

A Linear Time Delay Model for Studying Load Balancing Instabilities in Parallel Computations

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Abstract

A linear time-delay system is used to model load balancing in a cluster of computer nodes used for parallel computations. The linear model is analyzed for stability in terms of the delays in the transfer of information between nodes and the gains in the load balancing algorithm. This model is compared with an experimental implementation of the algorithm on a parallel computer network.

1 Introduction

In this work, a linear time-delay system is used to model load balancing in a cluster of computer nodes used for parallel computations. The linear model is analyzed for stability in terms of the delays in the transfer of information between nodes and the gains in the load balancing algorithm. This model is compared with an experimental implementation of the algorithm on a parallel computer network. Preliminary results by the authors appear in [1], however, a change has been made to the linear model in [1] which represents better fidelity and also experimental results are reported here.

Parallel computer architectures utilize a set of computational elements (CE) to achieve performance that is not attainable on a single processor, or CE, computer. A common architecture is the cluster of otherwise independent computers communicating through a shared network. To make use of parallel computing resources, problems must be broken down into smaller units that can be solved individually by each CE while exchanging information with CEs solving other problems.

The Federal Bureau of Investigation (FBI) National DNA Indexing System (NDIS) and Combined DNA Indexing System (CODIS) software are candidates for parallelization. New methods developed by Wang et al [3][4][5][17][18] lead naturally to a parallel decomposition of the DNA database search problem while providing orders of magnitude improvements in performance over the current release of the CODIS software. The projected growth of the NDIS database and in the demand for searches of the database necessitates migration to a parallel computing platform.

Effective utilization of a parallel computer architecture requires the computational load to be distributed more or less evenly over the available CEs. The qualifier “more or less” is used because the communications required to distribute the load consume both computational resources and network bandwidth. A point of diminishing returns exists.

Distribution of computational load across available resources is referred to as the *load balancing* problem in the literature. Various taxonomies of load balancing algorithms exist. Direct methods examine the global distribution of computational load and assign portions of the workload to resources before processing begins. Iterative methods examine the progress of the computation and the expected utilization of resources, and adjust the workload assignments periodically as computation progresses. Assignment may be either deterministic, as with the dimension exchange/diffusion [8] and gradient methods, stochastic, or optimization based. A comparison of several deterministic methods is provided by Willeback-LeMain and Reeves [19].

To adequately model load balancing problems, several features of the parallel computation environment should be captured (1) The workload awaiting processing at each CE; (2) the relative performances of the CEs; (3) the computational requirements of each workload component; (4) the delays and bandwidth constraints of CEs and network components involved in the exchange of workloads, and (5) the delays imposed by CEs and the network on the exchange of measurements. A queuing theory [15] approach is well-suited to the modeling requirements and has been used in the literature by Spies [16] and others. However, whereas Spies assumes a homogeneous network of CEs and models the queues in detail, the present work generalizes queue length to an expected waiting time, normalizing to account for differences among CEs, and aggregates the behavior of each queue using a continuous state model. The present work focuses upon the effects of delays in the exchange of information among CEs, and the constraints these effects impose on the design of a load balancing strategy.

Section 2 presents our approach to modeling the computer network and load balancing algorithms to incorporate the presence of delay in communicating

between nodes and transferring tasks. Section 3 contains an analysis of the stability properties of the linear models, while Section 4 presents simulations of the model. Section 5 presents experimental data from an actual implementation of a load balancing algorithm and finally, Section 6 is a summary and conclusion of the present work and a discussion of future work.

2 A Dynamic Model of Load Balancing

In this section, linear dynamic time-delay model is developed to model load balancing among a network of computers. To introduce the basic approach to load balancing, consider a computing network consisting of n computers (nodes) all of which can communicate with each other. At start up, the computers are assigned an equal number of tasks. However, when a node executes a particular task it can in turn generate more tasks so that very quickly the loads on various nodes become unequal. To balance the loads, each computer in the network sends its queue size $q_j(t)$ to all other computers in the network. A node i receives this information from node j *delayed* by a finite amount of time τ_{ij} , that is, it receives $q_j(t - \tau_{ij})$. Each node i then uses this information to compute its local estimate¹ of the average number of tasks in the queues of the n computers in the network. In this work, the simple estimator $\left(\sum_{j=1}^n q_j(t - \tau_{ij})\right)/n$ ($\tau_{ii} = 0$) which is based on the most recent observations is used. Node i then compares its queue size $q_i(t)$ with its estimate of the network average as $\left(q_i(t) - \left(\sum_{j=1}^n q_j(t - \tau_{ij})\right)/n\right)$ and, if this is greater than zero, the node sends some of its tasks to the other nodes while if it is less than zero, no tasks are sent (see Figure 1). Further, the tasks sent by node i are received by node j with a delay h_{ij} . The controller (load balancing algorithm) decides how often and fast to do load balancing (transfer tasks among the nodes) and how many tasks are to be sent to each node.

¹It is an estimate because at any time, each node only has the delayed value of the number of tasks in the other nodes.

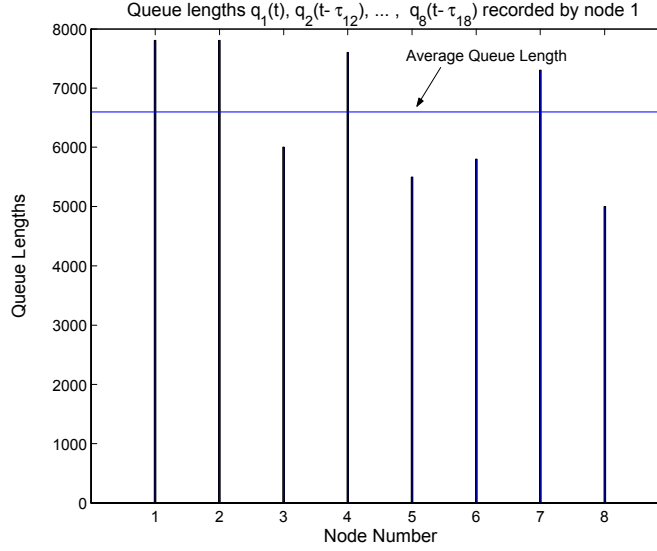


Figure 1: Graphical description of load balancing. This bar graph shows the load for each computer vs. node of the network. The thin horizontal line is the average load as estimated by node 1. Node 1 will transfer (part of) its load only if it is above its estimate of the average. Also, it will only transfer to nodes that it estimates are below the node average.

