Linear-time invariant Positive Systems: Stabilization and the Servomechanism Problem

by

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Abstract

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Positive systems, which carry the well known property of confining the state, output, and/or input variables to the nonnegative orphant, are of great practical importance, as the nonnegative property occurs quite frequently in numerous applications and in nature. These type of systems frequently occur in hydrology where they are used to model natural and artificial networks of reservoirs; in biology where they are used to describe the transportation, accumulation, and drainage processes of elements and compounds like hormones, glucose, insulin, and metals; and in stocking, industrial, and engineering systems where chemical reactions, heat exchanges, and distillation processes take place [30].

The interest of this dissertation is in two key problems: positive stabilization and the positive servomechanism problem. In particular, this thesis outlines the necessary and sufficient conditions for the stabilization of positive linear time-invariant (LTI) systems using state feedback control, along with providing an algorithm for constructing such a stabilizing regulator. Moreover, the results on stabilization also encompass the two problems of the positive separation principle and stabilization via observer design. The second, and most emphasized, problem of this dissertation considers the positive servomechanism problem for both single-input single-output (SISO) and multi-input multi-output (MIMO) stable positive LTI systems. The study of the positive servomechanism

problem focuses on the tracking problem of nonnegative constant reference signals for unknown/known stable SISO/MIMO positive LTI systems with nonnegative unmeasurable/measurable constant disturbances via switching tuning clamping regulators (TcR), linear quadratic clamping regulators (LTQcR), and ending with MPC control. Finally, all theoretical results on the positive servomechanism problem are justified via numerous experimental results on a waterworks system.

Dla Mamy i Taty,

wszystko co tu jest to dla Was!

... it is not success that makes your stronger, but the failures along the way ...

to Prof. Davison from the bottom of my heart ... Thank you!

Contents

Chapter 1

Introduction

Positive systems are systems where a natural restriction of nonnegativity occurs on states, outputs, and possibly inputs for all time.

For example, consider a patient that is sedated during a surgical procedure. Depending on the type of surgery, various organs, tissues, cells, or simply compartments - or states in the systems control sense - are under the infusion of an anesthetic (see Figure 1.1), which ultimately controls the patients pain sensitivity, movements, and overall consciousness. Normally, the amount of anesthesia must be continuously infused during the surgical procedure to guarantee ideal conditions. The latter process of anesthetic infusion represents a positive system; consisting of states, inputs, outputs, and various disturbances that affect the overall outcome of a successful surgery. The amount of anesthetic that enters each compartment must be positive, the levels within the compartments, or states, are positive, and the decaying effects of the substance are also positive. Simply put, in the natural biological sense, negative amounts cannot be present. We are restricted to have positive amounts of anesthesia within the human body as negative amounts do not translate to any physical interpretation. This constraint of positivity, or more precisely nonnegativity, defines a **positive system.**

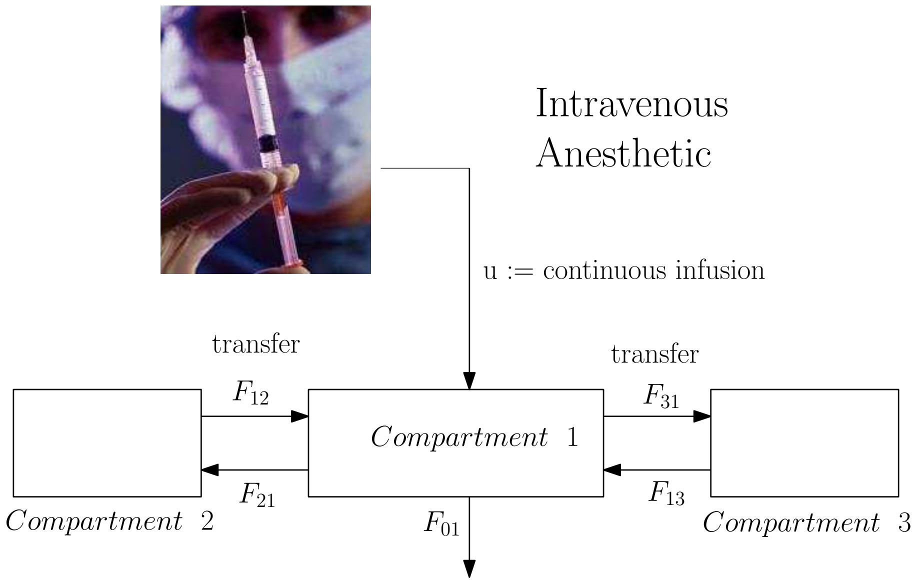


Figure 1.1: Intravenous anesthesia infusion.

The practical importance of these systems is widely visible, as the nonnegative property occurs quite frequently in numerous applications and in nature. Positive systems are often found in biology where they are used to describe the transportation, accumulation, and drainage processes of elements and compounds like hormones, glucose, and metals [30] - leading to applications of anesthesia infusion as described above, insulin control in diabetic patients, heartbeat rhythm control for athletes, amongst others. However, positive systems do not only radiate within biology, as they are present in hydrology, engineering, and industrial systems which involve water transportation, chemical reactions, heat exchangers, and distillation columns. In fact, positive systems are so widely visible in applications and in nature that it has been pointed out by Luenberger in 1979 that "it is for positive systems that dynamic systems theory assumes one of its most potent forms" [55].

This dissertation focuses on positive linear time-invariant (LTI) systems. In particular the problem of state and output feedback stabilization for positive LTI systems is discussed and solved from a view point previously not encountered in the literature. The problem of stabilization is also extended to include results of the positive separation principle of LTI systems, i.e. where the design of a state feedback stabilizing gain and an observer feedback gain is considered as two separate problems. After the problem of positive stabilization is finalized, the focal point of the dissertation shifts toward finding necessary and sufficient conditions for reference tracking and disturbance rejection of stable positive LTI systems via robust control strategies, i.e. the *servomechanism problem* for positive LTI systems. Once the necessary and sufficient conditions are established, results on finding adequate control methodologies that solve the servomechanism problem are outlined. Finally, all theoretical results are verified via experimentation on a waterworks positive system consisting of industrialized components.

1.1 Literature Review

This thesis studies two key problems: positive stabilization and the positive servomechanism problem, both of these problems are in one way or another related to or have a connection with one or more topics discussed in this section.

Positive systems, due to their inherent nonnegative constraint, have appeared in numerous fields of economics [48, 49]; engineering [55], [8], [13]; pharmokinetics and tracer kinetics [2], [79]; ecosystems and population modeling [50], [39], [64]; hydrology [68]; medicine and biology [40], [41], [2]; and many others. It is thus no surprise that the interest in positive systems has continued its growth both in the practical and theoretical fields.

The study of positive systems, in particular nonnegative matrices, can be traced back to the work of Perron [71] and Frobenius [31] in the early twentieth century; but it was not until the work of David G. Luenberger, who initiated a unifying approach of systems control and positive systems [55], that more control theorists took notice. Ever since Luenberger's famous book (1979), the interest in positive systems in control theory has evolved dramatically stretching to topics of reachability, controllability, switching systems, 2D systems, stabilization, and optimal control - all of these topics, and others, are reviewed next.

1.1.1 Nonnegative and Metzler matrices

The study of nonnegative and Metzler matrices dates back to Frobenius [31] and Perron [71], as mentioned above; however, over the years the interest in both nonnegative and Metzler matrices has grown considerably. One of the most referenced books, which presents various results on nonnegative and Metzler matrices, has been written by Gantmacher [32]. Other more recent texts covering similar, new, and extended topics are [10], [63], and [35]. The interest in nonnegative and Metzler matrices is obvious as its application to positive systems is direct and is outlined in Chapter 2.

1.1.2 Stability and real dominant equilibria

Stability is one of the most important topics discussed in systems control and it is no different in the case of positive systems. Many of the results on stability have been known since 1912 and are obtained and derived from the work of Frobenius [31]. One of the most crucial results is that of dominant stable eigenvalues, e.g. we now know that the dominant eigenvalue of a continuous-time positive system is real and unique. This latter link between matrices and positive systems has been made in [55], while numerous other results tying stability, equilibria, and positive systems have also been reported in [64] and [65].

1.1.3 Reachability and controllability

As in the case of stability, reachability and controllability have become extremely important throughout various disciplines in systems control. Reachability in positive systems considers the problem of reaching nonnegative states from other nonnegative states [30].

The reachability problem for positive systems dates back to the early nineteen eighties where [56] presented results on complete reachability for discrete time positive systems. The latter result was later verified and extended by [66], [74], and [28]. The extension to multiple inputs has been primarily done by [74], [29], and [83], and references therein. Other extensions to the reachability cone and general reachability can also be found in [14], [51], and [69]. An interesting survey on reachability and controllability in positive systems has been presented by Caccetta and Rumchev [11].

1.1.4 Positive Realization

The positive realization problem has now been thoroughly investigated for several decades. The problem of positive realization is one in which a system transfer function can be transformed into its state representation (A, B, C, D) such that the resulting state space system is positive [30] (for more precise definitions of positive realization the interested reader is encouraged to refer to the references provided below). It has been pointed out by [30] that the problem of realization can be traced back to the work on compartmental systems by [57] in the 1970's; since that time, the interest in positive realizations has grown considerably. In particular, [69] and [6] have outlined a set of necessary and sufficient conditions for positive realization; however, the latter citations are by no means the only ones that considered positive realizations, as [1], [46], [7], [85] and references therein (just to mention a few) have also considered this topic.

Positive realization still continues to be an active research interest among the systems control community and the interested reader is referred to [8] and [13] for recent results.

1.1.5 Switching positive systems

Another worthy topic of discussion for positive systems is that of switching positive systems. This topic is fairly new to positive systems, but has grown great roots within other disciplines of systems control, e.g. see the survey paper [24] and a recent book [52] for more details. The study of switching positive systems has thus far been primarily restricted to stability and reachability, e.g. see [77], [76], [58], and [59]. In this dissertation, the topic of switching control strategies comes up fairly often, but the topic of switching control, as presented within the servomechanism problem, has yet to be implemented elsewhere.

1.1.6 2D positive systems

Two dimensional (2D) positive systems have also garnered a lot of attention and although they have not been of interest in this thesis there are numerous complementing results for 2D systems to stability, reachability, realization, and the like, see for example the results presented in [84], [43], and [44].

1.1.7 Stabilization and observer design

The problem of stabilization and positive observer design has been previously studied, but not using the algorithmic approach presented in this dissertation. The interest in positive stabilization has grown over the years. For example, observer design for compartmental systems for reducible and irreducible structures has been introduced in [25] (although not completed, see Appendix), and finalized in [4]. The problem of observer design for positive LTI systems has also been investigated in [15]; however the results for the MIMO case are incomplete, as the example in the Appendix illustrates. Very recently (during the time of research of this thesis), results on the existence conditions for positive state stabilization using LMI's and special linear programming approaches have been summarized in [33] and [73], respectively. Another interesting recent observation has been obtained by [88], where the authors deal with observer design and output dynamic observer stabilization for interval positive systems via an LMI approach. Although the latter three citations cover and provide the solution to the positive stabilization problem, in this thesis we cover the results from a vertex enumeration algorithm which has not been previously covered. Moreover, in this thesis we present results, in particular on the separation principle that have not been considered in the same context, as in this thesis, in the literature. Several other intriguing results on observer design via Sylvester's approach have been portrayed in [3, 4]. It must be pointed out that results of positive stabilization for SISO positive LTI systems has been captured in [15], [23], [3]. In this thesis (Chapter 3), several of the results of [15], [23], [3] will be restated and extended upon, e.g., to output positive stabilization and the positive separation principle. Another very interesting and easy to implement sufficient result via quadratic programming has been presented in [42]. Some other results on stability control, pole-assignment, adaptive type schemes, and some general stability feedback control not mentioned above can be found in [26], [9], [37], [78] [75], and for optimal control in [38] and [47].

Although various research aspects tied to this thesis and to general positive systems have been outlined above, the one topic omitted was that of the "robust servomechanism problem". The discussion of this topic and its background has been deferred to Chapter 2.

1.2 Overview of the Thesis

This dissertation studies two key problems: positive stabilization and the positive servomechanism problem. However, before these two problems are tackled Chapter 2 provides detailed background work related to the thesis. This chapter first defines common terminology used throughout the thesis while outlining numerous definitions, statements, theorems, and common results related to both positive and compartmental systems. The focus of Chapter 2 then shifts to a discussion of tuning regulators, feedforward controllers, and the servomechanism problem for linear time invariant systems. Finally, a complete discussion of singular perturbation is presented. The results on singular perturbation are used throughout the thesis; in particular, in Chapter 4 onward.

Next, Chapter 3 considers positive stabilization. Although the problem of positive stabilization has been previously studied, it has never been presented from the viewpoint of the algorithms outlined in this thesis. Thus, Chapter 3 not only provides the necessary and sufficient conditions to the positive feedback stabilization problem, but also supplies the stabilizing feedback gain via a vertex based algorithm. Unlike standard LTI systems, due to the nonnegativity constraint, the solution to the positive systems stabilization problem uses linear programming methods and enumeration procedures and not algebraic results as in the LTI case. Aside from providing a complete solution to the positive state stabilization problem, Chapter 3 tackles the stabilization problem for a special class of positive systems, which are highly visible in compartmental systems. The solution to the latter problem is attained via a very simple and computationally efficient algorithm. As a natural progression from the outcome of positive state feedback stabilization, results are extended to the positive output feedback case, which previously was an open problem.

Unfortunately positive feedback stabilization assumes that the state is made available to the designer; however, in numerous situations the state is not measurable and the separation principle must be considered. The goal in the separation principle is of course to stabilize an unstable system via output feedback techniques, while having no knowledge of the states. In this situation, an estimate of the states must be produced, or in a systems control sense, an observer must be created - this approach, i.e. the positive separation principle, is also considered in Chapter 3. Namely, results on stabilizing the overall system while finding the stabilizing gain matrix and observer gain matrix in separation are presented. Particularly, Chapter 3 shows that the process of separation is feasible for positive single-input single-output (SISO) systems, but in the case of positive multi-input multi-output case a new resolution is needed.

The major theoretical portion of this dissertation deals with the *positive servomechanism problem* which is outlined for the SISO case in Chapter 4 and for the MIMO case in Chapter 6. In short, the essence of the problem behind the servomechanism problem is summarized next:

> Find a controller, for a positive LTI system under constant disturbances, that
>
> (a) guarantees closed loop *stability*;
>
> (b) ensures the plant is *nonnegative* for all time, i.e. the states and the outputs are nonnegative for all time; and
>
> (c) ensures *tracking* of a given set of reference signals.
>
> (d) In addition, assuming that a controller has been found so that conditions (a), (b), (c) are satisfied; then for all perturbations of the nominal plant model which maintain properties (a) and (b), it is desired that the controller can still achieve asymptotic tracking and regulation, i.e. property (c) still holds.

A general understanding of the above problem can be traced back to the anesthesia example. Namely, the system is the person or more precisely the organs, tissues, and flows of anesthesia within the body. Stability can be captured as the need of having the anesthetic bounded by some maximum value that will prevent any ill effects to the patient. The tracking signal can be thought of as the amount of anesthesia present within the system that will result in an ideal state of unconsciousness of the patient during surgery, while the nonnegativity of the anesthetic is clear from our previous discussions of positivity, i.e. the amount of anesthetic cannot fall below zero and we can only infuse positive amounts with no extraction of an anesthetic possible. The final point (d) is extremely relevant to real life applications since more often than not, the patient's body undergoes changes, or perturbations, during surgery; yet, it is highly desirable to still maintain ideal conditions for infusing anesthesia into the patients system.

The research results of Chapter 4 - Chapter 9 concentrate on the positive servomechanism problem. In particular,

- Chapter 4 considers the positive servomechanism problem under tuning regulators for the case of SISO positive systems where the mathematical model is known or unknown;

- Chapter 5 considers the positive servomechanism problem under linear quadratic control for the case of SISO positive systems;

- Chapter 6 considers the positive servomechanism problem under tuning regulators for the case of MIMO positive systems where the mathematical model is known or unknown;

- Chapter 7 considers the positive servomechanism problem under linear quadratic control for the case of MIMO positive systems;

- Chapter 8 considers the positive servomechanism problem under model predictive control (MPC) for the case of both SISO and MIMO positive systems; and

- Chapter 9 puts all theoretical results of Chapters 4 - 7 to the test by running experimental results on a positive water works setup.

Thus, the results presented in Chapter 4 - Chapter 9 consider the tracking and disturbance rejection problem for SISO and MIMO systems, from both the theoretical and experimental point of view. Mainly, existence conditions are provided, along with the actual control laws, that solve the positive servomechanism problem for constant tracking and (un)measurable disturbance signals for positive systems assuming that the mathematical model of the system can be described by an LTI model, but may be *unknown*. The motivation for studying unknown plants is that in many industrial systems and "real world" systems, the mathematical model has not been identified, but from the physics of the system it is known that the plant is positive. The control strategy considered here is conservative in the case of unknown plants (via the use of tuning regulators), and becomes more aggressive (via the use of linear quadratic regulators and MPC control) as more information is made available to the designer of the controller.

Finally all theoretical work is validated in Chapter 9 by an experimental waterworks real-time control setup, consisting of industrialized components with numerous constraints. The experimental setup consists of four water tanks with various connections in between them. The system is clearly positive as the amount of water within a tank cannot fall below zero.

In conclusion, the final chapter (Chapter 10) of the thesis summarizes the results and outlines several directions, while the Appendix presents several counter examples to previously published work on positive stabilization.

Chapter 2

Background

This chapter of the thesis outlines common terminology used throughout the chapters and presents preliminary and background work related to positive LTI systems, which will be needed throughout the pages of this thesis.

This chapter is broken down into several main sections. Section 2.1 defines all common terms and symbols used throughout the thesis. An overview of positive systems and compartmental systems follows. Next, tuning regulators and feedforward control is reviewed and finally Section 2.4 discusses singular perturbation theory, as outlined in [45].

2.1 Terminology

Let the set $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$, the set $\mathbb{R}^n_+ := \{x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n \mid x_i \in \mathbb{R}_+,\ \forall i = 1, ..., n\}$. Similarly, let $\mathbb{R}_- := \{x \in \mathbb{R} \mid x \leq 0\}$, and the set $\mathbb{R}^n_- := \{x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n \mid x_i \in \mathbb{R}_-,\ \forall i = 1, ..., n\}$. If exclusion of 0 from the sets will be necessary, then we'll denote the sets in the standard way $\mathbb{R}_+ \setminus \{0\}$ ($\mathbb{R}^n_+ \setminus \{0\}$). The set $(\mathbb{R}_+ \setminus \{0\})^n$ will commonly be referred to as $interior(\mathbb{R}^n_+)$. The set of eigenvalues of a matrix $\mathcal{A}$ will be denoted as $\sigma(\mathcal{A})$. The ij^{th} entry of a matrix $\mathcal{A}$ will be denoted as a_{ij}. A matrix $\mathcal{A} \in \mathbb{R}^{n \times n}$ is *Hurwitz* or *stable* when all the eigenvalues (λ) of $\mathcal{A}$ are in the open left half of the complex plane $\mathbb{C}$ (i.e., the real parts of all eigenvalues are negative). A *nonnegative* matrix $\mathcal{A}$ has all of its entries greater than or equal to 0, i.e. $a_{ij} \in \mathbb{R}_+$. A *Metzler* matrix $\mathcal{A}$ is a matrix for which all off-diagonal elements of $\mathcal{A}$ are nonnegative, i.e. $a_{ij} \in \mathbb{R}_+$ for all $i \neq j$. A *compartmental* matrix $\mathcal{A}$ is a matrix that is Metzler, and where the sum of the components within a column is less than or equal to zero, i.e. $\sum_{i=1}^{n} a_{ij} \leq 0$ for all $j = 1, 2, ..., n$. A matrix $\mathcal{A}$ that is compartmental, but also satisfies $\sum_{i=1}^{n} a_{ij} < 0$ for all $j = 1, 2, ..., n$ will be referred to as a *strictly compartmental* matrix. Note that all strictly compartmental matrices are stable due to their structure and Gerschgorin's Theorem. A *permutation matrix* is a square $(n \times n)$ matrix that has been obtained by permuting the rows of an identity matrix according to some permutation of the numbers 1 to n. A *monomial matrix* is a matrix that can be expressed as a product of a diagonal matrix and a permutation matrix, which has the property that there is exactly one nonzero entry in each row and each column. For convenience, lower case letters when used in appropriate context, e.g., $b,\ c,\ d, ...$, will represent scalars or vectors, while upper case letters, e.g., $A,\ B,\ C, ...$, will represent matrices. A vector $a > 0$ (or $<, \geq, \leq, =$) component-wise means that each component of a is greater than zero, i.e., $a > 0$ means $\forall\ i\ a_i > 0$. The notation $\mathcal{A}^i$ represents the principal submatrix of $\mathcal{A}$ with the $i-th$ row and $i-th$ column removed (throughout the thesis we will refer to a principal submatrix simply as a submatrix). A signal[1] $q(t,\epsilon) \in \mathbb{R}^n = O(\epsilon)$ if for all $i = 1, ..., n$

$$lim_{\epsilon \to 0} \frac{q_i(t,\epsilon)}{\epsilon} \leq \beta_i$$

uniformly for all $t \in [0, \infty)$ and where $\beta_i \in \mathbb{R}$ is a constant. A set $\mathcal{P} \subset \mathbb{R}^n$ is called a *convex polyhedron* if

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax = b,\ Cx \geq d\} \tag{2.1}$$

[1] Throughout the thesis, if it is clear from the context, a signal's dependence on time t may be dropped.

for some matrix $A \in \mathbb{R}^{q_1 \times n}$ and $C \in \mathbb{R}^{q_2 \times n}$, and some vectors $b \in \mathbb{R}^{q_1}$ and $d \in \mathbb{R}^{q_2}$. A closed and bounded polyhedron will be referred to as a *polytope*. Finally, the term *bidirectional* refers to a signal that can take on nonnegative and positive values.

2.2 Positive Linear Systems and Compartmental Systems

In this section we give an overview of both *positive linear systems* [55], [30], and a very important subset of positive linear systems known as *compartmental systems* [30], [41]. The inclusion of compartmental systems within this subsection will be made because in general compartmental systems are stable, a property of great significance throughout the chapters (Chapters 4 - Chapter 8).

We first define a positive linear system [30] in the traditional sense.

Definition 2.2.1. A linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{2.2}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{r \times n}$, and $D \in \mathbb{R}^{r \times m}$ is considered to be a *positive linear system* if for every nonnegative initial state and for every nonnegative input the state of the system and the output remain nonnegative.

The above definition states that any trajectory starting at an initial condition $x_0 \in \mathbb{R}^n_+$ will not leave the positive orthant, and moreover, that the output also remains nonnegative. For convenience, if for all time a state x satisfies $x \in \mathbb{R}^n_+$, or the output y satisfies $y \in \mathbb{R}^r_+$, or the input u satisfies $u \in \mathbb{R}^m_+$, then we'll say that the state, output, or input maintains *nonnegativity*. Notice that Definition 2.2.1 states that the input to the system must be nonnegative, a restriction that in applications is not always feasible;

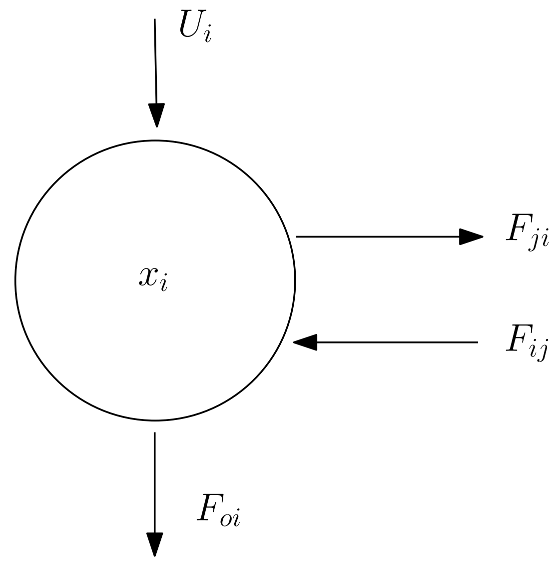
we'll return to this in the sequel.

It turns out that Definition 2.2.1 has a very nice interpretation in terms of the matrix quadruple (A, B, C, D).

Theorem 2.2.1 ([30], pg.14). A linear system (2.2) is positive if and only if the matrix A is a Metzler matrix, and B, C, and D are nonnegative matrices.

Next we introduce compartmental systems.

A compartmental system consists of n interconnected compartments or reservoirs (an example of such a system consisting solely of mass flows is given in Figure 2.1 for a model of one reservoir).



$$\dot{x}_i = \sum_{j \neq i} (-F_{ji} + F_{ij}) - F_{oi} + U_i$$

Figure 2.1: Model of one compartment.

In this figure, x_i represents the current state of reservoir i, U_i is the external inflow coming into reservoir i; F_{oi} is the flow exiting reservoir i, hence the notation to the outside (o) from reservoir (i) is used; F_{ji} is the outflow into reservoir j from reservoir i; F_{ij} is the inflow into reservoir i from reservoir j. For the mathematical description of a flow considered in this thesis see equation (2.3). The assumptions made for linear compartmental systems are that the variables just described satisfy:

$$\begin{aligned} U_i &= b_{i1}u_1 + b_{i2}u_2 + \cdots + b_{im}u_m \\ F_{oi} &= \beta_{oi}x_i \\ F_{ji} &= \beta_{ji}x_i \\ F_{ij} &= \beta_{ij}x_j. \end{aligned} \tag{2.3}$$

where $u_i \in \mathbb{R}$, $i = 1, ..., m$ denote the inputs that compose U_i.

Note that in a true compartmental system, by definition, all variables are nonnegative, i.e.

$$x_i, U, F_{oi}, F_{ij}, F_{ji} \in \mathbb{R}_+ \text{ or } b_{i1}, u_1, b_{i2}, u_2, \ldots, b_{im}, u_m, \beta_{io}, \beta_{ij}, \beta_{ji} \in \mathbb{R}_+.$$

See [41] for a more in depth treatment of (2.3).

