sign but gives rise to backward diffusion (an intrinsically unstable process) on part of the domain. At first glance, the evolution equation will resemble the Kuramoto-Sivashinshy equation [13]:

$$u_t + u_{xxxx} + u_{xx} + \tfrac{1}{2}u_x^2 = 0. \tag{41}$$

It has been established that steady solutions of this equation have a stability threshold dependent on the size of the domain. Essentially the second-derivative terms are destabilising. The fourth-derivative terms are diffusive (stabilising) and damp high frequency oscillations most efficiently. There exists a critical wavelength for disturbances above which the stabilising terms cannot overcome the destabilising effects. On an infinite domain, solutions to this equation are always unstable, but on a finite domain, stability is a possibility depending on the domain size. We expect similar considerations to apply to the rimming flow problem. In addition, we note that, in particular for the case where $H_0 < H_c$, the flow is neutrally stable and it is precisely such a state that might be susceptible to small effects which could push the flow from being neutrally stable to being unstable (or even asymptotically stable). Examination of the stability of steady solutions to an improved higher-order lubrication approximation evolution equation will be the subject of a follow-up paper.

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