As just explained, each node controller (load balancing algorithm) has only *delayed* values of the queue lengths of the other nodes, and each transfer of data from one node to another is received only after a finite time delay. An important issue considered here is to study the effect of these delays on system performance. Specifically, the continuous time models developed here represent our effort to capture the effect of the delays in load balancing techniques and were developed so that system theoretic methods could be used to analyze them.

2.1 Dynamic Model

The basic mathematical model of a given computing node for load balancing is given by

$$\begin{aligned}
\frac{dx_i(t)}{dt} &= \lambda_i - \mu_i + u_i(t) - \sum_{j=1}^n p_{ij} \frac{t_{p_i}}{t_{p_j}} u_j(t - h_{ij}) \\
y_i(t) &= x_i(t) - \frac{\sum_{j=1}^n x_j(t - \tau_{ij})}{n} \\
u_i(t) &= -K_i y_i(t) \\
p_{ij} &\geq 0, p_{jj} = 0, \sum_{i=1}^n p_{ij} = 1
\end{aligned} \tag{1}$$

where in this model we have

- n is the number of nodes.
- $x_i(t)$ is the *expected waiting time* experienced by a task inserted into the queue of the i^{th} node. With $q_i(t)$ the number of *tasks* in the i^{th} node and t_{p_i} the average time needed to process a task on the i^{th} node, the expected (average) waiting time is then given by $x_i(t) = q_i(t)t_{p_i}$. Note that $x_j/t_{p_j} = q_j$ is the number of tasks in the node 1 queue. If these tasks were transferred to node i , then the waiting time transferred is $q_j t_{p_i} = x_j t_{p_i}/t_{p_j}$, so that the fraction t_{p_i}/t_{p_j} converts waiting time on node j to waiting time on node i .
- λ_i is the rate of generation of waiting time on the i^{th} node caused by the addition of tasks (rate of increase in x_i)
- μ_i is the rate of reduction in waiting time caused by the service of tasks at the i^{th} node and is given by $\mu_i \equiv (1 \times t_{p_i})/t_{p_i} = 1$ for all i .
- $u_i(t)$ is the rate of removal (transfer) of the tasks from node i at time t by the load balancing algorithm at node i .
- $p_{ij}u_j(t)$ is the rate that node j sends waiting time (tasks) to node i at time t where $p_{ij} \geq 0$, $\sum_{i=1}^n p_{ij} = 1$ and $p_{jj} = 0$. That is, the transfer from node j of expected waiting time (tasks) $\int_{t_1}^{t_2} u_j(t)dt$ in the interval of time $[t_1, t_2]$ to the other nodes is carried out with the i^{th} node being sent the fraction $p_{ij} \frac{t_{p_i}}{t_{p_j}} \int_{t_1}^{t_2} u_j(t)dt$ where the fraction t_{p_i}/t_{p_j} converts the task from waiting time on node j to waiting time on node i . As $\sum_{i=1}^n \left(p_{ij} \int_{t_1}^{t_2} u_j(t)dt \right) = \int_{t_1}^{t_2} u_j(t)dt$, this results in a removing *all* the waiting time $\int_{t_1}^{t_2} u_j(t)dt$ from node j .

- The quantity $-p_{ij}u_j(t-h_{ij})$ is the rate of increase (rate of transfer) of the expected waiting time (tasks) at time t from node j by (to) node i where h_{ij} ($h_{ii} = 0$) is the time delay for the task transfer from node j to node i .
- The quantities τ_{ij} ($\tau_{ii} = 0$) denote the time delay for communicating the expected waiting time x_j from node j to node i .
- The quantity $x_i^{avg} = \left(\sum_{j=1}^n x_j(t-\tau_{ij})\right)/n$ is the estimate² by the i^{th} node of the average waiting time of the network and is referred to as the *local average* (local estimate of the average).

In this model, all rates are in units of the *rate of change of expected waiting time*, or *time/time* which is dimensionless). The j^{th} node receives the fraction $\int_{t_1}^{t_2} p_{ji}u_i(t)dt$ of transferred waiting time $\int_{t_1}^{t_2} u_i(t)dt$ delayed by the time h_{ij} . As all nodes are taken to be identical, the p_{ji} for each sending node i is specified as constant and equal to

$$\begin{aligned} p_{ji} &= p = 1/(n-1) \text{ for } j \neq i \\ p_{ii} &= 0 \end{aligned}$$

where it is clear that $p_{ij} \geq 0$, $\sum_{i=1}^n p_{ij} = 1$.

A delay is experienced by transmitted tasks before they are received at the other node. The control law $u_i(t) = -K_i y_i(t)$ states that if at the i^{th} node, $u_i(t) = -K_i y_i(t) < 0$ ($y_i(t) > 0$), then it sends data to the other nodes. However, if $y_i(t) < 0$, the controller (load balancing algorithm) $u_i(t) = -K_i y_i(t) > 0$ so that the node is *instantaneously* taking on waiting time (tasks) from the other nodes before those tasks are removed from the other nodes' queues. As explained above, the actual network does not do this; if $y_i(t) < 0$, then the local node does not initiate any transfer of tasks. That is, the actual network has a control law of the form $u_i(t) = -K_i \text{sat}(y_i(t))$ where $\text{sat}(y) = y$ if $y \geq 0$ and $\text{sat}(y) = 0$ if $y < 0$. However, in spite of this fact, the system model (1) is used because it can be completely analyzed with regards to stability and it does capture the oscillatory behavior of the $y_i(t)$.

