

## LINEAR TIME ALGORITHMS FOR HAMILTONIAN PROBLEMS ON (CLAW,NET)-FREE GRAPHS\*

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**Abstract.** We prove that claw-free graphs, containing an induced dominating path, have a Hamiltonian path, and that 2-connected claw-free graphs, containing an induced doubly dominating cycle or a pair of vertices such that there exist two internally disjoint induced dominating paths connecting them, have a Hamiltonian cycle. As a consequence, we obtain linear time algorithms for both problems if the input is restricted to (claw,net)-free graphs. These graphs enjoy those interesting structural properties.

**Key words.** claw-free graphs, (claw,net)-free graphs, Hamiltonian path, Hamiltonian cycle, dominating pair, dominating path, linear time algorithms

**AMS subject classifications.** 05C38, 05C45, 05C85, 05C75, 68R10

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**1. Introduction.** Hamiltonian properties of claw-free graphs have been studied extensively in the last couple of years. Different approaches have been made, and a couple of interesting properties of claw-free graphs have been established (see [1, 2, 3, 5, 6, 13, 14, 15, 16, 19, 22, 23, 25, 26]). The purpose of this work is to consider the algorithmic problem of finding a Hamiltonian path or a Hamiltonian cycle efficiently. It is not hard to show that both the Hamiltonian path problem and the Hamiltonian cycle problem are NP-complete, even when restricted to line graphs [28]. Hence, it is quite reasonable to ask whether one can find interesting subclasses of claw-free graphs for which efficient algorithms for the above problems exist.

Already in the eighties, Duffus, Jacobson, and Gould [12] defined the class of (claw,net)-free (CN-free) graphs, i.e., graphs that contain neither an induced claw nor an induced net (see Figure 1.1). Although this definition seems to be rather restrictive, the family of CN-free graphs contains a couple of graph families that are of interest in their own right. Examples of those families are unit interval graphs, claw-free asteroidal triple-free (AT-free) graphs, and proper circular arc graphs. In their paper [12], Duffus, Jacobson, and Gould showed that this class of graphs has the nice property that every connected CN-free graph contains a Hamiltonian path and every 2-connected CN-free graph contains a Hamiltonian cycle. Later, Shepherd [27] proved that there is an  $O(n^6)$  algorithm for finding such a Hamiltonian path/cycle in CN-free graphs. Note also that CN-free graphs are exactly the Hamiltonian-hereditary

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graphs [10], i.e., the graphs for which every connected induced subgraph contains a Hamiltonian path.

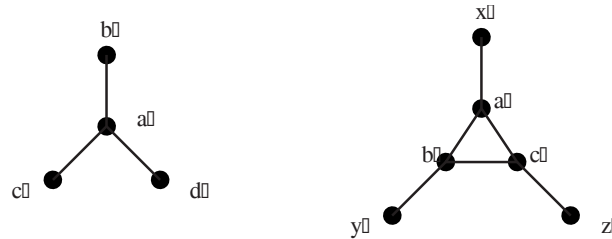
In this paper we give a constructive existence proof and present linear time algorithms for the Hamiltonian path and Hamiltonian cycle problems on CN-free graphs. The important structural property that we exploit for this is the existence of an induced *dominating path* in every connected CN-free graph (Theorem 2.3). The concept of a dominating path was first used by Corneil, Olariu, and Stewart [8] in the context of AT-free graphs. They also developed a simple linear time algorithm for finding such a path in every AT-free graph [7]. As we show in Theorem 2.3, for the class of CN-free graphs, a linear time algorithm for finding an induced dominating path exists as well. This property is of interest for our considerations since we prove that all claw-free graphs that contain an induced dominating path have a Hamiltonian path (Theorem 3.1). The proof implies that, given a dominating path, one can construct a Hamiltonian path for a claw-free graph in linear time.

For 2-connected claw-free graphs, we show that the existence of a dominating pair is sufficient for the existence of a Hamiltonian cycle. (A *dominating pair* is a pair of vertices such that every induced path connecting them is a dominating path.) Again, given a dominating pair, one can construct a Hamiltonian cycle in linear time (Theorem 5.6). This already implies, for example, a linear time algorithm for finding a Hamiltonian cycle in claw-free AT-free graphs, since every AT-free graph contains a dominating pair and it can be found in linear time [9]. Unfortunately, CN-free graphs do not always have a dominating pair. For example, an induced cycle with more than six vertices is CN-free but does not have such a pair of vertices. Nevertheless, 2-connected CN-free graphs have another nice property: they have a good pair or an induced doubly dominating cycle. An *induced doubly dominating cycle* is an induced cycle such that every vertex of the graph is adjacent to at least two vertices of the cycle. A *good pair* is a pair of vertices, such that there exist two internally disjoint induced dominating paths connecting these vertices. We prove that the existence of an induced doubly dominating cycle or a good pair in a claw-free graph is sufficient for the existence of a Hamiltonian cycle (Theorems 5.1 and 5.5). Moreover, given an induced doubly dominating cycle or a good pair of a claw-free graph, a Hamiltonian cycle can be constructed in linear time. In section 4 we present an  $O(m + n)$  time algorithm which, for a given 2-connected CN-free graph, finds either a good pair or an induced doubly dominating cycle.

For terms not defined here, we refer to [11, 17]. In this paper we consider finite connected undirected graphs  $G = (V, E)$  without loops and multiple edges. The cardinality of the vertex set is denoted by  $n$ , whereas the cardinality of the edge set is denoted by  $m$ .

A *path* is a sequence of vertices  $(v_0, \dots, v_l)$  such that all  $v_i$  are distinct and  $v_i v_{i+1} \in E$  for  $i = 0, \dots, l - 1$ ; its *length* is  $l$ . An *induced path* is a path where  $v_i v_j \in E$  if and only if  $i = j - 1$  and  $j = 1, \dots, l$ . A *cycle* (*k-cycle*) is a path  $(v_0, \dots, v_k)$  ( $k \geq 3$ ) such that  $v_0 = v_k$ ; its *length* is  $k$ . An *induced cycle* is a cycle where  $v_i v_j \in E$  if and only if  $|i - j| = 1$  (modulo  $k$ ). A *hole*  $H_k$  is an induced cycle of length  $k \geq 5$ .

The *distance*  $dist(v, u)$  between vertices  $v$  and  $u$  is the smallest number of edges in a path joining  $v$  and  $u$ . The *eccentricity*  $ecc(v)$  of a vertex  $v$  is the maximum distance from  $v$  to any vertex in  $G$ . The *diameter*  $diam(G)$  of  $G$  is the maximum eccentricity of a vertex in  $G$ . A pair  $v, u$  of vertices of  $G$  with  $dist(v, u) = diam(G)$  is called a *diametral pair*.

FIG. 1.1. The claw  $K(a; b, c, d)$  and the net  $N(a, b, c; x, y, z)$ .

For every vertex we denote by  $N(v)$  the set of all neighbors of  $v$ ,  $N(v) = \{u \in V : \text{dist}(u, v) = 1\}$ . The *closed neighborhood* of  $v$  is defined by  $N[v] = N(v) \cup \{v\}$ . For a vertex  $v$  and a set of vertices  $S \subseteq V$ , the minimum distance between  $v$  and vertices of  $S$  is denoted by  $\text{dist}(v, S)$ . The *closed neighborhood*  $N[S]$  of a set  $S \subseteq V$  is defined by  $N[S] = \{v \in V : \text{dist}(v, S) \leq 1\}$ .

We say that a set  $S \subseteq V$  *dominates*  $G$  if  $N[S] = V$ , and  $S$  *doubly dominates*  $G$  if every vertex of  $G$  has at least two neighbors in  $S$ . An induced path of  $G$  which dominates  $G$  is called an *induced dominating path*. A shortest path of  $G$  which dominates  $G$  is called a *dominating shortest path*. Analogously one can define an *induced dominating cycle* of  $G$ . A *dominating pair* of  $G$  is a pair of vertices  $v, u \in V$ , such that every induced path between  $v$  and  $u$  dominates  $G$ . A *good pair* of  $G$  is a pair of vertices  $v, u \in V$ , such that there exist two internally disjoint induced dominating paths connecting  $v$  and  $u$ .

