Linear Systems on Tropical Curves

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Outline

1. Introduction to Tropical Arithmetic and Tropical Functions
2. Abstract Tropical Curves (Think Metric Graph)
3. Tropical Riemann-Roch and Linear Systems
4. Connections with the Chip-Firing Game
5. Examples

Tropical Arithmetic

We work over the tropical semi-ring

$$(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$$

where $a \oplus b = \max(a, b)$ and $a \odot b = a + b$.

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We also have the tropical commutative and associative laws. Also,

$$a \oplus (-\infty) = a \quad \text{and} \quad b \odot 0 = b$$

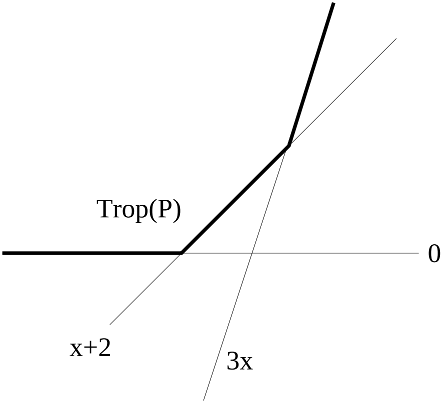
for any a and b, so we have additive and multiplicative identities.

Lastly, we have multiplicative inverses, but we do not have additive inverses.

Tropical Polynomials

We can form Tropical Polynomials such as

$$P = x^{\odot 3} \oplus 2 \odot x \oplus 0 = \max(3x, 2 + x, 0).$$

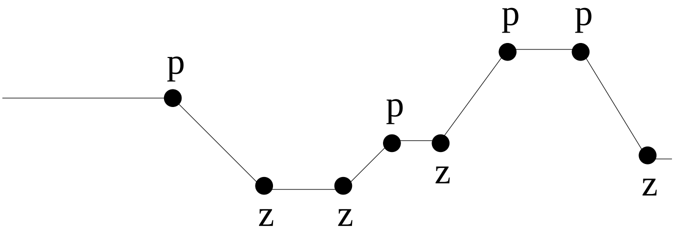


A tropical polynomial is a piecewise linear function with integer slopes, whose image is convex, and a finite number of linear pieces.

Tropical Rational Functions

A Tropical Rational Function is also a piecewise linear function of the same form, but the requirement of convexity is dropped.

The image of a Tropical Rational Function:



A zero of the Tropical Rational Function is a point where the slope increases, and a pole is a point where the slope decreases.

Notice that the image is convex at zeros, but is concave at poles.

Tropical Curves

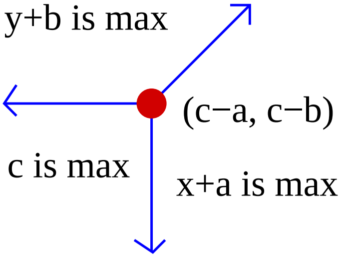
The Corner Locus of a Tropical Function is the set of all points where the slope changes (i.e. the maximum is achieved twice.)

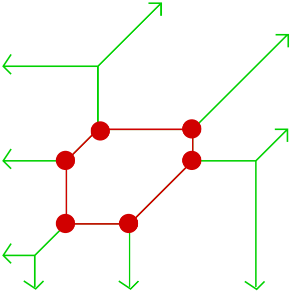
$1 - D$: the corner locus would be the set of zeros and poles.

$2 - D$: The corner locus looks like a Metric Graph (plus unbounded rays).

Tropical Line: $a \odot x \oplus b \odot y \oplus c$ and Tropical Cubic: $\bigoplus_{i+j \leq 3} x^i y^j$.

The Degree of the polynomial equals the # of rays in each direction.





Tropical Riemann-Roch

An Abstract Tropical Curve Γ is simply a Metric Graph, where we allow leaf edges to be of infinite length. The genus of Γ is $g(\Gamma) = |E| - |V| + 1$.