3 Stability Analysis of the Linear Model

A key issue here is whether or not the system model (1) is stable. It is well known that the presence of delays has a great influence on the stability of the system [2][11][12]. In addition to stability, performance is also an issue, that is, the system may be stable, but oscillate. This is undesirable as the network is wasting resources passing tasks back and forth between nodes rather than executing the tasks. In this section, the linear model (1) is analyzed for stability as a function of the control gains K_i .

²This is an only an estimate due to the delays.

To simplify the presentation of the stability analysis of (1), a three node model is considered with $K_1 = K_2 = K_3 = K$, $p = 1/2$, $\tau_{ij} = \tau$, $h_{ij} = 2\tau$ for $i \neq j$ for all $i, j = 1, 2, 3$ ($\tau_{ii} = h_{ii} = 0$). Letting $d_1 = \lambda_1 - \mu_1$, $d_2 = \lambda_2 - \mu_2$, and $d_3 = \lambda_3 - \mu_3$, the Laplace transform of the system equations (1) with zero initial conditions are then $(X_1(s) = \mathcal{L}\{x_1(s)\})$, etc.)

$$s \begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix} = \begin{bmatrix} 1 & -pe^{-2\tau s} & -pe^{-2\tau s} \\ -pe^{-2\tau s} & 1 & -pe^{-2\tau s} \\ -pe^{-2\tau s} & -pe^{-2\tau s} & 1 \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \\ U_3(s) \end{bmatrix} + \begin{bmatrix} D_1(s) \\ D_2(s) \\ D_3(s) \end{bmatrix}$$

$$\begin{bmatrix} U_1(s) \\ U_2(s) \\ U_3(s) \end{bmatrix} = -K \begin{bmatrix} (n-1)/n & -e^{-\tau s}/n & -e^{-\tau s}/n \\ -e^{-\tau s}/n & (n-1)/n & -e^{-\tau s}/n \\ -e^{-\tau s}/n & -e^{-\tau s}/n & (n-1)/n \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix}.$$

This is solved for $X(s)$ and the output is then given by

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \\ Y_3(s) \end{bmatrix} = \begin{bmatrix} (n-1)/n & -e^{-\tau s}/n & -e^{-\tau s}/n \\ -e^{-\tau s}/n & (n-1)/n & -e^{-\tau s}/n \\ -e^{-\tau s}/n & -e^{-\tau s}/n & (n-1)/n \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix}.$$

Performing the computations, the transfer function from the inputs $D_1(s), D_2(s), D_3(s)$ to the output $y_1(s) \triangleq x_1(s) - (x_1(s) + e^{-\tau s}x_2(s) + e^{-\tau s}x_3(s))/3$ is ($z \triangleq e^{-\tau s}$)

$$Y_1(s) = \frac{-6s - K(z^2 - 2)(z - 1)(z + 2)}{(3s + K(2 + z)(1 + 0.5z^2))(-3s + 2K(1 - z)(-1 + z^2))} D_1(s)$$

$$+ \frac{3sz + Kz^2(z - 1)(z + 2)}{(3s + K(2 + z)(1 + 0.5z^2))(-3s + 2K(1 - z)(z^2 - 1))} (D_2(s) + D_3(s))$$

A more compact representation is given by

$$Y_1(s) = \frac{b_1(s, z)}{a_1(s, z)a_2(s, z)} D_1(s) + \frac{zb_2(s, z)}{a_1(s, z)a_2(s, z)} (D_2(s) + D_3(s)) \quad (2)$$

where

$$b_1(s, z) = -6s - K(z^2 - 2)(z - 1)(z + 2)$$

$$a_1(s, z) = 3s + K(2 + z)(1 + 0.5z^2)$$

$$a_2(s, z) = -3s + 2K(1 - z)(-1 + z^2)$$

$$b_2(s, z) = 3s + Kz(z - 1)(z + 2)$$

The range of delay values τ for which (2) is stable is found by separately considering the stability of the transfer functions $1/a_1(s, z)$, $b_1(s, z)/a_2(s, z)$ and $b_2(s, z)/a_2(s, z)$.

3.1 Stability of $1/a_1(s, z)$

The idea here (see [6][7][13][14]) is to first note that the polynomial $a_1(s, e^{-s\tau})$ is stable for $\tau = 0$ as $a(s, e^{-s0}) = a(s, 1) = 3s + 4.5K \neq 0$ for $\text{Re}(s) \geq 0$. Next,

one considers the pair of polynomials

$$\begin{aligned} a_1(s, z) &\triangleq 3s + K(2 + z)(1 + z^2/2) \\ \tilde{a}_1(s, z) &\triangleq z^3 a_1(-s, 1/z) = -3z^3 s + K(2z^3 + z^2 + z + 1/2). \end{aligned}$$

If the system becomes unstable for some $\tau > 0$, then there is a first value of $\tau > 0$ for which $a_1(s, e^{-j\omega\tau})$ has zeros in s on the $j\omega$ axis. For this value of τ , there is a zero (s_0, z_0) of $a_1(s_0, z_0) = 0$ of the form $(s_0, z_0) = (j\omega_0, e^{-j\omega_0\tau})$, that is, $\text{Re}(s_0) = 0, |z_0| = 1$. As $-j\omega_0$ and $1/e^{-j\omega_0\tau}$ are simply the conjugates of $s_0 = j\omega_0, z_0 = e^{-j\omega_0\tau}$, (s_0, z_0) is also a zero of $\tilde{a}_1(s_0, z_0) = 0$ as seen by computing the complex conjugate of $a_1(s_0, z_0)$. As a consequence, the first value of τ that results in the system being unstable corresponds to there being a *common zero* (s, z) of

$$a_1(s, z) = 0, \tilde{a}_1(s, z) = 0 \quad (3)$$

with $\text{Re}(s_0) = 0, |z_0| = 1$. To find the common zeros of (3), the variable s is eliminated from $a_1(s, z) = 0$ and $\tilde{a}_1(s, z) = 0$ resulting in

$$R(z) \triangleq 1 + 2z + 2z^2 + 8z^3 + 2z^4 + 2z^5 + z^6 = 0.$$

The roots of $R(z)$ are

$$\left\{ -2.43475, -0.410721, 0.107602 \pm j0.574348, 0.315131 \pm j1.68207 \right\}. \quad (4)$$

For $a_1(s, e^{-j\omega\tau})$ to have a zero on the $j\omega$ axis for some $\tau > 0$, there must be a pair of the form $(s_0, z_0) = (j\omega_0, e^{-j\omega_0\tau})$ so that $\text{Re}(s_0) = 0, |z_0| = 1$. None of the roots (4) for z has magnitude 1 so there is no value of τ which results in $a_1(s, e^{-j\omega\tau})$ being zero on the $j\omega$ axis. As the system is stable for $\tau = 0$, i.e., $a_1(0, 1) \neq 0$, it follows that as τ is increased, no zeros of $a_1(s, e^{-j\omega\tau})$ can cross into the right-half plane and therefore $a_1(s, e^{-j\omega\tau})$ is stable for all delays $\tau \geq 0$.

