LINEAR SYSTEMS OF DIFFERENCE EQUATIONS WITH A REGULAR SINGULARITY

W. A. HARRIS, JR.[1]

1. **Introduction.** This paper is concerned with the linear system of difference equations

$$w(z+1) = A(z)w(z) \tag{1.1}$$

where w is a vector with n components and A is an n by n matrix which admits the generalized factorial series representation

$$A(z) = z^q \sum_{k=0}^{\infty} A_k z^{-[k]}, \qquad \operatorname{Re}\{z\} > \mu, \tag{1.2}$$

where $z^{-[k]} = \{z(z+1) \cdots (z+k-1)\}^{-1}$ and $z^{[0]} = 1$.

In an analogous manner to linear systems of differential equations with singularities (at $z = \infty$) we make the following definitions [**1**, p. 73], [**5**, p. 111].

The system $w(z+1) = A(z)w(z)$ is said to have a singularity of the first kind if $A(z)$ admits the factorial series representation

$$A(z) = I + \sum_{k=1}^{\infty} A_k z^{-[k]}, \qquad \operatorname{Re}\{z\} > \mu, \tag{1.3}$$

and otherwise a singularity of the second kind.

The system $w(z+1) = A(z)w(z)$ is said to have a regular singularity if there exists a fundamental matrix of the form

$$W(z) = S(z)z^R \tag{1.4}$$

such that $S(z)$ admits a generalized factorial series representation

$$S(z) = z^p \sum_{k=0}^{\infty} S_k z^{-[k]}, \qquad \operatorname{Re}\{z\} > \mu', \tag{1.5}$$

and R is a constant matrix.

Linear systems of difference equations with a *singularity of the first kind* have been extensively studied by the author [2] and such systems are known to have a *regular singularity*. However, the converse is not true. Indeed, a necessary and sufficient condition that a linear system of difference equations $w(z+1) = A(z)w(z)$ have a regular

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singularity is that there exists a nonsingular matrix $T(z)$ which admits a generalized factorial series representation such that the transformation $w=Tu$ yields a system $u(z+1)=B(z)\,u(z)$ which has a singularity of the first kind, i.e.

$$B(z) = T^{-1}(z+1)A(z)T(z) = I + \sum_{k=1}^{\infty} B_k z^{-[k]}.$$

If A, B and T admit generalized factorial series representations and $\det T(z)\not\equiv 0$, we shall denote the equivalence relation $B(z) = T^{-1}(z+1)A(z)T(z)$ by $A\sim B$.

Even though this condition is necessary and sufficient for the desired structure of a fundamental matrix, it cannot be used to resolve the question for a preassigned system. However, the author [3][2] has given an algorithm to determine whether a given system has a regular singularity which is contained in the following theorems.

THEOREM (HARRIS). *Let $A(z)$ admit a generalized factorial series representation, $A(z)=z^p\sum_{k=0}^{\infty} A_k z^{-[k]}$, $A_0\neq 0$, $\operatorname{Re}\{z\}>a$. A necessary condition that the system $w(z+1)=A(z)w(z)$ has a regular singularity is that $p\geqq 0$, and A_0 or A_0-I be nilpotent for $p>0$ or $p=0$ respectively.*

THEOREM (HARRIS). *Let $A(z)$ and $B(z)$ admit factorial series representations, $A(z)=\sum_{k=0}^{\infty} A_k z^{-[k]}$, $B(z)=\sum_{k=0}^{\infty} B_k z^{-[k]}$, $\operatorname{Re}\{z\}>a$, and let $A_0\neq 0$. A necessary and sufficient condition that $A\sim B$ such that $r=\operatorname{rank}(A_0-\rho I)>\operatorname{rank}(B_0-\rho I)$ for some ρ is that the polynomial*

$$\mathfrak{B}(\lambda) = \left\{z^{-r}\det[\lambda I + z(A(z)-\rho I)]\right\}\Big|_{z=\infty} = \sum_{k=0}^{n-r} \lambda^k \mathfrak{B}_k(A_0, A_1)$$

vanish identically in λ.