With the above description of one compartment, we can easily come up with the entire state space model for an overall system consisting of n interconnected compartments:

$$\begin{aligned} \dot{x}_i &= -(\beta_{oi} + \sum_{j \neq i} \beta_{ji})x_i + \sum_{j \neq i} \beta_{ij}x_j \\ &\quad + b_{i1}u_1 + b_{i2}u_2 + \cdots + b_{im}u_m. \end{aligned} \tag{2.4}$$

Setting

$$\alpha_i = \beta_{oi} + \sum_{j \neq i} \beta_{ji}, \tag{2.5}$$

results in

$$\begin{aligned}\dot{x}_i \quad = & \begin{bmatrix} -\alpha_1 & \beta_{12} & \beta_{13} & & \beta_{1n} \\ \beta_{21} & -\alpha_2 & \beta_{23} & \dots & \beta_{2n} \\ & \vdots & & & \vdots \\ \beta_{n1} & \beta_{n2} & \beta_{n3} & \dots & -\alpha_n \end{bmatrix} x \\ & + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ & \vdots & & \vdots \\ bn1 & b_{n2} & \dots & b_{nm} \end{bmatrix} u, \end{aligned} \tag{2.6}$$

where $x = [x_1 \ x_2 \ \dots \ x_n]^T$ and $u = [u_1 \ u_2 \ \dots \ u_m]^T$. For convenience, a compartmental matrix will be denoted by

$$A_c = \begin{bmatrix} -\alpha_1 & \beta_{12} & \beta_{13} & & \beta_{1n} \\ \beta_{21} & -\alpha_2 & \beta_{23} & \dots & \beta_{2n} \\ & \vdots & & & \vdots \\ \beta_{n1} & \beta_{n2} & \beta_{n3} & \dots & -\alpha_n \end{bmatrix} \tag{2.7}$$

and a compartmental B matrix will be denoted by

$$B_c = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ & \vdots & & \vdots \\ bn1 & b_{n2} & \dots & b_{nm} \end{bmatrix}$$

as was done for positive linear systems. Notice that, by definition, the summation of all elements within a column of A is less than or equal to zero, i.e.

$$-\alpha_i + \sum_{j \neq i} \beta_{ji} = -\beta_{oi} \leq 0,$$

where equality holds if there is no outflow lost to the outside environment. Note that any matrix (2.7) that is strictly compartmental is stable due to its structure and Gerschgorin's Theorem.

Before we proceed, let's make a distinction, in the control systems sense, between *inflow*, *outflow* and *output*. When dealing with compartmental systems, the term *inflow* designates the movement of material[2] into the system, the term *outflow* designates the movement of material out of the system, and the *outputs* of the system are measurements on some compartment or a combination of compartments, and may have nothing to do with the material outflows from the system. The above clarification has been nicely captured in [41].

With the above paragraph in mind, the output equation for compartmental systems

$$y = Cx + Du$$

can be arbitrarily specified; however, to satisfy our positive linear system definition we'll assume $C \in \mathbb{R}_+^{r \times n}$ and assume $D \in \mathbb{R}_+^{r \times m}$.

Finally, the above description of a compartmental system is not unique; in fact there are other descriptions in the literature, see for example [30], [41]. One common addition made to the above description, pointed out in [30], is that the summation of a column of the matrix B_c must equal one, i.e.

$$\sum_{j=1}^{n} b_{ji} = 1, \quad \forall i = 1, \ldots, m,$$

[2]"material" here can designate anything; for example, liquid, voltage, current, hormones, glucose, etc.

where each $0 \leq b_{ji} \leq 1$. This assumption is very natural to make since one expects the maximum inflow not to exceed u_i, i.e.

$$\sum_{j=1}^{n} b_{ji} u_i \leq u_i, \quad \forall i = 1, \ldots, m,$$

with the summation is equal to one when the entire inflow enters the compartmental system.

2.3 Tuning Regulators and Feedforward Control

In this section we describe two controllers, the tuning regulator and the feedforward compensator, which solve the tracking problem for *unknown*[3] stable LTI systems under constant disturbances. To accomplish this, steady-state experiments are carried out on the system to determine various DC gain matrices of the system. The results of this section can be found in their entirety and in their general form in [17, 61]. The tuning regulator described within this subsection is nothing more but a generalization of the classical "on-line tuning" controller [80].

Consider the plant

$$\begin{aligned} \dot{x} &= Ax + Bu + E\omega \\ y &= Cx + Du + F\omega \\ e &:= y_{ref} - y \end{aligned} \tag{2.8}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^r$, the disturbance vector $\omega \in \mathbb{R}^{\tilde{\Omega}}$, and $y_{ref} \in \mathbb{R}^r$ is a desired tracking signal. Assume that the output y and the disturbance ω are measurable with $m = r$, and that the matrix A is Hurwitz.

[3]by unknown we mean that there is no knowledge of (A, B, C, D)

In the case of constant disturbances and constant tracking signals, the feedforward compensator that solves the "servomechanism problem", i.e. such that

(i) the closed loop system is asymptotically stable, and

(ii) for all tracking signals and disturbances $e \to 0$ as $t \to \infty$.

is given by

$$u = K_r y_{ref} + K_d \omega \tag{2.9}$$

with $K_r = (D - CA^{-1}B)^{-1}$ and $K_d = -(D - CA^{-1}B)^{-1}(F - CA^{-1}E)$.

Note, in order to find K_r and K_d we require the numerical values of (A, B, C, D, E, F). A procedure will be given shortly that outlines how one can obtain K_r and K_d without requiring the numerical values of the matrices. Of course, we assume that $D - CA^{-1}B$ is full rank so that an inverse exists.

In the case of constant disturbances (ω) and constant tracking (y_{ref}) signals, the tuning regulator that solves the "robust servomechanism problem" [17], i.e. such that

(i) the closed loop system is asymptotically stable,

(ii) for all tracking signals and disturbances $e \to 0$ as $t \to \infty$, and

(iii) property (ii) occurs for all plant perturbations which maintain closed loop stability,

is given by:

$$\begin{aligned} \dot{\eta} &= \epsilon e \\ u &= K_r \eta, \end{aligned} \tag{2.10}$$

where $\epsilon \in (0, \epsilon^*]$, with $\epsilon^* \in \mathbb{R}_+ \setminus \{0\}$, where ϵ^* ensures that

$$\begin{bmatrix} A & BK_r \\ -\epsilon^* C & -\epsilon^* DK_r \end{bmatrix}$$

is Hurwitz. Such an ϵ^* always exists by the results of [17].

We summarize the above discussion by a Theorem for the case of MIMO LTI systems.

Theorem 2.3.1 ([17]). Consider the system (2.8), under the assumption that $y_{ref} \in \mathbb{R}^r$ and $\omega \in \mathbb{R}^{\tilde{\Omega}}$ are constants. Then there exists an ϵ^* such that $\forall \epsilon \in (0, \epsilon^*]$ the tuning regulator (2.10) solves the "robust servomechanism problem" and the feedforward compansator (2.9) solves the "servomechanism problem" if and only if $rank(D - CA^{-1}B) = r$.

Throughout the chapters it is emphasized that no knowledge of the plant (2.8) matrices exists, thus it is worth pointing out a procedure which will supply us with the gain matrix $K_r = D - CA^{-1}B$ without the knowledge of the actual values of (A, B, C, D). We present the procedure next.

Procedure 2.3.1. It is assumed that the outputs of the system are measurable and the inputs are excitable with no disturbances acting on the plant, i.e. $\omega = 0$.

1. Apply an input vector $u = [0 \; ... \; 0 \; \overline{u}_i \; 0 \; ... \; 0]^T$ to (2.8), $\forall i = 1, ..., m$, with $\overline{u}_i \neq 0$.

2. Measure the corresponding steady-state value of the output vectors $y = \overline{y}_i \in \mathbb{R}^r$, $\forall i = 1, ..., m$, where $\overline{y}_i = [\overline{y}_i^1 \; \overline{y}_i^2 \; ... \; \overline{y}_i^r]^T$.

3. Solve the equation:

$$K_1 \begin{bmatrix} \overline{u}_1 & 0 & ... & 0 \\ 0 & \overline{u}_2 & ... & 0 \\ & & \ddots & \\ 0 & 0 & ... & \overline{u}_m \end{bmatrix} = \begin{bmatrix} \overline{y}_1^1 & \overline{y}_2^1 & ... & \overline{y}_m^1 \\ \overline{y}_1^2 & \overline{y}_2^2 & ... & \overline{y}_m^2 \\ & & \ddots & \\ \overline{y}_1^r & \overline{y}_2^r & ... & \overline{y}_m^r \end{bmatrix}$$

   for $K_1 = D - CA^{-1}B$.

Note, in the case of a tuning regulator and the feedforward compensator we need the inverse of K_1; this is easily obtained if $rank(K_1) = r$.

Next we present a procedure to obtain the gain matrix $F - CA^{-1}E$ without any knowledge of the plant.

Procedure 2.3.2. It is assumed that the outputs of the system are measurable and the disturbances are excitable with the inputs set to zero, i.e. $u = 0$.

1. Apply a disturbance vector $\omega = [0 \ ... \ 0 \ \overline{\omega}_i \ 0 \ ... \ 0]^T$ to (2.8), $\forall i = 1, ..., \tilde{\Omega}$, with $\overline{\omega}_i$ having a non-zero steady-state value.

2. Measure the corresponding steady-state value of the output vectors $y = \overline{y}_i \in \mathbb{R}^r$, $\forall i = 1, ..., \tilde{\Omega}$, where $\overline{y}_i = [\overline{y}_i^1 \ \overline{y}_i^2 \ ... \ \overline{y}_i^r]^T$.

3. Solve the equation:

$$K_2 \begin{bmatrix} \overline{\omega}_1 & 0 & ... & 0 \\ 0 & \overline{\omega}_2 & ... & 0 \\ & & \ddots & \\ 0 & 0 & ... & \overline{\omega}_{\tilde{\Omega}} \end{bmatrix} = \begin{bmatrix} \overline{y}_1^1 & \overline{y}_2^1 & ... & \overline{y}_{\tilde{\Omega}}^1 \\ \overline{y}_1^2 & \overline{y}_2^2 & ... & \overline{y}_{\tilde{\Omega}}^2 \\ & & \ddots & \\ \overline{y}_1^r & \overline{y}_2^r & ... & \overline{y}_{\tilde{\Omega}}^r \end{bmatrix}$$

   for $K_2 = (F - CA^{-1}E)$.

In the above experiment it is assumed that the measurable disturbance can be excited in $\tilde{\Omega}$ independent ways. This is often the case in practice [17], e.g. in the control of distillation columns, the input feed composition is a measurable disturbance which can be excited; in a system of water tanks, an additional flow of water can be added to the various tanks, etc. Occasionally, however, it may be possible to measure a disturbance, but not excite it. In these cases, the operating records of the measurable disturbances can be monitored, e.g. in commercial heat exchangers [21].

Remark 2.3.1. Note that both K_r and K_d can be found via Procedure 2.3.1 and Procedure 2.3.2 by open-loop tests on the physical plant. In the case of the tuning regulator (2.10) the choice of $\epsilon > 0$ to use is found by "on-line tuning", i.e. the controller (2.10) is applied to the system to be controlled, and a 1-dimensional search on $\epsilon > 0$ is carried out to obtain the *best output response for the system*; Theorem 2.3.1 guarantees that such a stabilizing controller can always be found.

2.4 Singular Perturbation

This section has been added for completeness and covers singular perturbation results which are needed in order to prove various results throughout the thesis. The following discussion has been taken from [45], Chapter 11 and Chapter 4.

The standard singular perturbation model can be described as

$$\begin{aligned} \dot{q} &= f(t, q, z, \epsilon), \qquad q(t_0) = q_0 \\ \epsilon\dot{z} &= g(t, q, z, \epsilon), \qquad z(t_0) = z_0 \end{aligned} \tag{2.11}$$

where the functions f and g are continuously differentiable in their arguments $(t, q, z, \epsilon) \in [0, \infty) \times D_q \times D_z \times [0, \epsilon_0]$, with $D_q \subset \mathbb{R}^n$ and $D_z \subset \mathbb{R}^s$ being open and connected sets. By setting $\epsilon = 0$, we obtain

$$0 = g(t, q, z, 0), \tag{2.12}$$

where we designate the real root [4] of (2.12) as

$$z = h(t, q). \tag{2.13}$$

To obtain a reduced model, we substitute (2.13) into (2.11) resulting in

$$\dot{q} = f(t, q, h(t, q), 0), \qquad q(t_0) = q_0. \tag{2.14}$$

[4] without loss of generality, we assume there is only one root

The reduced model is sometimes referred to as the *slow* model, while (2.12) is referred to as the *quasi-steady-state model*, because z may rapidly converge to a root of (2.12).

Now denote the solution of (2.14) by $\overline{q}(t)$ and define

$$\overline{z}(t) = h(t, \overline{q}(t)),$$

which describes the quasi-steady-state behavior of z when $q = \overline{q}$.

In order to present a very important result on singular perturbations, we need to perform a change of variables first

$$p = z - h(t, q), \tag{2.15}$$

which shifts the quasi-steady-state of z to the origin. In the new variables (q, p) the full problem is

$$\begin{array}{rcl} \dot{q} & = & f(t, q, p + h(t, q), \epsilon), \qquad q(t_0) = q_0 \\ \epsilon \dot{p} & = & g(t, q, p + h(t, q), \epsilon) - \epsilon \dfrac{\partial h}{\partial t} \\ & & -\epsilon \dfrac{\partial h}{\partial q} f(t, q, p + h(t, q), \epsilon), \\ & & p(t_0) = z_0 - h(t_0, q_0) \end{array} . \tag{2.16}$$

Next, we set

$$\epsilon \frac{dp}{dt} = \frac{dp}{d\tau}; \quad \text{hence,} \ \frac{d\tau}{dt} = \frac{1}{\epsilon}$$

and use $\tau = 0$ as the initial value at $t = t_0$. In the new time scale, (2.16) becomes

$$\begin{array}{rcl} \dot{q} & = & f(t, q, p + h(t, q), \epsilon), \qquad q(t_0) = q_0 \\ \dfrac{dp}{d\tau} & = & g(t, q, p + h(t, q), \epsilon) - \epsilon \dfrac{\partial h}{\partial t} \\ & & -\epsilon \dfrac{\partial h}{\partial q} f(t, q, p + h(t, q), \epsilon), \\ & & p(t_0) = z_0 - h(t_0, q_0) \end{array} . \tag{2.17}$$

By setting $\epsilon = 0$, the latter equation reduces to

$$\frac{dp}{d\tau} = g(t, q, p + h(t, q), 0), \quad p(t_0) = z_0 - h(t_0, q_0), \tag{2.18}$$

which is commonly referred to as the *boundary-layer model*.

We will also make use of the autonomous system

$$\frac{dp}{d\tau} = g(t_0, q_0, p + h(t_0, q_0), 0), \; p(t_0) = z_0 - h(t_0, q_0) \tag{2.19}$$

which has an equilibrium at $p = 0$, and has been derived from (2.18) by setting $t = t_0$ and $q = q_0$. Define the solution of (2.19) as $\hat{p}(\tau)$.

Before we state the singular perturbation result on an infinite interval of time, we recall the following theorem on Lyapunov stability:

Theorem 2.4.1 ([45] pg.152). Let $x = 0$ be an equilibrium point for

$$\dot{x} = f(t, x)$$

and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : [0, \infty) \times D \to \mathbb{R}$ be a continuously differentiable function such that

$$W_1(x) \leq V(t, x) \leq W_2(x) \tag{2.20}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x) \tag{2.21}$$

$\forall t \geq 0$ and $\forall x \in D$, where $W_1(x)$, $W_2(x)$ and $W_3(x)$ are continuous positive definite functions on D. Then, $x = 0$ is uniformly asymptotically stable. Moreover, if r and c are chosen such that $B_r = \{\|x\| \leq r\} \subset D$ and $c < min_{\|x\|=1} W_1(x)$, then every trajectory starting in $\{x \in B_r \mid W_2(x) \le c\}$ satisfies

$$\|x\| \le \beta(\|x(t_0)\|, t - t_0), \qquad t \ge t_0 \ge 0$$

for some class KL function[5] β.

The following theorem presents the singular perturbation result of interest in this thesis.

Theorem 2.4.2 ([45] pg.439). Consider the singular perturbation system (2.11). Assume that the following conditions are satisfied for all

$$[t, q, z - h(t, q), \epsilon] \in [0, \infty) \times D_q \times D_p \times [0, \epsilon_0]$$

for some domains $D_x \subset \mathbb{R}^n$ and $D_p \subset \mathbb{R}^s$, which contain their respective origins:

1. On any compact subset of $D_x \times D_y$, the functions f, g, their first partial derivatives with respect to (q, z, ϵ), and the first partial derivative of g with respect to t are continuous and bounded, $h(t, q)$ and $[\partial g(t, q, z, 0)/\partial z]$ have bounded first partial derivatives with respect to their arguments, and $[\partial f(t, q, h(t, q), 0)/\partial q]$ is Lipschitz in q, uniformly in t;

2. the origin is an exponentially stable equilibrium point of the reduced system (2.14); i.e. there is a Lyapunov function $V(t, x)$ that satisfies the conditions of Theorem 2.4.1 for (2.14) for $(t, q) \in [0, \infty) \times D_q$ and $\{W_1(q) \le c\}$ is a compact subset of D_q;

3. the origin is an exponentially stable equilibrium point of the boundary-layer model (2.18), uniformly in (t, q). Let $\mathcal{R}_p \subset D_p$ be the region of attraction of (2.19) and Γ_p be a compact subset of $\mathcal{R}_y$.

[5] see [45] for an overview of class KL functions

Then, for each compact set $\Gamma_q \subset \{W_2(x) \leq \xi c, 0 < \xi < 1\}$ there is a positive constant ϵ_1 such that for all $t_0 \geq 0$, $q_0 \in \Gamma_q$, $z_0 - h(t_0, q_0) \in \Gamma_p$, and $0 < \epsilon < \epsilon_1$, the singular perturbation problem (2.16) has a unique solution $q(t,\epsilon)$, $z(t,\epsilon)$ on $[t_0, \infty)$, and

$$q(t,\epsilon) - \overline{q}(t) = O(\epsilon)$$

$$z(t,\epsilon) - h(t, \overline{q}(t)) - \hat{p}(t/\epsilon) = O(\epsilon)$$

uniformly in $t \in [0, \infty)$.

We will also make use of the standard theorem on the continuity of solutions in terms of parameters, which we recall below.

Theorem 2.4.3 ([45] pg.97). Let $f(t,x,\lambda)$ be continuous in (t,x,λ) and locally Lipshitz in x (uniformly in t and λ) on $[t_0,t_1] \times D \times \{\|\lambda - \lambda_0\| \leq c\}$, where $D \subset \mathbb{R}^n$ is an open connected set. Let $y(t,\lambda_0)$ be a solution of $\dot{x} = f(t,x,\lambda_0)$ with $y(t_0,\lambda_0) = y_0 \in D$. Suppose $y(t,\lambda_0)$ is defined and belongs to D for all $t \in [t_0,t_1]$. Then, given $\epsilon_2 > 0$, there is a $\delta > 0$ such that if

$$\|z_0 - y_0\| < \delta \text{ and } \|\lambda - \lambda_0\| < \delta$$

then there is a unique solution $z(t,\lambda)$ of $\dot{x} = f(t,x,\lambda)$ defined on $[t_0,t_1]$, with $z(t_0,\lambda) = z_0$, and $z(t,\lambda)$ satisfies

$$\|z(t,\lambda) - y(t,\lambda_0)\| < \epsilon_2, \qquad \forall t \in [t_0,t_1]$$

This concludes the preliminary and background work needed in order to proceed with the remaining chapters.

Chapter 3

Positive Stabilization

The problem of stabilization, using feedback gain matrices for linear time-invariant systems, dates back several decades; the main question, of course, being able to find conditions, and a controller if at all possible, such that the closed loop eigenvalues of the system are in the open left half plane of the imaginary axis. In this chapter we approach the positive stabilization problem from the viewpoint of a vertex enumeration method. The problem of positive stabilization is still actively researched and we refer the reader to Section 1.1.7 for a summary of current and past results related to the positive stabilization problem.

Throughout the following pages, the necessary and sufficient conditions for the existence of a feedback controller are presented. The results of this chapter outline the solution to the stabilization problem via first attacking the single-input single-output case, and secondly transforming the knowledge of the single-input single-output systems to that of multi-input multi-output systems. Provided within the pages of this chapter are numerous examples illustrating the key differences between single-input single-output systems and multi-input multi-output systems. In addition to the stabilization problem, all results, via duality, are extended to the creation of positive observers and output gain feedback controllers. Very interesting results on the separation principle (stabilization via observer design) for positive linear time-invariant systems are also described.

The chapter is divided into three sections. The first section is introductory where all definitions of interest are presented. The second section outlines feedback stabilization, observer design, and the separation principle for single-input single-output (SISO) systems. The final third section considers similar problems as that of section two, but for the multi-input multi-output (MIMO) case. Numerous examples are provided throughout the chapter to illustrate all theoretical results.

3.1 Preliminaries

This section presents preliminary results needed for the remainder of the chapter.

First, the system of interest is provided. Throughout the remainder of this chapter, unless otherwise stated, we will consider the following positive linear time-invariant (LTI) system:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{3.1}$$

where $A \in \mathbb{R}^{n\times n}$ and is Metzler, $B \in \mathbb{R}_+^{n\times m}$ and is full rank, $C \in \mathbb{R}_+^{r\times n}$ and is full rank, $D \in \mathbb{R}_+^{r\times m}$ and is full rank if nonzero, and the initial condition $x_0 \in \mathbb{R}_+^n$. Notice that the system of interest satisfies the standard definition of a positive linear time-invariant system as presented by Theorem 2.2.1.

We now introduce several definitions, which will become vital in the sequel. First, we define stabilization and output stabilization of positive LTI systems (3.1); thereafter, we define completely stabilizable systems and completely output stabilizable systems.

Definition 3.1.1. A positive system (3.1) is *stabilizable* if there exists a feedback gain matrix $K_s \in \mathbb{R}^{m\times n}$, for the control law:

$$u = -K_s x,$$

such that the closed loop system:

$$\dot{x} = (A - BK_s)x$$

results in the closed loop matrix $A - BK_s$ being stable and Metzler.

In similar fashion we can extend the definition to output stabilization.

Definition 3.1.2. A positive system (3.1) is *output stabilizable*, with $D = 0$, if there exists an output feedback gain matrix $K_o \in \mathbb{R}^{m\times r}$, for the control law:

$$u = -K_o y,$$

such that the closed loop system:

$$\dot{x} = (A - BK_o C)x$$

results in the closed loop matrix $A - BK_oC$ being stable and Metzler.

In the above definitions only state nonnegativity was taken into account (this is obvious by recalling the Metzler property of Theorem 2.2.1); however, if the D matrix is not zero the above definitions do not guarantee that the gain matrices will ensure both state x and output y nonnegativity; thus, to take the output into account two additional definitions are provided.

Definition 3.1.3. A positive system (3.1) is *completely stabilizable* if there exists a feedback gain matrix $K_s \in \mathbb{R}^{m \times n}$, for the control law:

$$u = -K_s x,$$

such that the closed loop system:

$$\begin{aligned} \dot{x} &= (A - BK_s)x \\ y &= (C - DK_s)x \end{aligned}$$

results in y and x being nonnegative for all time $t \geq 0$, and

$$x \rightarrow 0, \text{as } t \rightarrow \infty,$$

i.e. the closed loop matrix $A - BK_s$ is stable and Metzler and the matrix $C - DK_s$ is nonnegative.

In similar fashion the latter definition can be extended to completely output stabilizable systems.

Definition 3.1.4. A positive system (3.1) is *completely output stabilizable* if there exists an output feedback gain matrix $K_o \in \mathbb{R}^{m \times r}$, for the control law:

$$u = -K_o y,$$

such that the closed loop system:

$$\begin{aligned} \dot{x} &= Ax - BK_o y \\ y &= Cx - DK_o y \end{aligned}$$

results in y and x being nonnegative for all time $t \geq 0$, and

$$x \to 0, \text{as } t \to \infty,$$

i.e. the closed loop matrix $A - BK_oC$ is stable and Metzler.

Assumption 3.1.1. Note that in complete output stabilizability the control law $u = -K_oy$ results in:

$$\begin{aligned} y &= Cx + D(-K_oy) \\ y &= Cx - DK_oy \\ y + DK_oy &= Cx \\ (I + DK_o)y &= Cx \\ y &= (I + DK_o)^{-1}Cx. \end{aligned}$$

For the remainder of the chapter, we will assume that K_o will be chosen in such a way that the above inverse exists. This is not a restrictive assumption since the inverse exists for generic K_o.

The above definitions ensure that regardless of the control input (positive or negative), the plant will maintain nonnegativity of states (and/or outputs) for all initial conditions $x_0 \in \mathbb{R}_+^n$ of the plant. It is worth pointing out that in applications, nonnegativity of states (and/or outputs) occurs quite often; however, the need for the input u to be also nonnegative, as in the original definition (Definition 2.2.1), may not always be a necessity, as was also pointed out in [23]. Thus, throughout this chapter, we do not restrict ourselves to nonnegative inputs.

3.2 Necessary and Sufficient Conditions: SISO case

This section tackles the stabilization, output stabilization, and the extensions to complete stabilization and complete output stabilization for SISO positive LTI systems (3.1) with $m = r = 1$. In addition, stabilization and complete stabilization results are afterward extended to observer design, and the separation principle.

State and Output Stabilization

The results of this subsection are a summary and an extension of the work of [15], [23], [3]; in particular, in this subsection we provide controllers and several new conditions for state stabilization and direct output stabilization for unstable SISO positive LTI systems.

We first point out that for an unstable Metzler matrix A, no $k_s \in \mathbb{R}_-^{1\times n}$ (or $k_o \in \mathbb{R}_-$ for output stabilization) exists that would make $A - bk_s$ (or equivalently $A - k_o bc$) Metzler and stable, i.e. a controller

$$u = -k_s x, \quad k_s \in \mathbb{R}_-^{1\times n}$$

or

$$u = -k_o x, \quad k_o \in \mathbb{R}_-$$

cannot stabilize a linear system (3.1) with A Metzler unstable. In order to point out this observation we will invoke a result from [70].

Lemma 3.2.1. A Metzler matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$ is stable if and only if

$$\exists d \in \text{ interior}(\mathbb{R}_+^n) \text{ such that } -\mathcal{A}d \in \text{ interior}(\mathbb{R}_+^n). \tag{3.2}$$

Moreover, if $\mathcal{A}$ is stable then every principal submatrix of A is also stable.

We are now ready to state the observation that nonpositive gain matrices cannot solve the stabilization problem. The implication of the Lemma below is quite significant, as we are stating that a strictly nonnegative input cannot stabilize a single-input positive LTI system.

Lemma 3.2.2.