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Examples (Finite portions of Genus 2):

A Chip Configuration C of Γ is a formal linear combination of points of Γ:

$$C = \sum_P c_P P \qquad \text{(only finitely many } c_P\text{'s are nonzero).}$$

The Canonical Chip Configuration K is the sum

$$K = K(\Gamma) = \sum_{V \in \Gamma} (\deg(V) - 2)V.$$

A certain rank function $r(C)$ satisfies Riemann-Roch: (Baker-Norine '07)

$$r(C) - r(K - C) = \deg C + 1 - g(\Gamma).$$

Tropical Linear Systems

Given a tropical rational function f, we let $ord_P(f)$ denote the sum of the outgoing slopes locally at point P with respect to the function f.

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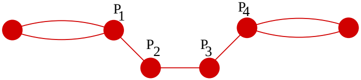
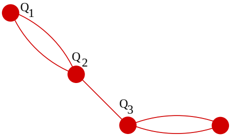
Examples: $g_1 =$ 

, $g_2 =$ 

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Then $(g_1) = -P_1 + P_2 + P_3 - P_4$. and $(g_2) = -2Q_1 + Q_2 + Q_3$.

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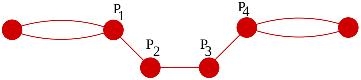
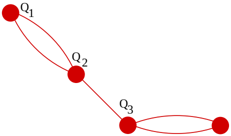
Examples: $g_1 =$ 

, $g_2 =$ 

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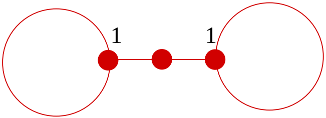
Can also think of these transformations as weighted chip-firing moves.

The Tropical Linear System of C (following Gathmann-Kerber):

$$|C| = \{C' \geq 0 : C' = C + (f) \text{ for some tropical rational funciton } f\}.$$

Tropical Linear Systems (Example Continued)

For $\Gamma =$ 

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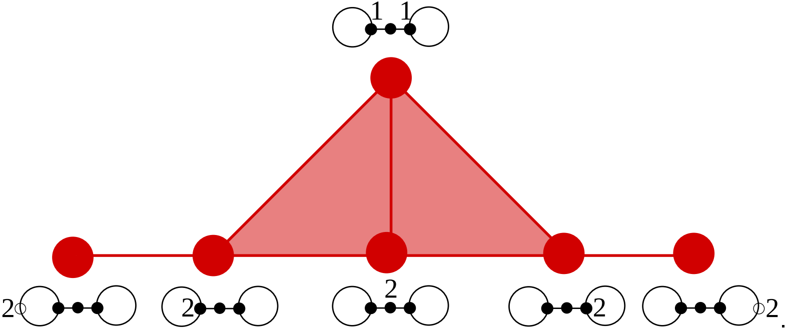

The Linear System $|C|$ contains six 0-cells, seven 1-cells and two 2-cells.

$|C|$ and $R(C)$ as polyhedral cell complexes

Recall $|C| = \{C' \geq 0 : C' = C + (f)$ for some tropical rational function $f\}$.

Let $R(C) = \{f : C + (f) \geq 0\}$. This is a tropical semi-module of functions.

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Let $\mathbb{1}$ denote the set of constant functions on Γ. (Note that if f is constant, then the chip configuration $(f) = 0$.)

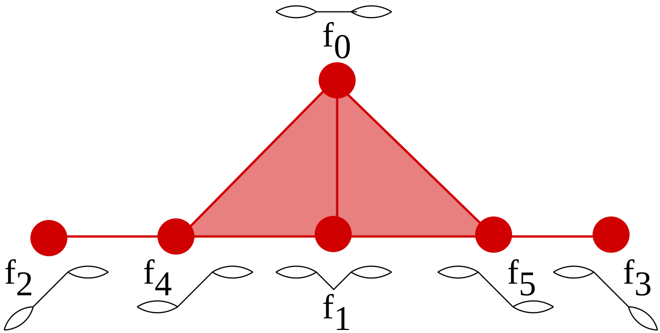
In fact, there is the natural homeomorphism:

$$\begin{array}{ccc} R(C)/\mathbb{1} & \longrightarrow & |C| \\ f & \mapsto & C + (f). \end{array}$$

So a linear system can be described also by tropical rational functions modulo tropical multiplication (i.e. translation by adding a a constant function). Only local slope changes matter, not the function values.