In this paper we shall derive necessary conditions for a given system $w(z+1)=A(z)w(z)$ to have a regular singularity based on the characteristic polynomial of the matrix $A(z)$. These results parallel recent results of D. A. Lutz [6] for linear differential systems with a regular singular point.

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2. **Statement of results.** It is natural and convenient to associate with the matrix $A(z)$ the matrix $\hat{A}(z)$ defined by

$$\hat{A}(z) = A(z) - I. \tag{2.1}$$

[2] This paper assumes that $T^{-1}(z)$ also admits a generalized factorial series representation; a fact subsequently proved by the author and H. L. Turrittin [4].

DEFINITION. *The symmetric function of rank k of a matrix A is the coefficient of λ^{n-k} in the polynomial*

$$\mathfrak{U}(\lambda) = \det(\lambda I + A) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n.$$

For a system $u(z+1) = B(z)u(z)$ with a singularity of the first kind we have: *the symmetric function of rank k of the matrix $B(z)$ satisfies the order condition $b_k(z) = O(z^{-k})$.*

Since, for every system $w(z+1) = A(z)w(z)$ with a regular singularity, we have $A \sim B$ where the system $u(z+1) = B(z)u(z)$ has a singularity of the first kind, i.e. $A(z) = T(z+1)B(z)T^{-1}(z)$, it is natural to expect that the orders of the n symmetric functions of A are somewhat restricted. This is indeed the case and we state:

THEOREM 1. *If the system $w(z+1) = A(z)w(z)$ with*

$$A(z) = z^q \sum_{k=0}^{\infty} A_k z^{-[k]}, \qquad A_0 \neq 0, \ \operatorname{Re}\{z\} > \mu$$

has a regular singularity, then the symmetric functions of A satisfy

$$a_n(z) = O(z^{(n-1)q-2} + z^{-n}), \qquad a_k(z) = O(z^{kq-2} + z^{-k}),$$
$$k = 1, \cdots, n-1.$$

We shall show by example that this result is sharp for all k, n and $q \geqq 0$.

In the same manner we can also state

THEOREM 2. *If the system $w(z+1) = A(z)w(z)$ with*

$$A(z) = z^q \sum_{k=0}^{\infty} A_k z^{-[k]}, \qquad A_0 \neq 0, \ \operatorname{Re}\{z\} > \mu$$

has a regular singularity and we write

$$A(z) = A(z) - I = z^q \sum_{k=0}^{\infty} A_k z^{-[k]},$$

then

(i) $\hat{A}_0^k = 0$ *for some $k \leqq n$, i.e. $\hat{A}_0$ is nilpotent and*

(ii) *trace* $(\hat{A}_0^k \hat{A}_1) = 0$ *for $k = 0, 1, \cdots, n-1$ in case $q \geqq 1$ and for $k = 1, 2, \cdots, n-1$ in case $q = 0$.*

Even though the order conditions for the symmetric functions given in Theorem 1 are sharp, they are clearly not sufficient for a regular singularity. However, we do have the following partial converse to Theorem 1.

THEOREM 3. *If*

$$A(z) = z^q \sum_{k=0}^{\infty} A_k z^{-[k]}, \qquad A_0 \neq 0,\ q \geqq 0,\ \mathrm{Re}\{z\} > \mu$$

and the symmetric functions of $A(z)$ *satisfy* $a_k(z) = O(z^{kq-2} + z^{-k})$, $k = 1, 2, \cdots, n$, *then*

(i) $\hat{A}_0^n = 0$,

and if $\hat{A}_0^{n-1} \neq 0$, *then*

(ii) trace $(\hat{A}_0^k \hat{A}_1) = 0$, $k = 0, \cdots, n-1$ *for* $q \geqq 1$ *and* $k = 1, \cdots, n-1$ *for* $q = 0$.