The *claw* is the induced complete bipartite graph  $K_{1,3}$ , and for simplicity, we refer to it by  $K(a; b, c, d)$  (see Figure 1.1). The *net* is the induced six-vertex graph  $N(a, b, c; x, y, z)$  shown in Figure 1.1. A graph is called *CN-free* or, equivalently, (*claw, net*)-free if it contains neither an induced claw nor an induced net. An *asteroidal triple* of  $G$  is a triple of pairwise nonadjacent vertices, such that for each pair of them there exists a path in  $G$  that does not contain any vertex in the neighborhood of the third one. A graph is called *AT-free* if it does not contain an asteroidal triple. Finally, a *Hamiltonian path* or *Hamiltonian cycle* of  $G$  is a path or cycle, respectively, containing all vertices of  $G$ .

**2. Induced dominating path.** In this section we give a constructive proof for the property that every connected CN-free graph contains an induced dominating path. In fact, we show that there is an algorithm that finds such a path in linear time. To prove the main theorem of this section we will need the following two lemmas.

LEMMA 2.1 (see [12]). *Let  $P = (x_1, \dots, x_k)$  be an induced path of a CN-free graph  $G$ , and let  $v$  be a vertex of  $G$  such that  $\text{dist}(v, P) = 2$ . Then any neighbor  $y$  of  $v$  with  $\text{dist}(y, P) = 1$  is adjacent to  $x_1$  or to  $x_k$ .*

LEMMA 2.2. *Let  $P$  be an induced path connecting vertices  $v$  and  $u$  of a CN-free graph  $G$ . Let also  $s$  be a vertex of  $G$  such that  $s \notin N[P]$  and  $\text{dist}(v, s) \leq \text{dist}(v, u)$ . Then*

1. *for every shortest path  $P'$  connecting  $v$  and  $s$ ,  $P' \cap P = \{v\}$  holds, and*
2. *if there is an edge  $xy$  of  $G$  such that  $x \in P \setminus \{v\}$  and  $y \in P' \setminus \{v\}$ , then both  $x$  and  $y$  are neighbors of  $v$ .*

*Proof.* Let  $y$  be the vertex of  $P' \setminus \{v\}$  which is closest to  $s$  and has a neighbor  $x$  on  $P \setminus \{v\}$ ; clearly,  $y \neq s$ . Let  $s', v'$  be the neighbors of  $y$  on the subpaths of  $P'$  connecting  $y$  with  $s$  and  $y$  with  $v$ , respectively. Since  $s' \notin N[P]$ , by Lemma 2.1, vertex

$y$  must be adjacent to  $v$  or to  $u$ . If  $yu \in E$ , then  $v'u \in E$ , too (otherwise, we have a claw  $K(y; s', v', u)$ ). But now  $dist(v, u) \leq dist(v, v') + 1 = dist(v, y) < dist(v, s) \leq dist(v, u)$ , and a contradiction arises. Therefore,  $y$  is adjacent to  $v$ , and since  $y \notin P$ , the paths  $P$  and  $P'$  have only the vertex  $v$  in common. Moreover, to avoid a claw  $K(y; s', v, x)$ , vertex  $x$  has to be adjacent to  $v$ .  $\square$

**THEOREM 2.3.** *Every connected CN-free graph  $G$  has an induced dominating path, and such a path can be found in  $O(n + m)$  time.*

*Proof.* Let  $G$  be a connected CN-free graph. One can construct an induced dominating path in  $G$  as follows. Take an arbitrary vertex  $v$  of  $G$ . Using breadth first search (BFS), find a vertex  $u$  with the largest distance from  $v$  and a shortest path  $P$  connecting  $u$  with  $v$ . Check whether this path  $P$  dominates  $G$ . If so, we are done. Now, assume that the set  $S = V \setminus N[P]$  is not empty. Again, using BFS, find a vertex  $s$  in  $S$  with largest distance from  $v$  and a shortest path  $P'$  connecting  $v$  with  $s$ . Create a new path  $P^*$  by joining  $P$  and  $P'$  in the following way:  $P^* = (P \setminus \{v\}, P' \setminus \{v\})$  if there is a chord  $xy$  between the paths  $P$  and  $P'$  (see Lemma 2.2), and  $P^* = (P \setminus \{v\}, P')$ , otherwise. By Lemma 2.2, the path  $P^*$  is induced. It remains to show that this path dominates  $G$ .

Assume there exists a vertex  $t \in V \setminus N[P^*]$ . First, we claim that  $t$  is dominated neither by  $P$  nor by  $P'$ . Indeed, if  $t \in (N[P] \cup N[P']) \setminus N[P^*]$ , then necessarily  $tv \in E$  and  $v \notin P^*$ , i.e., neighbors  $x \in P$  and  $y \in P'$  of  $v$  are adjacent. Therefore, we get a net  $N(v, y, x; t, s', u')$ , where  $s'$  and  $u'$  are the vertices at distance two from  $v$  on paths  $P'$  and  $P$ , respectively. Note that vertices  $s', u'$  exist because  $dist(v, s) \geq 2$ .

Thus,  $t$  is dominated neither by  $P$  nor by  $P'$ . Moreover, from the choice of  $u$  and  $s$  we have  $2 \leq dist(v, t) \leq dist(v, s) \leq dist(v, u)$ . Now let  $P''$  be a shortest path, connecting  $t$  with  $v$ , and let  $z$  be a neighbor of  $v$  on this path. Applying Lemma 2.2 twice (to  $P, P''$  and to  $P', P''$ ), we obtain a subgraph of  $G$  depicted in Figure 2.1. We have three shortest paths  $P, P', P''$ , each of length at least 2 and with only one common vertex  $v$ . These paths can have only chords of type  $zx, zy, xy$ . Any combination of them leads to a forbidden claw or net. This contradiction completes the proof of the theorem. Evidently, the method described above can be implemented to run in linear time.  $\square$

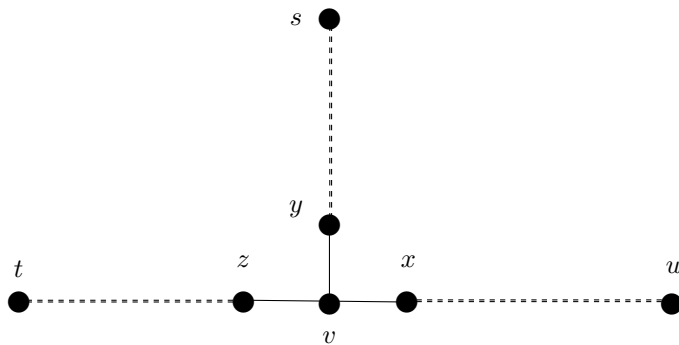


FIG. 2.1.

It is not clear whether CN-free graphs can be recognized efficiently. But, to apply our method for finding an induced dominating path in these graphs, we do not need to know in advance that a given graph  $G$  is CN-free. Actually, our method can be

applied to any graph  $G$ . It either finds an induced dominating path or returns either a claw or a net of  $G$ , showing that  $G$  is not CN-free.

**COROLLARY 2.4.** *There is a linear time algorithm that for a given (arbitrary) connected graph  $G$  either finds an induced dominating path or outputs an induced claw or an induced net of  $G$ .*

*Proof.* Let  $G$  be a graph. For an arbitrary vertex  $v$  of  $G$ , we find a vertex  $u$  with the largest distance from  $v$  and a shortest path  $P$  connecting  $u$  with  $v$ . If  $P$  dominates  $G$ , then we are done. Else, we find a vertex  $s \in V \setminus N[P]$  with the largest distance from  $v$  and a shortest path  $P'$  connecting  $v$  with  $s$ . If there are vertices in  $P' \setminus \{v\}$  which have a neighbor on  $P \setminus \{v\}$ , we take the vertex  $y$  that is closest to  $s$  and check whether  $y$  is adjacent to  $v$  and  $u$ . If it is adjacent neither to  $u$  nor to  $v$ , then  $G$  has a net or a claw (see the proof of Lemma 2.1). If  $yu \in E$  or  $yv \in E$  and a neighbor  $x$  of  $y$  on  $P \setminus \{v\}$  is not adjacent to  $v$ , then  $G$  has a claw (see Lemma 2.2). Now, if we did not yet find a forbidden subgraph, then the only possible chord between the paths  $P$  and  $P'$  is  $xy$  with  $xv, yv \in E$ , and we can create an induced path  $P^*$  as described in the proof of Theorem 2.3. Hence, it remains to check whether  $P^*$  dominates  $G$ . If there exists a vertex  $t \in V \setminus N[P^*]$ , then again we will find a net or a claw in  $G$  (see Theorem 2.3). It is easy to see that the total time bound of all these operations is linear.  $\square$

**3. Hamiltonian path.** In what follows we show that for claw-free graphs the existence of an induced dominating path is a sufficient condition for the existence of a Hamiltonian path. The proof for this result is constructive, implying that, given an induced dominating path, one can find a Hamiltonian path efficiently.