Back To Barbell Example

In terms of tropical rational functions, we obtain the following labeling of the polyhderal complex's vertices instead:



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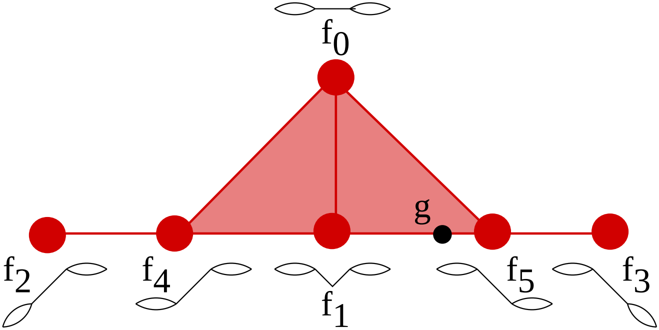

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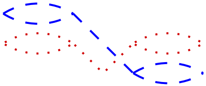
$g = f_1 \oplus (+1/4 \odot f_5) =$ 

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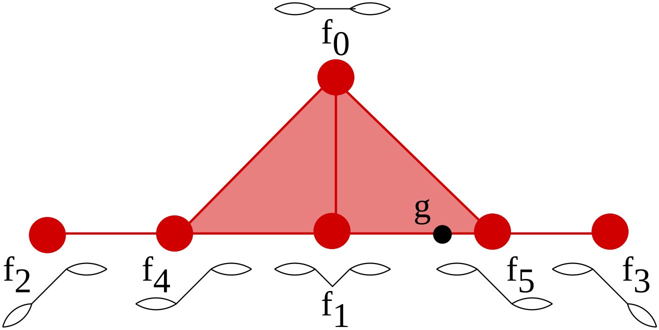

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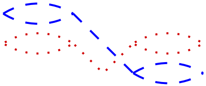
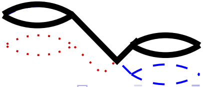
$g = f_1 \oplus (+1/4 \odot f_5) =$  .

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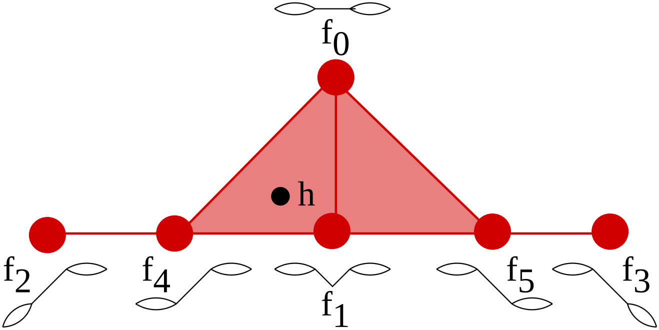

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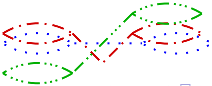
$h = f_0 \oplus (+1/4 \odot f_1) \oplus (+1/3 \odot f_4) =$ 

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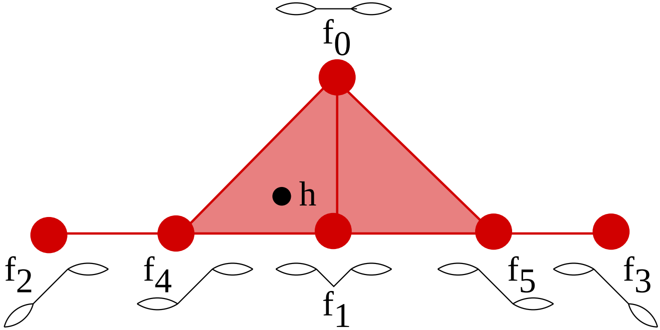

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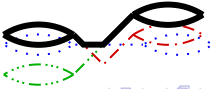
$h = f_0 \oplus (+1/4 \odot f_1) \oplus (+1/3 \odot f_4) =$  .

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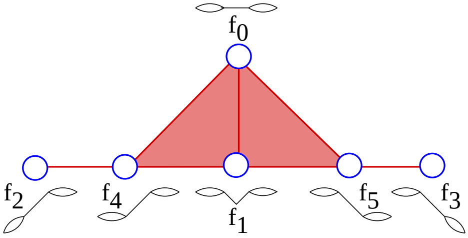

In particular, every tropical rational function on Γ is the tropical convex hull of the 0-cells $\{f_0, f_1, \ldots, f_5\}$.