3.2 Stability of $b_1(s, z)/a_2(s, z)$

The objective here is determine for what values of the delay $\tau \geq 0$ the transfer function

$$\frac{b_1(s, z)}{a_2(s, z)} = \frac{-6s - 2K(-1 + 0.5z^2)(z^2 + z - 2)}{-3s + 2K(1 - z)(z^2 - 1)}$$

is stable. Note that the polynomial $a_2(s, z)$ is unstable for $\tau = 0$ as

$$a_2(s, e^{-\tau s})|_{\tau=0} = -3s + 2K(1 - z)(z^2 - 1)|_{z=1} = -3s$$

However,

$$\lim_{\tau \rightarrow 0} \frac{b_1(s, e^{-\tau s})}{a_2(s, e^{-\tau s})} = \lim_{z \rightarrow 1} \frac{-6s - 2K(-1 + 0.5z^2)(z - 1)(z + 2)}{-3s + 2K(1 - z)(z^2 - 1)} = 2 \neq \infty.$$

In fact,

$$\begin{aligned}
\lim_{s \rightarrow 0} \frac{b_1(s, e^{-s\tau})}{a_2(s, e^{-s\tau})} &= \lim_{s \rightarrow 0} \frac{-6s + K(-4 + 2e^{-s\tau} + 4e^{-2s\tau} - e^{-3s\tau} - e^{-4s\tau})}{-3s + 2K(-1 + e^{-s\tau} + e^{-2s\tau} - e^{-3s\tau})} \\
&= \lim_{s \rightarrow 0} \frac{-6 + K(-2\tau e^{-s\tau} - 8\tau e^{-2s\tau} + 3\tau e^{-3s\tau} + 4\tau e^{-4s\tau})}{-3 + 2K(-\tau e^{-s\tau} - 2\tau e^{-2s\tau} + 3\tau e^{-3s\tau})} \\
&= -2 - K\tau \neq \infty \text{ for } K \geq 0, \tau \geq 0
\end{aligned} \tag{5}$$

That is, there is a pole-zero cancellation so that transfer function $b_1(s, e^{-s\tau})/a_2(s, e^{-s\tau})$ does not have a pole at $s = 0$ for all $\tau \geq 0$. To determine the delay values for which the transfer function is stable, the rational function

$$c_1(s, z) = \frac{a_2(s, z)}{b_1(s, z)} = \frac{-3s + 2K(1 - z)(z^2 - 1)}{-6s - 2K(-1 + 0.5z^2)(z - 1)(z + 2)}$$

is checked for its *zeros* (i.e., $c_1(s, e^{-js\tau}) = 0$) in the right-half plane as a function of the delay τ . To do so, an auxiliary rational function $\tilde{c}_2(s, z)$ is defined as follows

$$\tilde{c}_1(s, z) = c_1(-s, 1/z) = \frac{-3s + 2K(1 - z)(z^2 - 1)}{-6s - 2K(-1 + 0.5z^2)(z - 1)(z + 2)}.$$

If the system becomes unstable for some $\tau > 0$, then there is a first value of τ for which $c_1(s, e^{-js\tau})$ has zeros on the $j\omega$ axis. On the $j\omega$ axis, these zeros must be of the form $s_0 = j\omega_0, z_0 = e^{-j\omega_0\tau}$ (i.e., $\text{Re}(s_0) = 0, |z_0| = 1$) and, the complex conjugate of $c_1(s_0, z_0)$ is $\tilde{c}_1(s_0, z_0)$ so that $\tilde{c}_1(s_0, z_0) = 0$ too. The first value of τ that results in $c_1(s, e^{-js\tau})$ having a zero on the $j\omega$ axis must correspond to a common zero of

$$c_1(s_0, z_0) = 0, \tilde{c}_1(s_0, z_0) = 0 \tag{6}$$

which satisfies $\text{Re}(s_0) = 0, |z_0| = 1$. Eliminating s from (6) gives

$$R(z) = \frac{-2z(z + 1)^2(z - 1)^2(z - e^{j\pi/3})(z - e^{-j\pi/3})}{-1 - z + 4z^2 + 2z^3 - 8z^4 + 4z^5 + 4z^6 - 4z^7} = 0.$$

The roots are then $\{0, -1, -1, 1, 1, e^{j\pi/3}, e^{-j\pi/3}\}$ and solving $c_2(s, z) = 0$ for these particular values of z gives

$$\begin{aligned}
\{(s_i, z_i)\} &= \left\{ (2K/3, 0), (0, -1), (0, -1), (0, 1), (0, 1), \right. \\
&\quad \left. (j\frac{4K}{3}\sin(\pi/3), e^{j\pi/3}), (-j\frac{4K}{3}\sin(\pi/3), e^{-j\pi/3}) \right\}
\end{aligned}$$

In order to correspond to a zero of $c_1(s, e^{-js\tau})$, a common zero (s_0, z_0) must also satisfy

$$z_0 = e^{-s_0\tau}. \tag{7}$$

The pair $(-2K/3, 0)$ can never satisfy (7) and therefore $s = -2K/3$ does not correspond to a zero of $c_1(s, e^{-js\tau})$. Similarly, $(s, z) = (0, -1)$ can never satisfy