Note that if $n \geqq 2$, the terms z^{-n} and z^{-k} may be dropped in the order conditions except for $k = 1$ when $q = 0$.

We also have the following characterization of the order conditions.

COROLLARY. *Let* $\hat{A}_0^{n-1} \neq 0$. *Then the symmetric functions of* $A(z)$ *satisfy the order conditions* $a_k = O(z^{kq-2})$ *if and only if there exists a transformation matrix* $P(z) = I + P_1 z^{-1}$ *such that*

$$\begin{aligned} P^{-1}(z+1)A(z)P(z) &= z^q A_0 + O(z^{q-2}), \qquad q \neq 1 \\ &= zA_0 + I + O(z^{-1}), \quad q = 1. \end{aligned}$$

3. **Preliminary lemma.** A useful tool for the proof of these theorems is the following lemma which is a generalization to matrices of factorial series of the fact that an analytic function $f(z) \neq 0$ with at most a pole at $z = \infty$ can be written in the form $f(z) = z^\alpha g(z)$ where $g(z)$ is analytic at $z = \infty$ and $g(\infty) \neq 0$.

LEMMA. *Let* $T(z)$ *admit a factorial series representation*

$$T(z) = \sum_{k=0}^{\infty} T_k z^{-[k]}, \qquad T_0 \neq 0,\ \mathrm{Re}\{z\} > a.$$

Then $T(z)$ *can be represented in the form*

$$T(z) = P(z) z^{-D} Q(z)$$

where $P(z)$ *is a polynomial in* z^{-1}, det $P(z) \equiv 1$, $Q(z)$ *admits a factorial series representation*

$$Q(z) = \sum_{k=0}^{\infty} Q_k z^{-[k]}, \qquad \det Q_0 \neq 0,$$

and

$$z^{-D} = \mathrm{diag}(z^{-d_1}, z^{-d_2}, \cdots, z^{-d_n})$$

where $0 = d_1 \leqq d_2 \leqq \cdots \leqq d_n$ *are integers.*

For a proof of this Lemma, see Harris [**3**, p. 257].

4. **Proof of Theorem 1.** If the system $w(z+1)=A(z)w(z)$ has a regular singularity, there exists a fundamental matrix of the form $W(z)=S(z)z^R$ and without loss of generality we may assume that

$$S(z) = \sum_{k=0}^{\infty} S_k z^{-[k]}, \qquad S_0 \neq 0, \ \operatorname{Re}\{z\} > a.$$

Hence $S(z)$ has the representation given in the preceding Lemma,

$$S(z) = P(z)z^{-D}Q(z).$$

Thus

$$\begin{aligned} A(z) \sim B(z) &= P^{-1}(z+1)A(z)P(z), \\ B(z) \sim C(z) &= (z+1)^D B(z) z^{-D}, \\ C(z) \sim H(z) &= Q^{-1}(z+1)C(z)Q(z) = I + O(z^{-1}). \end{aligned}$$

Since Q_0 is nonsingular, $C(z)=I+O(z^{-1})$. Writing

$$B(z) = (z+1)^{-D}C(z)z^D = z^{-D}(1+z^{-1})^{-D}C(z)z^D$$

and noting that $(1+z^{-1})^{-D}=I+O(z^{-1})$, we have that $\det(\lambda I+\hat{B}) =\lambda^n+b_1\lambda^{n-1}+ \cdots +b_n$, where

$$b_k(z) = O(z^{-k}), \qquad k = 1, \cdots, n. \tag{4.1}$$

We have

$$\begin{aligned} A(z) &= P(z)\{B(z) + (P^{-1}(z)P(z+1) - I)B(z)\}P^{-1}(z) \\ &= P(z)\{B(z) + F(z)\}P^{-1}(z). \end{aligned}$$

Since $P^{-1}(z)P(z+1)-I=O(z^{-2})$, $F(z)=O(z^{q-2})$ and

$$\det(\lambda I + \hat{A}) = \det(\lambda I + \hat{B} + F) = \det(\lambda I + \hat{B}) + \sum (\lambda I + \hat{B}, F),$$

where $\sum(\lambda I+\hat{B}, F)$ represents the sum of all determinants formed from k rows of $\lambda I+\hat{B}$ and $n-k$ rows of F with natural ordering for $0 \leqq k < n$.