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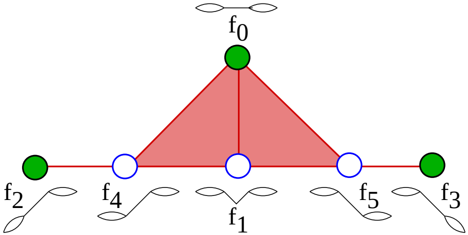

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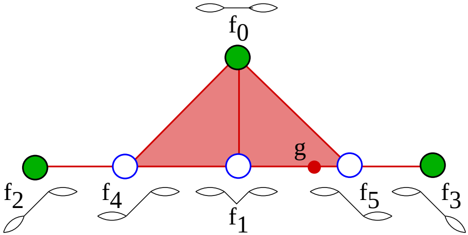

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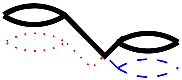

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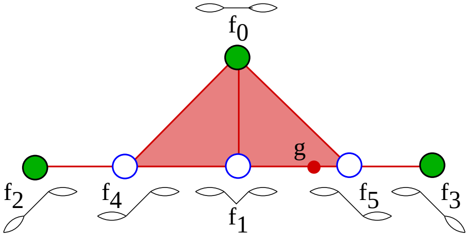

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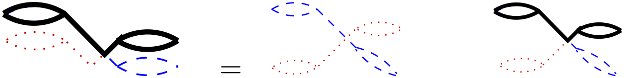

Main Results

Theorem (HMY 2009) $R(C)$ is a finitely generated tropical semimodule.

If $C' \in |C|$, with $C' = C + (f)$, is in the cell with vertices $C_1, C_2, \ldots, C_k$ (with corresponding $f_1, f_2, \ldots, f_k$), then

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In particular, $R(C)/\mathbb{1} \cong |C|$ is finitely generated by the 0-cells of $|C|$.

Theorem (HMY 2009) The 0-cells of $|C|$, as well as all other d-cells, can be described explicitly.

Dimension of a cell

Definition. A point $P \in \Gamma$ is **smooth** if it has valence two and is not a vertex (i.e. the interior of an edge).

Definition. The **support** of a chip configuration C is the set of points of Γ with nonzero coefficients in C.

Let $I(\Gamma, C') = \Gamma \setminus (\mathrm{Supp}\ C' \cap \{\mathrm{Smooth\ points}\})$.

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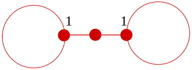
Corollary (HMY 2009) The 0-cells, i.e. a set of generators for $R(C)/\mathbb{1}$, correspond to the C''s whose smooth support does not disconnect Γ.

The extremals lie inside this set: They are the functions f precisely such that no two proper subgraphs Γ_1 and Γ_2 of Γ covering Γ (i.e. $\Gamma_1 \cup \Gamma_2 = \Gamma$) can both fire on the chip configuration $C + (f)$.

For $\Gamma =$ 

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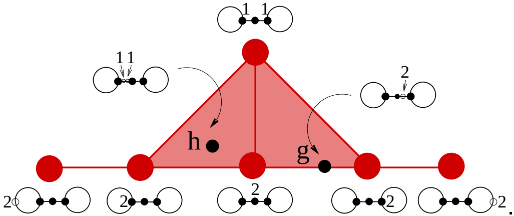
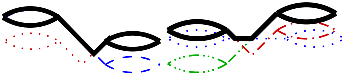
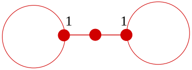




Notice that removal of the smooth support of C' (for C' a 0-cell) does not disconnect the graph Γ.

Another return to the barbell

For $\Gamma =$ 

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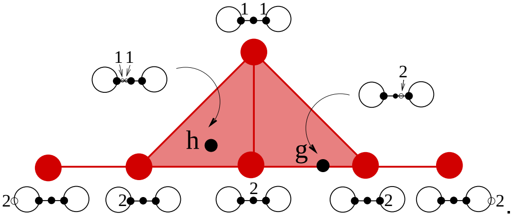