(i) There does not exist a nonpositive gain matrix $k_s \in \mathbb{R}_-^{1\times n}$ ($k_o \in \mathbb{R}_-$) such that an unstable Metzler matrix $A \in \mathbb{R}^{n\times n}$, an input matrix $b \in \mathbb{R}_+^n$, and an output matrix $c \in \mathbb{R}_+^{1\times n}$ can be stabilized (output stabilized).

(ii) Assume $\tilde{k}_s \in \mathbb{R}^{1\times n}$ stabilizes an unstable Metzler matrix $A \in \mathbb{R}^{n\times n}$ and an input matrix $b \in \mathbb{R}_+^n$; then this implies that there also exists a stabilizing gain matrix $k_s \in \mathbb{R}_+^{1\times n}$ that can do the same.

(iii) Assume $\tilde{k}_s \in \mathbb{R}^{1\times n}$ completely stabilizes an unstable Metzler matrix $A \in \mathbb{R}^{n\times n}$, an input matrix $b \in \mathbb{R}_+^n$, an output matrix $c \in \mathbb{R}_+^{1\times n}$ and $\tilde{d} \in \mathbb{R}_+ \setminus \{0\}$; then this implies that there also exists a completely stabilizing gain matrix $k_s \in \mathbb{R}_+^{1\times n}$ that can do the same.

Proof. First, the proof of (i) is presented.

It is sufficient to show that for any unstable Metzler matrix A and any nonnegative matrix $A^+ \in \mathbb{R}_+^{n\times n}$ the summation $A + A^+$ cannot satisfy (3.2). The reason for the introduction of A^+ is due to the fact that regardless of the choice of $k_s \in \mathbb{R}_-^{1\times n}$ (or $k_o \in \mathbb{R}_-$) the multiplication of bk_s (or bk_oc) results in

$$-bk_s \in \mathbb{R}_+^{n\times n}\ (-bk_oc \in \mathbb{R}_+^{n\times n}), \quad k_s \in \mathbb{R}_-^{1\times n},\ k_o \in \mathbb{R}_-,\ b \in \mathbb{R}_+^n.$$

Thus, by Lemma 3.2.1 if $A + A^+$ is a stable Metzler matrix, then there exists a $d \in$ interior($\mathbb{R}_+^n$) such that

$$\begin{aligned} 0 &< -d_1a_{i1} - ... - d_na_{in} - d_1a_{i1}^+ - ... - d_na_{in}^+ \\ &\leq -d_1a_{i1} - ... - d_na_{in} \quad \forall i = 1, ..., n. \end{aligned}$$

However, since A is unstable we have that for all $d \in$ interior$(\mathbb{R}_+^n)$ there exists an i such that $-d_1 a_{i1} - ... - d_n a_{in} \leq 0$, resulting in $0 < 0$, a contradiction.

The second statement (ii) can be proved in a similar fashion as (i) above.

Assume that $\tilde{k}_s \in \mathbb{R}^{1\times n}$ stabilizes an unstable Metzler matrix $A \in \mathbb{R}^{n\times n}$ and an input matrix $b \in \mathbb{R}_+^n$. First, by (i) at least one element of $\tilde{k}_s$ must be positive. Next assume, without loss of generality, that the first r elements of $\tilde{k}_s$ are negative while all elements after r are greater than zero. It now follows by Lemma 3.2.1 that there exists a $d \in$ interior$(\mathbb{R}_+^n)$ such that

$$\begin{aligned} 0 \;<\; & -d_1 a_{i1} - ... - d_n a_{in} - d_1(-b_i \tilde{k}_s^1) - ... \\ & - d_r(-b_i \tilde{k}_s^r) - d_{r+1}(-b_i \tilde{k}_s^{r+1}) - ... - d_n(-b_i \tilde{k}_s^n), \end{aligned}$$

for all $i = 1, ..., n$. However, from the above inequality we can set k_s equal to $\tilde{k}_s$ with all $\tilde{k}_s^i \in \mathbb{R}_-$ set to zero, i.e.

$$\begin{aligned} 0 \;<\; & -d_1 a_{i1} - ... - d_n a_{in} - d_1(-b_i \tilde{k}_s^1) - ... \\ & - d_r(-b_i \tilde{k}_s^r) - d_{r+1}(-b_i \tilde{k}_s^{r+1}) - ... - d_n(-b_i \tilde{k}_s^n) \\ \leq\; & -d_1 a_{i1} - ... - d_n a_{in} - d_1(-b_i \times 0) - ... \\ & - d_r(-b_i \times 0) - d_{r+1}(-b_i k_s^{r+1}) - ... - d_n(-b_i k_s^n), \end{aligned}$$

with

$$[k_s^1 \; ... \; k_s^r \; k_s^{r+1} \; ... \; k_s^n] = [0 \; ... \; 0 \; \tilde{k}_s^{r+1} \; ... \; \tilde{k}_s^n].$$

This completes the proof for (ii).

Finally, in statement (iii) k_s has the additional constraint that it must satisfy:

$$c - \tilde{d} k_s \geq 0,$$

component-wise. However, by the same argument as used in (ii) we can simply set all negative components of the completely stabilizing gain k_s to zero from which the result follows. $\square$

Remark 3.2.1. Note that Lemma 3.2.2 (i) can be extended to the MIMO case with the gain matrix K_s (see Definition 3.1.1), but Lemma 3.2.2 (ii) and (iii) cannot (more on this issue will be presented in the sequel).

The implication of Lemma 3.2.2 is that we can safely assume that a gain matrix, if it exists, will not be strictly nonpositive (based on our definition of stabilization), and moreover, by the second statement of Lemma 3.2.2, we can directly deal with nonnegative stabilizing gain matrices for the SISO case. What is astounding in Lemma 3.2.2 (i) is the fact that one cannot stabilize a positive SISO LTI system with strictly nonnegative gain feedback control (the same will be true in the MIMO case as Remark 3.2.1 already hinted), i.e. if the control law is of the form:

$$u = -k_s x,$$

with at least one element of k_s being strictly positive, then there always exists an initial condition $x_0 \in \mathbb{R}^n_+$ which would result in a negative input control. The only way that we could guarantee a strictly positive control input (for unstable systems) is if the gain matrix k_s would have nonpositive entries, which Lemma 3.2.2 (i) shows is impossible if stabilization is the goal.

We note that the concept of dealing with nonpositive gain matrices k_s (more precisely with nonnegative inputs) for the stabilization problem has been pointed out before, as for example [23] illustrates for the special case of Metzler matrices with maximal eigenvalue(s) at the origin.

With the latter observations, we next invoke the results of [15] to obtain necessary and sufficient conditions for the stabilization of unstable SISO positive LTI systems. The next

Theorem summarizes the SISO result of [15] (although we present it as a stabilization problem and not a positive observer design as in [15]), and with the aid of Lemma 3.2.1, contributes an additional observation. Before the Theorem is presented, we point out an additional result originally given in [55].

Lemma 3.2.3. Let $A \in \mathbb{R}^{n\times n}$ be a Metzler matrix. Let $A_1^+,\ A_2^+ \in \mathbb{R}_+^{n\times n}$ be such that $A_1^+ \leq A_2^+$ component-wise. If $A - A_1^+$ and $A - A_2^+$ are Metzler, then the maximal eigenvalue of $A - A_1^+$ is larger or equal to the maximal eigenvalue of $A - A_2^+$.

The latter Lemma will play a key role in the proof of the next result.

Theorem 3.2.1. Given a Metzler matrix $A \in \mathbb{R}^{n\times n}$ and a nonnegative input matrix $b \in \mathbb{R}_+^n$ of rank one, let $k_s = [k_s^1\ ...\ k_s^n] \in \mathbb{R}_+^{1\times n}$ be defined by

$$\begin{aligned} k_s^i &> \frac{a_{ii}}{b_i},\ \text{if } b_j = 0, \forall j \neq i \\ k_s^i &= \min_{j\neq i, b_j\neq 0}\{\frac{a_{ji}}{b_j}\},\ \text{else.} \end{aligned}$$

Then:

(i) there exists a matrix $k \in \mathbb{R}^{1\times n}$ such that $A - bk$ is a stable Metzler matrix if and only if $A - bk_s$ is a stable Metzler matrix;

(ii) moreover, if $b_i \neq 0$ and $b_j = 0$ for all $j \neq i$, $j, i \in \{1, ..., n\}$, then there always exists a $k \in \mathbb{R}_+^{1\times n}$ such that $A - bk$ is a stable Metzler matrix if and only if the submatrix A^i is stable.

Proof. The combination of Lemma 3.2.2(i), Lemma 3.2.3, and the proof in [3] prove the first statement (i).

The proof for the second statement (ii) is given next.

Let us first assume, without loss of generality, that $b_1 \neq 0$.

($\Rightarrow$) If there exists a $k \in \mathbb{R}_+^{1\times n}$ such that $A - bk$ is a stable Metzler matrix, then since

$b_j = 0$ for all $j \neq 1$, we have that the submatrices $(A - bk)^1$ and A^1 are equal, and by Lemma 3.2.1 A^1 must be stable.

($\Leftarrow$) Now since the submatrix A^1 is stable then there exists a $d' = [d_2 \ldots d_3] \in$ interior$(\mathbb{R}_+^{n-1})$ such that $-A^1 d' \in$ interior$(\mathbb{R}_+^{n-1})$ by Lemma 3.2.1, i.e. for all $j \in \{2, \ldots, n\}$

$$-(a_{j2}d_2 + a_{j3}d_3 + \ldots + a_{jn}d_n) > \delta_j.$$

Let $\delta = \underset{i=2,\ldots,n}{min}\{\delta_j\}$ and $a = \underset{j=2,\ldots,n}{max}\{a_{j1}\}$. Now, if $a = 0$, then let $d_1 = 1$; otherwise, let $0 < d_1 < \frac{\delta}{a}$. With the latter choice we can satisfy (3.2) for all rows $2, \ldots, n$. What remains is the first row. However, our first result with d set as above and $k_s^i = 0$ for $i \neq 1$ results in:

$$\begin{aligned} -(a_{11} - k_s^1 b_1)d_1 - (a_{12}d_2 + \ldots + a_{1n}d_n) \geq \\ k_s^1 b_1 d_1 - (|a_{11}|\, d_1 + a_{12}d_2 + \ldots + a_{1n}d_n) > 0; \end{aligned}$$

thus, we can always set

$$k_s^1 > \frac{|a_{11}|\, d_1 + a_{12}d_2 + \ldots + a_{1n}d_n}{b_1 d_1} \geq 0$$

and

$$k_s^i = 0, \quad i = 2, \ldots, n,$$

which gives us the needed result of

$$-(a_{11} - k_s^1 b_1)d_1 - (a_{12}d_2 + \ldots + a_{1n}d_n) > 0$$

and by Lemma 3.2.1 we have the desired outcome.

$\square$

The latter Theorem provides a very easy way of checking the stabilization existence condition; moreover, if the condition is met, a stabilizing gain matrix k_s is provided. Let us next illustrate the power of the latter Theorem via an example.

Example 3.2.1. Consider the following three-dimensional unstable single input positive system:

$$\dot{x} = \begin{bmatrix} -1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 1 & 0.5 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u.$$

We now invoke Theorem 3.2.1 (i) to calculate our gain matrix $k_s = [k_s^1 \; k_s^2 \; k_s^3]$:

1. for $i = 1$
$$k_s^1 = \min_{j \neq 1, b_j \neq 0} \{\frac{a_{31}}{b_3}\} = 2$$

2. for $i = 2$
$$k_s^2 = \min_{j \neq 2, b_j \neq 0} \{\frac{a_{12}}{b_1}, \frac{a_{32}}{b_3}\} = 1$$

3. for $i = 3$
$$k_s^3 = \min_{j \neq 3, b_j \neq 0} \{\frac{a_{13}}{b_1}\} = 1,$$

resulting in

$$k_s = [2 \; 1 \; 1].$$

The closed loop system matrix with the obtained k_s is

$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 1 & 0.5 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -0.5 \end{bmatrix},$$

with eigenvalues $\{-3, \ -1, \ -0.5\}$.

$\triangle$

The next result deals with systems in controllable canonical form.

Corollary 3.2.1. Every unstable positive system (3.1) in controllable canonical form is not stabilizable.

Proof. Directly follows from Theorem 3.2.1 (ii), i.e.

$$A^n = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & \vdots & & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

which is not a stable matrix. $\square$

The next example brings out the latter Corollary.

Example 3.2.2. Consider the following unstable three-dimensional SISO positive system:

$$\dot{x} \ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 10 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u.$$

We now invoke Corollary 3.2.1 directly to state that the system is not stabilizable. Notice

$$A^3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

clearly not stable.

$\triangle$

Note that from the latter discussion, we clearly can conclude that stabilization in the LTI sense does not translate to stabilization in the positive LTI sense.

We now extend the results of stabilization to the output stabilization case.

Corollary 3.2.2. Given a Metzler matrix $A \in \mathbb{R}^{n\times n}$, a nonnegative input matrix $b \in \mathbb{R}^n_+$, and a nonnegative output matrix $c \in \mathbb{R}^{1\times n}_+$, then

(i) if $b_i c_i \neq 0$ for some $i \in \{1, ..., n\}$ and $b_k c_j = 0$, for all $j, k \neq i$, $j, k \in \{1, ..., n\}$, then there exists a $k_o \in \mathbb{R}$ such that $A - k_o bc$ is a stable Metzler matrix if and only if the submatrix A^i is stable;

(ii) else, let

$$k_o = \underset{j\neq i, b_i c_j \neq 0}{min} \{\frac{a_{ij}}{b_i c_j}\},$$

then there exists a $k \in \mathbb{R}$ such that $A - kbc$ is a stable Metzler matrix if and only if $A - k_o bc$ is a stable Metzler matrix.

Before the proof is presented, we point out that in Corollary 3.2.2(i) one can easily find k_o by doing a one-dimensional search in the positive direction (by Lemma 3.2.2 (i)).

We are now ready to prove Corollary 3.2.2.

Proof. In order to prove the first statement (i) we will assume without loss of generality that $b_1 c_1 \neq 0$. The proof follows the same procedure as that of Theorem 3.2.1 (ii) and is included for completeness.

($\Rightarrow$) If there exists a $k \in \mathbb{R}_+$ such that $A - kbc$ is a stable Metzler matrix, then since $b_1 c_j = 0$ and $b_j c_1 = 0$, i.e. $c_j = 0/b_j = 0$, for all $j \neq 1$, we have that the submatrices $(A - kbc)^1$ and A^1 are equal, and by Lemma 3.2.1 A^1 must be stable.

($\Leftarrow$) Now since the submatrix A^1 is stable then there exists a $d' = [d_2 \ ... \ d_3] \in$ interior$(\mathbb{R}^{n-1}_+)$

such that $-A^1 d' \in$ interior$(\mathbb{R}_+^{n-1})$ by Lemma 3.2.1, i.e. for all $j \in \{2, ..., n\}$

$$-(a_{j2}d_2 + a_{j3}d_3 + ... + a_{jn}d_n) > \delta_j.$$

Let $\delta = \underset{i=2,...,n}{min}\{\delta_j\}$ and $a = \underset{j=2,...,n}{max}\{a_{j1}\}$. Now if $a = 0$, let $d_1 = 1$; otherwise, let $0 < d_1 < \frac{\delta}{a}$, we can satisfy (3.2) for all rows $2, ..., n$. What remains is the first row. However, our first result with d set as above results in:

$$\begin{aligned} -(a_{11} - k_o b_1 c_1)d_1 - (a_{12}d_2 + ... + a_{1n}d_n) \geq \\ k_o b_1 c_1 d_1 - (|a_{11}|\, d_1 + a_{12}d_2 + ... + a_{1n}d_n) > 0; \end{aligned}$$

thus, we can always set

$$k_0 > \frac{|a_{11}|\, d_1 + a_{12}d_2 + ... + a_{1n}d_n}{b_1 c_1 d_1} \geq 0$$

which gives us the needed result of

$$-(a_{11} - k_o b_1 c_1)d_1 - (a_{12}d_2 + ... + a_{1n}d_n) > 0$$

and by Lemma 3.2.1 we have the desired outcome.

We now shift to point (ii). First, we know that k_o must satisfy:

$$0 < k_o \leq \underset{j \neq i, b_i c_j \neq 0}{min}\{\frac{a_{ij}}{b_i c_j}\}.$$

where $k_o > 0$ by Lemma 3.2.2(i) and

$$k_o \leq \underset{j \neq i, b_i c_j \neq 0}{min}\{\frac{a_{ij}}{b_i c_j}\}$$

due to the Metzler restriction placed on the closed-loop matrix. Now define $A_1^+ = \tilde{k}_o bc$, $\quad A_2^+ = k_o bc$ in Lemma 3.2.3, with $\tilde{k}_o \leq k_o \Rightarrow A_1^+ \leq A_2^+$ component-wise. It is now clear by Lemma 3.2.3 that the result follows. $\square$

We now return to Example 3.2.1 and illustrate Corollary 3.2.2.

Example 3.2.3. Consider the following unstable three-dimensional SISO positive system:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 1 & 0.5 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u \\ y &= [1 \;\; 0 \;\; 1]x \end{aligned}$$

We now invoke Corollary 3.2.2 to calculate our gain k_o:

$$k_o = \underset{j \neq i, b_i c_j \neq 0}{min} \{ \frac{a_{13}}{b_1 c_3}, \; \frac{a_{31}}{b_3 c_1} \} = 1.$$

The closed loop system matrix with the obtained k_o is

$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 1 & 0.5 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -0.5 \end{bmatrix},$$

with two stable eigenvalues $(-2.0515 \pm 0.8750i)$ and one unstable eigenvalue (0.6031), i.e. the system is not output stabilizable.

--- $\triangle$

The next logical step in the sequence is the extension of Theorem 3.2.1 and Corollary 3.2.2 to complete stabilization and complete output stabilization. The results are outlined below.

Corollary 3.2.3. Given the system (3.1) with a Metzler matrix $A \in \mathbb{R}^{n\times n}$, a nonnegative input matrix $b \in \mathbb{R}^n_+$ of rank one, a nonnegative output matrix $c \in \mathbb{R}^{1\times n}_+$ of rank one, and a nonnegative output input scalar $\tilde{d} \in \mathbb{R}_+ \setminus \{0\}$, let $k_s = [k_s^1 \ ... \ k_s^n] \in \mathbb{R}^{1\times n}_+$ be defined by

$$\begin{aligned} k_s^i &> \frac{a_{ii}}{b_i}, \text{ if } b_j = 0, \ \forall j \neq i \\ k_s^i &= \min_{j\neq i, b_j \neq 0} \{\frac{a_{ji}}{b_j}\}, \text{ else.} \end{aligned}$$

under the constraint

$$k_s^i \leq \frac{c_i}{\tilde{d}}, \ \forall i = 1, ..., n. \tag{3.3}$$

Then, (3.1) is completely stabilizable with a gain matrix $k \in \mathbb{R}^{1\times n}$ if and only if (3.1) is completely stabilizable with the gain matrix k_s defined above.

Proof. Here the proof follows the proof of Theorem 3.2.1, and the extra condition (3.3) follows from the fact that we need the output

$$y = cx + \tilde{d}u$$

to be positive for all time. Thus,

$$y \geq 0, \ \forall t \geq 0$$

if and only if

$$\begin{aligned} cx + \tilde{d}u &= cx + \tilde{d}(-k_s x) \\ &= (c - \tilde{d}k_s)x \\ &\geq 0, \end{aligned}$$

which is true if and only if

$$c - \tilde{d}k_s \geq 0,$$

component-wise, i.e.

$$k_s^i \leq \frac{c_i}{\tilde{d}}, \ \forall i = 1, ..., n.$$

□

It is worth pointing out that Theorem 3.2.1 (ii) cannot be extended to the complete stabilization case. An example illustrates this point below.

Example 3.2.4. Consider the following two-dimensional SISO positive system:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 5 & 0 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= [1 \ \ 0]x + u. \end{aligned}$$

Here, we notice that

$$A^1 = -1,$$

which is clearly stable and thus A is stabilizable; however, the system is not completely stabilizable, as in order to obtain complete stabilization we need to, simultaneously, satisfy the constraints:

$$k_s^1 > 5, \quad k_s^1 \leq 1,$$

with the first condition coming from stabilization and the second from output nonnegativity.

△

The next result extends Corollary 3.2.2 to complete output stabilization.

Corollary 3.2.4. Given the system (3.1) with a Metzler matrix $A \in \mathbb{R}^{n\times n}$, a nonzero nonnegative input matrix $b \in \mathbb{R}^n_+$, a nonzero nonnegative output matrix $c \in \mathbb{R}^{1\times n}_+$, and a nonzero nonnegative output input scalar $\tilde{d} \in \mathbb{R}_+ \setminus \{0\}$, then let

$$\overline{k}_o \quad = \quad \min_{j\neq i, b_i c_j \neq 0}\{\frac{a_{ij}}{b_i c_j}\}, \tag{3.4}$$

under the extra constraint

$$0 < \overline{k}_o < \frac{1}{\tilde{d}}.$$

Then there exists a $k \in \mathbb{R}$ such that (3.1) is completely output stabilizable if and only if

$$k_o = \frac{\overline{k}_o}{1 - \tilde{d}\overline{k}_o} > -\frac{1}{\tilde{d}}$$

completely output stabilizes (3.1).

Proof.

($\Leftarrow$) Trivial, set $k = k_o$.

($\Rightarrow$) Consider

$$y = cx + \tilde{d}u$$

with $u = -k_o y$, resulting in

$$\begin{aligned} y &= cx - \tilde{d}k_o y \\ y + \tilde{d}k_o y &= cx \\ y(1 + \tilde{d}k_o) &= cx \\ y &= \frac{c}{1 + \tilde{d}k_o}x \end{aligned} \tag{3.5}$$

implying that

$$\begin{aligned} (1+\tilde{d}k_o) &> 0 \text{ see Assumption 3.1.1} \\ k_o &> \frac{-1}{\tilde{d}}. \end{aligned}$$

Next,

$$\begin{aligned} \dot{x} &= Ax + bu \\ &= Ax + b(-k_o y) \\ &= Ax - k_o b\left(\frac{c}{1+\tilde{d}k_o}x\right) \\ &= \left(A - \frac{k_o}{1+k_o\tilde{d}}bc\right)x, \end{aligned}$$

with

$$\overline{k}_o = \frac{k_o}{1+k_o\tilde{d}},$$

which by Lemma 3.2.2 must be greater than 0, i.e. $\overline{k}_o > 0$. The second inequality,

$$\overline{k}_o < \frac{1}{\tilde{d}}$$

comes from the fact that if $\overline{k}_o \geq \frac{1}{\tilde{d}}$, then

1. (i) if $\overline{k}_o = \frac{1}{\tilde{d}}$, then k_o does not exist since

$$k_o = \frac{\overline{k}_o}{1-\tilde{d}\overline{k}_o}.$$

2. (ii) if $\overline{k}_o > \frac{1}{\tilde{d}}$, then we have the contradiction

$$-\frac{1}{\tilde{d}} < k_o = \frac{\overline{k}_o}{1-\tilde{d}\overline{k}_o} < \frac{\overline{k}_o}{-\tilde{d}\overline{k}_o} = -\frac{1}{\tilde{d}},$$

i.e.

$$-\frac{1}{\tilde{d}} < -\frac{1}{\tilde{d}}.$$

Thus, indeed we must satisfy

$$0 < \overline{k}_o < \frac{1}{\tilde{d}},$$

with condition (3.4) being justified by Corollary 3.2.2.

□

Finally, one cannot overlook that in the case of $\tilde{d} = 0$ complete stabilizability and stabilizability are equivalent (similarly for complete output stabilizability and output stabilizability).

The results presented thus far provide us with the necessary and sufficient conditions for both (complete) stabilization and (complete) output stabilization of SISO positive LTI systems. We next shift our focus to the presentation of results for the design of Luenberger observers for positive SISO LTI systems.

Observer Design

The following result has been covered in [15] and [3]; its inclusion in this subsection is strictly for completeness. However, below, we note an interesting observation that was not covered in the latter citations. Additionally, we come back to observer design in the sequel; thus, it's inclusion here is a necessity.

Theorem 3.2.2 ([15]). Given a Metzler matrix $A \in \mathbb{R}^{n\times n}$ and a nonnegative output matrix $c \in \mathbb{R}_+^{1\times n}$, let $l_o = [l_1 \; ... \; l_n]^T \in \mathbb{R}_+^n$ be defined by

$$\begin{aligned} l_i &> \frac{a_{ii}}{c_i}, \text{ if } c_j = 0, \forall j \neq i \text{ else} \\ l_i &= \min_{j\neq i, c_j \neq 0} \{\frac{a_{ij}}{c_j}\}. \end{aligned}$$

Then there exists a nonnegative matrix $l \in \mathbb{R}^n_+$ such that $A - lc$ is a stable Metzler matrix if and only if $A - l_o c$ is a stable Metzler matrix.

We note that as in the case of the stabilization theorem, we can add to the latter result the following lemma.

Lemma 3.2.4. Given a Metzler matrix $A \in \mathbb{R}^{n \times n}$ and a nonnegative output matrix $c \in \mathbb{R}^{1\times n}_+$, if $c_i \neq 0$ and $c_j = 0$ for all $j \neq i$, $j, i \in \{1, ..., n\}$, then there exists an $l \in \mathbb{R}^n_+$ such that $A - lc$ is a stable Metzler matrix if and only if the submatrix A^i is stable.

We have now presented necessary and sufficient results for both stabilization and the existence of Luenberger observers. What remains to be shown is the design of observer based stabilization. The interest of course being that regardless of the input into the system, we would like to maintain nonnegativity of the states and outputs for the plant; this we present in the sequel.

State and Observer based Stabilization

In LTI systems the separation property applies, i.e. first designing a stabilizing matrix k, then designing, independently, an observer by finding the matrix gain l, and finally combining the two to produce an adequate controller. We would like to determine if this is also possible in the case of positive LTI systems, something that to date has not been covered anywhere in the literature. The problem, of course, is ensuring that we do not violate any nonnegativity constraint on the states or outputs of our plant. Here we present the details behind the separation property for positive LTI systems.