(7) either. The two common zeros at $(0, 1)$ correspond to having a zero of $c_1(s, e^{-js\tau})$ at $s = 0$ on the $j\omega$ axis, but it has already been shown in (5) that the transfer function $b_1(s, e^{-\tau s})/a_2(s, e^{-\tau s})$ does not have a pole at $s = 0$. Finally, putting the common zeros $(\pm j\frac{4K}{3} \sin(\pi/3), e^{\pm j\pi/3})$ into (7) requires solving

$$e^{j\pi/3} = e^{-j\tau\frac{4K}{3} \sin(\pi/3)} \text{ or } e^{-j5\pi/3} = e^{-j\tau\frac{4K}{3} \sin(\pi/3)}$$

giving

$$\tau = \frac{5\pi}{4K \sin(\pi/3)}.$$

That is, for $K < K_{\max} \triangleq 5\pi/(4\tau \sin(\pi/3))$, the system is stable and for $K = K_{\max}$ the system will have poles on the $j\omega$ axis at $\pm j\frac{4K}{3} \sin(\pi/3)$ and is therefore unstable.

3.3 Stability of $b_2(s, z)/a_2(s, z)$

Finally, the stability of the transfer function

$$\frac{b_2(s, z)}{a_2(s, z)} = \frac{3s + Kz(z-1)(z+2)}{-3s + 2K(1-z)(-1+z^2)}$$

as a function of the delay τ is determined. As the polynomial $a_2(s, z)$ is unstable for $\tau = 0$, one checks the transfer function itself at $\tau = 0$, that is,

$$\lim_{\tau \rightarrow 0} \frac{b_2(s, e^{-\tau s})}{a_2(s, e^{-\tau s})} = \lim_{z \rightarrow 1} \frac{3s + K(-2z + z^2 + z^3)}{-3s + 2K(1-z)(z^2-1)} = -1 \neq \infty.$$

Also,

$$\begin{aligned} \lim_{s \rightarrow 0} \frac{b_2(s, e^{-s\tau})}{a_2(s, e^{-s\tau})} &= \lim_{s \rightarrow 0} \frac{3s + K(-2e^{-s\tau} + e^{-2s\tau} + e^{-3s\tau})}{-3s + 2K(-1 + e^{-s\tau} + e^{-2s\tau} - e^{-3s\tau})} \\ &= \lim_{s \rightarrow 0} \frac{3 + K(2\tau e^{-s\tau} - 2\tau e^{-2s\tau} - 3\tau e^{-3s\tau})}{-3 + 2K(-\tau e^{-s\tau} - 2\tau e^{-2s\tau} + 3\tau e^{-3s\tau})} \\ &= -1 + K\tau \neq \infty \text{ for } K \geq 0, \tau \geq 0 \end{aligned} \quad (8)$$

so that this transfer function does *not* have a pole at $s = 0$ for all $\tau \geq 0$.

To determine the delay values for which the transfer function is stable, the rational function

$$c_2(s, z) = \frac{a_2(s, z)}{b_2(s, z)} = \frac{-3s + 2K(1-z)(-1+z^2)}{3s + Kz(z-1)(z+2)}$$

is checked for its *zeros* (i.e., $c_1(s, e^{-js\tau}) = 0$) in the right-half plane as a function of the delay τ . As explained in the previous subsection, an auxiliary rational function $\tilde{c}_2(s, z)$ is defined as

$$\tilde{c}_2(s, z) = c_2(-s, 1/z) = \frac{z(3sz^3 - 2K(-1+z)^2(1+z))}{-3sz^3 - 2K(-1+z)(0.5+z)}$$

and the rational functions

$$c_1(s, z) = 0, \tilde{c}_1(s, z) = 0$$

are solved for their common zeros. Eliminating s gives

$$R(z) = \frac{-z(1+z)^2(z-1)(1-z+z^2)}{(-1.33224+z)(0.433748+z)(1.18764+z)(0.728556-0.289146z+z^2)} = 0.$$

The roots are then $\{0, -1, -1, 1, e^{j\pi/3}, e^{-j\pi/3}\}$. Solving $c_2(s, z) = 0$ for these particular values of z gives

$$\{(s_i, z_i)\} = \left\{ (2K/3, 0), (0, -1), (0, -1), (0, 1), \right. \\ \left. (j\frac{4K}{3}\sin(\pi/3), e^{j\pi/3}), (-j\frac{4K}{3}\sin(\pi/3), e^{-j\pi/3}) \right\}$$

Again, each of these common zeros must also satisfy $z = e^{-s\tau}$ which leaves the common zeros $(\pm j\frac{4K}{3}\sin(\pi/3), e^{\pm j\pi/3})$ as the only viable candidates. As in the previous subsection, this results in

$$\tau = \frac{5\pi}{4K\sin(\pi/3)}.$$

That is, for $K < K_{\max} \triangleq 5\pi/(4\tau\sin(\pi/3))$, the system is stable and for $K = K_{\max}$ the system will have poles on the $j\omega$ axis at $\pm j\frac{4K}{3}\sin(\pi/3)$ and is therefore unstable.

In summary, the system (2) is stable for

$$0 \leq K < K_{\max} = \frac{5\pi}{4\tau\sin(\pi/3)}$$

4 Simulations

Experimental procedures to determine the delay values are given in [9] and summarized in [10]. These give representative values for a Fast Ethernet network with three nodes of $\tau_{ij} = \tau = 200 \mu\text{sec}$ for $i \neq j$, $\tau_{ii} = 0$, and $h_{ij} = 2\tau = 400 \mu\text{sec}$ for $i \neq j$, $h_{ii} = 0$. The initial conditions were $x_1(0) = 0.6$, $x_2(0) = 0.4$ and $x_3(0) = 0.2$. The inputs were set as $\lambda_1 = 3\mu_1$, $\lambda_2 = 0$, $\lambda_3 = 0$, $\mu_1 = \mu_2 = \mu_3 = 1$. The t_{p_i} 's were taken to be equal. Figure (2) is a block diagram of one node of the system.