**THEOREM 3.1.** *Every connected claw-free graph  $G$  containing an induced dominating path has a Hamiltonian path. Moreover, given an induced dominating path, a Hamiltonian path of  $G$  can be constructed in linear time.*

*Proof.* Let  $G = (V, E)$  be a connected claw-free graph and let  $P = (x_1, \dots, x_k)$  ( $k \geq 1$ ) be an induced dominating path of  $G$ . If  $k = 1$ , vertex  $x_1$  dominates  $G$  and, since  $G$  is claw-free, there are no three independent vertices in  $G - \{x_1\}$ . (By  $G - \{x_1\}$  we denote a subgraph of  $G$  induced by vertices  $V \setminus \{x_1\}$ .) If  $G - \{x_1\}$  is not connected, then, again because  $G$  is claw-free, it consists of two cliques  $C_0, C_1$  and a Hamiltonian path of  $G$  can easily be constructed. If  $G - \{x_1\}$  is connected, we can construct a Hamiltonian path as follows. First, we construct a maximal path  $P_1 = (y_1, \dots, y_l)$ , i.e., all vertices that are not in  $P_1$  are neither connected to  $y_1$  nor to  $y_l$ . Let  $R$  be the set of all remaining vertices. If  $R = \emptyset$ , we are done. If there is any vertex in  $R$ , it follows that  $y_1 y_l \in E$  since otherwise there are three independent vertices in  $G - \{x_1\}$ . Furthermore, any two vertices of  $R$  are joined by an edge, since otherwise they would form an independent triple with  $y_1$  (and with  $y_l$  as well). Hence,  $R$  induces a clique. Since  $G - \{x_1\}$  is connected, there has to be an edge from a vertex  $v_R \in R$  to some vertex  $y_i \in P_1$  ( $1 < i < l$ ). Now we can construct a Hamiltonian path  $P$  of  $G$ :  $P = (x_1, y_{i+1}, y_{i+2}, \dots, y_l, y_1, y_2, \dots, y_i, v_R, \tilde{R})$ , where  $\tilde{R}$  stands for an arbitrary permutation of the vertices of  $R \setminus \{v_R\}$ .

For  $k \geq 2$  we first construct a Hamiltonian path  $P_2$  for  $G' = G(N[x_1] \setminus \{x_2\})$  as described above, using  $x_1$  as the dominating vertex. At least one endpoint of  $P_2$  is adjacent to  $x_2$  since if  $G' - \{x_1\}$  is not connected,  $x_2$  has to be adjacent to all vertices of either  $C_0$  or  $C_1$  (otherwise, there is a claw in  $G$ ), and if  $G' - \{x_1\}$  is connected, the construction gives a path ending in  $x_1$  which is, of course, adjacent to  $x_2$ . To construct a Hamiltonian path for the rest of the graph we define for each vertex  $x_i$  ( $i \geq 2$ ) of  $P$  a set of vertices  $C_i = N(x_i) \setminus \bigcup_{j=1}^{i-1} N[x_j]$ . Each set  $C_i$  forms a clique of  $G$  since if two

vertices  $u, v \in C_i$  are not adjacent, then the set  $u, v, x_i, x_{i-1}$  induces a claw. Hence we can construct a path  $P^* = (P_2, x_2, P_2^C, x_3, P_3^C, x_4, \dots, x_{k-1}, P_{k-1}^C, x_k, P_k^C)$ , where  $P_i^C$  stands for an arbitrary permutation of the vertices of  $C_i \setminus \{x_{i+1}\}$ . This path  $P^*$  is a Hamiltonian path of  $G$  because it obviously is a path, and, since  $P$  is a dominating path, each vertex of  $G$  has to be either on  $P$ ,  $P_2$ , or in one of the sets  $C_i$ .

For the case  $k = 1$  both finding the connected components of  $G - \{x_1\}$  and constructing the path  $P_1$  can easily be done in linear time. For  $k \geq 2$  we just have to make sure that the construction of the sets  $C_i$  can be done in  $O(n + m)$ , and this can be realized easily within the required time bound.  $\square$

**THEOREM 3.2.** *Every connected CN-free graph  $G$  has a Hamiltonian path, and such a path can be found in  $O(n + m)$  time.*

*Proof.* By Theorem 2.3, every connected CN-free graph has an induced dominating path  $P$ , and it can be found in linear time. Using the path  $P$ , by Theorem 3.1, one can construct a Hamiltonian path of  $G$  in linear time.  $\square$

Analogously to Corollary 2.4, we can state the following.

**COROLLARY 3.3.** *There is a linear time algorithm that for a given (arbitrary) connected graph  $G$  either finds a Hamiltonian path or outputs an induced claw or an induced net of  $G$ .*

*Proof.* The proof follows from Corollary 2.4 and the proof of Theorem 3.1.  $\square$

**4. Induced dominating cycle, dominating shortest path, or good pair.**

In this section we show that every 2-connected CN-free graph  $G$  has an induced doubly dominating cycle or a good pair. Moreover, we present an efficient algorithm that, for a given 2-connected CN-free graph  $G$ , finds either a good pair or an induced doubly dominating cycle.

**LEMMA 4.1.** *Every hole of a connected CN-free graph  $G$  dominates  $G$ .*

**COROLLARY 4.2.** *Let  $H$  be a hole of a connected CN-free graph  $G$ . Every vertex of  $V \setminus H$  is adjacent to at least two vertices of  $H$ .*

A subgraph  $G'$  of  $G$  (doubly) dominates  $G$  if the vertex set of  $G'$  (doubly) dominates  $G$ .

**LEMMA 4.3.** *Every induced subgraph of a connected CN-free graph  $G$  which is isomorphic to  $S_3$  or  $S_3^-$  (see Figure 4.1) dominates  $G$ .*

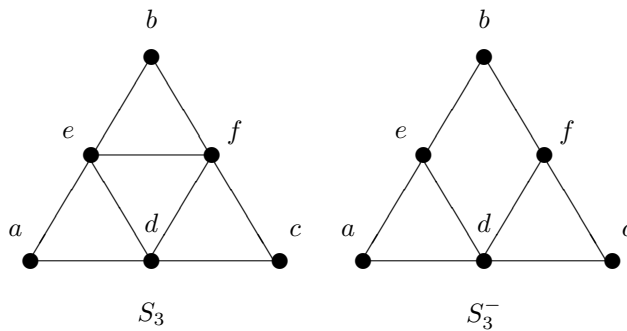


FIG. 4.1.

*Proof.* Let  $G$  contain an induced subgraph isomorphic to  $S_3^-$ , and assume that it does not dominate  $G$ . Then, there must be a vertex  $s$  such that  $dist(s, S_3^-) = 2$ . Let  $x$  be a neighbor of  $s$  from  $N[S_3^-]$ . If  $x$  is adjacent neither to  $a$ , nor to  $b$ , nor to  $c$  (see Figure 4.1), then  $G$  contains a claw (e.g., if  $xf \in E$ , then a claw  $K(f; b, c, x)$  arises). Thus, without loss of generality,  $x$  has to be adjacent to  $a$  or  $b$ .