Recall that a Luenberger observer is of the form:

$$\dot{\hat{x}} = A\hat{x} + bu + l(y - \hat{y}), \tag{3.6}$$

where $\hat{y} = c\hat{x} + \tilde{d}u$. Now in our system plant, we assume that $u = -k_s\hat{x}$, where k_s has been found so that $A - bk_s$ is Metzler and stable, and $\hat{x}$ is the estimate of the state x, i.e.

$$\begin{aligned} \dot{x} &= Ax + bu && (3.7) \\ &= (A - bk_s)x + bk_s(x - \hat{x}), && (3.8) \end{aligned}$$

so if $e_x := x - \hat{x} \geq 0$ component-wise and $k_s \geq 0$, then $x \in \mathbb{R}^n_+$ for all time. Note that from Lemma 3.2.2 and Theorem 3.2.1, we can always find a nonnegative gain matrix k_s so long as the system is stabilizable. Also,

$$\begin{aligned} \dot{e}_x &= Ax + bu - A\hat{x} - bu - l(y - \hat{y}) && (3.9) \\ &= (A - lc)e_x. && (3.10) \end{aligned}$$

Now, if initially e_x is nonnegative, then the problem is solved. This causes no problem as the initial condition of $\hat{x}$ is at our disposal and we can simply choose it to be zero.

If in the latter discussion $\tilde{d} \neq 0$, then we must ensure that k_s has been chosen in such a way that the system of interest is completely stabilizable, i.e. with $\tilde{d} \neq 0$ we have the extra constraint on k_s:

$$\begin{aligned} y &= cx + \tilde{d}u \\ &= cx - \tilde{d}k_s\hat{x} + \tilde{d}k_s x - \tilde{d}k_s x \\ &= (c - \tilde{d}k_s)x + \tilde{d}k_s(x - \hat{x}) \\ &= (c - \tilde{d}k_s)x + \tilde{d}k_s e, \end{aligned}$$

which results in the extra constraint placed by complete stabilizability

$$c - \tilde{d}k_s \geq 0.$$

In conclusion, the separation property of positive LTI systems can be carried out, with the assumption that $\hat{x}_0$ (initial condition of $\hat{x}$) is zero, the system is positive observable (i.e. $A - lc$ is Metzler stable), and either stabilizable (if $\tilde{d} = 0$) or completely stabilizable (if $\tilde{d} \neq 0$). We note, however, that the closed loop poles of $A - lc$ cannot be as freely chosen as in SISO LTI systems, i.e. we normally would like to have "fast" observer poles, but this may not be as freely done in the positive LTI SISO case.

This completes the study of stabilization, scalar output stabilization and observer based stabilization for SISO systems.

3.3 Necessary and Sufficient Conditions: MIMO case

This section extends the results of the previous subsection of SISO positive LTI systems to the MIMO case.

State Stabilization

This subsection will outline the differences between MIMO and SISO cases by outlining certain special cases of MIMO systems that encapsulate the solution that was observed in the latter section, and by providing the main necessary and sufficient results of the chapter. However, we first turn to Lemma 3.2.2 and Remark 3.2.1 by extending the result to the MIMO case. Namely, in Lemma 3.2.2 and Remark 3.2.1, it was noted that in SISO and in MIMO positive systems the gain matrix k_s (K_s) cannot be nonpositive. The following Corollary is a direct extension of Lemma 3.2.2 for the MIMO case.

Corollary 3.3.1.

(i) There does not exist a nonpositive gain matrix $K_s \in \mathbb{R}_-^{m\times n}$ ($K_o \in \mathbb{R}_-^{m\times r}$) such that an unstable Metzler matrix $A \in \mathbb{R}^{n\times n}$, an input matrix $B \in \mathbb{R}_+^{n\times m}$, and an output matrix $C \in \mathbb{R}_+^{r\times n}$ can be stabilized (output stabilized); moreover,

(ii) any column k_s^i of K_s that is nonpositive can be replaced by some column $\overline{k}_s^i$ which contains at least one strictly positive entry.

Proof. See the proof of Lemma 3.2.2 (i) and replace k_s with K_s. The result follows directly for both (i) and (ii). $\square$

Although we have extended Lemma 3.2.2 (i) to the MIMO case above, the same cannot be done with Lemma 3.2.2 (ii) and (iii). An example is provided below to illustrate this point.

Example 3.3.1. Consider the system:

$$\dot{x} = \begin{bmatrix} 0.5 & 0.51 & 0 \\ 9 & -\frac{18}{700} & 0 \\ 4 & 0.5 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 3 & 0 \\ 2 & 1 \end{bmatrix} u. \tag{3.11}$$

In this case, it turns out that there does not exist a nonnegative K_s (we will come back to the reason for this, later in the chapter) which stabilizes (3.11), yet the gain matrix:

$$K_s = \begin{bmatrix} -1 & -0.005 & 0 \\ 6 & 0.51 & 0 \end{bmatrix}$$

yields $\sigma(A - BK_s) = \{-5.5, \ -0.0107, \ -1\}$, with the closed loop system being Metzler:

$$\begin{bmatrix} 0.5 & 0.51 & 0 \\ 9 & -\frac{18}{700} & 0 \\ 4 & 0.5 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 3 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & -0.005 & 0 \\ 6 & 0.51 & 0 \end{bmatrix} = \begin{bmatrix} -5.5 & 0 & 0 \\ 12 & -0.0107 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

To disprove Lemma 3.2.2 (iii) for the MIMO case we can take the latter example with

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and obtain the same closed loop matrix $A - BK_s$ with the additional constraint

$$C - DK_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & -0.005 & 0 \\ 6 & 0.51 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0.005 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$\triangle$

Through the latter example we already see a main distinguishing feature between SISO and MIMO systems, i.e. in SISO systems we can concentrate on nonnegative gain matrices, while in MIMO systems the same is not true.

For the remainder of this subsection, the concentration will be on the stabilization and output stabilization problems. The treatment of complete stabilization and complete output stabilization will follow.

Next, we will utilize Lemma 3.2.1 to provide the necessary and sufficient condition for stabilization of positive LTI systems via the use of bilinear inequalities. For the rest of the chapter, we will assume that system (3.1) is unstable, i.e. the A matrix is not Hurwitz.

Corollary 3.3.2. System (2.2) is stabilizable by some gain matrix $K \in \mathbb{R}^{m \times n}$ if and only if there exists a matrix $\mathcal{K} \in \mathbb{R}^{m \times n}$ and a vector $d \in \ interior(\mathbb{R}^n_+)$, such that $A - B\mathcal{K}$ is Metzler and the following bilinear matrix inequality problem

$$-(A - B\mathcal{K})d \in \ interior(\mathbb{R}^n_+) \tag{3.12}$$

is feasible.

Proof. From Lemma 3.2.1 we have that a Metzler matrix $\mathcal{A} \in \mathbb{R}^{n \times n}$ is stable if and only if

$$\exists d \in \ \text{interior}(\mathbb{R}^n_+) \text{ such that } -\mathcal{A}d \in \ \text{interior}(\mathbb{R}^n_+).$$

by letting $\mathcal{A} = A - B\mathcal{K}$ the result follows. $\square$

The above Corollary, stemming from Lemma 3.2.1, definitely provides the necessary and sufficient results without having to look at the spectrum of the closed loop system. Unfortunately, in general, the feasibility of bilinear matrix inequalites has been shown by Toker and Özbay [82] to be an $\mathcal{N}P$-hard problem. Thus, in the remainder of this subsection, it will be shown how Corollary 3.3.2 can be reduced to a more subtle and checkable solution.

Before we present the main result of this subsection, a special case of MIMO positive LTI systems, where the B matrix in (3.1) has *unitary rows* and is full rank, i.e. each row b_i, $i = 1, ..., n$, of B has one non-zero entry

$$\exists\ j \in \{1, ...m\} \ \ b_{ij} \neq 0 \text{ and } \forall\ k \neq j \ \ b_{ik} = 0$$

will be considered.

Theorem 3.3.1. Consider system (3.1), where B has *unitary rows* and rank(B)=m. Let $K_s \in \mathbb{R}^{m\times n}$ be defined by:

1. if $b_{ji} = 0, \forall j \neq i$, then

$$\begin{aligned} k_s^{ii} &> \frac{a_{ii}}{b_{ii}}, \text{ and} \\ k_s^{ij} &= \frac{a_{ij}}{b_{ii}} \end{aligned} \tag{3.13}$$

2. else

$$k_s^{rj} = \min_{j\neq i, b_{ir}\neq 0} \{\frac{a_{ij}}{b_{ir}}\}. \tag{3.14}$$

Then there exists a matrix $K \in \mathbb{R}^{m\times n}$ such that $A - BK$ is a stable Metzler matrix if and only if $A - BK_s$ is a stable Metzler matrix.

Proof.

($\Leftarrow$) Simply set $K = K_s$.

($\Rightarrow$) Assume that there exists a gain matrix K which stabilizes. Without loss of generality assume that the unitary columns of B, i.e. a column b_j is unitary if it contains one non-zero entry

$$\exists\ i \in \{1, ...n\} \quad b_{ji} \neq 0 \text{ and } \forall\ k \neq i \quad b_{ki} = 0,$$

are $1, .., q$, with $q \leq m$. Due to (3.13) and the assumption of unitary rows we can now express the system as:

$$\dot{x} = Ax - B_1 K_s^1 x - B_2 K_s^2 x,$$

with B_1 containing only unitary columns and B_2 containing nonunitary columns with appropriate dimensions. More precisely, we have (3.15).

$$\begin{aligned}
\dot{x} \; &= \begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{q1} & a_{q2} & \ldots & a_{qn} \\ a_{(q+1)1} & a_{(q+1)2} & & a_{(q+1)n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} x - \begin{bmatrix} b_{11} & 0 & & 0 \\ 0 & b_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b_{qq} \\ 0 & 0 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 0 \end{bmatrix} \begin{bmatrix} k_s^{11} & \frac{a_{12}}{b_{11}} & \ldots & \frac{a_{1n}}{b_{11}} \\ \frac{a_{21}}{b_{22}} & k_s^{22} & & \frac{a_{2n}}{b_{22}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{q1}}{b_{qq}} & \frac{a_{q2}}{b_{qq}} & & k_s^{qq} \end{bmatrix} x \\
&\quad - \begin{bmatrix} 0 & 0 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 0 \\ b_{(q+1)(q+1)} & b_{(q+1)(q+2)} & \ldots & b_{(q+1)m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n(q+1)} & b_{n(q+2)} & \ldots & b_{nm} \end{bmatrix} \begin{bmatrix} k_s^{(q+1)1} & k_s^{(q+1)2} & \ldots & k_s^{(q+1)n} \\ \vdots & \vdots & \ddots & \vdots \\ k_s^{m1} & k_s^{m2} & & k_s^{mn} \end{bmatrix} x \\
&= \begin{bmatrix} a_{11} - b_{11}k_s^{11} & 0 & & 0 \\ 0 & a_{22} - b_{22}k_s^{22} & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \\ a_{(q+1)1} & a_{(q+1)2} & & a_{(q+1)n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} x - \begin{bmatrix} 0 & 0 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 0 \\ b_{(q+1)(q+1)} & b_{(q+1)(q+2)} & \ldots & b_{(q+1)m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n(q+1)} & b_{n(q+2)} & \ldots & b_{nm} \end{bmatrix} \times \\
&\qquad \begin{bmatrix} k_s^{(q+1)1} & k_s^{(q+1)2} & \ldots & k_s^{(q+1)n} \\ \vdots & \vdots & \ddots & \vdots \\ k_s^{m1} & k_s^{m2} & & k_s^{mn} \end{bmatrix} x. \qquad (3.15)
\end{aligned}$$

Let

$$B_2 = \begin{bmatrix} b_{(q+1)(q+1)} & b_{(q+1)(q+2)} & \dots & b_{(q+1)m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n(q+1)} & b_{n(q+2)} & \dots & b_{nm} \end{bmatrix}. \tag{3.16}$$

Now clearly the upper $(q \times n)$ matrix consists of a Metzler and stable matrix and zero entries; thus, if there exists a

$$K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$$

which stabilizes with some K_1 then there definitely exists a

$$\overline{K} = \begin{bmatrix} K_s^1 \\ K_2 \end{bmatrix}$$

which stabilizes, by the argument above, where

$$K_s^1 = \begin{bmatrix} k_s^{11} & \frac{a_{12}}{b_{11}} & \dots & \frac{a_{1n}}{b_{11}} \\ \frac{a_{21}}{b_{22}} & k_s^{22} & & \frac{a_{2n}}{b_{22}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{q1}}{b_{qq}} & \frac{a_{q2}}{b_{qq}} & & k_s^{qq} \end{bmatrix}.$$

We now show that K_2 can be replaced by K_s^2.

By Lemma 3.2.1, we can concentrate on the stabilization of the lower submatrix

$$A^{1,\dots,q};$$

thus, we take

$$\overline{K} = \begin{bmatrix} & K_s^1 & \\ K_{21} & & K_{22} \end{bmatrix}$$

where $K_2 = [K_{21} \quad K_{22}]$, and since $\overline{K}$ stabilizes then

$$A^{1,\ldots,q} - B_2K_2 \tag{3.17}$$

must be stable and Metzler. Therefore, there exists a $d_2 \in interior(\mathbb{R}^n_+)$, by Lemma 3.2.1, such that

$$-(A^{1,\ldots,q} - B_2K_2)d_2 \in interior(\mathbb{R}^n_+).$$

Now, clearly each entry of K_2 cannot exceed the constraint (3.14) due to the Metzler property, thus either each entry is equal to that of (3.14) or is less. If an entry is less, then by the same procedure as used in the proof of Lemma 3.2.2 with (3.17) under consideration, it follows that K_2 can be replaced by K_s^2, completing the proof.

□

Notice the close resemblance of Theorem 3.3.1 to that of the results of Theorem 3.2.1 for SISO systems. In fact, the results for SISO systems are just a special case of Theorem 3.3.1 due to the fact that, clearly, the b matrix in the SISO case has *unitary rows*. Although, at first, it may appear that Theorem 3.3.1 is restrictive (w.r.t. the B matrix), it has direct application in compartmental systems. Theorem 3.3.1 uses the idea of unitary rows, which intuitively just means that there is at most one input controlling a state (or compartment). One can argue that it may be redundant, in compartmental systems, to have more than one controlling input per compartment.

We illustrate the results of Theorem 3.3.1 in the next example.

Example 3.3.2. Consider the following unstable positive system:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u.$$

Clearly, the rows of B are unitary so we may proceed to use Theorem 3.3.1 to find a legitimate K_s if it exists. By (3.13) and (3.14) we end up with:

1. for k_s^{11}, $r=1$, $j=1$
$$k_s^{11} = \min_{i\neq 1, b_{i1}\neq 0} \left\{ \frac{a_{31}}{b_{31}} = 1 \right\} = 1$$

2. for k_s^{12}, $r=1$, $j=2$
$$k_s^{12} = \min_{i\neq 2, b_{i1}\neq 0} \left\{ \frac{a_{12}}{b_{11}} = 1,\ \frac{a_{32}}{b_{31}} = 0 \right\} = 0$$

3. for k_s^{13}, $r=1$, $j=3$
$$k_s^{13} = \min_{i\neq 3, b_{i1}\neq 0} \left\{ \frac{a_{13}}{b_{11}} = 1 \right\} = 1$$

4. for $[k_s^{21}\ k_s^{22}\ k_s^{23}]$ (by Equation (3.13)) we have
$$k_s^{21} = \frac{a_{21}}{b_{22}},\ k_s^{22} > 0,\ k_s^{23} = \frac{a_{23}}{b_{22}},$$

resulting in
$$K_s = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix},$$
where $k_s^{22} > 0$ is chosen arbitrarily to equal one. The resultant closed loop matrix is:
$$A_c = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix},$$
which is clearly Metzler and stable.

△

Theorem 3.3.1 provides the necessary and sufficient (checkable) conditions for a special *row unitary* MIMO positive LTI system to be stable; we now return to the general solution. Our first goal will be to utilize Corollary 3.3.2 for the case when each column of the B matrix contains at least two (or more) nonzero entries, i.e. the columns of B are all *nonunitary*.

Lemma 3.3.1. Consider system (3.1), where rank(B)=m and each column of B is *nonunitary*. Let $\overline{K} \in \mathbb{R}_+^{m\times n}$, and define n polytopes P_i, $i = 1, ..., n$ by the set of inequalities:

$$\begin{array}{rcl}
a_{1i} - b_{11}\overline{k}_{1i} - ... - b_{1m}\overline{k}_{mi} & \geq & 0 \\
\vdots & & \vdots \\
a_{(i-1)i} - b_{(i-1)1}\overline{k}_{1i} - ... - b_{(i-1)m}\overline{k}_{mi} & \geq & 0 \\
a_{ii} - b_{i1}\overline{k}_{1i} - ... - b_{im}\overline{k}_{mi} & \leq & 0 \\
a_{(i+1)i} - b_{(i+1)1}\overline{k}_{1i} - ... - b_{(i+1)m}\overline{k}_{mi} & \geq & 0 \\
\vdots & & \vdots \\
a_{ni} - b_{n1}\overline{k}_{1i} - ... - b_{nm}\overline{k}_{mi} & \geq & 0.
\end{array}$$

Let V_i be the accompanying set of vertices for P_i.

Then the system (3.1) is stabilizable with a nonnegative gain if and only if there exists a set of vertices $k_s^i \in V_i$, such that

$$A - BK_s, \tag{3.18}$$

where $K_s = [k_s^1 \; ... \; k_s^n]$, is stable.

Before we prove the result a few remarks are in order.

Remark 3.3.1. We note that in Lemma 3.3.1 we restricted $K_s \in \mathbb{R}_+^{m\times n}$, but there is no need to make that restriction and, in fact, any lower bound $k_l \in \mathbb{R}_-$ for the elements of K_s can be chosen!

We point out that the finite set of inequalities in Lemma 3.3.1 are a result of Metzler properties of the $A - B\overline{K}$ matrix. Also, the ii^{th}-inequalities carry a less or equal to zero symbol due to the fact that for positive LTI systems all submatrices must be stable, including the ii^{th} entries (see Lemma 3.2.1). Thus, we can automatically drop any vertices that fall on the hyperplane generated by the ii^{th}-inequality. If we did choose a vertex on the hyperplane generated by the ii^{th}-inequality, then the ii^{th} entry of the closed loop matrix would result in 0, clearly not stable.

Additionally, any vertex that has only negative components can also be dropped, by a similar argument that was made in the SISO case for the stabilization vector k_s and stabilization constant k_o, which had to be nonnegative, and by Corollary 3.3.1 (ii).

It is to be noted that a stabilizing gain matrix K_s, if it exists, can be found by enumeration, on checking the finite list of K_s vertices which result from examining each of the vertices of (3.18).

We are now ready to prove the result.

Proof. First, since by assumption B is full rank and has at least two nonzero elements in each column, P_i is indeed a polytope and the set of vertices V_i exists.

Next, by contradiction assume that there exists a K such that $A - BK$ is stable, yet no K_s (see (3.18)) stabilizes. Since a K exists, then $(A-BK)^T$ must be stable and there exists a $d \in interior(\mathbb{R}^n_+)$, by Corollary 3.3.2, such that:

$$-(A - BK)^T d = -(A^T - K^T B^T)d \in interior(\mathbb{R}^n_+). \tag{3.19}$$

Assume, without loss of generality, that the first column (k_1) of K does not belong to V_1, i.e. $k_1 \notin V_1$. Now taking equation (3.19), we can set up the following linear programming problem:

$$\max_{k_1} -(a_1 - k_1 B^T)d$$

subject to the constraints:

$$\begin{aligned} a_{11} - b_{11}k_{11} - ... - b_{1m}k_{m1} &\leq 0 \\ a_{21} - b_{21}k_{11} - ... - b_{2m}k_{m1} &\geq 0 \\ \vdots \quad & \quad \vdots \\ a_{n1} - b_{n1}k_{11} - ... - b_{nm}k_{m1} &\geq 0 \end{aligned}$$

where $k_1 = [k_{11} \ ...k_{m1}]$, $a_1 = [a_{11} \ ... \ a_{n1}]$, and the maximizing function coming from the condition of Corollary 3.3.2. It is now well known that the maximum of a linear programming problem occurs at a vertex, i.e. it is both necessary and sufficient to just check the vertices of P_1. We can continue this process for each P_i, $i = 2, ..., n$, resulting in a contradiction that no K_s (see (3.18)) stabilizes. This completes the proof. □

We now illustrate the importance and power of Lemma 3.3.1 via returning to Example 3.3.1.

Example 3.3.3. Consider the system of Example 3.3.1. We first restrict the gain matrix K_s to be nonnegative, which will illustrate that indeed no nonnegative gain matrix K_s can solve the stabilization problem, as pointed out in Example 3.3.1. Afterward, we allow the gain matrix to take on negative values, bounded from below, arbitrarily by -1, and thus resulting in the stabilization of system (3.11). Let us first define the polytopes $P_1, \ P_2, \ P_3$.

P_1 :

$$\begin{array}{lcc} 0.5 - \overline{k}_{21} & \leq & 0 \\ 9 - 3\overline{k}_{11} & \geq & 0 \\ 4 - 2\overline{k}_{11} - \overline{k}_{21} & \geq & 0 \\ \overline{k}_{i1} & \in & \mathbb{R}_+, \ i \in \{1, 2\} \end{array}$$

The resultant $V_1 = \{[0 \ 4]^T\}$. Notice that by the statement of Remark 3.3.1, i.e. no vertices on the hyperplane $0.5 - \overline{k}_{21} = 0$ qualify as possible candidates, we can omit all other vertices. Next, we define P_2 and P_3:

P_2 :

$$\begin{array}{rcl} 0.51-\overline{k}_{22} & \geq & 0 \\ -\frac{18}{700}-3\overline{k}_{12} & \leq & 0 \\ 0.5-2\overline{k}_{12}-\overline{k}_{22} & \geq & 0 \\ \overline{k}_{i2} & \in & \mathbb{R}_{+},\ i\in\{1,2\} \end{array}$$

and P_3 :

$$\begin{array}{rcl} 0-\overline{k}_{23} & \geq & 0 \\ 0-3\overline{k}_{13} & \geq & 0 \\ -1-2\overline{k}_{13}-\overline{k}_{23} & \leq & 0 \\ \overline{k}_{i3} & \in & \mathbb{R}_{+},\ i\in\{1,2\}. \end{array}$$

The resultant sets of vertices are: $V_2=\{[0\ 0.5]^T,\ [0\ 0]^T,\ [0.25\ 0]^T\}$ and $V_3=\{[0\ 0]^T\}$.

We now can show by enumeration that the most advantageous gain matrix generated from $V_1,\ V_2,\ V_3$ is:

$$K_s=\begin{bmatrix} 0 & 0 & 0 \\ 4 & 0.5 & 0 \end{bmatrix},$$

which results in closed loop eigenvalues of $\{-3.526,0,-1\}$. Notice that indeed a nonnegative gain matrix does not stabilize system (3.11). We now lift the restriction of nonnegative gain matrices and instead bound all entries of K_s to be greater or equal to -1. The definition of $P_1,\ P_2,\ P_3$ is identical, except for the bound condition. The new sets of vertices are:

$$\begin{array}{rcl} V_1 & = & \{[-1\ 6]^T\} \\ V_2 & = & \{[0.25\ 0]^T,\ [-0.005\ 0.51]^T\} \\ V_3 & = & \{[0\ 0]^T\}, \end{array}$$

which results in the same K_s as in Example 3.3.1, where the closed loop system matrix is given by:

$$(A - BK_s) = \begin{bmatrix} -5.5 & 0 & 0 \\ 12 & -0.0107 & 0 \\ 0 & 0 & -1. \end{bmatrix}.$$

$\triangle$

The next step toward the full result of stabilization is to consider the case when the input B matrix has at least one column with just one nonzero entry, i.e. *unitary columns*[1]. Recall that we have already summarized this case in the SISO situation (see Theorem 3.2.1 (ii)) and thus we are only extending it to the MIMO case below.

Corollary 3.3.3. Consider system (3.1), where rank(B)=m and there exists a unitary column (b_i) of B. Without loss of generality assume $i = 1$. Then system (3.1) is stabilizable if and only if there exists a gain matrix $K_s^{2,\dots,n} \in \mathbb{R}^{m\times(n-1)}$ such that $(A - B[0_{n\times 1}\ K_s^{2,\dots,n}])^1$ is stable.

Proof.
($\Rightarrow$) Trivial.
($\Leftarrow$) Let the candidate stabilizing matrix be $K_s = [k_s^1\ K_s^{2,\dots,n}]$, where $K_s^{2,\dots,n} \in \mathbb{R}^{m\times(n-1)}$ stabilizes A^1, i.e.

$$A_c = \left[\begin{array}{c|c} a_{11} & a_{12}\ \dots\ a_{1n} \\ \hline a_{21} & \\ \vdots & A^1 \\ a_{n1} & \end{array}\right] - B[0_{n\times 1}\ K_s^{2\dots n}]$$

with A_c^1 stable. Next, by similar arguments as used in the proof of Corollary 3.2.2, we can set the first element k_s^{11} of k_s^1 arbitrarily (positive) large (all other elements of k_s^1 can be set to any value), as there are no constraints on its size; thus if A^1 is stabilizable then so is A. $\square$

One remark regarding the latter Corollary is in order, namely, the value of k_s^{11} may

[1]without loss of generality the case of zero columns is neglected

turn out to be very large, *although it always exists so long as A^1 is stabilizable.* Thus, the designer of the stabilizing matrix may want to place an upper bound $k_u \in \mathbb{R}_+$ on the value of k_s^{11}; however, the resultant bound, clearly, may not be sufficiently large enough to provide stability. As it stands, k_s^{11}, in Corollary 3.3.3, can be found via a one-dimensional search, which translates to setting (and possibly resetting) an upper bound $k_u \in \mathbb{R}_+$.

The next logical step is to combine the results of Lemma 3.3.1 and Corollary 3.3.3.

Theorem 3.3.2. Consider system (3.1), where rank(B)=m, there exists one (or more) row(s) of B which is nonunitary, and (without loss of generality) where columns $1, ..., j$ of B are unitary and columns $j+1, ..., m$ are not. Assume the elements of K_s are bounded from below and above by $k_l \in \mathbb{R}_-$ and $k_u \in \mathbb{R}_+$, respectively.

Let $\overline{K} \in \mathbb{R}^{m\times n}$. Set $\overline{k}_{rr} = k_u$ for $r = 1, ..., j$ and define n polytopes P_i, $i = 1, ..., n$ by the set of inequalities:

$$
\begin{array}{rcl}
a_{1i} - b_{11}\overline{k}_{1i} - ... - b_{1m}\overline{k}_{mi} & \geq & 0 \\
\vdots & & \vdots \\
a_{(i-1)i} - b_{(i-1)1}\overline{k}_{1i} - ... - b_{(i-1)m}\overline{k}_{mi} & \geq & 0 \\
a_{ii} - b_{i1}\overline{k}_{1i} - ... - b_{im}\overline{k}_{mi} & \leq & 0 \\
a_{(i+1)i} - b_{(i+1)1}\overline{k}_{1i} - ... - b_{(i+1)m}\overline{k}_{mi} & \geq & 0 \\
\vdots & & \vdots \\
a_{ni} - b_{n1}\overline{k}_{1i} - ... - b_{nm}\overline{k}_{mi} & \geq & 0.
\end{array}
$$

Let V_i be the accompanying set of vertices for P_i.