The simulation of the linear model was performed with three nodes ($n = 3$), $K_1 = K_2 = K_3 = K$, $p_{ij} = 1/2$, for all i, j , $\tau_{ij} = \tau$, $h_{ij} = 2\tau$ for $i \neq j$, $\tau_{ii} = 0$, $h_{ii} = 0$ for $i = 1, 2, 3$ and $\tau = 200 \mu\text{sec}$. The maximum value for the gain using these parameter values is

$$K_{\max} = \frac{5\pi}{4\tau\sin(\pi/3)} = \frac{5\pi}{4(200 \times 10^{-6})\sin(\pi/3)} = 22672$$

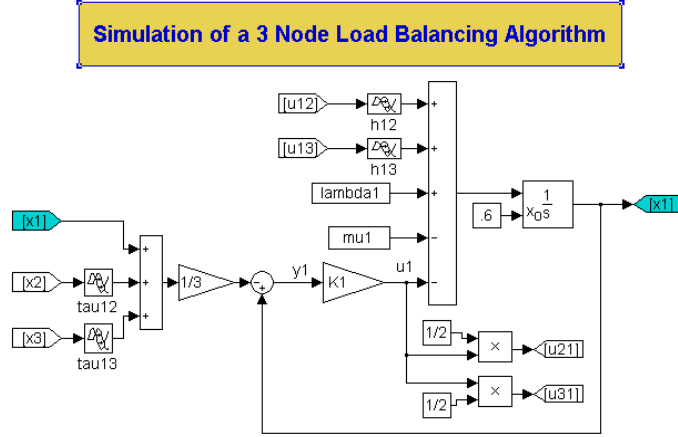


Figure 2: Simulation Block Diagram for Node 1

Figures 3 and 4 show the responses $y_1(t)$, $y_2(t)$, $y_3(t)$ with the gain $K = 1000$ and $K = 5000$, respectively. Note the increase in oscillatory behavior as the gain is increased. To compare with the experimental results given in Figure 8, Figure 5 shows the output responses with the gains set as $K_1 = 6667$, $K_2 = 4167$, $K_3 = 5000$, respectively. In each of the plots, the effect of delay of $200\mu\text{sec}$ coming into play at $t = 200\mu\text{sec}$ is evident. Note that the responses in 4 with the higher gain die out slower and oscillate more compared to the responses in 3. This is due to the delays in that, if the delays are set to zero, then the response with $K = 5000$ dies out fastest as expected.

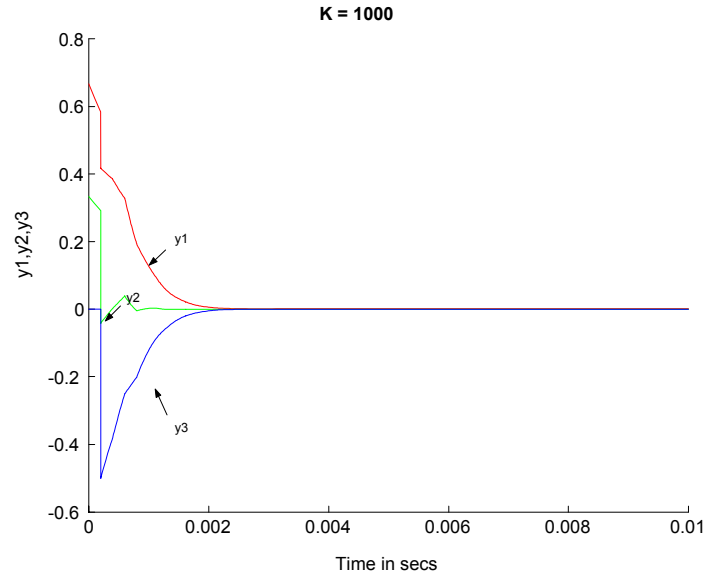


Figure 3: Linear output responses with $K = 1000$.

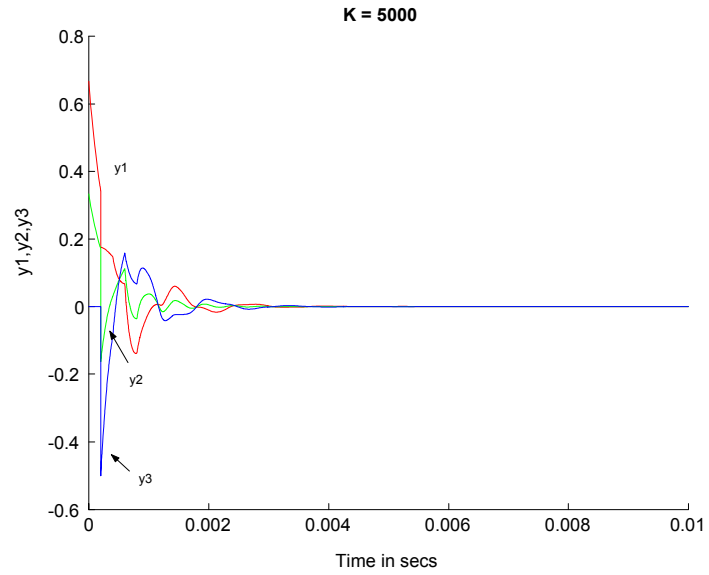


Figure 4: Linear output responses with $K = 5000$.