If  $xa \in E$ , then  $x$  is adjacent neither to  $b$  nor to  $c$ , since otherwise we will get a claw  $(K(x; a, b, s)$  or  $K(x; a, c, s))$ . To avoid a net  $N(a, e, d; x, b, c)$  vertex  $x$  must be adjacent to  $e$  or  $d$ . But, if  $ex \in E$ , then  $xd \in E$  too. (Otherwise, we will have a claw  $K(e; b, d, x)$ .) Analogously, if  $xd \in E$ , then also  $xe \in E$ . Hence,  $x$  is adjacent to both  $e$  and  $d$ , and a net  $N(x, e, d; s, b, c)$  arises.

Now, we may assume that  $x$  is adjacent to  $b$  and not to  $a, c$ . To avoid a claw  $K(b; x, e, f)$ ,  $x$  must be adjacent to  $e$  or  $f$ . But again,  $xe \in E$  if and only if  $xf \in E$ . (Otherwise, we get a net  $N(x, b, e; s, f, a)$  or  $N(x, b, f; s, e, c)$ .) Hence  $x$  is adjacent to both  $e$  and  $f$  and a claw  $K(x; s, e, f)$  arises.

Consequently,  $S_3^-$  dominates  $G$ . Similarly, every induced  $S_3$  (if it exists) dominates  $G$ .  $\square$

LEMMA 4.4. *Let  $P$  be an induced path connecting vertices  $v$  and  $u$  of a connected CN-free graph  $G$ . Let  $s$  be a vertex of  $G$  such that  $s \notin N[P]$  and  $dist(v, s) \leq dist(v, u)$ ,  $dist(u, s) \leq dist(v, u)$ . Then  $G$  has an induced doubly dominating cycle, and such a cycle can be found in linear time.*

*Proof.* Let  $P_v$  and  $P_u$  be shortest paths connecting vertex  $s$  with  $v$  and  $u$ , respectively. Both these paths as well as the path  $P$  have lengths at least 2. Since  $dist(v, s) \leq dist(v, u)$  and  $dist(u, s) \leq dist(u, v)$ , by Lemma 2.2, we have  $P \cap P_v = \{v\}$  and  $P \cap P_u = \{u\}$ . Moreover, if there is a chord between  $P$  and  $P_v$ , then it is unique and both its endvertices are adjacent to  $v$ . The same holds for  $P$  and  $P_u$ ; both endvertices of the chord (if it exists) are adjacent to  $u$ .

Now, without loss of generality, we suppose that  $dist(s, u) \leq dist(s, v)$ . Then, from  $u \notin N[P_v]$  and Lemma 2.2 we deduce that  $P_u \cap P_v = \{s\}$  and between paths  $P_v$  and  $P_u$  at most one chord is possible, namely, the one with both endvertices adjacent to  $s$ . Consequently, we have constructed an induced subgraph of  $G$  shown in Figure 4.2 (only chords  $s's''$ ,  $v'v''$  and  $u'u''$  are possible).

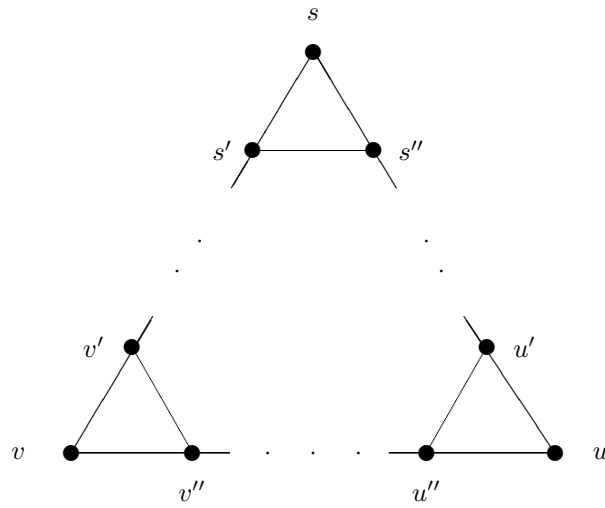


FIG. 4.2.

If the lengths of all three paths  $P, P_v, P_u$  are at least 3, then it is easy to see that  $G$  has a hole  $H_k$  ( $k \geq 6$ ). Furthermore, if at least one of these paths has length greater than or equal to 4, or two of them have lengths 3, then  $G$  must contain a hole  $H_k$  ( $k \geq 5$ ). It remains to consider two cases: lengths of both  $P_v$  and  $P_u$  are 2 and

the length of  $P$  is 3 or 2. Clearly, in both of these cases the graph  $G$  contains either a hole  $H_k$  ( $k \in \{5, 6, 7\}$ ) or an induced subgraph isomorphic to  $S_3^-$  or  $S_3$ . By Corollary 4.2, every hole of  $G$  doubly dominates  $G$ .

Let  $G$  contain an  $S_3^-$  with vertex labeling shown in Figure 4.1. We claim that the induced cycle  $(e, b, f, d, e)$  dominates  $G$  or  $G$  contains a hole  $H_6$ . Indeed, if a vertex  $s$  of  $G$  does not belong to  $S_3^-$ , then, by Lemma 4.3, it is adjacent to a vertex of  $S_3^-$ . Suppose that  $s$  is adjacent to none of  $e, b, f, d$ . Then, without loss of generality,  $sa \in E$ , and we obtain an induced subgraph of  $G$  isomorphic either to a net  $N(e, a, d; b, s, c)$  or to  $H_6 = (s, a, e, b, f, c, s)$ , depending on whether vertices  $s$  and  $c$  are adjacent. Hence, we may assume that  $(e, b, f, d, e)$  dominates  $G$ , and since  $G$  is claw-free, this cycle is doubly dominating.

Now let  $G$  contain an  $S_3$  with vertex labeling shown in Figure 4.1. We will show that every vertex of  $G$  is adjacent to at least two vertices of the cycle  $(e, f, d, e)$  or  $G$  contains a hole  $H_5$ . Suppose vertex  $s$  of  $G$  is adjacent to none of  $e, d$ . Then, by Lemma 4.3,  $s$  is adjacent to at least one of  $a, b, c, f$ . Let  $sf \in E$ . To avoid a claw, vertex  $s$  is adjacent to both  $b$  and  $c$ . But then a hole  $H_5 = (s, b, e, d, c, s)$  arises. Assume that  $sf \notin E$  and, without loss of generality,  $sa \in E$ . To avoid a net  $N(a, e, d; s, b, c)$ ,  $s$  must be adjacent to  $b$  or  $c$ . In both cases a hole  $H_5$  occurs.

Clearly, the construction of an induced doubly dominating cycle of  $G$  given above takes linear time.  $\square$

**THEOREM 4.5.** *There is a linear time algorithm that, for a given connected CN-free graph  $G$ , either finds an induced doubly dominating cycle or gives a dominating shortest path of  $G$ .*

*Proof.* Let  $G$  be a connected CN-free graph. One can construct an induced doubly dominating cycle or a dominating shortest path of  $G$  as follows (compare with the proof of Theorem 2.3). Take an arbitrary vertex  $v$  of  $G$ . Find a vertex  $u$  with the largest distance from  $v$  and a shortest path  $P$  connecting  $u$  with  $v$ . Check whether  $P$  dominates  $G$ . If so, we are done;  $P$  is a dominating shortest path of  $G$ . Assume now that the set  $S = V \setminus N[P]$  is not empty. Find a vertex  $s$  in  $S$  with the largest distance from  $v$  and a shortest path  $P_v$  connecting  $v$  with  $s$ . Create again a new path  $P^*$  by “joining” shortest paths  $P$  and  $P_v$  as in the proof of Theorem 2.3. We have proven there that  $P^*$  dominates  $G$ . Now let  $P_u$  be a shortest path between  $s$  and  $u$ . If  $dist(s, u) \leq dist(v, u)$  or both  $dist(s, u) > dist(v, u)$  and  $v \notin N[P_u]$ , then Lemma 4.4 can be applied to get an induced doubly dominating cycle of  $G$  in linear time. Therefore, we may assume that  $dist(s, u) > dist(v, u) \geq dist(v, s)$  and  $v \in N[P_u]$ . Now we show that the shortest path  $P_u$  dominates  $G$ . If  $v$  lies on the path  $P_u$ , then  $P^* = P_u$  and we are done. Otherwise, let  $x$  be a neighbor of  $v$  in  $P_u$ . Note that  $dist(v, s) > 1$  and so  $x \neq s, u$ . Since  $G$  is claw-free,  $v$  is adjacent to a neighbor  $y \in P_u$  of  $x$ . Assume, without loss of generality, that  $x$  is closer to  $s$  than  $y$ . If we show that  $dist(v, s) = 1 + dist(x, s)$  and  $dist(v, u) = 1 + dist(y, u)$ , then again, by the proof of Theorem 2.3, the path  $P_u$  will dominate  $G$  (as a path obtained by “joining” two shortest paths that connect  $v$  with  $u$  and  $v$  with  $s$ , respectively). By the triangle condition, we have  $dist(u, s) < dist(v, u) + dist(v, s)$  (strict inequality because  $v \notin P_u$ ) and  $dist(v, s) \leq 1 + dist(x, s)$ ,  $dist(u, v) \leq 1 + dist(y, u)$ . Consequently,  $dist(v, u) + dist(v, s) > dist(u, s) = dist(u, y) + 1 + dist(x, s) \geq dist(v, u) - 1 + 1 + dist(v, s) - 1 = dist(v, u) + dist(v, s) - 1$ . That is,  $dist(u, s) = dist(v, u) + dist(v, s) - 1$  and  $dist(v, s) = 1 + dist(x, s)$ ,  $dist(u, v) = 1 + dist(y, u)$ .  $\square$