Then, there exists a stabilizing matrix, for (3.1), with elements bounded by (k_l, k_u) if and only if there exists a set of vertices $k_s^i \in V_i$, such that $A - BK_s$, where $K_s = [k_s^1 \; ... \; k_s^n]$, is stable.

The proof for the above Theorem is omitted, as it is a direct consequence of the set of results provided thus far.

We are now ready to outline an algorithm that will generate a stabilizing gain matrix for (3.1) or let us know that (3.1) is not stabilizable.

Algorithm 3.3.1 (Stabilization of MIMO positive LTI systems). Consider an unstable Metzler matrix A and a nonnegative matrix B, with $\text{rank}(B) = m$.

1. If each row of B is unitary, apply the results of Theorem 3.3.1 to find a stabilizing gain matrix K_s and return "(A,B)-stabilizable" and the gain matrix K_s; if not stabilizable, then return "(A,B)-not stabilizable", and stop; otherwise (i.e. if there exists a row (or more) of B that is not unitary) proceed to the next step.

2. Provide lower and upper bounds on the entries of the gain matrix K_s: $(k_l,\ k_u)$. Proceed to the next step.

3. If column i is unitary, then set entry $k_s^{ii} = k_u$ and use the results of Theorem 3.3.2 to find the sets of vertices $V_1,\ V_2, ..., V_n$. Proceed to next step.

4. Find a stabilizing gain matrix $K_s = [k_s^1\ ...\ k_s^n]$, where $k_s^i \in V_i$, for $i = 1, ..., n$, using enumeration. If successful return "(A,B)-stabilizable" and the gain matrix is "K_s", otherwise return 'not stabilizable under bounds (k_l, k_u)".

In the above algorithm, the bounds on the entries of K_s are used simply to avoid dealing with unboundedness, and have been already discussed after Lemma 3.3.1 and Corollary 3.3.3. Clearly it is practically infeasible to have infinite gains and thus the designer can choose appropriate bounds according to their application. The latter algorithm can be slightly modified by separating the system into three components: one subsystem with unitary columns and rows, one subsystem with just unitary columns, and the final subsystem with nonunitary columns, i.e.

$$\dot{x} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} x - \begin{bmatrix} B_{11} & 0 & 0 \\ 0 & B_{22} & 0 \\ 0 & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} x,$$

where, without loss of generality, we can assume that rows and columns $1, ..., k$ of B are unitary, columns $k+1, ..., j$ of B are nonunitary, and columns $j+1, ..., m$ are unitary but the corresponding rows are nonunitary. The latter separation allows the algorithm to deal with (possibly) smaller stabilizing subroutines dependent on the number of unitary columns, i.e. the larger amount of unitary columns the smaller the subsystem to stabilize, this is justified by Corollary 3.3.3. Additionally, due to the results of Theorem 3.3.1 and Remark 3.3.1 one can add several subroutines to speed up the enumeration process for each component of K_s.

We point out that in the case of output stabilization, algorithm 3.3.1 can be repeated, except we now have to deal with K_o at once and not column by column, i.e. the Metzler constraints coming from the closed loop matrix $A_{co} = A - BK_oC$ result in only one polytope P and thus one accompanying set of vertices V. The extension to output stabilization can be found below.

Corollary 3.3.4. Consider system (3.1), where rank(B)=m, rank(C)=r, and $D = 0$. Assume the elements of K_o are bounded from below and above by $k_l \in \mathbb{R}_-$ and $k_u \in \mathbb{R}_+$, respectively. Define:

$$B = \begin{bmatrix} - & b_1 & - \\ - & b_2 & - \\ & \vdots & \\ - & b_n & - \end{bmatrix} \text{ and } C = \begin{bmatrix} | & | & & | \\ c_1 & c_2 & \dots & c_n \\ | & | & & | \end{bmatrix},$$

where b_i is the i^{th} row of B and c_j is the j^{th} column of C.

Let $\overline{K} \in \mathbb{R}^{m \times r}$ and initially set to 0. Define the polytope P by the set of inequalities:

$$\begin{aligned} a_{ii} - b_i \overline{K} c_i &\leq 0 \\ a_{ij} - b_i \overline{K} c_j &\geq 0 \ \ \forall\, i \neq j \end{aligned}$$

Let V represent the accompanying set of vertices for P.

Then, there exists an output stabilizing matrix, for (3.1), with elements bounded by (k_l, k_u) if and only if there exists a vertex $K_o \in V$ such that $A - BK_oC$ is stable.

Proof. Omitted, see proof of Lemma 3.3.1 in combination with Theorem 3.3.2. □

Remark 3.3.2. The extension to positive observer design is trivial to capture due to the result of duality, i.e. we can consider the observer gain L design problem for:

$$A - LC$$

as a stabilization problem for:

$$A^T - C^T L^T$$

in the usual LTI sense. The details for the design of the observer gain L are omitted.

Next, we illustrate the power of Algorithm 3.3.1 via a four dimensional example.

Example 3.3.4. Consider the following positive system:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & -2 & 2 & 0 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} u.$$

By Algorithm 3.3.1, we first check if the system is stable. Since entry $a_{11} = 0$, by Lemma 3.2.1, we know that A cannot be stable. Next, we provide the algorithm with $(k_l,\ k_u) = (-10, 10)$ (arbitrary). Moving to step 3 of the algorithm, we notice that column 3 is unitary, so we set $k_{33} = 10$. Afterward, we find the corresponding sets of vertices $V_1,\ V_2,\ V_3,\ V_4$, with Remark 3.3.1 in mind. First, we find P_1, which is defined by the following inequalities:

$$
\begin{aligned}
0 - \overline{k}_{11} - \overline{k}_{21} &\leq 0 \\
1 - 2\overline{k}_{11} &\geq 0 \\
2 - \overline{k}_{11} - \overline{k}_{21} - \overline{k}_{31} &\geq 0 \\
1 - \overline{k}_{21} &\geq 0
\end{aligned}
$$

and yields

$$
V_1 = \left\{ \begin{bmatrix} 0.5 \\ 0 \\ 1.5 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0 \\ -10 \end{bmatrix}, \begin{bmatrix} 0.5 \\ -0.5 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix}, \begin{bmatrix} 0.5 \\ -0.5 \\ -10 \end{bmatrix} \right\}.
$$

By Remark 3.3.1 the last four elements of V_1 can be removed as they lie on the inequality

$$
\overline{k}_{11} + \overline{k}_{21} \geq 0;
$$

thus yielding

$$
V_1 = \{[0.5\ 0\ 1.5]^T,\ [0.5\ 0\ -10]^T\}.
$$

In a similar manner, we can find the remaining sets:

$$
\begin{aligned}
V_2 &= \{[11\ -10\ 0]^T,\ [11\ -10\ -10]^T\} \\
V_3 &= \{[1\ 0\ 10]^T\} \\
V_4 &= \{[0\ 2\ -1]^T,\ [-10\ 12\ -1]^T,\ [-10\ 12\ -10]^T,\ [0\ 2\ -10]\}.
\end{aligned}
$$

Now by enumeration we obtain:

$$K_s = \begin{bmatrix} 0.5 & 11 & 1 & 0 \\ 0 & -10 & 0 & 2 \\ 1.5 & 0 & 10 & -1 \end{bmatrix},$$

which results in the closed loop matrix:

$$A_c = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & -2 & 2 & 0 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 11 & 1 & 0 \\ 0 & -10 & 0 & 2 \\ 1.5 & 0 & 10 & -1 \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0 & 0 \\ 0 & -24 & 0 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 12 & 0 & -3 \end{bmatrix},$$

which is Metzler and is clearly stable.

$\triangle$

Thus far the results presented have been for stabilization and output stabilization, we now shift to complete stabilization and complete output stabilization. First, complete stabilization is captured.

Corollary 3.3.5. Consider system (3.1), where rank(B)=m, rank(C)=r, and $D \neq 0$. Assume the elements of K_s are bounded from below and above by $k_l \in \mathbb{R}_-$ and $k_u \in \mathbb{R}_+$, respectively.

Let $\overline{K} \in \mathbb{R}^{m\times n}$ and initially set to 0. Define n polytopes P_i, $i = 1, ..., n$ by the set of inequalities:

$$\begin{array}{rcl}
a_{1i} - b_{11}\overline{k}_{1i} - ... - b_{1m}\overline{k}_{mi} & \geq & 0 \\
\vdots & & \vdots \\
a_{(i-1)i} - b_{(i-1)1}\overline{k}_{1i} - ... - b_{(i-1)m}\overline{k}_{mi} & \geq & 0 \\
a_{ii} - b_{i1}\overline{k}_{1i} - ... - b_{im}\overline{k}_{mi} & \leq & 0 \\
a_{(i+1)i} - b_{(i+1)1}\overline{k}_{1i} - ... - b_{(i+1)m}\overline{k}_{mi} & \geq & 0 \\
\vdots & & \vdots \\
a_{ni} - b_{n1}\overline{k}_{1i} - ... - b_{nm}\overline{k}_{mi} & \geq & 0 \\
 & & \\
c_{1i} - d_{11}\overline{k}_{1i} - ... - d_{1m}\overline{k}_{mi} & \geq & 0 \\
\vdots & & \vdots \\
c_{ri} - d_{r1}\overline{k}_{1i} - ... - d_{rm}\overline{k}_{mi} & \geq & 0
\end{array}$$

Let V_i be the accompanying set of vertices for P_i.

Then, there exists a completely stabilizing matrix, for (3.1), with elements bounded by (k_l, k_u) if and only if there exists a set of vertices $k_s^i \in V_i$, such that $A - BK_s$, where $K_s = [k_s^1 \; ... \; k_s^n]$, is stable.

Proof. The proof is identical to that of Lemma 3.3.1 with the additional constraints coming from

$$C - DK_s \geq 0$$

to yield a closed loop nonnegative output matrix. □

The final result of this subsection is regarding complete output stabilization, which is quite different and much more complex than that of the SISO case. Complete output stabilizability asks for: stabilizability of the system, and nonnegativity of states and outputs for all time. If we take Assumption 3.1.1 into account, then when solving the output stabilization problem, we must also incorporate the nonnegativity constraint:

$$(I - DK_o)^{-1}C \geq 0 \;\; \text{component-wise.}$$

MIMO State and Observer based Stabilization

In this subsection, we once again return to the problem of observer based stabilization. In the SISO subsection, we have shown that the separation property applies; however, we will show that in general the same may not hold for MIMO positive systems. The results are very similar to that of SISO, thus for the most part we omit detail.

Once again, recall the structure of the observer:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}). \tag{3.20}$$

Now, our system plant with the assumption that $u = -K_s\hat{x}$, where K_s has been found by stabilization and $\hat{x}$ is the estimate of the state x, is

$$\dot{x} \;=\; (A - BK_s)x + BK_s(x - \hat{x}), \tag{3.21}$$

so if $e_x = x - \hat{x} \in \mathbb{R}^n_+$ component-wise and $BK_s \geq 0$ component-wise, then $x \in \mathbb{R}^n_+$ for all time. We recall that

$$\dot{e}_x \;=\; (A - LC)e_x, \tag{3.22}$$

and if initially $e_x \in \mathbb{R}^n_+$, then $e_x \in \mathbb{R}^n_+$ for all time. Thus, we only have to make certain that $BK_s \geq 0$ component-wise and set the initial condition of $\hat{x}$ to zero.

We can now conclude that unless K_s is already nonnegative, BK_s may not necessarily be nonnegative component-wise; thus, in the case of observer based stabilization, an extra linear programming condition, $BK_s \geq 0$ component-wise, must be added to Algorithm 3.3.1 in order to guarantee that (3.21) is satisfied.

This completes the study of stabilization and observer based stabilization for MIMO positive LTI systems.

This chapter has presented fundamental results on stabilization of positive LTI systems. In particular, we have outlined checkable (vertex) conditions for stabilization. The results in this chapter are both necessary and sufficient, thus adding an additional dimension to the ever growing tools for positive systems.

Chapter 4

Servomechanism Problem: SISO tuning regulators

In the previous chapter our discussion was geared toward the problem of positive stabilization of unstable positive LTI systems. In this chapter our focus shifts to the tuning regulator problem for stable *unknown* SISO positive LTI systems under unmeasurable and measurable disturbances. In particular, we provide existence conditions, along with the actual control law, that solves the servomechanism problem for constant tracking and (un)measurable disturbance signals for positive LTI systems assuming that the mathematical model of the system can be described by an LTI model, but is unknown. The motivation for studying this problem is that in many applications, the mathematical model of the system may not be known, but from the physics of the system it is known that the plant is positive.

The chapter is organized as follows. Preliminaries are given first, where the details of the plant, all accompanying assumptions, and several introductory results are given. The servomechanism problem and its solution is presented next. In particular, the servomechanism problem for unmeasurable disturbances under nonnegative control is considered with the use of tuning regulators (TR). Thereafter, results on the servomechanism problem for measurable disturbances under nonnegative feedforward (FF) and tuning regulators (TR) are outlined. Finally, comments are made on the use of non-positive control inputs and implementation approaches for the servomechanism problem. Several examples illustrating the theory and the implementation approach conclude the chapter.

4.1 Preliminaries

The plant of interest is given first[1]:

$$\begin{aligned} \dot{x} &= Ax + bu + e_\omega \omega \\ y &= cx + du + f\omega \\ e &:= y_{ref} - y \end{aligned} \tag{4.1}$$

where A is an $n \times n$ Metzler stable matrix, $b \in \mathbb{R}^n_+$, $c \in \mathbb{R}^{1\times n}_+$, $d \in \mathbb{R}_+$, $e_\omega \in \mathbb{R}^n_+$, $f \in \mathbb{R}_+$; the signal $y_{ref} \in Y_{ref} \subset \mathbb{R}_+$ is a constant, as is $\omega \in \Omega \subset \mathbb{R}_+$. The sets Y_{ref} and Ω are defined in Assumption 4.1.1 below. Note that throughout the remainder of the thesis we assume that the output y is measurable.

First a result for $d - cA^{-1}b$ and $f - cA^{-1}e_\omega$ will be stated. In order to discuss the two gains, we recall an important result about stable Metzler matrices, which was presented in [55], but has also appeared in the literature on nonnegative matrices, see [36] for example.

Lemma 4.1.1. Let A be a Metzler matrix; then $-A^{-1}$ exists and is a nonnegative matrix if and only if A is Hurwitz.

We are now ready to present the result for $d - cA^{-1}b$ and $f - cA^{-1}e_\omega$.

[1] dependence on time may be dropped on occasion throughout the thesis

Lemma 4.1.2. Consider the system matrices of the plant (4.1). If $d - cA^{-1}b \neq 0$ then

$$d - cA^{-1}b > 0.$$

Moreover,

$$f - cA^{-1}e_{\omega} \geq 0.$$

Proof.

$$\begin{aligned} -A^{-1} &\in \mathbb{R}_{+}^{n\times n} && \text{by Lemma 4.1.1} \\ -cA^{-1}b &\in \mathbb{R}_{+} && \text{since } c \in \mathbb{R}_{+}^{1\times n},\ b \in \mathbb{R}_{+}^{n} \\ d - cA^{-1}b &\in \mathbb{R}_{+} && \text{since } d \in \mathbb{R}_{+} \end{aligned}$$

and since

$$d - cA^{-1}b \neq 0,$$

then

$$d - cA^{-1}b > 0.$$

The result for $f - cA^{-1}e_{\omega}$ follows in the same fashion.

$$\begin{aligned} -A^{-1} &\in \mathbb{R}_{+}^{n\times n} && \text{by Lemma 4.1.1} \\ -cA^{-1}e_{\omega} &\in \mathbb{R}_{+} && \text{since } c \in \mathbb{R}_{+}^{1\times n},\ e_{\omega} \in \mathbb{R}_{+}^{n} \\ f - cA^{-1}b &\in \mathbb{R}_{+} && \text{since } f \in \mathbb{R}_{+}. \end{aligned}$$

□

Next, we provide an important assumption which will be commonly used in the sequel. The assumption is needed in order to ensure that the steady state value of the input is nonnegative, under the choice of the reference signals and the unmeasurable disturbances of the plant. If this assumption was not true, then clearly one cannot attempt to satisfy any sort of nonnegativity of the input.

Assumption 4.1.1. Given the plant (4.1) assume that $d - cA^{-1}b \neq 0$. Also, assume the sets Ω and Y_{ref} are chosen such that for all $y_{ref} \in Y_{ref}$ and all $\omega \in \Omega$, *the steady state of the input*

$$u_{ss} := \frac{cA^{-1}e_\omega \omega - f\omega + y_{ref}}{d - cA^{-1}b} \tag{4.2}$$

has the property $u_{ss} > 0$.

The *steady state* x_{ss} and y_{ss} are defined next.

Definition 4.1.1. Consider the plant (4.1) under Assumption 4.1.1. Define

$$x_{ss} := -A^{-1}(bu_{ss} + e_\omega \omega) \tag{4.3}$$

and

$$y_{ss} := cx_{ss} + du_{ss} + f\omega. \tag{4.4}$$

Under Assumption 4.1.1 we can make the following claim.

Lemma 4.1.3. Consider the plant (4.1). If Assumption 4.1.1 holds, then

$$x_{ss} \geq 0 \text{ and } y_{ss} \geq 0.$$

Proof. It follows that

$$\begin{aligned} -A^{-1} &\in \mathbb{R}_+^{n\times n} && \text{by Lemma 4.1.1} \\ -A^{-1}(bu_{ss} + e_\omega \omega) &\in \mathbb{R}_+^n && \text{since } b \in \mathbb{R}_+^n, u_{ss} > 0, e_\omega \in \mathbb{R}_+^n, \omega \in \Omega \subset \mathbb{R}_+. \end{aligned}$$

Similarly for y_{ss}. □

Let us now shift to a discussion of Assumption 4.1.1.

4.1.1 Discussion of Assumption 4.1.1

Assumption 4.1.1 provides an algebraic expression for u_{ss}. It would be of great interest to actually know how large or small the disturbances can be in order for Assumption 4.1.1 to hold, i.e. what are the feasible sets Y_{ref} and Ω such that (4.2) holds. In this subsection, we consider the latter problem of finding the feasible sets Y_{ref} and Ω.

First, recall where (4.2) comes from, i.e.

$$\dot{x} = 0 = Ax_{ss} + bu_{ss} + e_\omega \omega \tag{4.5}$$

$$e = y_{ref} - y = 0 = cx_{ss} + du_{ss} + f\omega - y_{ref}. \tag{4.6}$$

Taking equation (4.5) and isolating it for x_{ss} we obtain:

$$x_{ss} = -A^{-1}bu_{ss} - A^{-1}e_\omega \omega. \tag{4.7}$$

Now, substituting equation (4.7) into equation (4.6) and isolating for u_{ss} results in:

$$u_{ss} = \frac{cA^{-1}e_\omega \omega - f\omega + y_{ref}}{d - cA^{-1}b}. \tag{4.8}$$

From above and the fact that we need $u_{ss} > 0$, we obtain

$$\begin{aligned} \frac{cA^{-1}e_\omega \omega - f\omega + y_{ref}}{d - cA^{-1}b} &> 0 \\ \text{or } cA^{-1}e_\omega \omega - f\omega + y_{ref} &> 0 \\ \text{or } y_{ref} - (f - cA^{-1}e_\omega)\omega &> 0 \end{aligned} \tag{4.9}$$

since, $d - cA^{-1}b > 0$ by Lemma 4.1.2 (provided $d - cA^{-1}b \neq 0$), resulting in:

$$(y_{ref},\ \omega) \in (Y_{ref},\ \Omega) := \{(\xi_1, \xi_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid \xi_1 - (f - cA^{-1}e_\omega)\xi_2 > 0\} \tag{4.10}$$

Remark 4.1.1. The last inequality (4.9) brings out numerous answers regarding the positive steady-state assumption on the input presented in Assumption 4.1.1; namely,

(a) if no disturbances are present then the steady-state u_{ss} assumption holds for all $y_{ref} > 0$.

(b) if the tracking signal is omitted and only positive disturbances are considered, then the assumption on u_{ss} will not hold, i.e. if $y_{ref} = 0$ and $\omega \neq 0$, then $u_{ss} < 0$ and the steady-state assumption will not be valid[2].

(c) in the case of unmeasurable/measurable disturbances, we can also deduce that if the disturbances are small in comparison to the tracking signal, i.e. $y_{ref} > (f - cA^{-1}e_\omega)\omega$, then the assumption on u_{ss} will hold true.

Note that if the system matrices are known, then one can use (4.10) directly to find Y_{ref} and Ω.

Assumption 4.1.2. Note that a necessary result, coming from equation (4.9), for Assumption 4.1.1 to hold is that $y_{ref} - f\omega > 0$; thus, without loss of generality, we can assume in the remainder of this chapter that $f = 0$.

4.2 Servomechanism Problem: unmeasurable disturbances and nonnegative input

With the plant and assumptions presented in the latter section, we now introduce *the servomechanism problem for SISO positive LTI systems* under unmeasurable disturbances,

[2] we are of course assuming that $f - cA^{-1}e_\omega \neq 0$

measurable reference tracking signals, and nonnegative input control. In this section, we also consider positive LTI systems under the assumption that the mathematical model of the system can be described by a positive LTI model (4.1), but is *unknown*[3].

The positive servomechanism problem of interest in this section is stated below.

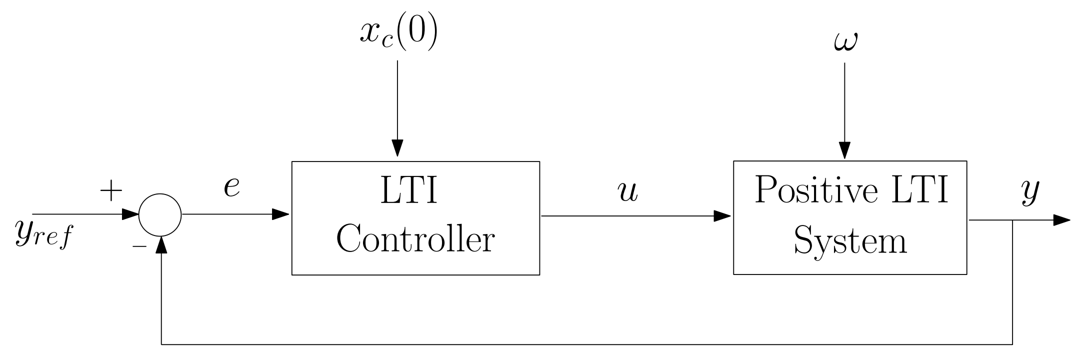


Figure 4.1: Closed-loop LTI system.

Problem 4.2.1. Consider the plant (4.1) where the disturbance ω is unmeasurable, the tracking signal y_{ref} is measurable, and the initial condition $x_0 \in \mathbb{R}^n_+$. Assume that Assumption 4.1.1 holds true.

Find an LTI controller connected as in the diagram (Figure 4.1) such that the closed-loop system satisfies

(a) asymptotic stability in the sense of Lyapunov with respect to the origin, and for every $y_{ref} \in Y_{ref}$, $\omega \in \Omega$, with the initial condition of the controller $x_c(0)$ having the property that $u(0) \in \mathbb{R}_+$

(b) the states $x(t) \geq 0$, output $y(t) \geq 0$, and input[4] $u(t) > 0 \; \forall \; t$; and

(c) tracking of the reference signal occurs, i.e. $e(t) = y_{ref} - y(t) \to 0$, as $t \to \infty$. In addition,

[3] i.e., no numerical values of A, B, C, D, E, F are known

[4] if $u(t) \geq 0$ for all time $t \in [0, \infty)$, then $x(t)$ and $y(t)$ will be nonnegative for all time by definition of positive systems

(d) assume that a controller has been found so that conditions (a), (b), (c) are satisfied; then for all perturbations of the nominal plant model which maintain properties (a) and (b), it is desired that the controller can still achieve asymptotic tracking and regulation, i.e. the controller is *robust* and property (c) still holds.

In Problem 4.2.1, it is to be noted in Assumption 4.1.1 that the two assumptions made are both necessary conditions for a solution to exist to Problem 4.2.1, since the condition $d - cA^{-1}b \neq 0$ is a necessary condition for there to exist a solution to the robust servomechanism problem [17], and since $u_{ss} > 0$ is clearly a necessary condition.

The following Tuning Regulator (TR) of interest will now be defined:

$$\begin{aligned} \dot{\eta} &= \epsilon(y_{ref} - y), \quad \eta(0) > 0 \text{ and fixed} \\ u &= \eta \end{aligned}, \tag{4.11}$$

with $\epsilon \in (0, \epsilon^*]$, $\epsilon^* > 0$ to be shown to exist. Notice that

$$u(0) = \eta(0) > 0.$$

Lemma 4.2.1. Set TR (4.11) as the LTI controller and plant (4.1) as the positive LTI system in Figure 4.1. Assume $d - cA^{-1}b \neq 0$. Then there exists an $\epsilon^+ > 0$ such that for all $\epsilon \in (0, \epsilon^+]$ the closed-loop matrix:

$$\begin{bmatrix} A & b \\ -\epsilon c & -\epsilon d \end{bmatrix} \tag{4.12}$$

is stable.

Proof. By [17] there exists an $\dfrac{\overline{\epsilon}}{d - cA^{-1}b}$ such that for all $\epsilon \in (0, \overline{\epsilon}/(d - cA^{-1}b)]$ the closed-loop matrix (4.12) is stable. Now, set

$$\epsilon^+ = \frac{\overline{\epsilon}}{d - cA^{-1}b}.$$

□

Notice that if TR (4.11) is the LTI controller and the plant (4.1) is the positive LTI system in Figure 4.1, then the signals $x,\ u,\ \eta,\ y,\ e$ depend on t and ϵ. For convenience, the dependence on ϵ will be understood, i.e., instead of $x(t,\epsilon)$ we'll simply write $x(t)$, and similarly for y, u, e, and η through the remainder of the thesis. Of course, if ϵ is fixed the dependence disappears.

The main result of this section is given next.

Theorem 4.2.1. Consider system (4.1). Then for all $x(0) \in \mathbb{R}^n_+$ there exists an $\epsilon^*(x(0), \eta(0)) > 0$ such that for all $\epsilon \in (0, \epsilon^*(x(0), \eta(0))]$ the controller (4.11) solves[5] Problem 4.2.1.