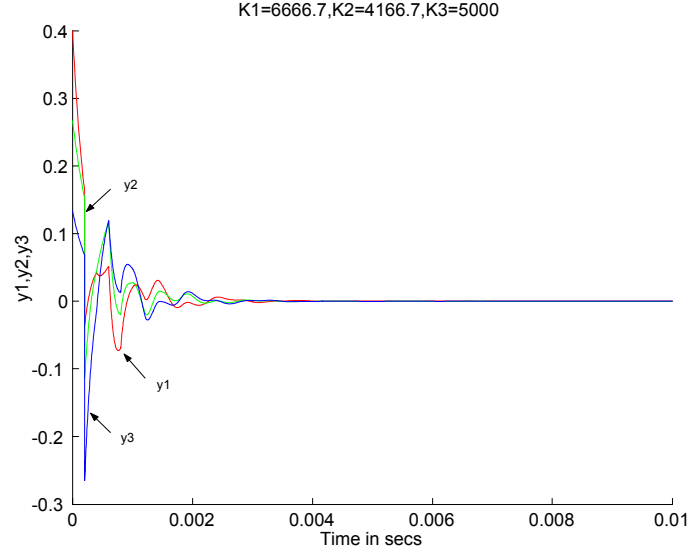


Figure 5: Linear simulation with $K1 = 6666.7$; $K2 = 4166.7$; $K3 = 5000$

5 Experimental Results

A parallel machine has been built to implement an experimental facility for evaluation of load balancing strategies. To date, this work has been performed for the FBI Laboratory to evaluate candidate designs of the parallel CODIS database. The design layout of the parallel database is shown in Figure 6.

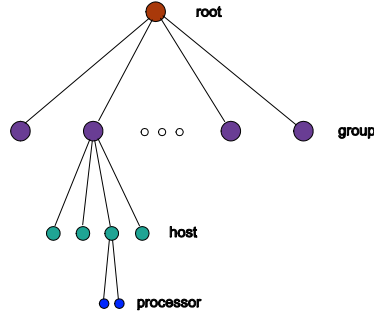


Figure 6: Hardware structure of the parallel database.

A root node communicates with k groups of computer networks. Each of these groups is composed of n nodes (hosts) holding identical copies of a portion of the database. (Any pair of groups correspond to different databases,

which are not necessarily disjoint. A specific record, or DNA profile, is in general stored in two groups for redundancy to protect against failure of a node.) Within each node, there are either one or two processors. In the experimental facility, the dual processor machines use 1.6 GHz Athlon MP processors, and the single processor machines use 1.33 GHz Athlon processors. All run the Linux operating system. Our interest here is in the load balancing in any one group of n nodes/hosts.

The database is implemented as a set of queues with associated search engine threads, typically assigned one per node of the parallel machine. Due to the structure of the search process, search requests can be formulated for any target DNA profile and associated with any node of the index tree. These search requests are created not only by the database clients; the search process also creates search requests as the index tree is descended by any search thread. This creates the opportunity for parallelism; search requests that await processing may be placed in any queue associated with a search engine, and the contents of these queues may be moved arbitrarily among the processing nodes of a group to achieve a balance of the load. This structure is shown in Figure 7.

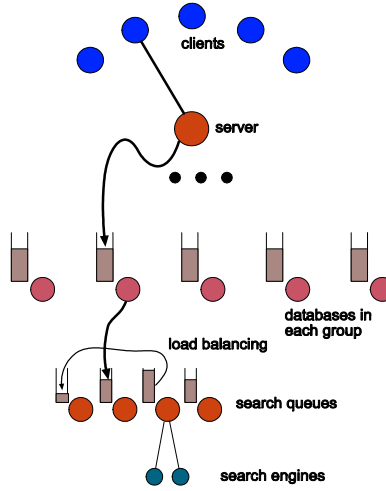


Figure 7: A depiction of multiple search threads in the database index tree. Here the server corresponds to the “root” in Figure 6. To even out the search queues, load balancing is done between the nodes (hosts) of a group. If a node has a dual processor, then it can be considered to have two search engines for its queue.

An important point is that the actual delays experienced by the network traffic in the parallel machine are *random*. Work has been performed to characterize the bandwidth and delay on unloaded and loaded network switches, in order to identify the delay parameters of the analytic models and is reported in [9][10]. The value $\tau = 200 \mu\text{sec}$ used for simulations represents an average

value for the delay and was found using the procedure described in [10]. The interest here is to compare the experimental data with that from the three models previously developed.

To explain the connection between the control gain K and the actual implementation, recall that the waiting time is related to the number of tasks as $x_i(t) = q_i(t)t_{p_i}$ where t_{p_i} is the average time to carry out a task. The continuous time control law is

$$u(t) = -Ky_i(t)$$

where $u(t)$ is the rate of decrease of waiting time $x_i(t)$ per unit time. Consequently, the gain K represents the rate of reduction of waiting time per second in the continuous time model. Also, $y_i(t) = \left(q_i(t) - \left(\sum_{j=1}^n q_j(t - \tau_{ij}) \right) / n \right) t_{p_i} = r_i(t)t_{p_i}$ where $r_i(t)$ is simply the number of tasks above the estimated (local) average number of tasks. With Δt the time interval between successive executions of the load balancing algorithm, the control law says that a fraction of the queue $K_z r_i(t)$ ($0 < K_z < 1$) is removed in the time Δt so the rate of reduction of *waiting time* is $-K_z r_i(t)t_{p_i}/\Delta t = -K_z y_i(t)/\Delta t$ so that

$$u(t) = -\frac{K_z y_i(t)}{\Delta t} \implies K = \frac{K_z}{\Delta t}. \quad (9)$$

This shows that the gain K is related to the actual implementation by how fast the load balancing can be carried out and how much (fraction) of the load is transferred. In the experimental work reported here, Δt actually varies each time the load is balanced. As a consequence, the value of Δt used in (9) is an average value for that run. The average time t_{p_i} to process a task is the same on all nodes (identical processors) and is equal $10\mu\text{sec}$ while the time it takes to transfer of load is about $50\mu\text{sec}$. The initial conditions were taken as $q_1(0) = 60000, q_2(0) = 40000, q_3(0) = 20000$ (corresponding to $x_1(0) = q_1(0)t_{p_i} = 0.6, x_2(0) = 0.4, x_3(0) = 0.2$). All of the experimental responses were carried out with constant $p_{ij} = 1/2$ for $i \neq j$.