Since all our proofs were constructive, we can conclude the following.



COROLLARY 4.6. *There is a linear time algorithm that, for a given (arbitrary) connected graph  $G$ , either finds an induced doubly dominating cycle, or gives a dominating shortest path, or outputs an induced claw or an induced net of  $G$ .*

LEMMA 4.7. *Let  $P = (v, x_2, \dots, x_{k-1}, u)$  be a dominating shortest path of a graph  $G$ . Then  $\max\{ecc(v), ecc(u)\} \geq diam(G) - 1$ .*

*Proof.* Let  $x, y$  be a diametral pair of vertices of  $G$ ; that is,  $diam(G) = dist(x, y)$ . If both  $x$  and  $y$  are on  $P$ , then necessarily  $\{x, y\} = \{v, u\}$  and therefore  $dist(v, u) = diam(G) = ecc(v) = ecc(u)$ . If  $x \in P$  and  $y \in N[P] \setminus P$ , then either  $x \neq v, u$  and  $diam(G) = dist(x, y) = dist(v, u)$  holds or, without loss of generality,  $v = x$  and  $ecc(v) = diam(G)$ . Finally, if both  $x$  and  $y$  are in  $N[P] \setminus P$  and  $dist(v, u) < dist(x, y)$ , then we may assume that at least one of  $x, y$  belongs to  $N(v)$ , say,  $x$ . Hence,  $dist(x, y) \leq 1 + dist(v, y) \leq 1 + ecc(v)$ ; that is,  $ecc(v) \geq diam(G) - 1$ .  $\square$

A pair of vertices  $u, v$  of  $G$  with  $dist(u, v) = ecc(u) = ecc(v)$  is called a *pair of mutually furthest vertices*.

COROLLARY 4.8. *For a graph  $G$  with a given dominating shortest path, a pair of mutually furthest vertices can be found in linear time.*

*Proof.* Let  $P = (v, x_2, \dots, x_{k-1}, u)$  be a dominating shortest path of  $G$  with  $ecc(v) \geq ecc(u)$ . Then, by Lemma 4.7,  $ecc(v) \geq diam(G) - 1$  holds. Denote by  $x$  a vertex of  $G$  such that  $dist(v, x) = ecc(v)$ . Note that both the eccentricity of  $v$  and a vertex furthest from  $v$  can be found in linear time by BFS. Now, if  $ecc(x) = ecc(v)$ , then  $v, x$  are mutually furthest vertices of  $G$ . Else,  $ecc(x) > ecc(v) \geq diam(G) - 1$  must hold and vertices  $x$  and  $y$ , where  $y$  is a vertex with  $dist(x, y) = ecc(x)$ , form a diametral pair of  $G$ ;  $dist(x, y) = ecc(x) = ecc(y) = diam(G)$ .  $\square$

In what follows we will use the fact that in a 2-connected graph every pair of vertices is joined by two internally disjoint paths. In order to actually find such a pair of paths, one can use Tarjan's linear time depth first search- (DFS)-algorithm for finding the blocks of a given graph. For the proof of Lemma 4.9, we refer to [21].

LEMMA 4.9. *Let  $G$  be a 2-connected graph, and let  $x, y$  be two different nonadjacent vertices of  $G$ . Then one can construct in linear time two induced, internally disjoint paths, both joining  $x$  and  $y$ .*

THEOREM 4.10. *There is a linear time algorithm that, for a given 2-connected CN-free graph  $G$ , either finds an induced doubly dominating cycle or gives a good pair of  $G$ .*

*Proof.* By Theorem 4.5, we get either an induced doubly dominating cycle or a dominating shortest path of  $G$  in linear time. We show that, having a dominating shortest path of a 2-connected graph  $G$ , one can find in linear time a good pair or an induced doubly dominating cycle. By Corollary 4.8, we may assume that a pair  $x, y$  of mutually furthest vertices of  $G$  is given. Let also  $P_1, P_2$  be two induced internally disjoint paths connecting  $x$  and  $y$  in  $G$ . They exist and can be found in linear time by Lemma 4.9 (clearly, we may assume that  $xy \notin E$ , because otherwise  $N[x] = V = N[y]$  and  $x, y$  together with a vertex  $z \in V \setminus \{x, y\}$  will form a doubly dominating triangle). If one of these paths, say,  $P_1$ , is not dominating, then there must be a vertex  $s \in V \setminus N[P_1]$ . Since  $x, y$  are mutually furthest vertices of  $G$ , we have  $dist(s, x) \leq dist(x, y)$ ,  $dist(s, y) \leq dist(x, y)$ . Hence, we are in the conditions of Lemma 4.4 and can find an induced doubly dominating cycle of  $G$  in linear time.  $\square$

COROLLARY 4.11. *There is a linear time algorithm that, for a given (arbitrary) 2-connected graph  $G$ , either finds an induced doubly dominating cycle, or gives a good pair, or outputs an induced claw or an induced net of  $G$ .*

**5. Hamiltonian cycle.** In this section we prove that, for claw-free graphs, the existence of an induced doubly dominating cycle or a good pair is sufficient for the existence of a Hamiltonian cycle. The proofs are also constructive and imply linear time algorithms for finding a Hamiltonian cycle.

**THEOREM 5.1.** *Every claw-free graph  $G$  that contains an induced doubly dominating cycle has a Hamiltonian cycle. Moreover, given an induced doubly dominating cycle, a Hamiltonian cycle of  $G$  can be constructed in linear time.*

*Proof.* Let  $DC = (x_1, \dots, x_k, x_1)$  ( $k \geq 3$ ) be an induced doubly dominating cycle of  $G$ . As before, we define  $C_i = N(x_i) \setminus \bigcup_{j=1}^{i-1} N[x_j]$  ( $2 \leq i \leq k$ ). Each set  $C_i$  forms a clique of  $G$ ; otherwise, we would have a claw. Furthermore,  $C_k = \emptyset$  holds, and the sets  $N[x_1], C_2, \dots, C_{k-1}$  form a partition of the vertex set of  $G$ . Note that any vertex adjacent to  $x_k$  and not to  $x_j$  ( $1 < j < k$ ) belongs to  $N[x_1]$ , since the cycle  $DC$  is doubly dominating. Let  $G' = G(N[x_1] \setminus \{x_2, x_k\})$  be the subgraph of  $G$  induced by  $N[x_1] \setminus \{x_2, x_k\}$ . If we show that there is a Hamiltonian path  $P$  in  $G'$  starting at a neighbor of  $x_k$  and ending at a neighbor of  $x_2$ , then we are done; the cycle  $(x_k, P, x_2, P_2^C, x_3, P_3^C, x_4, \dots, x_{k-1}, P_{k-1}^C, x_k)$ , is a Hamiltonian cycle of  $G$  (recall that  $P_i^C$  stands for an arbitrary permutation of the vertices of  $C_i \setminus \{x_{i+1}\}$ ).