If, for a particular determinant, m rows have been taken from F, $1 \leqq m \leqq n$, there will possibly be a nonzero coefficient of λ^k for $k \leqq n-m$ which will be the sum of products of $n-m-k$ elements from $\hat{B}$ and m elements from F. Since $\hat{B}=O(z^q)$ and $F=O(z^{q-2})$, the coefficient of λ^k will have an order not exceeding

$$(n-m-k)q + m(q-2) = (n-k)q - 2m \leqq (n-k)q - 2.$$

Hence, $\sum(\lambda I+\hat{B}, F)=f_1\lambda^{n-1}+f_2\lambda^{n-2}+\cdots+f_n$, where

$$f_k = O(z^{kq-2}), \qquad k = 1, \cdots, n. \tag{4.2}$$

Combining (4.1) and (4.2) we have

$$\det(\lambda I + \hat{A}) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$$

where $a_k(z)=b_k(z)+f_k(z)=O(z^{kq-2}+z^{-k})$ which gives the correct order estimates for a_k when $q=0$ and for a_k, $k=1, \cdots, n-1$ when $q\geqq 1$; but $a_n(z)=O(z^{nq-2})$, $n\geqq 2$. To obtain the order estimate $a_n(z)=O(z^{(n-1)q-2})$, $q\geqq 1$, we utilize a special property of systems with a regular singularity; namely, if $w(z+1)=A(z)w(z)$ has a regular singularity, there exists a fundamental matrix of the form $W(z)=S(z)\,z^R$ and hence

$$\det A(z) = \det S(z+1)\det(1+z^{-1})^R[\det S(z)]^{-1} = 1 + O(z^{-1}).$$

Consider

$$\begin{aligned}\det(\lambda I + A) &= \det[(\lambda+1)I + \hat{A}] \\ &= (\lambda+1)^n + a_1(\lambda+1)^{n-1} + \cdots + a_n.\end{aligned}$$

Hence $a_n=\det A-1-a_1-\cdots-a_{n-1}$ and using the preceding order estimates, we obtain $a_n(z)=O(z^{(n-1)q-2})$, $n\geqq 2$, which concludes the proof of Theorem 1.

5. **Proof of Theorem 2.** As in the proof of Theorem 1, we have $(q\geqq 0)$

$$B(z) \sim A(z) = P(z+1)B(z)P^{-1}(z) \tag{5.1}$$

and

$$C(z) \sim B(z) = z^{-D}(1+z^{-1})^{-D}C(z)^D \tag{5.2}$$

where $C(z)=I+O(z^{-1})$.

Since P_0 is nonsingular, from (5.1) we have

$$A_0 = P_0\hat{B}_0P_0^{-1}, \qquad A_1 = P_0\hat{B}_1P_0^{-1} - \hat{A}_0P_1P_0^{-1} + P_1P_0^{-1}\hat{A}_0. \tag{5.3}$$