Figure 8 is a plot of the responses $r_i(t) = q_i(t) - \left(\sum_{j=1}^n q_j(t - \tau_{ij}) \right) / n$ for $i = 1, 2, 3$ (recall that $y_i(t) = r_i(t)t_{p_i}$). The (average) value of the gains were ($K_z = 0.5$) $K_1 = 0.5/75\mu\text{sec} = 6667, K_2 = 0.5/120\mu\text{sec} = 4167, K_3 = 0.5/100\mu\text{sec} = 5000$. This figure compares favorably with Figure 5 of the linear model except for the time scale being off, that is, the experimental responses are slower. The explanation for this is that the gains here vary during the run because Δt (the time interval between successive executions of the load balancing algorithm) varies during the run. Further, this time Δt is *not* modeled in the continuous time simulations, only its average effect in the gains K_i . That is, the continuous time model does not stop processing jobs (at the average rate t_{p_i}) while it is transferring tasks to do the load balancing.

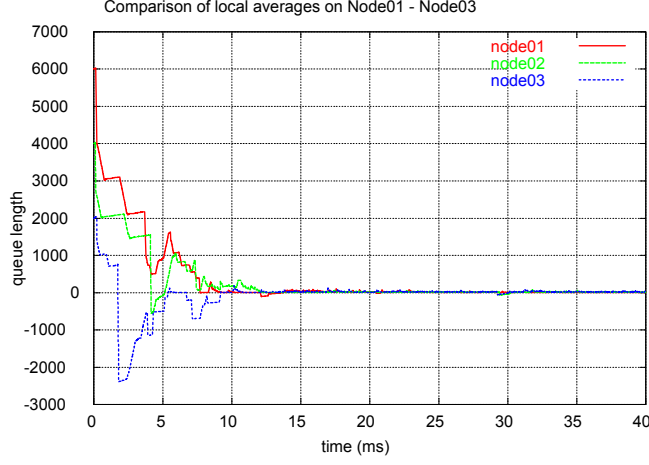


Figure 8: Experimental response of the load balancing algorithm. The average value of the gains are ($K_z = 0.5$) $K_1 = 6667, K_2 = 4167, K_3 = 5000$ with constant p_{ij} .

Figure 9 shows the plots of the response for the (average) value of the gains given by ($K_z = 0.2$) $K_1 = 0.2/125\mu\text{sec} = 1600, K_2 = 0.2/80\mu\text{sec} = 2500, K_3 = 0.2/70\mu\text{sec} = 2857$. Note that these gains are about half that of the previous case with a consequence that the response die out slower. The initial conditions were $q_1(0) = 60000, q_2(0) = 40000, q_3(0) = 20000$ ($x_1(0) = q_1(0)t_{pi} = 0.6, x_2(0) = 0.4, x_3(0) = 0.2$).

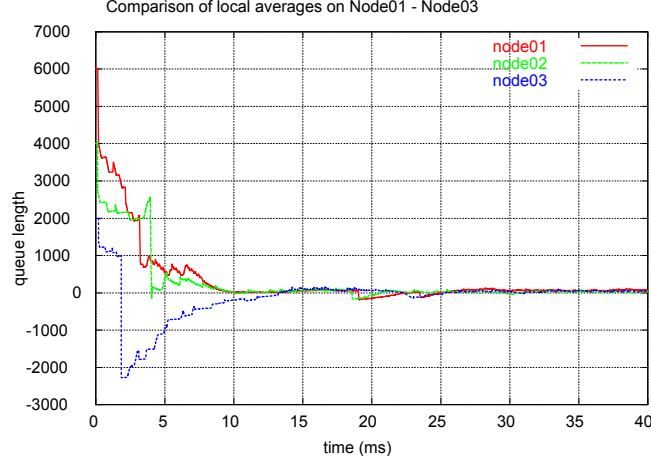


Figure 9: Experimental response of the load balancing algorithm. The average value of the gains are ($K_z = 0.2$) $K_1 = 16000, K_2 = 2500, K_3 = 2857$ with constant p_{ij} .

Figure 10 shows the plots of the response for the (average) value of the gains given by ($K_z = 0.3$) $K_1 = 0.3/125\mu\text{sec} = 2400, K_2 = 0.3/110\mu\text{sec} = 7273, K_3 = 0.3/120\mu\text{sec} = 2500$.

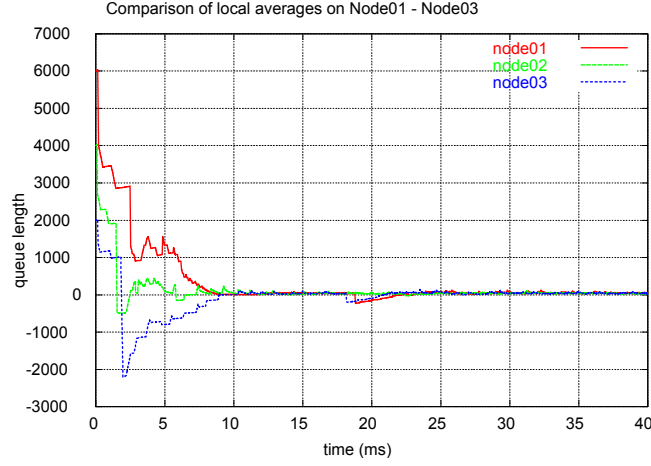


Figure 10: Experimental response of the load balancing algorithm. The average value of the gains are ($K_z = 0.3$) $K_1 = 2400, K_2 = 7273, K_3 = 2500$ with constant p_{ij} .

6 Summary and Conclusions

In this work, a load balancing algorithm was modeled in three ways using a linear time-delay model. Under the assumption of symmetric nodes and controllers (all intercommunication delays are identical and the controller gains identical) a systematic procedure was presented to determine the stability of the linear system by an explicit relationship between the delay values and the control gain. In particular, the delays create a limit on the size of the controller gains in order to ensure stability. Experiments were performed that indicate a correlation of the continuous time models with the actual implementation.

A consideration for future work is the fact that the load balancing operation involves processor time which is not being used to process tasks. Consequently, there is a trade-off between using processor time/network bandwidth and the advantage of distributing the load evenly between the nodes to reduce overall processing time.

An issue is that the delays in actuality are not constant and depend on such factors as network availability, the execution of the software, etc. An approach to modeling using a discrete-event/hybrid state formulation that accounts for block transfers that occur after random intervals may also be advantageous in analyzing the network.

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