Through the remainder of this section $\epsilon^*(x(0), \eta(0))$ will be denoted by simply ϵ^*.

The proof is given next.

Proof. The closed-loop system of the plant and the controller is given in Figure 4.2. The

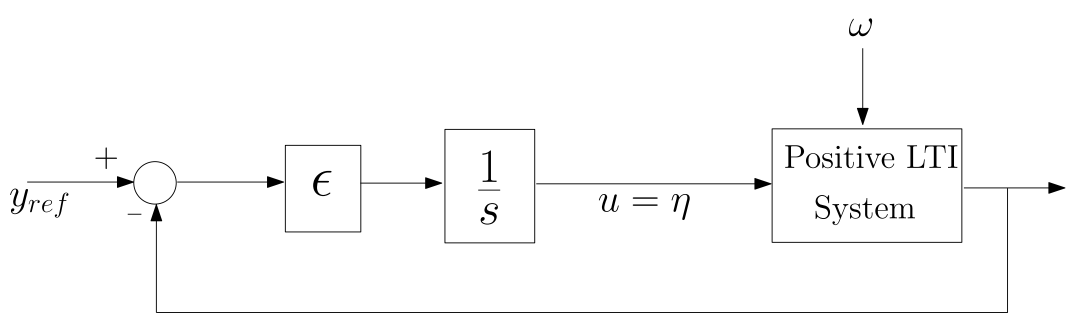


Figure 4.2: Closed-loop LTI system.

closed-loop system is LTI and by Lemma 4.2.1 there always exists an $\epsilon^+ > 0$ such that $\forall\ \epsilon \in (0, \epsilon^+]$ the closed-loop matrix is stable. This implies that all the conditions of Problem 4.2.1 will hold true if the nonnegativity condition (b) holds true. Through the remainder of the proof we assume $\epsilon \in (0, \epsilon^+]$.

Let us recall the two key assumptions:

[5]Notice that the result, along with all similar results in the thesis, states that we might have to find a separate $\epsilon*$ for different initial conditions $x(0)$, i.e. one ϵ^* will not necessarily work for all initial conditions $x(0)$, it may have to be re-tuned.

1. $u(0) = \eta(0) > 0$ (by the definition of the TR controller (4.11));

2. $u_{ss} > 0 \Rightarrow \eta_{ss} > 0$ (since $u = \eta$ and $u_{ss} > 0$).

First, by the definition of positive LTI systems we know that if $u(t) \geq 0$ for all t, then the states $x(t)$ and the outputs $y(t)$ also remain nonnegative for all t. Let us show now that there exists an $\epsilon^* \leq \epsilon^+$ such that for all $\epsilon \in (0, \epsilon^*]$, $u(t) > 0$ for all t under the two assumptions listed above. However, since

$$u(t) = \eta(t), \ \forall \ t$$

then it is sufficient to show that

$$\eta(t) > 0, \ \forall \ t.$$

In order to prove the above, we use the results of singular perturbation. The closed-loop system with the tuning regulator (4.11) is of the form:

$$\begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A & b \\ -\epsilon c & -\epsilon d \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix} + \begin{bmatrix} e_\omega & 0 \\ -\epsilon f & \epsilon \end{bmatrix} \begin{bmatrix} \omega \\ y_{ref} \end{bmatrix}. \tag{4.13}$$

Now, since the equilibrium (x_{ss}, η_{ss}), which depends on ω, y_{ref}, is independent of ϵ, we can transform the system as needed, i.e. let $z = x - x_{ss}$ and $q = \eta - \eta_{ss}$ in (4.13), resulting in the new system

$$\begin{bmatrix} \dot{q} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\epsilon d & -\epsilon c \\ b & A \end{bmatrix} \begin{bmatrix} q \\ z \end{bmatrix}. \tag{4.14}$$

Notice that if $q(t, \epsilon) + \eta_{ss} > 0$ for all t, then $\eta(t) > 0$ for all t.

Next, let us scale the derivatives by $\epsilon dt = d\tau$ (i.e. scaling of time $\epsilon t = \tau$) resulting in the transformed system

$$\begin{bmatrix} \overset{\odot}{q} \\ \epsilon\overset{\odot}{z} \end{bmatrix} = \begin{bmatrix} -d & -c \\ b & A \end{bmatrix} \begin{bmatrix} q \\ z \end{bmatrix}, \tag{4.15}$$

with $\epsilon\overset{\odot}{q} = \dot{q}$ and $\epsilon\overset{\odot}{z} = \dot{z}$.

Notice that if $q(\tau, \epsilon) + \eta_{ss} > 0$ for all τ, then $q(t, \epsilon) + \eta_{ss} > 0$ for all t, and consequently $\eta(t) > 0$ for all t. Therefore, it remains to show that indeed $q(\tau, \epsilon) + \eta_{ss} > 0$ for all τ.

Our model (4.15) now satisfies the singular perturbation model. In order to use the singular perturbation (SP) results, we must show that all assumptions of SP hold true. However, as (4.15) is linear and time invariant, and the boundry-layer model is

$$\dot{p} = Ap$$

with A stable, it suffices to show that the reduced model (given below) yields exponential stability. Now by setting $\epsilon = 0$ we obtain $z = h(q) = -A^{-1}bq$, and since A is Hurwitz, $h(q)$ exists and is unique. Next, by substituting $h(q)$ into $\overset{\odot}{q}$ we obtain the reduced model:

$$\overset{\odot}{q} \;=\; -dq + cA^{-1}bq = -(d - cA^{-1}b)q, \text{ where by Lemma 4.1.2 } (d - cA^{-1}b) > 0.$$

Denote the solution of $\overset{\odot}{q} = -(d - cA^{-1}b)q$ by $\overline{q}(\tau)$, which is clearly exponentially stable (as needed by SP) and monotonic. Thus, by SP we have:

$$q(\tau, \epsilon) - \overline{q}(\tau) = O(\epsilon)\ \forall\tau$$

uniformly in τ, where

$$\begin{aligned} \overline{q}(\tau) &= e^{-(d-cA^{-1}b)\tau}\overline{q}(0) \text{ and} \\ \overline{q}(\tau) + \eta_{ss} &= \eta_{ss} + e^{-(d-cA^{-1}b)\tau}\overline{q}(0) \end{aligned}$$

with $\overline{q}(0) = q(0) = \eta(0) - \eta_{ss}$ by definition. Now, since $\eta(0) > 0$, then there exists an $\epsilon^* \leq \epsilon^+$ such that $q(\tau, \epsilon) + \eta_{ss} > 0$ for all $\epsilon \in (0, \epsilon^*]$ since $\overline{q}(\tau) + \eta_{ss}$ is *monotonically* approaching η_{ss}.

This completes the proof. □

The latter problem can be interpreted as the tracking and disturbance rejection problem under nonnegative control for positive LTI plants experiencing unmeasurable constant nonnegative disturbances.

Before we depart this section, we consider a Corollary to Theorem 4.2.1 with the extra assumption that $0 \leq u(t) \leq \overline{u}$, $\overline{u} > 0$ fixed, for all $t \in [0, \infty)$ and where $\eta(0)$ in (4.11) is fixed such that $0 \leq \eta(0) \leq \overline{u}$ (recall $u = \eta$).

Corollary 4.2.1. Consider system (4.1) and controller (4.11) where $0 < \eta(0) < \overline{u}$, $0 < u_{ss} < \overline{u}$, $\overline{u} > 0$ fixed. Then for all $x(0) \in \mathbb{R}_+^n$ there exists an $\epsilon^*(x(0), \eta(0)) > 0$ such that for all $\epsilon \in (0, \epsilon^*(x(0), \eta(0))]$ the controller (4.11) solves Problem 4.2.1 with condition (b) replaced by

(b') the states $x(t) \geq 0$, output $y(t) \geq 0$ and input $0 < u(t) < \overline{u}$.

The latter Corollary simply states that we would like to bound our input signal both from below by zero and from above by some constant $\overline{u}$.

Proof. The proof follows a similar argument as the proof of Theorem 4.2.1. The main difference is in the two key assumptions:

1. $0 < u(0) = \eta(0) < \overline{u}$;

2. $0 < u_{ss} < \overline{u} \Rightarrow 0 < \eta_{ss} < \overline{u}$ (since $u = \eta$ and $0 < u_{ss} < \overline{u}$).

Now, recall the SP result of the previous proof under time scaling (i.e. $\epsilon t = \tau$):

$$q(\tau, \epsilon) - \overline{q}(\tau) = O(\epsilon) \ \forall \tau$$

uniformly in τ, where

$$\begin{aligned}\overline{q}(\tau) &= e^{-(d-cA^{-1}b)\tau}\overline{q}(0) \text{ and} \\ \overline{q}(\tau)+\eta_{ss} &= \eta_{ss}+e^{-(d-cA^{-1}b)\tau}\overline{q}(0)\end{aligned}$$

with $\overline{q}(0) = q(0) = \eta(0) - \eta_{ss}$ by definition. Now, since $0 < \eta(0) < \overline{u}$, then there exists an $\epsilon^* \leq \epsilon^+$ such that $q(\tau, \epsilon) + \eta_{ss} > 0$ for all $\epsilon \in (0, \epsilon^*]$ since $\overline{q}(\tau) + \eta_{ss}$ is *monotonically* approaching η_{ss}. Moreover, due to the signal being monotonic we can also conclude that $q(\tau, \epsilon) + \eta_{ss} < \overline{u}$. □

This completes the results of this section.

4.3 Servomechanism Problem: measurable disturbances and nonnegative input

In this subsection, we tackle the servomechanism problem under measurable disturbances for unknown positive LTI systems.

The goal of this section will be to take advantage of the measurable disturbances via the use of feedforward controllers and tuning regulators introduced in Section 2.3. Thus, along with a modified solution to Problem 4.2.1, we will introduce a second problem which solely solves the servomechanism problem without the robust property[6]. The new non-robust servomechanism problem is outlined below.

Problem 4.3.1. Consider the plant (4.1) where the disturbance ω is measurable, the tracking signal y_{ref} is measurable, and the initial condition $x_0 \in \mathbb{R}^n_+$. Assume that Assumption 4.1.1 holds true.

[6]introduced in Problem 4.2.1 (d)

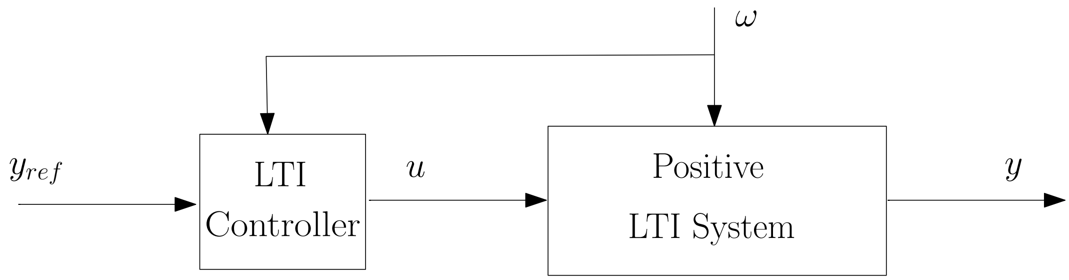


Figure 4.3: Open-loop LTI control.

Find an LTI controller connected as in Figure 4.3 such that the open-loop system for every $y_{ref} \in Y_{ref}$, and every $\omega \in \Omega$ satisfies

(a) the states $x(t) \geq 0$, output $y(t) \geq 0$, and input $u(t) \geq 0 \ \forall \ t$; and

(b) ensures tracking of the reference signal, i.e. $e(t) = y_{ref} - y(t) \rightarrow 0$, as $t \rightarrow \infty$.

Notice that the main difference between Problem 4.3.1 and Problem 4.2.1 is the restriction of $u(t) \geq 0$ instead of $u(t) > 0$ and the omission of point (d) from Problem 4.3.1, i.e. the need for robustness.

Next, we will show that Problem 4.3.1 can be solved by using a feedforward compensator, and thereafter, we show that Problem 4.2.1 under measurable disturbances can be solved by using a combination of a feedforward compensator and a tuning regulator. Note that the use of a feedforward compensator was not feasible in the previous section, as unmeasurable disturbances were considered.

Theorem 4.3.1. Consider system 4.1. The feedforward compensator (4.16) solves Problem 4.3.1 for all $x(0) \in \mathbb{R}^n_+$.

$$u = \frac{cA^{-1}e_\omega \omega - f\omega + y_{ref}}{d - cA^{-1}b} \tag{4.16}$$

Proof.

Condition (b) holds since u is set to u_{ss}. Condition (a) can be guaranteed if the control input $u(t) \geq 0$ for all time. However, the feedforward compensator (2.9) has the property that:

$$u = u_{ss} \geq 0,$$

which will guarantee nonnegativity of the states and outputs for all time. □

Notice that if in Problem 4.3.1 we set $Y_{ref} = \mathbb{R}_+$ and $\Omega = \mathbb{R}_+$, then we can set the feedforward controller to

$$\begin{aligned} u &= \frac{cA^{-1}e_\omega \omega - f\omega + y_{ref}}{d - cA^{-1}b} \text{ if } u_{ss} \geq 0 \\ u &= 0 \text{ else} \end{aligned} \tag{4.17}$$

resulting in the controller (4.17) turning itself off whenever a negative steady state of the input appears.

Feedforward compensators are an effective tool to solve the tracking and regulation problem, and have the advantage of providing "a fast speed of response" [17]. However in practice, due to possible changes to the parameters of the plant, lack of precise measurements, etc., feedforward controllers in general may lead to unsatisfactory tracking/regulation. Thus, in practice, if the disturbances are measurable, one may wish to apply both the tuning regulator (4.11) and the feedforward controller (4.16) simultaneously to control the system[7], i.e. we can use the control law:

$$u = u_{ff} + u_{tr}, \tag{4.18}$$

where u_{ff} is the feedforward compensator (4.16) and u_{tr} is the tuning regulator (4.11).

Remark 4.3.1. In this section and the previous, we considered nonnegative control. *It is vital to point out that if nonnegative control does not solve the tracking and disturbance*

[7] assuming $u_{ss} > 0$

rejection problem, then it does not mean that a bidirectional or nonpositive type of controller will not do the job. Namely, we may still be able to find a controller that has a bidirectional output and still maintains nonnegativity of the states and output. For example, if one considers cooling a house in the summer with a heater, then clearly the physics will not permit lowering of the temperature; however, an air conditioner will do the job. Thus we tackle the problem of using negative control in the next section.

4.4 Servomechanism Problem: Bidirectional Control

The results presented in the previous sections have provided solutions to the servomechanism problem for systems with measurable and unmeasurable disturbances using nonnegative control. In real life systems, nonnegativity of states occurs quite often; however, the need for the input u to be also nonnegative may not always be a necessity, as was also pointed out in [23]. In this section, we drop the assumption of nonnegativity of the input. Our focus will be to solve the servomechanism problem using bidirectional control, subject to nonnegativity of the states and outputs of the system.

A new steady state value is defined first. Let the steady state of the open loop positive LTI system (4.1) be defined as

$$\overline{x}_{ss} := -A^{-1}e_\omega \omega \tag{4.19}$$

Let us restate Assumption 4.1.1 under the assumption that only the steady state of the output and states (x_{ss}, $\overline{x}_{ss}$, and y_{ss}) must be nonnegative.

Assumption 4.4.1. Given the plant (4.1) assume that $d - cA^{-1}b \neq 0$. Also, assume the sets Ω and Y_{ref} are chosen such that for all $y_{ref} \in Y_{ref}$ and all $\omega \in \Omega$, the steady state of the output y_{ss} (4.4) and the states of the closed-loop x_{ss} (4.3) and open-loop $\overline{x}_{ss}$ are all positive.

We will assume in this section that Y_{ref} and Ω are defined as in Assumption 4.4.1.

Next, a restatement of Problem 4.2.1 under the latter assumption is outlined.

Problem 4.4.1. Consider the plant (4.1) where the disturbance ω is unmeasurable, the tracking signal y_{ref} is measurable, and the initial condition $x(0) = \overline{x}_{ss}$. Assume that Assumption 4.4.1 holds true.

Find an LTI controller connected as in the diagram (Figure 4.1) such that the closed-loop system satisfies

(a) asymptotic stability in the sense of Lyapunov with respect to the origin, and for every $y_{ref} \in Y_{ref}$, $\omega \in \Omega$, with the initial condition of the controller $x_c(0)$ having the property $u(0) \in \mathbb{R}_+$

(b) the states $x(t) \geq 0$ and output $y(t) \geq 0$, $\forall\, t$; and

(c) tracking of the reference signal occurs, i.e. $e(t) = y_{ref} - y(t) \to 0$, as $t \to \infty$. In addition,

(d) assume that a controller has been found so that conditions (a), (b), (c) are satisfied; then for all perturbations of the nominal plant model which maintain properties (a) and (b), it is desired that the controller can still achieve asymptotic tracking and regulation, i.e. the controller is *robust* and property (c) still holds.

The solution to the latter problem is given next. The result is stated as a corollary to Theorem 4.2.1.

Corollary 4.4.1. Consider system (4.1) under Assumption 4.4.1. Then there exists an ϵ^* such that for all $\epsilon \in (0, \epsilon^*]$ the tuning regulator (4.11) solves Problem 4.4.1.

Notice that a necessary result for Corollary 4.4.1 above is the need to have $\omega \neq 0$, as if $\omega = 0$, then the results for Problem 4.2.1 suffice, i.e. bidirectional control only needs to be considered when $\omega \neq 0$.

This completes this section.

4.5 Implementation

In the remainder of this chapter, implementation of the theory presented in the previous sections is considered. In particular, we deal with the situation when the assumption $u_{ss} > 0$ made in Theorem 4.2.1 does not hold, and in this case we consider tuning clamping regulators (TcR) and reset tuning clamping regulators (RTcR) to prevent the input signals from going negative.

First, however, we consider two results. The first result pertains to Figure 4.4, while the second to Figure 4.5.

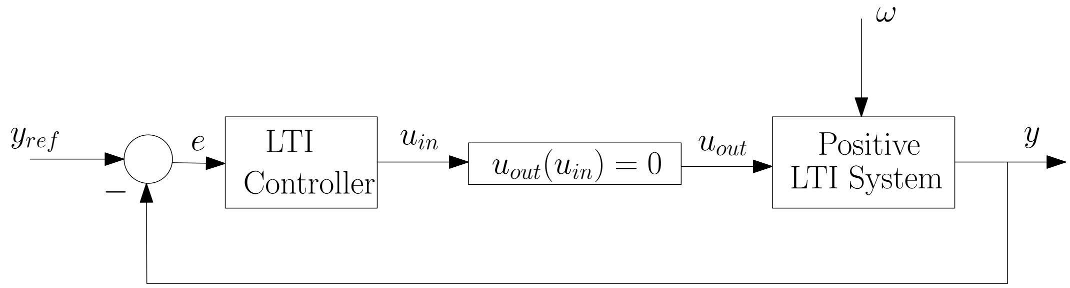


Figure 4.4: Implementation system 1.

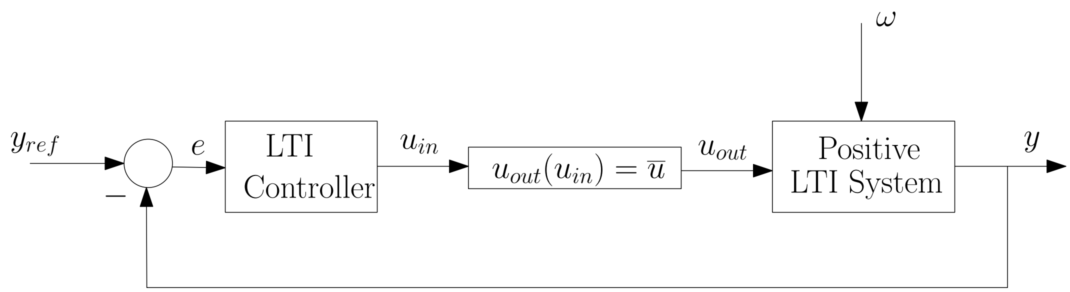


Figure 4.5: Implementation system 2.

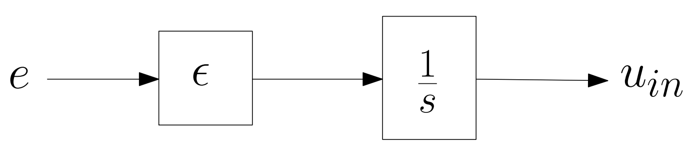


Figure 4.6: LTI controller.

Theorem 4.5.1. Consider the system of Figure 4.4 where the positive LTI system is represented as (4.1) and where the block diagram of Figure 4.6 represents the LTI controller. Assume $y_{ref} \in \mathbb{R}_+$, $\omega \in \mathbb{R}_+$, and $d - cA^{-1}b \neq 0$.

Then, for all $\epsilon > 0$, and all $x(0) \in \mathbb{R}^n_+$ there exists a $t^*(x(0), \epsilon) \geq 0$ such that for all $t \in [t^*(x(0), \epsilon), \infty)$

$$u_{in}(t) > 0$$

if and only if

$$u_{ss} > 0.$$

Proof. The overall system is:

$$\begin{aligned} \dot{x} &= Ax + e_\omega \omega \\ \dot{u}_{in} &= \epsilon(y_{ref} - y) \\ y &= cx + f\omega. \end{aligned}$$

As $t \to \infty$ we have

$$x \to \overline{x}_{ss} = -A^{-1}e_\omega \omega \text{ since A is stable.}$$

Recall,

$$x_{ss} = -A^{-1}(bu_{ss} + e_\omega \omega).$$

Thus, we can rewrite $\overline{x}_{ss}$ as

$$\overline{x}_{ss} = x_{ss} + A^{-1}bu_{ss}.$$

Next, since $x \to \overline{x}_{ss}$ as $t \to \infty$ then

$$\begin{aligned}
\dot{u}_{in} &= \epsilon(y_{ref} - y) \\
\dot{u}_{in} &\to \epsilon(y_{ref} - c\overline{x}_{ss} - f\omega) \text{ as } t \to \infty \\
\epsilon(y_{ref} - c\overline{x}_{ss} - f\omega) &= \epsilon(y_{ref} - c\overline{x}_{ss} - d(0) - f\omega) \\
&= \epsilon(y_{ref} - c(x_{ss} + A^{-1}bu_{ss}) - d(0) - f\omega) \\
&= \epsilon(y_{ref} - c(x_{ss} + A^{-1}bu_{ss}) - d(u_{ss} - u_{ss}) - f\omega) \\
&= \epsilon(y_{ref} - cx_{ss} - du_{ss} - f\omega) + \epsilon(d - cA^{-1}b)u_{ss}) \\
&= 0 + \epsilon(d - cA^{-1}b)u_{ss}
\end{aligned}$$

Thus, clearly since the derivative tends to $\epsilon(d - cA^{-1}b)u_{ss} > 0 \iff u_{ss} > 0$ there exists a $t^*(x(0), \epsilon) \geq 0$ such that for all $t \in [t^*(x(0), \epsilon), \infty)$

$$u_{in}(t) > 0$$

$\square$

We now come back to the system of Figure 4.5 and present a corollary to the latter theorem.

Corollary 4.5.1. Consider the system of Figure 4.5 where the positive LTI system is represented as (4.1) and where the block diagram of Figure 4.6 represents the LTI controller. Assume $y_{ref} \in \mathbb{R}_+$, $\omega \in \mathbb{R}_+$, and $d - cA^{-1}b \neq 0$, and let $\overline{u} > 0$ be a fixed constant.

Then, for all $\epsilon > 0$, and all $x(0) \in \mathbb{R}^n_+$ there exists a $t^*(x(0), \epsilon) \geq 0$ such that for all $t \in [t^*(x(0), \epsilon), \infty)$

$$u_{in}(t) < \overline{u}$$

if and only if

$$u_{ss} < \overline{u}.$$

The proof follows similar guidelines as in the proof of Theorem 4.5.1 and is omitted.

Next, from the results of Theorem 4.2.1 and Theorem 4.5.1 we propose a tuning clamping regulator (TcR), which clamps all negative input signals at zero or employs the TR whenever the input signal is positive. The TcR is presented below.

$$\begin{array}{rcl} \dot{\eta} & = & \epsilon(y_{ref} - y), \quad \eta(0) = 0 \\ u & = & k(\eta)\eta \end{array}, \tag{4.20}$$

where

$$k(\eta) \;=\; \begin{cases} 0 & \eta \leq 0 \\ 1 & \eta > 0 \end{cases}$$

The introduction of the TcR is necessary as in Theorem 4.2.1 we do not know how small the ϵ^* should be; moreover, since we are dealing with unknown systems we may not know a priori if the steady state of the input u_{ss} is actually positive. However, by Theorem 4.5.1 we can deduce that under the TcR controller if $u_{ss} \leq 0$, then our controller will "shut itself off" in finite time and remain shut off and if $u_{ss} > 0$ we know that there exists a time $t^*(x(0), \epsilon)$ such that the trajectory will return to the linear region. We note that under the circumstances of unknown plants this is the best that any LTI controller placed in the feedback loop can do.

The next result takes into account the information presented by Corollary 4.2.1, Theorem 4.5.1, and Corollary 4.5.1.

In the TcR controller a two step process was proposed. Basically, if the control input results in a negative value, we simply clamp the control input and allow the tuning regulator to continue working until it finally comes back to positivity (which was shown to occur by Theorem 4.5.1). A potential problem with the above approach is that the servo compensator (integrator) may output excessively large values for the servo state (η) and thus result in possibly undesirable "reset windup" like behavior in the response of the system. Below, we provide a control strategy, for Problem 4.2.1 with the same assumptions of unknown LTI plant, and unmeasurable disturbances, that incorporates anti-reset windup under a saturation constraint. This controller will be referred to as the Reset Tuning clamping Regulator (RTcR).

$$\begin{array}{ll} & \text{if } ((0 < u < \overline{u}) \text{ or} \\ \dot{\eta} = \epsilon(y_{ref} - y) & (u = 0 \text{ and } e > 0) \text{ or} \\ & (u = \overline{u} \text{ and } e < 0)) \\ & \\ \dot{\eta} = 0 & \text{else,} \end{array} \tag{4.21}$$

with

$$u = k(\eta)\eta, \;\; 0 < \eta(0) < \overline{u} \text{ and fixed}$$

where

$$k(\eta) \;\; = \;\; \begin{cases} 0 & \eta \leq 0 \\ 1 & 0 < \eta < \overline{u} \\ \overline{u}/\eta & \eta \geq \overline{u} \end{cases}$$

Both the TcR and the RTcR controllers use the theoretical results of this chapter to implement controllers that will be used in (simulated) examples in this chapter and on experimental results in Chapter 9.