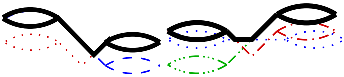

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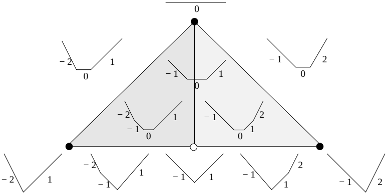
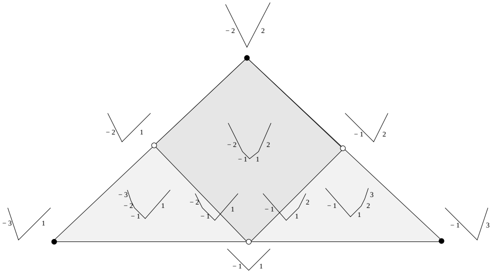
Chip configurations corresponding to tropical rational functions g and h correspond to the interiors of 1-cells and 2-cells.

Removal of their breakpoints disconnects the graph into 2 and 3 pieces.

Final Examples: Genus One Circle Graph

Take the circle $\Gamma = S^1$ on one vertex and a chip configuration of degree d. E.g. $d = 3$ or 4:



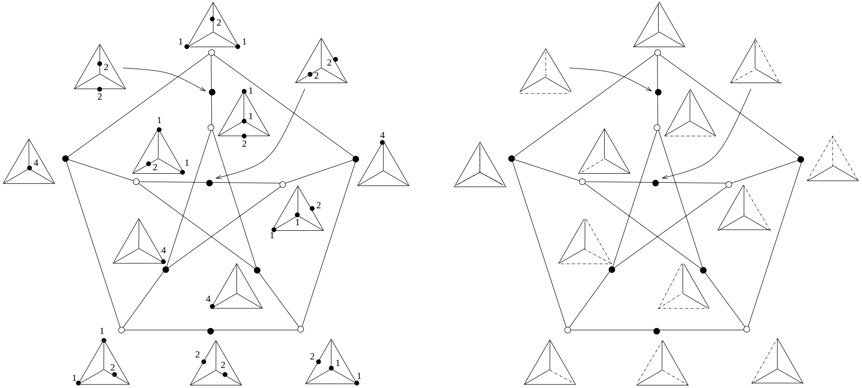


Black Vertices correspond to Extremals. $|C|$ is a subdivision of a $(d-1)$-simplex.

In the case of $d = 4$, $|C|$ is a cone over the triangle that is shown. The cone point is the constant function, and is another extremal.

Final Examples: Complete Graph on 4 Vertices

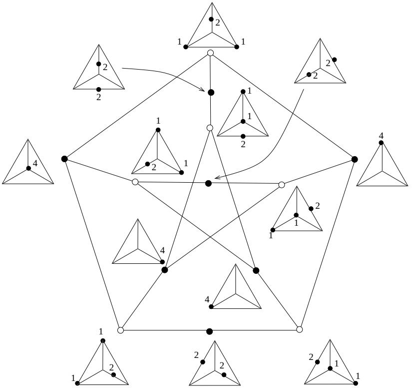
For $\Gamma = K_4$ with edges of equal length and K the canonical chip configuration with 1 at all four vertices: $|K|$ is a cone over the Petersen graph from point K.

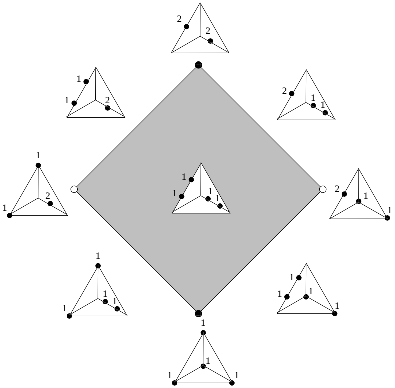


Theorem (HMY) For any Γ, the fine subdivision of $\text{link}(K, |K|)$ contains the fine subdivision of the Bergman complex $B(M^*(\Gamma))$ as a subcomplex.

Final Examples: Complete Graph on 4 Vertices (Continued)

Fourteen 0-cells, seven (black vertices) of which (not K) are extremal.





This is a 2-dimensional cell complex: including K (at the bottom), here is a close-up of a quadrilateral cell. In particular, $|K|$ is not simplicial.

Open Questions

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Thanks for Listening!

Linear Systems on Tropical Curves (with Christian Haase and Josephine Yu), `arXiv:math.AG/0909.3685`