Since  $G'$  is a connected graph, by Theorem 3.1, there exists a Hamiltonian path  $P' = (s, y_1, \dots, y_l, t)$  of  $G'$ . Assume that  $x_k s, x_k t \notin E$ . Then, to avoid a claw  $K(x_1; x_k, s, t)$ , vertices  $s$  and  $t$  have to be adjacent, giving a new Hamiltonian path  $P''$  of  $G'$  starting at  $x_1$  and ending at a vertex  $y$ . If  $y$  is adjacent neither to  $x_k$  nor to  $x_2$ , then a claw  $K(x_1; x_k, x_2, y)$  occurs. (Note that in case  $k = 3$ , i.e.,  $x_k x_2 \in E$ ,  $y$  is adjacent to at least one of  $x_k, x_2$  because the cycle  $DC = (x_1, x_2, x_3, x_1)$  is doubly dominating.) Without loss of generality,  $yx_2 \in E$  and the path  $P''$  is a desired path of  $G'$ .

So, we may assume that  $x_k$  is adjacent to  $t$  or  $s$ . Analogously,  $x_2$  is adjacent to one of  $t, s$ . If  $x_k, x_2$  are adjacent to different vertices, then we are done; the path  $P'$  starts at a neighbor of  $x_k$  and ends at a neighbor of  $x_2$ . Otherwise, let both  $x_k$  and  $x_2$  be adjacent to  $t$  and not to  $s$ . Then a claw  $K(x_1; x_k, x_2, s)$  arises when  $k > 3$ , or we get a contradiction with the property of  $DC = (x_1, x_2, x_3, x_1)$  to be a doubly dominating cycle.  $\square$

**COROLLARY 5.2.** *Every claw-free graph, containing an induced dominating cycle of length at least 4, has a Hamiltonian cycle, and, given that induced dominating cycle, one can construct a Hamiltonian cycle in linear time.*

Let  $G = (V, E)$  be a graph, and let  $P = (x_1, \dots, x_k)$  be an induced dominating path of  $G$ .  $P$  is called an *enlargeable* path if there is some vertex  $v$  in  $V \setminus P$  that is either adjacent to  $x_1$  or to  $x_k$  but not to both of them and, additionally, to no other vertex in  $P$ . Consequently, an induced dominating path  $P$  is called *nonenlargeable* if such a vertex does not exist. Obviously, every graph  $G$  that has an induced dominating path has a nonenlargeable induced dominating path as well. Furthermore, given an induced dominating path  $P$ , one can find in linear time a nonenlargeable induced dominating path  $P'$  by simply scanning the neighborhood of both  $x_1$  and  $x_k$ . For the next theorem we will need an auxiliary result.

**LEMMA 5.3.** *Let  $G$  be a claw-free graph, and let  $P = (x_1, x_2, \dots, x_k)$  ( $k > 2$ ) be an induced nonenlargeable dominating path of  $G$  such that there is no vertex  $y$  in  $G$  with  $N(y) \cap P = \{x_1, x_k\}$ . Then there is a Hamiltonian path in  $G$  that starts in  $x_1$  and ends in  $x_k$  and, given the path  $P$ , one can construct this Hamiltonian path in linear time.*

*Proof.* Let  $C_i = N(x_i) \setminus \bigcup_{j=1}^{i-1} N[x_j]$  ( $i \geq 2$ ). Since  $P$  is nonenlargeable,  $C_k$  is empty. Using the method described in the proof of Theorem 3.1, we can easily construct a path, starting in  $x_1$  and ending in  $x_k$ , that contains all vertices of  $C_2, \dots, C_{k-1}$ . This implies that we have to worry only about how to insert the vertices of the neighborhood of  $x_1$  into this path. We have to consider two cases.

*Case 1.*  $H = G(N(x_1) \setminus \{x_2\})$  consists of two connected components  $C_0, C_1$ .

Since  $G$  is claw-free, both  $C_0$  and  $C_1$  induce cliques in  $G$ . Furthermore,  $x_2$  is adjacent to all vertices of at least one of  $C_0$  and  $C_1$ , say,  $C_1$ , because otherwise we have a claw in  $G$ .

Let  $y$  be an arbitrary vertex of  $C_0$ . Since  $P$  is nonenlargeable,  $y$  has at least one neighbor on  $P \setminus \{x_1\}$ , and let  $x_j$  be the one with smallest index. By the preconditions of our lemma,  $j \neq k$ . If  $j > 2$ , then  $y$  has to be adjacent to  $x_{j+1}$  as well, since otherwise  $K(x_j; y, x_{j-1}, x_{j+1})$  is a claw. Furthermore,  $y$  is adjacent to all vertices  $c_j \in C_j$ , since otherwise  $K(x_j; y, x_{j-1}, c_j)$  is a claw. Hence, when constructing the Hamiltonian path, we can simply add  $y$  to  $C_j$ .

Now we consider the set  $Y$  of all vertices  $y$  of  $C_0$  with  $yx_2 \in E$ . Suppose there is a vertex  $c_2$  in  $C_2$  with  $c_2 \neq x_3$ . If there is a vertex  $c_1 \in C_1$  that is nonadjacent to vertex  $c_2$ , then there is an edge from every vertex  $c_0 \in Y$  to  $c_2$ ; otherwise,  $K(x_2; c_0, c_1, c_2)$  is a claw of  $G$ . This implies that we can construct a Hamiltonian path with the required properties. If, on the other hand, all vertices of  $C_1$  are adjacent to all vertices of  $C_2$ , we can construct such a path by starting in  $x_1$ , traversing through  $Y, x_2, C_1, C_2$ , and proceeding as before. Now suppose that there is no vertex  $c_2 \in C_2$  with  $c_2 \neq x_3$ . In this case either all vertices  $c_0 \in Y$  or all vertices  $c_1 \in C_1$  have to be adjacent to  $x_3$ , because otherwise  $K(x_2; c_0, c_1, x_3)$  is a claw. Suppose, without loss of generality, that all vertices of  $Y$  are adjacent to  $x_3$ . Then we construct the path by starting in  $x_1$ , traversing through  $C_1, x_2, Y, x_3$ , and proceeding as before.

*Case 2.*  $H = G(N(x_1) \setminus \{x_2\})$  induces a connected graph.

If  $x_2$  is not adjacent to any of the vertices in  $H$ , then  $H$  has to be a clique and we can apply the method described in case 1.

Suppose now that  $x_2$  is adjacent to some vertex in  $H$ . First, we construct a Hamiltonian path  $P' = (y_1, \dots, y_l)$  in  $H$ , which is done as in the proof of Theorem 3.1, since there is no independent triple in  $H$ . Now we claim that either  $x_2$  is adjacent to one of  $y_1$  or  $y_l$ , or  $P'$  does in fact induce a Hamiltonian cycle of  $H$  implying again the existence of a path with an end-vertex adjacent to  $x_2$ . Indeed, suppose  $x_2$  is not adjacent to any of the endvertices of  $P'$ . Then, since  $G$  is claw-free,  $y_1$  has to be adjacent to  $y_l$ , because otherwise  $K(x_1; y_1, y_l, x_2)$  would induce a claw in  $G$ . Hence  $P'$  induces a Hamiltonian cycle in  $H$ .

Using  $P'$ , we can easily construct a Hamiltonian path in  $N[x_1]$  starting in  $x_1$  and ending in  $x_2$ . The rest of the Hamiltonian path of  $G$  can be constructed as before.  $\square$

In fact, we can prove a slightly stronger result. Let  $A = N(x_1) \setminus \bigcup_{j=2}^k N[x_j]$ ,  $B = N(x_k) \setminus \bigcup_{j=1}^{k-1} N[x_j]$ , and, as usual,  $C_i = N(x_i) \setminus \bigcup_{j=1}^{i-1} N[x_j]$  ( $i \geq 2$ ). Each of these sets forms a clique of  $G$ .