Finally, it must be pointed out that clamping, saturation, and reset-windup type controllers can also be used to obtain nonpositive inputs, under the assumption that the steady state of the input u_{ss} is negative.

We now turn our attention to several examples.

4.6 Examples

In this section, we illustrate the results presented in this chapter via several examples.

The first example describes the monitoring and controlling the depth of anesthesia in surgery and illustrates all results of the chapter. The problem has been originally considered by Haddad et al. [37].

Example 4.6.1. The use of propofol as an intravenous anesthetic is common for both induction and maintenance of general anesthesia [27]. An effective patient model for the disposition of propofol is based on a three-compartmental mammillary model, see Figure 4.7 [53], [2]. The three-compartmental mammillary system provides a pharmacokinetic model for a patient, describing the distribution of propofol into the central compartment and the other various tissue groups of the body. The mass balance equations for the compartmental system yield [37]:

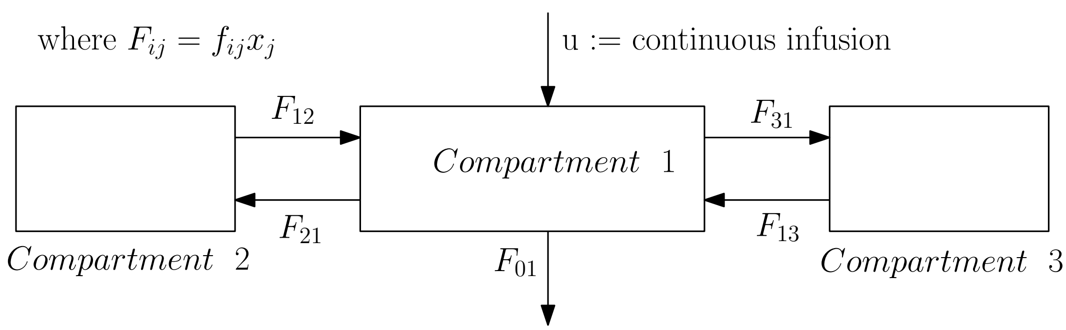


Figure 4.7: Three compartmental mammillary model.

$$
\begin{aligned}
\dot{x}_1 &= -(f_{01} + f_{21} + f_{31})x_1 + f_{12}x_2 + f_{13}x_3 + u \\
\dot{x}_2 &= f_{21}x_1 - f_{12}x_2 \\
\dot{x}_3 &= f_{31}x_1 - f_{13}x_2
\end{aligned}
\tag{4.22}
$$

where the states are masses in *grams* of propofol in the respective compartments. The input u is the infusion rate in $grams/min$ of the anesthetic propofol into the first compartment. The rate constant $f_{11} \geq 0$ is in min^{-1} and represents the elimination rate from the central compartment, while the rate constants $f_{ij} \geq 0$, which are also in min^{-1}, characterize drug transfer between compartments. It has been pointed out in [37] that the rate constants, although nonnegative, can be uncertain due to patient gender, weight, pre-existing disease, age, and concomitant medication. It has also been pointed out in [37], [86] that 2.5-6 $\mu g/ml$ blood concentration levels of propofol are required during the maintenance stage in general anesthesia.

In [37] the assumption made was that a 70 kg patient was treated with propofol concentration levels of $4\mu g/mol$, which led to the desired tracking value for $x_1 = 44.52mg$. It has also been pointed out that the values of f_{ij} in (4.22) may be uncertain and difficult to estimate; this however causes no problem since a mathematical model of the system is not required.

Our system matrices for (4.22) become:

$$A = \begin{bmatrix} -(f_{01} + f_{21} + f_{32}) & f_{12} & f_{31} \\ f_{21} & -f_{12} & 0 \\ f_{31} & 0 & -f_{13} \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For our simulations, we will use the the parameters in Table 4.1, presented in [34]. Note that these parameters are not used in our controller design for this example. The table presents two sets of data; in order to show that our controller works for uncertain systems, we will alternate between the two sets. It will be assumed that the desired set point is $y_{ref} = 45$.

In addition to the model above, we assume that a disturbance exists, and that it affects the input to the first compartment, see Figure 4.8. With the disturbance in place,

Table 4.1: Pharmacokinetic parameters [34]

Data	f_{01}	f_{21}	f_{12}	f_{31}	f_{13}
1	0.152	0.207	0.092	0.040	0.0048
2	0.119	0.114	0.055	0.041	0.0033

our system model becomes:

$$\dot{x} = \begin{bmatrix} -(f_{01}+f_{21}+f_{31}) & f_{12} & f_{13} \\ f_{21} & -f_{12} & 0 \\ f_{31} & 0 & -f_{13} \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} e_\omega \\ 0 \\ 0 \end{bmatrix} \omega, \quad y = [1\ 0\ 0]x.$$

The simulation given by Figure 4.9 and Figure 4.10 show the response of $y = x_1$ and

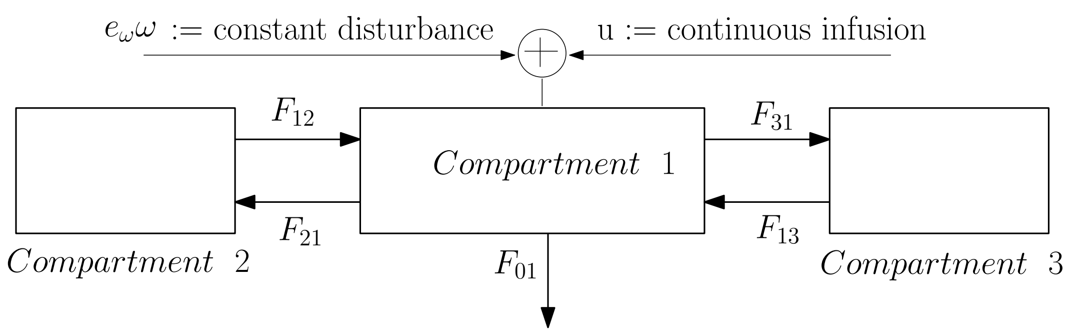


Figure 4.8: Three compartmental mammillary model with disturbance.

u with $\epsilon = 0.1$ for the TR controller (4.11) where $\eta(0) = 10^{-10}$ (the reason the initial condition is so small, but positive, is to abide to the TR initial condition constraint of $\eta(0) > 0$), and where for $t \in [0, 50min)$ data 1, from Table 4.1, is used with $e_\omega\omega = 0.5$; at $t = 50min$ the system switches to data 2 with $e_\omega\omega = 0.5$, and finally at $t = 100min$ the system undergoes a further disturbance with $e_\omega\omega = 1.5$. It is seen that satisfactory tracking and regulation occurs.

$\triangle$

Our next example considers a set of water tanks.

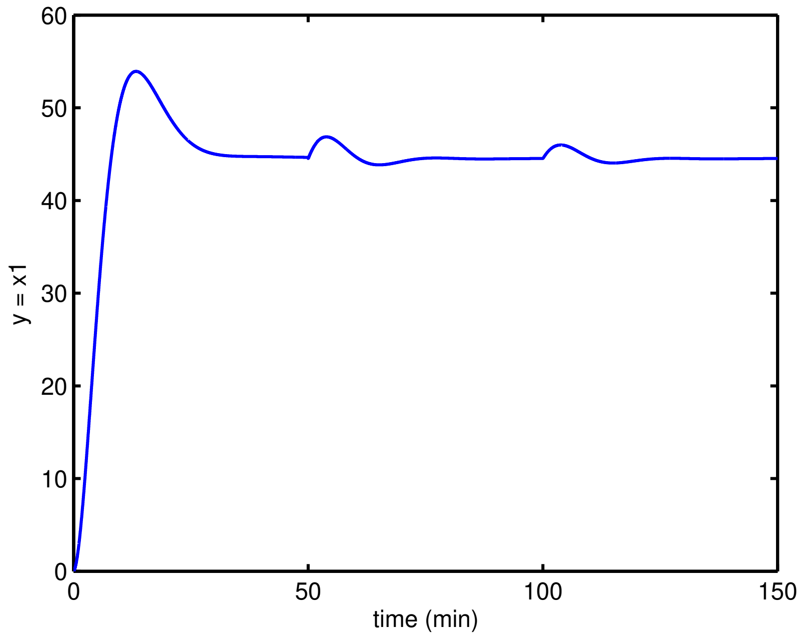


Figure 4.9: Output response for Example 4.6.1.

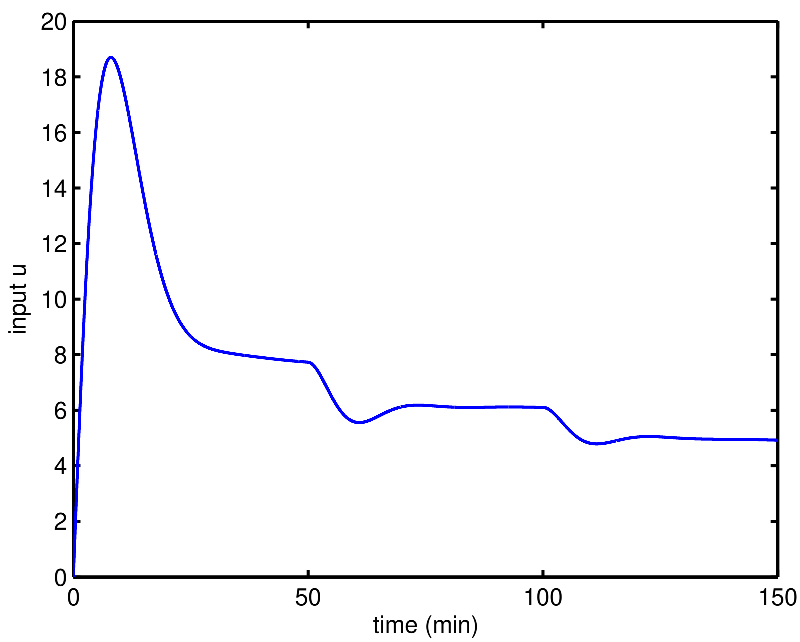


Figure 4.10: Control input for Example 4.6.1.

Example 4.6.2. The following plant, which is a stable compartmental system, has been taken from [30] pg.105. Consider the reservoir network of Figure 4.11, with u as the input flow rate of water and ω an input flow disturbance. The system is of dimension 6, as we assume the pump dynamics can be neglected. As pointed out in [30], the dynamics of each reservoir can be captured by a single differential equation:

$$\dot{x}_i = -\alpha_i x_i + v,\ \alpha_i > 0,\ i = 1, ..., 6,$$

where x_i represents the depth of the water in each reservoir.

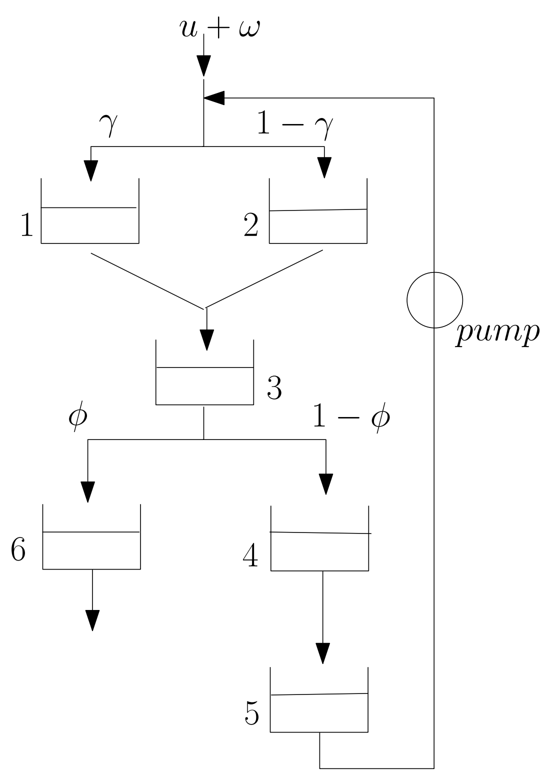


Figure 4.11: System set up for Example 4.6.2.

Consider the case where $\gamma = 0.5$, $\phi = 0.9$, $\alpha_1 = 2$, $\alpha_2 = 1.7$, $\alpha_3 = 1.5$, $\alpha_4 = 1$, $\alpha_5 = 2$, and $\alpha_6 = 2$. This results in the following system:

$$\dot{x} = \begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & -1.7 & 0 & 0 & 0 & 0 \\ 2 & 1.7 & -1.5 & 0 & 0 & 0 \\ 0 & 0 & 0.15 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1.35 & 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} (u + \omega),$$

$$y = [0\ 0\ 0\ 0\ 0\ 1]x.$$

Assume now that we would like to track the reference input $y_{ref} = 1$, subject to the disturbance $\omega = 0.5$. For simulation purposes we assume $x_0 = [2\ 4\ 1\ 0.5\ 0.5\ 2]$. In this case, the application of the TcR controller (4.20) with $\epsilon = 0.5$, solves the tracking problem. Note that condition (4.10) holds for this problem; however, the information was not used in order to implement the controller (4.20). Figure 4.12 illustrates both the output y and the input u. The plots of the states x are omitted, however, it is easy to deduce that they are nonnegative as $u \geq 0$, $\forall t \geq 0$.

△

The final example of this chapter illustrates the servomechanism problem under a negative control input.

Example 4.6.3. The system considered in this example has been taken from [5].

The interior temperature of an electrically cooled oven is to be controlled by varying the input u to the jacket. Let the heat capacities of the oven interior and of the jacket be c_2 and c_1, respectively, let the interior and exterior jacket surface areas be a_1 and a_2, and let the radiation coefficient of the interior and exterior jacket surfaces be r_1 and r_2. If the external temperature is T_0, the jacket temperature T_1 and the oven interior temperature

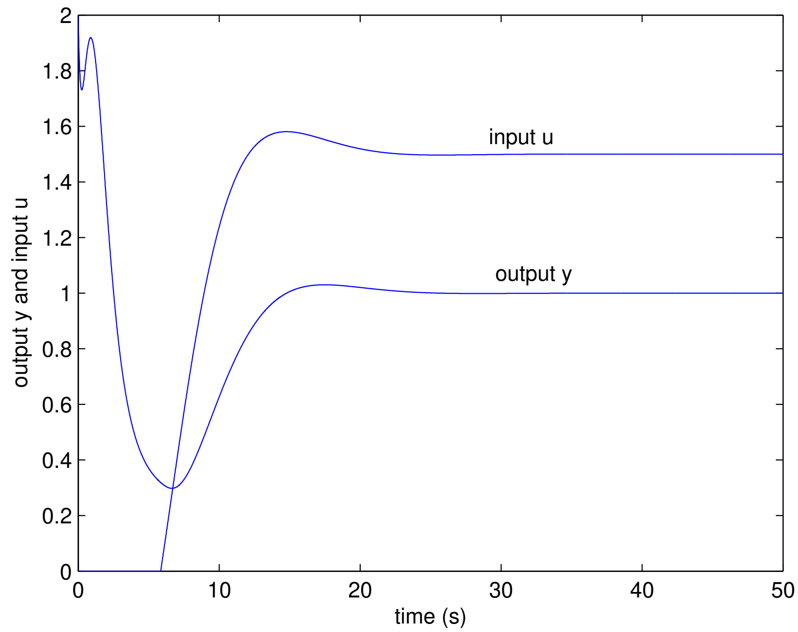


Figure 4.12: Output and input response for Example 4.6.2.

is T_2, then the behaviour for the jacket is described by:

$$c_1\dot{T}_1 = -a_2r_2(T_1 - T_0) - a_1r_1(T_1 - T_2) + u + a_3\omega$$

where ω is a disturbance, and for the oven interior:

$$c_2\dot{T}_2 = a_1r_1(T_1 - T_2).$$

By setting the state variables to be the excess of temperature over the exterior, i.e.

$$x_1 := T_1 - T_0 \quad \text{and} \quad x_2 := T_2 - T_0$$

results in the system:

$$\dot{x} = \begin{bmatrix} \frac{-(a_2 r_2 + a_1 r_1)}{c_1} & \left(\frac{a_1 r_1}{c_1}\right) \\ \left(\frac{a_1 r_1}{c_2}\right) & \left(\frac{-a_1 r_1}{c_2}\right) \end{bmatrix} x + \begin{bmatrix} \frac{1}{c_1} & \frac{a_3}{c_1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ \omega \end{bmatrix}$$

Assume that the values of the constants above are chosen such that $c_1 = c_2 = 1$, $a_1 = a_2 = a_3 = 1$ and $r_1 = r_2 = 1$, and that the disturbance vectors are $e_\omega \omega = [3\ 0]^T$ and $f\omega = 0$, then:

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 3 \\ 0 \end{bmatrix} \omega, \ y = [1 \ \ 0]x.$$

We will now show that if the temperature excess in the jacket is initially zero, then we can maintain nonnegativity of the states and output, along with obtaining tracking control by using the tuning regulator. In this example we'll assume that we want to regulate y to $y_{ref} = 1$, i.e. the desired temperature excess should be 1°C higher in the jacket than the outside.

First, it is easy to show that the A matrix is stable, i.e.
$\sigma(A) = \{-2.618, -0.382\}$, and that the assumptions of Corollary 4.4.1 hold; in particular, that $\overline{x}_{ss} > 0$. We will also introduce clamping with nonpositive control. In this case, the application of controller (4.11) with $\epsilon = 10$ will suffice. Figure 4.13 and Figure 4.14 illustrate the states x and the input u, for the case of $y_{ref} = 1$, with zero initial conditions.

△

4.7 Conclusion

In this chapter we have discussed the tuning regulator robust servomechanism problem for SISO positive LTI systems. We note that the necessary and sufficient conditions

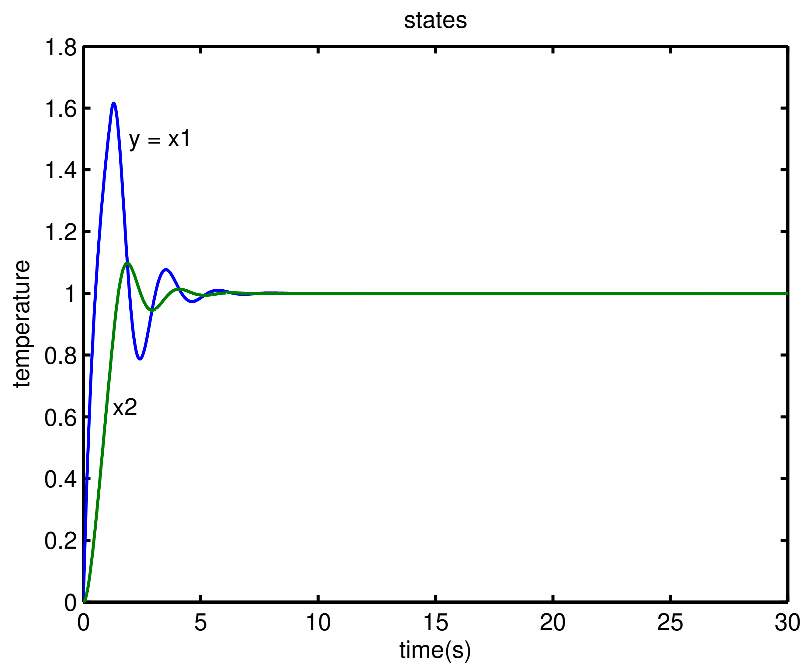


Figure 4.13: State response for Example 4.6.3.

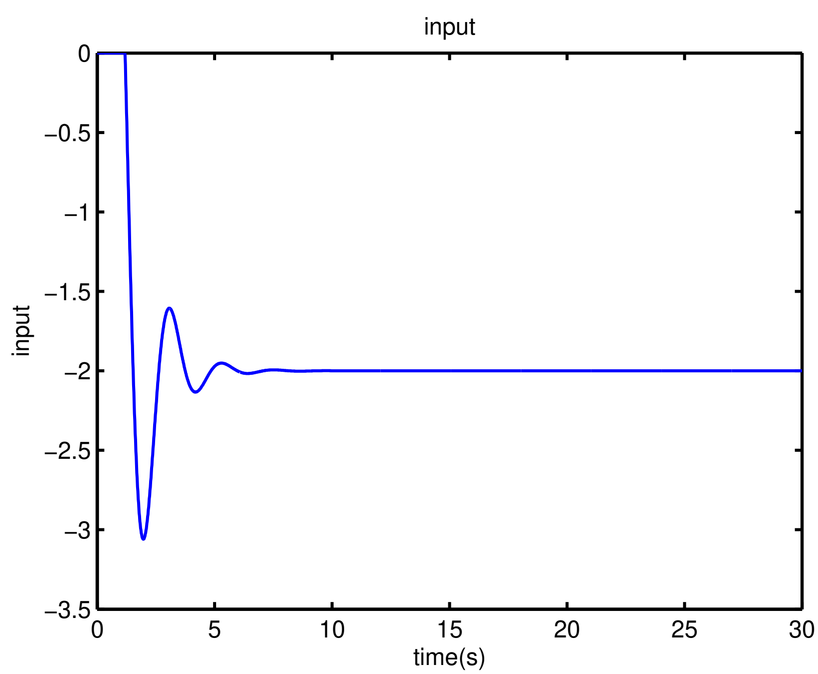


Figure 4.14: Input response for Example 4.6.3.

for the positive servomechanism problem are: $d - cA^{-1}b \neq 0$ (which can be checked by Procedure 2.3.1) and $u_{ss} > 0$. The condition of $d - cA^{-1}b \neq 0$ is equivalent to stating that the zeros of (c, A, b, d) exclude 0 since we are tracking/rejecting constant disturbances.

It must be pointed out that the results presented in this chapter for unknown mathematical models are highly realistic and robust. This is illustrated via experimental results at the conclusion of the dissertation.

Chapter 5

Servomechanism Problem: SISO LTQcR

In this chapter the servomechanism problem which was outlined by Problem 4.2.1 is considered. In particular, we consider Problem 4.2.1 under a nonnegative optimal control approach. Unlike in the previous chapter, here we assume that the plant model is known (i.e. we know the numerical values of $(A,\ b,\ c,\ d)$); in this case, we show, via a simulation example, that our results for tracking and disturbance rejection may be significantly improved over those presented for unknown models of Chapter 4 with the use of a Linear Tuning Quadratic clamping Regulator (LTQcR). However, unlike the case of standard LTI systems, we show that an arbitrarily fast response, as for example in perfect control type behaviour [20] for minimum phase systems, may not be attained.

The chapter is organized as follows. The system plant and the problem statement are restated, and the Linear Tuning Quadratic Regulator (LTQR) approach is defined. Section 5.2 provides the main results of the chapter based on nonnegative LTQR control. The chapter then shifts to an implementation discussion of a Linear Tuning Quadratic clamping Regulator (LTQcR) and a Reset Linear Tuning Quadratic clamping Regulator (RLTQcR), after which, a comparison of the TcR and the RLTQcR controllers is carried out. Concluding remarks finalize the chapter.

5.1 Preliminaries

In this section, we restate the details of the plant and the problem of interest introduced in Chapter 4.

Throughout this chapter we consider the following plant:

$$\begin{aligned} \dot{x} &= Ax + bu + e_\omega \omega \\ y &= cx + du + f\omega \\ e &:= y - y_{ref} \end{aligned} \tag{5.1}$$

where A is an $n \times n$ Metzler stable matrix, $b \in \mathbb{R}_+^n$, $c \in \mathbb{R}_+^{1\times n}$, $d \in \mathbb{R}_+$, $e_\omega \in \mathbb{R}_+^n$, $f \in \mathbb{R}_+$; the signal $y_{ref} \in Y_{ref} \subset \mathbb{R}_+$ is a constant, as is $\omega \in \Omega \subset \mathbb{R}_+$. The sets Y_{ref} and Ω are defined as in Assumption 4.1.1 of the previous chapter. The plant being considered is identical to the one presented in Chapter 4 equation (4.1) and is reintroduced here for ease of reference. The only difference made is in the definition of the error $e = y - y_{ref}$. The reason for doing this is to ease in the proof of the main result of this chapter.

With the above plant restated, we outline the problem of interest. Unlike in the previous chapter, where the control strategy was nonnegative and/or bidirectional, here we deal with nonnegative inputs only.

The positive servomechanism problem of interest in this section is stated below.

Problem 5.1.1. Consider the plant (5.1) where the disturbance ω is unmeasurable, the tracking signal y_{ref} and the state x are measurable, and the initial condition $x_0 \in \mathbb{R}_+^n$. Assume that Assumption 4.1.1 holds true.

Find an LTI controller connected as in the diagram (Figure 5.1) such that the closed-loop system satisfies

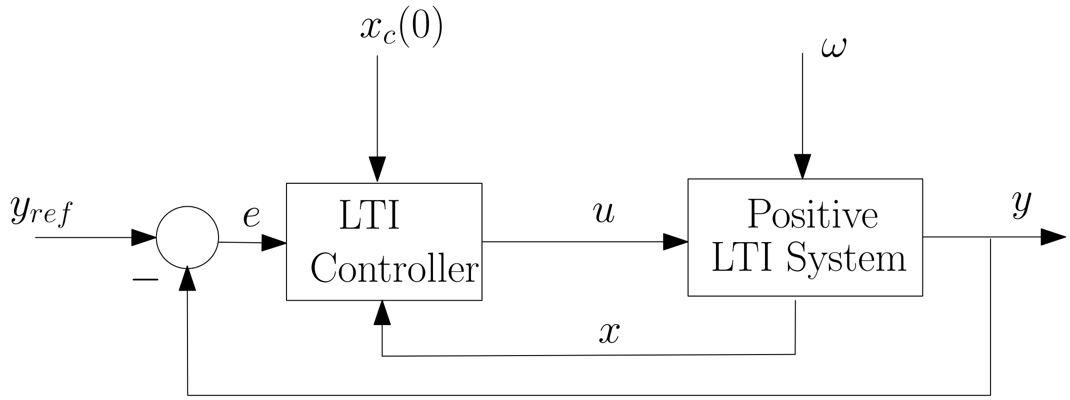


Figure 5.1: Closed-loop LTI system.

(a) asymptotic stability in the sense of Lyapunov with respect to the origin, and for every $y_{ref} \in Y_{ref}$, $\omega \in \Omega$, with the initial condition of the controller $x_c(0)$ having the property $u(0) \in \mathbb{R}_+$

(b) the states $x(t) \geq 0$, output $y(t) \geq 0$, and input $u(t) > 0 \; \forall \; t$; and

(c) tracking of the reference signal occurs, i.e. $e(t) = y(t) - y_{ref} \to 0$, as $t \to \infty$. In addition,

(d) assume that a controller has been found so that conditions (a), (b), (c) are satisfied; then for all perturbations of the nominal plant model which maintain properties (a) and (b), it is desired that the controller can still achieve asymptotic tracking and regulation, i.e. the controller is *robust* and property (c) still holds.