From (5.2) we see that $\hat{B}(z)=z^{-D}G(z)z^D$, where $G(z)=O(z^{-1})$. Thus the ijth element of $\hat{B}(z)$ satisfies

$$\hat{b}_{ij}(z) = O(z^{d_j-d_i-1}).$$

Since the d_i are nondecreasing, all the elements on and below the diagonal are zero for $\hat{B}_0$ if $q=0$, and for $\hat{B}_0$ and $\hat{B}_1$ if $q\geqq 1$. Thus $\hat{B}_0$ and hence also $\hat{A}_0$ is nilpotent and trace $(\hat{B}_0^k\,\hat{B}_1)=0$, $k=0, \cdots, n-1$

for $q \geqq 1$. For $q=0$, write $\hat{B}_0=(b_{ij}^0)$, $\hat{B}_1=(b_{ij}^1)$ and note that $b_{ij}^1 \neq 0$, $i>j$, implies $d_k=d_i$ for $j \leqq k<i$ and $\hat{b}_{ki}=O(z^{-1})$ and hence $b_{ki}^0=0$, $j \leqq k<i$. Since $b_{ij}^0=0$, $i \geqq j$, we have for $k=1, \cdots, n-1$

$$\operatorname{trace}(\hat{B}_0^k\hat{B}_1) = \sum_{i=1}^{n} \sum_{i<i_1<\cdots<i_k} b_{i\ i_1}^0 \cdots b_{i_{k-1}i_k}^0 b_{i_k i}^1 = 0.$$

Using equation (5.3) we have

$$A_0^k A_1 = P_0\hat{B}_0^k\hat{B}_1P_0^{-1} - A_0^{k+1}P_1P_0^{-1} + A_0^kP_1P_0^{-1}A_0.$$

Thus, trace $(\hat{A}_0^k\,\hat{A}_1)=$ trace $(\hat{B}_0^k\,\hat{B}_1)=0$, and Theorem 2 is proved.

6. **Proof of Theorem 3.** $\hat{A}$ satisfies its characteristic equation. Hence using the order conditions on the symmetric functions we obtain $z^{-nq}\hat{A}^n=O(z^{-2})$, $q \geqq 1$ and $\hat{A}^n-(\operatorname{trace} \hat{A})\hat{A}^{n-1}=O(z^{-2})$, $q=0$, or $\hat{A}_0^n=0$ and

$$\begin{aligned} A_0^{n-1}A_1 + A_0^{n-2}A_1A_0 + \cdots + A_1A_0^{n-1} &= 0, \quad q \geqq 1, \\ &= (\operatorname{trace} A_1)A_0^{n-1}, \quad q = 0. \end{aligned} \tag{6.1}$$

Since $\hat{A}_0^n=0$ and $\hat{A}_0^{n-1}\neq 0$ by hypothesis, there exists a nonsingular matrix G such that $\hat{A}_0$ has Jordan canonical form $N=G^{-1}\hat{A}_0G$ with 1 on the superdiagonal and 0 elsewhere. Setting $G_1=G^{-1}\hat{A}_1G$, equation (6.1) becomes

$$\begin{aligned} N^{n-1}G_1 + N^{n-2}G_1N + \cdots + G_1N^{n-1} &= 0, \quad q \geqq 1, \\ &= (\operatorname{trace} A_1)N^{n-1}, \quad q = 0. \end{aligned}$$

A simple computation using the special form of N shows that trace $(N^k\,G_1)=0$, $k=0, 1, \cdots, n-1$ for $q \geqq 1$ and $k=1, \cdots, n-1$ for $q=0$; but trace $(N^k\,G_1)=$ trace $(\hat{A}_0^k\,\hat{A}_1)$ and Theorem 3 is proved.

REMARK. The equation (6.1) is always satisfied if the order conditions $a_k=O(z^{kq-2}+z^{-k})$ are satisfied. However, if $\hat{A}_0$ is not nilpotent of maximum rank, this equation does not imply that trace $(\hat{A}_0^k\,\hat{A}_1)=0$, $k=1, \cdots, n-2$.