**LEMMA 5.4.** *Let  $G$  be a claw-free graph, and let  $P = (x_1, x_2, \dots, x_k)$  ( $k > 2$ ) be an induced dominating path of  $G$  such that there is no vertex  $y$  in  $G$  with  $N(y) \cap P = \{x_1, x_k\}$ . Let also  $P$  be enlargeable but only to one end, e.g.,  $A = \emptyset, B \neq \emptyset$ , and assume that there exists an edge  $zb$  with  $z \in C_{k-1} \setminus \{x_k\}$  and  $b \in B$ . Then there is a Hamiltonian path in  $G$  that starts in  $x_1$  and ends in  $x_k$  and, given the path  $P$ , one can construct this Hamiltonian path in linear time.*

*Proof.* First, we can easily construct a path, starting in  $x_1$  and ending in  $x_{k-1}$ , that contains all vertices of  $C_2, \dots, C_{k-2}$ . Then we attach to this path a path which starts at  $x_{k-1}$ , goes through  $C_{k-1}, B$  using all their vertices, and ends in  $x_k$ . Finally, we insert the vertices of the neighborhood of  $x_1$  into the obtained path as we have done in the proof of Lemma 5.3.  $\square$

**THEOREM 5.5.** *Let  $G$  be a 2-connected claw-free graph with a good pair  $u, v$ . Then  $G$  has a Hamiltonian cycle and, given the corresponding induced dominating paths, one can construct a Hamiltonian cycle in linear time.*

*Proof.* Let  $P_1 = (u = x_1, \dots, v = x_k), P_2 = (u = y_1, \dots, v = y_l)$  be the induced dominating paths, corresponding to the good pair  $u, v$ . By the definition of a good pair both  $k$  and  $l$  are greater than 2. We may also assume that, for any induced dominating path  $P = (a_1, \dots, a_s)$  of  $G$  with  $s > 2$ , no vertex  $y \in V \setminus P$  exists such that  $N(y) \cap P = \{a_1, a_s\}$ . Otherwise,  $P$  together with  $y$  would form an induced dominating cycle of length at least 4, and we can apply Corollary 5.2 to construct a Hamiltonian cycle of  $G$  in linear time.

Let  $A_1$  be the set of vertices  $a_1$  that are adjacent to  $x_1$  but to no other vertex of  $P_1$ ; let  $B_1$  be the set of vertices  $b_1$  that are adjacent to  $x_k$  but to no other vertex of  $P_1$ .  $A_2$  and  $B_2$  are defined accordingly for  $P_2$ . Of course, each of the sets  $A_1, A_2, B_1, B_2$  forms a clique of  $G$ .

First we assume that one of these paths, say,  $P_1$ , is nonenlargeable, i.e.,  $A_1 = \emptyset, B_1 = \emptyset$ . In this case we do the following. We remove the inner vertices of  $P_2$  from  $G$  and get the graph  $G - (P_2)$ , where  $(P_2)$  denotes the inner vertices of  $P_2$ . Then, using  $P_1$ , we create a Hamiltonian path in  $G - (P_2)$  that starts at  $u$  and ends at  $v$  (Lemma 5.3), and we add  $(P_2)$  to this path to create a Hamiltonian cycle of  $G$ .

We can use this method for creating a Hamiltonian cycle of  $G$  whenever we have two internally disjoint paths  $P, P'$  of  $G$  both connecting  $u$  with  $v$  such that one of them is an induced dominating and nonenlargeable path of the graph obtained from  $G$  by removing the inner vertices of the other path.

Now we suppose that both paths  $P_1, P_2$  are enlargeable. Because of symmetry we have to consider the following three cases.

*Case 1.* There exist a vertex  $a_1 \in A_1 \setminus A_2$  and a vertex  $b_1 \in B_1 \setminus B_2$ .

In this case there must be edges from  $a_1, b_1$  to inner vertices  $y_i, y_j$  of  $P_2$ . Consequently, we can form a new path  $P'_2$  by starting in  $u$  and traversing through  $A_1, y_i, \dots, y_j, B_1, v$ , where  $(y_i, \dots, y_j)$  is the subpath of  $P_2$  between  $y_i$  and  $y_j$ . Evidently,  $P'_2$  contains all vertices of  $B_1, A_1$  and is internally disjoint from  $P_1$ , which is nonenlargeable in  $G - (P'_2)$ .

*Case 2.*  $B_1 = B_2$  and either  $A_1 = A_2$  or there exists a vertex  $a_1 \in A_1 \setminus A_2$ .

In this case none of the vertices of  $B := B_1 = B_2$  (if  $B \neq \emptyset$ ) has a neighbor in  $P_1 \cup P_2$  other than  $v$ . As  $G$  is 2-connected, for some vertex  $b \in B$  there has to be a vertex  $z \in V \setminus (P_1 \cup P_2 \cup B)$  with  $zb \in E$ . Since  $P_2$  dominates  $G$  and  $z \notin B$ , vertex  $z$  must be adjacent to a vertex  $y \in P_2 \setminus \{v\}$ . If  $z$  is only adjacent to  $y_1 = u$  but to no other vertex of  $P_2$ , then  $z$  necessarily belongs to  $A_2$  and we can form a new path  $P'_1$  by starting in  $u$ , using all vertices of  $A_2, B$  and ending in  $v$ . Again,  $P'_1$  is internally disjoint from  $P_2$  and  $P_2$  is nonenlargeable in  $G - (P'_1)$ . If  $N(z) \cap P_2 = \{u, v\}$ , then we can apply Corollary 5.2.

Therefore, we may assume that  $z$  is adjacent to an inner vertex  $y$  of  $P_2$ . Now, if there exists a vertex  $a_1 \in A_1 \setminus A_2$ , then  $a_1$  is adjacent to some vertex  $y'$  of  $(P_2)$  and we can construct a new path  $P'_2$  by using  $u, A_1, y', \dots, y, z, B, v$ . (If  $B$  was empty, then  $P'_2$  ends at  $\dots, y', \dots, y_{l-1}, v$ .) This path is internally disjoint from  $P_1$ , which

is nonenlargeable in  $G - (P'_2)$ . If  $A_1 = A_2$ , then from the discussion above we may assume that either  $A := A_1 = A_2$  is empty or there is a vertex  $z' \in V \setminus (P_1 \cup P_2 \cup A)$  which is adjacent to a vertex of  $A$  and has a neighbor  $y'$  in  $(P_2)$ . Hence, we can construct a path  $P'_2$  by using  $u, A, z', y', \dots, y, z, B, v$ , which is internally disjoint from  $P_1$ . (If  $z' = z$ , then  $P'_2$  is constructed by using  $u, A, z, B, v$ .)

*Case 3.*  $A_2$  is strictly contained in  $A_1$ , and  $B_1$  is strictly contained in  $B_2$ .

Consider vertices  $b \in B_1, z \in B_2 \setminus B_1$ , and  $z' \in A_1 \setminus A_2$  and cliques  $C_i = N(x_i) \setminus \bigcup_{j=1}^{i-1} N[x_j]$  ( $i \geq 2$ ). If  $zz' \in E$ , then we can construct a new path  $P'_2$  by using  $u, A_1, z, B_1, v$ . This path is internally disjoint from  $P_1$ , which is nonenlargeable in  $G - (P'_2)$ .

Since  $z' \notin A_2$ , there must be a neighbor  $y' \in (P_2)$  of  $z'$ . If vertex  $b$  is adjacent to some vertex in  $C_{k-1} \setminus \{v\}$ , then we construct a new path  $P'_2$  by using  $u, A_1, y', \dots, v$ . It will be internally disjoint from  $P_1$ , which is enlargeable only to one end (at  $x_k = v$ ) in  $G - (P'_2)$ . We are now in the conditions of Lemma 5.4 and can construct a Hamiltonian path of  $G - (P'_2)$  that starts in  $u$  and ends in  $v$ . Adding  $(P'_2)$  to this path, we obtain a Hamiltonian cycle of  $G$ .

So, we may assume that  $zz' \notin E$  for any vertex  $z' \in A_1 \setminus A_2$  and that vertex  $b$  is not adjacent to any vertex of  $C_{k-1} \setminus \{v\}$ . From this we conclude also that  $z \notin C_{k-1}$ . But since  $z \notin B_1$ , there must be a neighbor  $x_j \in (P_1)$  of  $z$ . We choose vertex  $x_j \in (P_1)$  with the smallest  $j$ . Clearly,  $1 < j < k - 1$  and  $z \in C_j$ .