7. **Proof of Corollary.** The necessity can be proved as in Theorem 1 and is omitted. To prove sufficiency consider the equation

$$\begin{aligned} &(I + P_1(z+1)^{-1})^{-1}A(z)(I + P_1z^{-1}) \\ &\qquad = z^qA_0 + z^{q-1}\{A_0P_1 - P_1A_0 + A_1\} + O(z^{q-2}). \end{aligned}$$

Thus, the sufficiency reduces to showing that the equation $\hat{A}_0P_1-P_1\hat{A}_0+\hat{A}_1=0$ has a solution. It is well known that if $\hat{A}_0$ is nilpotent of maximum rank, i.e. $\hat{A}_0^n=0$, $\hat{A}_0^{n-1}\neq 0$, then trace $(\hat{A}_0^k \hat{A}_1) =0$, $k=0, 1, \cdots, n-1$ is necessary and sufficient for a solution of this equation (for a proof of this fact with this formulation see Wasow [7, pp. 102–104]). Since these conditions are satisfied by Theorem 3, the Corollary is proved.

8. **Example.** Let N be a maximum rank nilpotent in Jordan form as given in §6 and R a constant diagonal matrix. Then the system $u(z+1)=B(z)u(z)$ where $B=z^qN+I+z^{-1}R$ has a regular singularity. This is easily seen since $(z+1)^DB(z)z^{-D}=I+O(z^{-1})$ where $D =\operatorname{diag}(0, q+1, 2(q+1), \cdots, (n-1)(q+1))$.

For any constant matrix E, let $P(z)$ be a solution to the equation

$$P(z+1) = (I+Ez^{-2})^{-1}P(z), \qquad P(z) = I+O(z^{-1})$$

(this is a special case of a singularity of the first kind, see Harris [2]).

The system $w(z+1)=A(z)w(z)$ has a regular singularity if $A(z)$ is defined as

$$A(z) = P^{-1}(z+1)B(z)P(z). \tag{8.1}$$

It follows that $\hat{A}(z)=P^{-1}(z)[\hat{B}(z)+z^{-2}EB(z)]P(z)$ and hence

$$\det(\lambda I + A) = \det(\lambda I + \hat{B} + z^{-2}EB(z)).$$

Choose the first $n-1$ rows of E to be zero and the nth row to be $(1, 1, \cdots, 1, 0)$, $R=\operatorname{diag}(0, \cdots, 0, 1)$ and note that $\hat{B}+z^{-2}EB =z^qN+z^{q-2}EN+z^{-2}E+z^{-1}R$.

If $D_n(\lambda)=\det(\lambda I+\hat{A})$ where $\hat{A}$ is n by n, then considering n as a variable, $n\geqq 2$, it follows that

$$D_{n+1}(\lambda) = \lambda D_n(\lambda) + (-1)^{n+1}z^{nq-2}\lambda + (-1)^{n+2}z^{nq-2}$$

and hence by induction that

$$D_n(\lambda) = \lambda^n + (z^{q-2}+z^{-1})\lambda^{n-1} + \sum_{k=2}^{n-1}(-1)^{k+1}(z^{kq-2}+z^{(k-1)q-2})\lambda^{n-k} + (-1)^{n+1}z^{(n-1)q-2},$$

and hence the order conditions are sharp for all k, n and $q\geqq 0$.

References

1. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.

2. W. A. Harris, Jr., *Linear systems of difference equations*, Contributions to Differential Equations **1** (1963), 489–518.

3. ———, *Equivalent classes of difference equations*, Contributions to Differential Equations **2** (1963), 253–264.

4. W. A. Harris, Jr. and H. L. Turrittin, *Reciprocals of inverse factorial series*, Funkcial. Ekvac. **6** (1964), 37–46.

5. P. Hartman, *Ordinary differential equations*, Wiley, New York, 1964.

6. D. A. Lutz, *On systems of linear differential equations having regular singular solutions*, J. Differential Equations **3** (1967), 311–322.

7. W. Wasow, *Simplification of turning point problems for systems of linear differential equations*, Trans. Amer. Math. Soc. **106** (1963), 100–114.

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