First we define a new induced path  $P'_1 := (P_1 \setminus \{x_{j+1}, \dots, x_{k-1}\}) \cup \{z\}$  and cliques  $A'_1 := N(u) \setminus \bigcup_{x \in P'_1 \setminus \{u\}} N[x], B'_1 := N(v) \setminus \bigcup_{x \in P'_1 \setminus \{v\}} N[x]$ . We have  $z' \in A'_1$ , since otherwise from the construction of  $P'_1, z'$  would be adjacent to  $z$ , and that is impossible.

Note that vertex  $x_{j+1}$  is dominated by the path  $P_2$ . If it is adjacent to only vertex  $v$  from  $P_2$ , then  $j + 1 = k - 1$  and a claw  $K(v; x_{k-1}, y_{l-1}, b)$  arises. Therefore,  $x_{j+1}$  must be adjacent to an inner vertex  $y$  of  $P_2$ . Now we define a new path  $P'_2$  by using  $u, A'_1, y', \dots, y, x_{j+1}, C_{j+1}, x_{j+2}, \dots, C_{k-1}, v$ . It is internally disjoint from  $P'_1$  and contains all vertices of  $A'_1$  and  $C_i$  ( $j + 1 \leq i \leq k - 1$ ). It is clear from the construction that the path  $P'_1$  dominates the graph  $G - (P'_2)$ . (Every vertex which was not dominated by the path  $P'_1$  in  $G$  belongs to some set  $C_i$  ( $j + 1 \leq i \leq k - 2$ )).

It remains to show that the path  $P'_1$  is nonenlargeable in  $G - (P'_2)$ . Assume by way of contradiction that it is enlargeable. Since  $A'_1 \subset (P'_2)$ , this is possible only if  $B'_1 \neq \emptyset$ . Let  $p$  be a vertex of  $B'_1$ . Then  $p$  does not belong to  $B_1$ , since otherwise it should be adjacent to  $z$ , which is contained in  $(P'_1)$ . (Recall that  $B_1, B_2$  are cliques,  $B_1 \subset B_2$ , and  $z \in B_2 \setminus B_1$ .) Now, from  $p \in B'_1 \setminus B_1$  we conclude that the neighbors of  $p$  in  $P_1 \setminus \{v\}$  are only vertices from  $\{x_{j+1}, \dots, x_{k-1}\}$ , i.e.,  $p$  belongs to a set  $C_s$  for some  $s \geq j + 1$ . Consequently, a contradiction to  $C_s \subset (P'_2)$  arises.

It is not hard to see that the above method can be implemented to run in linear time.  $\square$

**THEOREM 5.6.** *Every 2-connected claw-free graph  $G$  that contains a dominating pair has a Hamiltonian cycle, and, given a dominating pair, a Hamiltonian cycle can be constructed in linear time.*

*Proof.* Let  $v, u$  be a dominating pair of a 2-connected graph  $G$ . If  $vu \notin E$ , then by Lemma 4.9, there exist two internally disjoint induced paths connecting  $v$  and  $u$ . Both these paths dominate  $G$ , and, therefore,  $u, v$  is a good pair of  $G$ . Thus, the statement holds by Theorem 5.5.

Now let  $vu \in E$ . Define sets  $A := N(u) \setminus N[v], B := N(v) \setminus N[u]$ , and  $S := N(v) \cap N(u)$ . Since  $G$  is claw-free, the sets  $A$  and  $B$  are cliques of  $G$ . Notice also that

sets  $A, B, S$ , and  $\{v, u\}$  form a partition of the vertex set of  $G$ .

If there is an edge  $ab$  in  $G$  such that  $a \in A$  and  $b \in B$ , then vertices  $a, u, v, b$  induce a 4-cycle which dominates  $G$ . Hence, we can apply Corollary 5.2 to get a Hamiltonian cycle of  $G$ . Therefore, assume that no such edge exists. But since  $G$  is 2-connected, there must be edges  $ax, by$  with  $x, y \in S, a \in A$ , and  $b \in B$ . We distinguish between two cases. Let  $G_S$  denote the subgraph of  $G$  induced by  $S$ .

*Case 1.*  $G_S$  is disconnected.

Then, it consists of two cliques  $S_1$  and  $S_2$ . Now, if vertices  $x, y$  are in different components of  $G_S$ , say,  $x \in S_1$  and  $y \in S_2$ , then  $(u, P^{A \setminus \{a\}}, a, x, P^{S_1 \setminus \{x\}}, v, P^{B \setminus \{b\}}, b, y, P^{S_2 \setminus \{y\}}, u)$  is a Hamiltonian cycle of  $G$ . ( $P^M$  stands for an arbitrary permutation of the vertices of a set  $M$ .) If  $x, y$  are in one component, say,  $S_1$ , then  $(u, P^{A \setminus \{a\}}, a, x, P^{S_1 \setminus \{x, y\}}, y, b, P^{B \setminus \{b\}}, v, P^{S_2}, u)$  is a Hamiltonian cycle of  $G$ .

*Case 2.*  $G_S$  is connected.

Then, by Theorem 3.1, there exists a Hamiltonian path  $P = (s, y_1, \dots, y_l, t)$  of  $G_S$ . Assume that  $as, at \notin E$ . Then, to avoid a claw  $K(u; a, s, t)$ , vertices  $s$  and  $t$  have to be adjacent, giving a Hamiltonian cycle  $HC := (s, y_1, \dots, y_l, t, s)$  of  $G_S$ . Vertices  $x$  and  $y$  split this cycle into two paths  $P_1 = (x, \dots, y)$  and  $P_2 = HC \setminus P_1$ . Hence, a cycle  $(u, P^{A \setminus \{a\}}, a, P_1, b, P^{B \setminus \{b\}}, v, P_2, u)$  is a Hamiltonian cycle of  $G$ .

Now, we may assume that  $a$  is adjacent to  $s$  or  $t$ . Analogously,  $b$  is adjacent to one of  $t, s$ . If  $a, b$  are adjacent to different vertices, say,  $as, bt \in E$ , then  $(u, P^{A \setminus \{a\}}, a, P, b, P^{B \setminus \{b\}}, v, u)$  is a Hamiltonian cycle of  $G$ . Finally, if  $a, b$  are adjacent only to  $s$  (similarly, to  $t$ ), then  $(u, P \setminus \{s\}, v, P^{B \setminus \{b\}}, b, s, a, P^{A \setminus \{a\}}, u)$  is a Hamiltonian cycle of  $G$ .  $\square$

**THEOREM 5.7.** *Every 2-connected CN-free graph  $G$  has a Hamiltonian cycle, and such a cycle can be found in  $O(n + m)$  time.*

*Proof.* The proof follows from Theorems 4.10, 5.1, and 5.5.  $\square$

**COROLLARY 5.8.** *There is a linear time algorithm that for a given (arbitrary) 2-connected graph  $G$  either finds a Hamiltonian cycle or outputs an induced claw or an induced net of  $G$ .*

**COROLLARY 5.9.** *A Hamiltonian cycle of a 2-connected (claw, AT)-free graph can be found in  $O(n + m)$  time.*

*Remark.* Corollary 5.8 implies that every 2-connected unit interval graph has a Hamiltonian cycle, which is, of course, well known (see [24, 20]). The interesting difference of the above algorithm compared to the existing algorithms for this problem on unit interval graphs is that it does not require the creation of an interval model. It also follows from Corollaries 3.3 and 5.8 that both the Hamiltonian path problem and the Hamiltonian cycle problem are linear time solvable on proper circular arc graphs. Note that previously known algorithms for these problems had time bounds  $O(m + n \log n)$  [18].

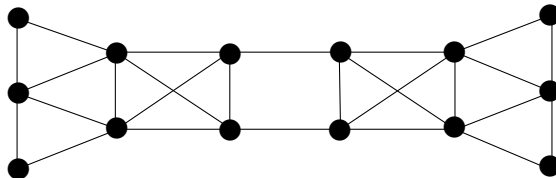


FIG. 5.1. Claw-free graph, containing a dominating pair and a net.

It should also be mentioned that Theorems 3.1 and 5.5 do cover a class of graphs that is not contained in the class of CN-free graphs. Figure 5.1 shows a graph that is claw-free, does contain a dominating/good pair and, consequently, a dominating path, but, obviously, it is neither AT-free nor net-free.

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