In the previous chapter, necessary conditions for Problem 5.1.1 were obtained ($d - cA^{-1}b \neq 0$ and $u_{ss} > 0$), below we make a remark regarding these conditions.

Remark 5.1.1. Throughout this chapter we will assume that the above necessary conditions hold true (i.e. we assume Assumption 4.1.1 holds true for the remainder of this chapter). Additionally, all definitions introduced in Chapter 4 carry over to this chapter.

The main distinction between Chapter 4 and the present chapter is the fact that here we assume the system model is *known*, i.e. the matrices (A, b, c, d) are given to the designer. This assumption will allow us to discuss less restrictive control strategies in the hope of improving the transient response and settling time of those given in the previous chapter. In particular, our focus will be to concentrate on a linear quadratic approach to Problem 5.1.1.

5.2 Servomechanism Problem: LTQR approach

In this section, the solution to Problem 5.1.1 under the LTQR controller is presented.

The Linear Tuning Quadratic Regulator (LTQR) of interest in this section and throughout this chapter is defined next.

The LTQR controller is given by:

$$\begin{aligned} \dot{\eta} &= y - y_{ref} \\ u &= [K_x \quad K_\eta] \begin{bmatrix} x \\ \eta \end{bmatrix}, \ u(0) > 0 \text{ and fixed} \end{aligned} , \tag{5.2}$$

where $K_x \in \mathbb{R}^{1 \times n}$ and $K_\eta \in \mathbb{R}$ are found by solving the expensive control problem [20], [16]:

$$\int_0^\infty \eta^T \eta + \rho^2 u^T u d\tau \tag{5.3}$$

where $\rho > 0$, for the system:

$$\begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A & 0 \\ c & 0 \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix} + \begin{bmatrix} b \\ d \end{bmatrix} u \tag{5.4}$$

$$\eta = [0 \quad 1] \begin{bmatrix} x \\ \eta \end{bmatrix} \tag{5.5}$$

with $y_{ref} = 0$ and $\omega = 0$.

For convenience, since we will be interested in letting $\rho \to \infty$, we re-write (5.3) as

$$\int_0^\infty \epsilon^2 \eta^T \eta + u^T u d\tau \tag{5.6}$$

where $\epsilon > 0$. In this case, since it has been assumed that A is stable and that $d - cA^{-1}b \neq 0$, it directly follows that the system (5.4) has the property that

$$rank \begin{pmatrix} A & b \\ c & d \end{pmatrix} = n + 1$$

which implies it is stabilizable, and by inspection, (5.4) is detectable. This implies that there exists a unique stabilizing optimal state feedback controller for (5.4) given by:

$$u = [K_x \ K_\eta] \begin{bmatrix} x \\ \eta \end{bmatrix}.$$

The latter control law can also be presented in a slightly different fashion, where $y_{ref} \neq 0$ and $\omega \neq 0$, which yields the same gain matrix as that of the control strategy above. Thus, we can replace the latter with

$$\int_0^\infty e^T e + \rho^2 \dot{u}^T \dot{u} d\tau \tag{5.7}$$

where $\rho > 0$, or

$$\int_0^\infty \epsilon^2 e^T e + \dot{u}^T \dot{u} d\tau, \tag{5.8}$$

where $\epsilon > 0$ for the system:

$$\begin{aligned} \begin{bmatrix} \ddot{x} \\ \dot{e} \end{bmatrix} &= \begin{bmatrix} A & 0 \\ c & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ e \end{bmatrix} + \begin{bmatrix} b \\ d \end{bmatrix} \dot{u} \\ e &= [0 \ \ 1] \begin{bmatrix} \dot{x} \\ e \end{bmatrix} \end{aligned}$$

and where

$$\dot{u} = [K_x \ K_\eta] \begin{bmatrix} \dot{x} \\ e \end{bmatrix}.$$

The above approach to the cheap control problem for LTI systems under constant tracking and disturbance signals has been justified in [16].

The main result under the LTQR is presented next.

Theorem 5.2.1. Consider system (5.1). Then for all $x(0) \in \mathbb{R}^n_+$ there exists an $\epsilon^*(x(0), u(0)) > 0$ such that for all $\epsilon \in (0, \epsilon^*(x(0), u(0))]$ the controller (5.2) solves Problem 5.1.1.

Through the remainder of this section, as was done in the previous Chapter, $\epsilon^*(x(0), \eta(0))$ will be denoted by simply ϵ^*.

The proof is given next.

Proof. The closed-loop system of the plant and the controller is given in Figure 5.2. The

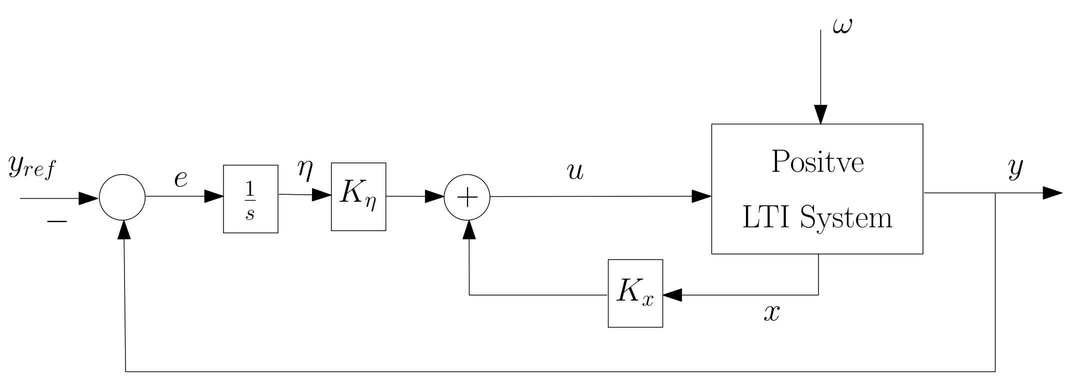


Figure 5.2: Closed-loop LTI system.

resulting closed-loop system is LTI and by the existence of a unique linear quadratic gain $K_\epsilon := [K_x \ K_\eta]$ there always exists an $\epsilon^+ > 0$ such that $\forall \ \epsilon \in (0, \epsilon^+]$ the closed-loop matrix is stable. This implies that all the conditions of Problem 4.2.1 will hold true if the nonnegativity condition (b) holds true. Through the remainder of the proof we assume $\epsilon \in (0, \epsilon^+]$, and without loss of generality let us assume ϵ^+ is fixed.

Let us recall the two key assumptions:

1. $u(0) > 0$ (by the definition of the LTQR controller (5.2));

2. $u_{ss} > 0$ (by the choice of Y_{ref} and Ω).

First, by the definition of positive LTI systems we know that if $u(t) \geq 0$ for all t, then the states $x(t)$ and the outputs $y(t)$ also remain nonnegative for all t. Let us show now that there exists an $\epsilon^* \leq \epsilon^+$ such that for all $\epsilon \in (0, \epsilon^*]$, $u(t) > 0$ for all t under the two assumptions listed above.

Under the set up of the closed-loop system the steady states x_{ss}, u_{ss} are independent of ϵ; therefore, using the results of the Appendix (Section 5.4), we can shift the closed-loop system by setting $z = x - x_{ss}$ and $q = u - u_{ss}$, to obtain the resultant closed-loop system:

$$\begin{bmatrix} \dot{z} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} A & b \\ \epsilon(\overline{K}_x A - c) & \epsilon(\overline{K}_x b - d) \end{bmatrix} \begin{bmatrix} z \\ q \end{bmatrix} \tag{5.9}$$

where $[K_x \ K_\eta] = \epsilon[\overline{K}_x(\epsilon) \quad -1]$, where $lim_{\epsilon \to 0} \overline{K}_x(\epsilon)$ exists[1].

For convenience, rewrite

$$\begin{bmatrix} \dot{q} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \epsilon(\overline{K}_x b - d) & \epsilon(\overline{K}_x A - c) \\ b & A \end{bmatrix} \begin{bmatrix} q \\ z \end{bmatrix}, \tag{5.10}$$

and note that if $q(t, \epsilon) + u_{ss} > 0$ for all t, then $u(t) > 0$ for all t.

Next, let us scale the derivatives by $\epsilon dt = d\tau$ (i.e. scaling of time $\epsilon t = \tau$) resulting in the transformed system

[1] The dependence of $\overline{K}_x(\epsilon)$ on ϵ will be dropped for the remainder of the thesis.

$$\begin{bmatrix} \overset{\odot}{q} \\ \epsilon\overset{\odot}{z} \end{bmatrix} = \begin{bmatrix} (\overline{K}_x b - d) & (\overline{K}_x A - c) \\ b & A \end{bmatrix} \begin{bmatrix} q \\ z \end{bmatrix}, \tag{5.11}$$

with $\epsilon\overset{\odot}{q} = \dot{q}$ and $\epsilon\overset{\odot}{z} = \dot{z}$.

Notice that if $q(\tau,\epsilon)+u_{ss} > 0$ for all τ, then $q(t,\epsilon)+u_{ss} > 0$ for all t and consequently $u(t) > 0$ for all t. Therefore, it remains to show that indeed $q(\tau,\epsilon)+u_{ss} > 0$ for all τ.

We now observe that we have transformed our system (5.11) into that of a singular perturbation model (SP).

Next, we must show that all assumptions of singular perturbation hold true. However, since (5.11) is linear and time invariant, and we are only interested in u, it suffices to show that the reduced model (see below) yields exponential stability; all other assumptions clearly hold, including the continuous differentiability with respect to ϵ of $\overline{K}_x$ around the origin (see Section 5.4 for details) and the stability of the boundry-layer model, which is

$$\dot{p} = Ap$$

with A stable.

Next, by setting $\epsilon = 0$ in (5.11) we obtain

$$z = h(q) = -A^{-1}bq,$$

as A is Hurwitz, $h(q)$ exists and is unique, and by substituting $h(q)$ into $\overset{\odot}{q}$ we obtain the reduced model:

$$\begin{aligned} \overset{\odot}{q} &= (\overline{K}_x b - d)q - (\overline{K}_x A - c)A^{-1}bq \\ &= -(d - cA^{-1}b)q \text{ where by Lemma 4.1.2 } (d - cA^{-1}b) > 0. \end{aligned}$$

Denote the solution of $\overset{\odot}{q} = -(d - cA^{-1}b)q$ by $\overline{q}(\tau)$, which is clearly exponentially stable (as needed by SP) and monotonic. Thus, by SP we have, as in Chapter 4:

$$q(\tau, \epsilon) - \overline{q}(\tau) = O(\epsilon) \ \forall \tau$$

uniformly in τ, where

$$\begin{aligned} \overline{q}(\tau) &= e^{-(d-cA^{-1}b)\tau}\overline{q}(0) \text{ and} \\ \overline{q}(\tau) + u_{ss} &= u_{ss} + e^{-(d-cA^{-1}b)\tau}\overline{q}(0) \end{aligned}$$

with $\overline{q}(0) = q(0) = u(0) - u_{ss}$ by definition. Now, since $u(0) > 0$, then there exists an $\epsilon^* \leq \epsilon^+$ such that $q(\tau, \epsilon) + u_{ss} > 0$ for all $\epsilon \in (0, \epsilon^*]$ since $\overline{q}(\tau) + u_{ss}$ is *monotonically* approaching u_{ss}.

This completes the proof. □

The latter problem can be interpreted as the tracking and disturbance rejection problem under nonnegative LTQR control for positive LTI plants experiencing unmeasurable constant nonnegative disturbances.

Before we depart this section, we consider a Corollary to Theorem 5.2.1 with the extra assumption that $0 \leq u(t) \leq \overline{u}$, $\overline{u} > 0$ fixed, for all $t \in [0, \infty)$.

Corollary 5.2.1. Consider system (5.1) and controller (5.2) where $0 < u(0) < \overline{u}$, $0 < u_{ss} < \overline{u}$, $\overline{u} > 0$ fixed. Then for all $x(0) \in \mathbb{R}^n_+$ there exists an $\epsilon^*(x(0), u(0)) > 0$ such that for all $\epsilon \in (0, \epsilon^*(x(0), u(0))]$ the controller (5.2) solves Problem 5.1.1 with condition (b) replaced by

(b') the states $x(t) \geq 0$, output $y(t) \geq 0$ and input $0 < u(t) < \overline{u}$.

The latter Corollary simply states that we would like to bound our input signal both from below by zero and from above by some constant $\overline{u}$. The proof is omitted as it follows similar guidelines as the results of Chapter 4 (see Corollary 4.2.1).

A very interesting question now arises with the use of the controller that we proposed; namely, how large can we make ϵ? Although our controller can be used to solve Problem 5.1.1, we do not have direct knowledge of how large our ϵ can be. In fact, we point out in the next subsection that if we make ϵ too large then the control objective can in fact no longer be satisfied.

5.3 Implementation

In the remainder of this chapter, implementation of the theory presented in the previous sections is considered. In particular, we consider linear tuning quadratic clamping regulators (LTQcR) and reset linear tuning quadratic clamping regulators (RLTQcR) to prevent the input signals from going negative.

First, however, we consider two results. The first result pertains to Figure 5.3, while the second to Figure 5.4.

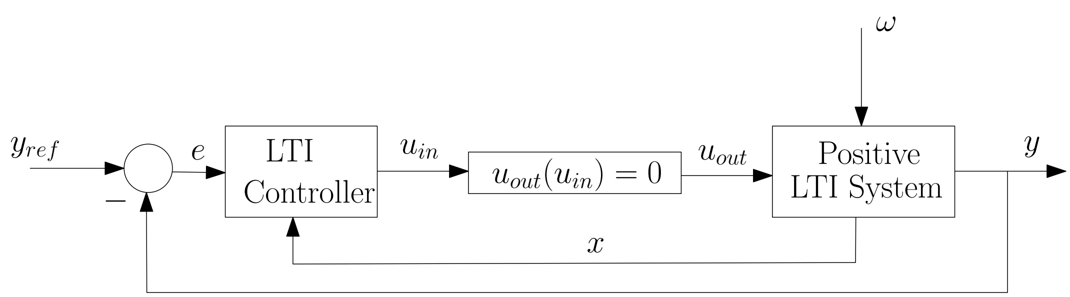


Figure 5.3: Implementation system 1.

Theorem 5.3.1. Consider the system of Figure 5.3 where the positive LTI system is represented as (5.1) and where the block diagram of Figure 5.5 represents the LTI controller. Assume $y_{ref} \in \mathbb{R}_+$, $\omega \in \mathbb{R}_+$, and $d - cA^{-1}b \neq 0$.

Then, for all $\epsilon > 0$, and all $x(0) \in \mathbb{R}^n_+$ there exists a $t^*(x(0), \epsilon) \geq 0$ such that for all $t \in [t^*(x(0), \epsilon), \infty)$

$$u_{in}(t) > 0$$

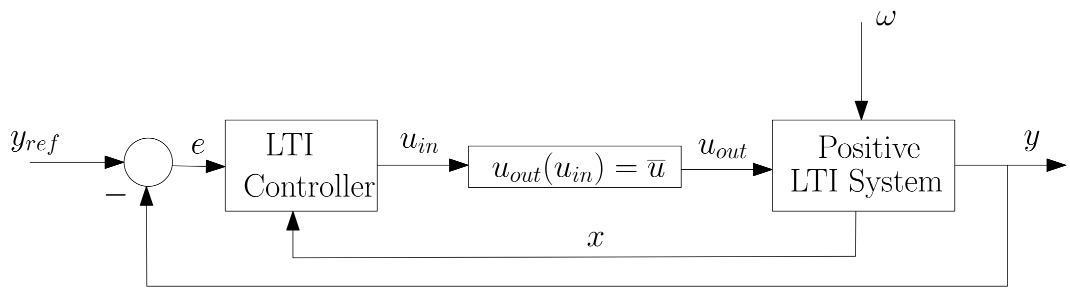


Figure 5.4: Implementation system 2.

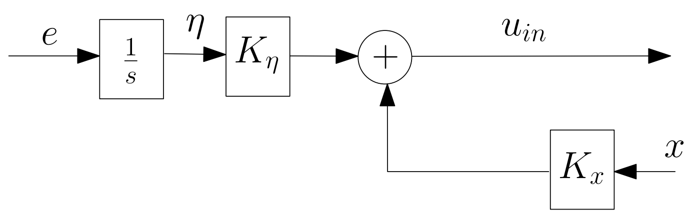


Figure 5.5: LTI controller.

if and only if

$$u_{ss} > 0.$$

Proof. The overall system is:

$$\begin{aligned}
\dot{x} &= Ax + e_\omega \omega \\
\dot{u}_{in} &= K_x(Ax + e_\omega \omega) + K_\eta(y - y_{ref}) \\
y &= cx + f\omega.
\end{aligned}$$

As $t \to \infty$ we have

$$x \to \overline{x}_{ss} = -A^{-1}e_\omega \omega \text{ since A is stable.}$$

Recall,

$$x_{ss} = -A^{-1}(bu_{ss} + e_\omega \omega).$$

Thus, we can rewrite $\overline{x}_{ss}$ as

$$\overline{x}_{ss} = x_{ss} + A^{-1}bu_{ss}.$$

Next, since $x \to \overline{x}$ as $t \to \infty$ then

$$\begin{aligned}
\dot{u}_{in} &= K_x\dot{x} + K_\eta\dot{e} \\
&= K_x(Ax + e_\omega\omega) + K_\eta(y - y_{ref}) \\
\dot{u}_{in} &\to K_x(A\overline{x}_{ss} + e_\omega\omega) + K_\eta(c\overline{x}_{ss} + d(0) - y_{ref}) \text{ as } t \to \infty
\end{aligned}$$

$$\begin{aligned}
K_x(A\overline{x}_{ss} + e_\omega\omega)\ + & \\
K_\eta(c\overline{x}_{ss} + d(0) - y_{ref}) &= K_x(0) + K_\eta(c(x_{ss} + A^{-1}bu_{ss}) + d(0) - y_{ref}) \\
&= K_\eta(c(x_{ss} + A^{-1}bu_{ss}) + d(u_{ss} - u_{ss}) - y_{ref}) \\
&= K_\eta(cx_{ss} + du_{ss} - y_{ref}) - K_\eta(d - cA^{-1}b)u_{ss}) \\
&= 0 + -K_\eta(d - cA^{-1}b)u_{ss} \\
&= \epsilon(d - cA^{-1}b)u_{ss} \text{ where from Section 5.4 } K_\eta = -\epsilon
\end{aligned}$$

Thus, clearly since the derivative tends to $\epsilon(d - cA^{-1}b)u_{ss} > 0 \iff u_{ss} > 0$ there exists a $t^*(x(0), \epsilon) \geq 0$ such that for all $t \in [t^*(x(0), \epsilon), \infty)$

$$u_{in}(t) > 0.$$

$\square$

We now come back to the system of Figure 5.4 and present a corollary to the latter theorem.

Corollary 5.3.1. Consider the setup of Figure 5.4 where the positive LTI system is represented as (5.1) and where the block diagram of Figure 5.5 represents the LTI controller. Assume $y_{ref} \in \mathbb{R}_+$, $\omega \in \mathbb{R}_+$, and $d - cA^{-1}b \neq 0$.

Then, for all $\epsilon > 0$, and all $x(0) \in \mathbb{R}^n_+$ there exists a $t^*(x(0), \epsilon) \geq 0$ such that for all $t \in [t^*(x(0), \epsilon), \infty)$

$$u_{in}(t) < \overline{u}$$

if and only if

$$u_{ss} < \overline{u}.$$

The proof follows similar guidelines as the proof of Theorem 5.3.1 and is omitted.

Next, from the results of Theorem 5.2.1 and Theorem 5.3.1 we propose a linear tuning quadratic clamping regulator (LTQcR), which clamps all signals at zero or employs the LTQR whenever the input signal is positive. The LTQcR is presented below.

$$\begin{aligned} \dot{\eta} &= y - y_{ref} \\ u &= max\{[K_x \quad K_\eta] \begin{bmatrix} x \\ \eta \end{bmatrix}, 0\},\ u(0) > 0 \text{ and fixed} \end{aligned}, \tag{5.12}$$

where K_x and K_η are defined as in the LTQR (5.2).

The introduction of the LTQcR is necessary as in Theorem 5.2.1 we do not know how small the ϵ^* should be due to unmeasurable disturbances, thus, the input u_{ss} is actually unknown. However, by Theorem 5.3.1 we can deduce that under the LTQcR controller if $u_{ss} \leq 0$, then our controller will "shut itself off" in finite time and remain shut off and if $u_{ss} > 0$ we know that there exists a time $t^*(x(0), \epsilon)$ such that the trajectory will return to the linear region.

The next result takes into account the information presented by Corollary 5.2.1, Theorem 5.3.1, and Corollary 5.3.1.

In the case of the RTcR controller we implemented an "anti-reset" wind-up like behaviour. The proposed RLTQcR controller will serve the same purpose.

$$\begin{aligned} \dot{\eta} &= y - y_{ref} \text{ if } (u > 0) \text{ or } (u = 0 \text{ and } e \leq 0) \\ \dot{\eta} &= 0 \text{ otherwise} \\ u &= max\{[K_x \quad K_\eta] \begin{bmatrix} x \\ \eta \end{bmatrix}, 0\},\ u(0) > 0 \text{ and fixed} \end{aligned}, \tag{5.13}$$

where, once again, K_x and K_η are defined as in the LTQR (5.2). The addition of $\overline{u} > 0$ as in the RTcR can easily be incorporated and thus taking advantage of Corollary 5.3.1 (see Remark 5.3.1).

Both the LTcQR and RLTcQR controllers use the theoretical results of this chapter to implement controllers that will be used in a simulated example in this chapter and on experimental results in Chapter 9.

We now turn our attention to perfect LTI control and an illustrative example.

5.3.1 Perfect Control

In this section a discussion of perfect control [20] for positive LTI systems is considered. Our focus will be to show, via an example, that one may not be able to obtain arbitrarily good approximate error regulation with respect to (5.1) and (5.8), or more precisely we may not be able to satisfy the conditions:

- $lim_{\epsilon \to \infty}\ e(t, \epsilon) = 0,\ t > 0$
- $e(t, 0) = 0,\ t > 0$
- $e(t, \epsilon)$ has no unbounded peaking as $\epsilon \to \infty$,

just as in the case of LTI systems [20]. In particular, we would like to provide an example of where the controller with clamping (LTQcR, proposed in the previous subsection) cannot achieve arbitrarily fast response as $\epsilon \to \infty$. The main goal of the example is to show that we cannot simply take an LTI approach, where we design a gain matrix K_ϵ, then clamp the system, and expect things to be stable and have error regulation.

We provide an example of a fluid exchange system where our controller with clamping fails to solve the servomechanism problem 5.1.1, as $\epsilon \to \infty$. It is worth pointing out that the system in the example is minimum phase, which implies that in the LTI sense perfect control can be attained!

Let us consider the example of interest next.

Example 5.3.1. The following system (shown in Figure 5.6) has been introduced in the previous chapter. The details of its setup can be found in Example 4.6.2.

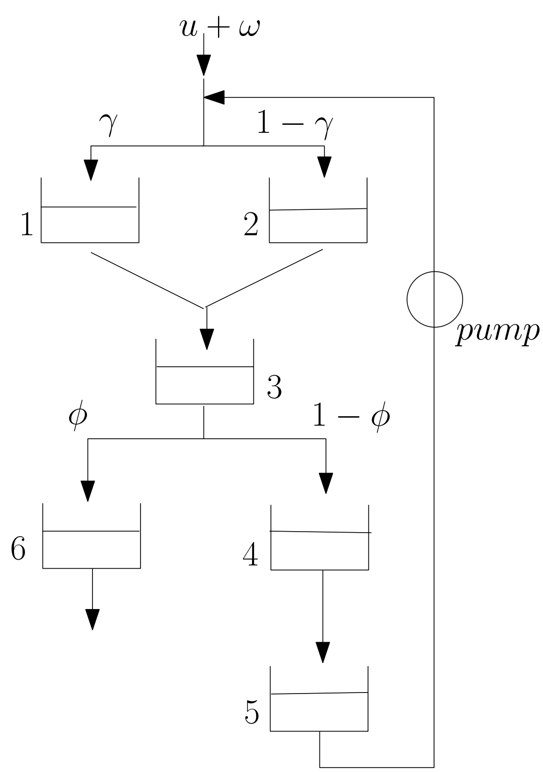


Figure 5.6: System set up for Example 5.3.1.

Consider the case where $\gamma = 0.5$, $\phi = 0.7$, $\alpha_1 = 0.8$, $\alpha_2 = 0.7$, $\alpha_3 = 0.5$, $\alpha_4 = 1$, $\alpha_5 = 2$, $\alpha_6 = 0.8$, and $\omega = 0$. Note that all the rates are measured in L/s. With the variables defined above we obtain the following system:

$$\dot{x} = \begin{bmatrix} -0.8 & 0 & 0 & 0 & 2 & 0 \\ 0 & -0.7 & 0 & 0 & 0 & 0 \\ 0.8 & 0.7 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.15 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0.35 & 0 & 0 & -0.8 \end{bmatrix} x + \begin{bmatrix} .5 \\ .5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u. \tag{5.14}$$

$$y = [0\ 0\ 0\ 0\ 0\ 1]x.$$

It is easy to show that the above compartmental system is stable, as

$$\sigma(A) = \{-0.8, \ -0.2112, \ -0.9924 \pm 0.5249i, \ -2.1039, \ -0.7000\}.$$

Also, the system is minimum phase (required to achieve perfect control [20]), where the zeros are: $-2, \ -0.74667, \ -1$. Assume that the initial condition $x_0 = 0$, and that we would like to track the reference input $y_{ref} = 1$. By applying the LQcR controller with $\epsilon = 1e3$ we illustrate that the output y clearly is not tracking the desired reference (see figure 5.7 for the output response). The input is shown in Figure 5.8. The above example answers an interesting question, namely, perfect control using controller (5.12) for minimum phase systems is not necessarily possible to obtain. Note: although the choice of $\epsilon = 1 \times 10^3$ does not work, if one reduces this value (say to 1×10^2) Problem 5.1.1 now becomes solvable. The plots are shown in Figure 5.9 and 5.10; notice that although the response is satisfactory for the problem statement, the result is excessively oscillatory.

We now return to the same example under the RLTQcR (5.13).

We provide the response of the output and input in Figure 5.11 and Figure 5.12 for the same choice of $\epsilon = 1e3$.

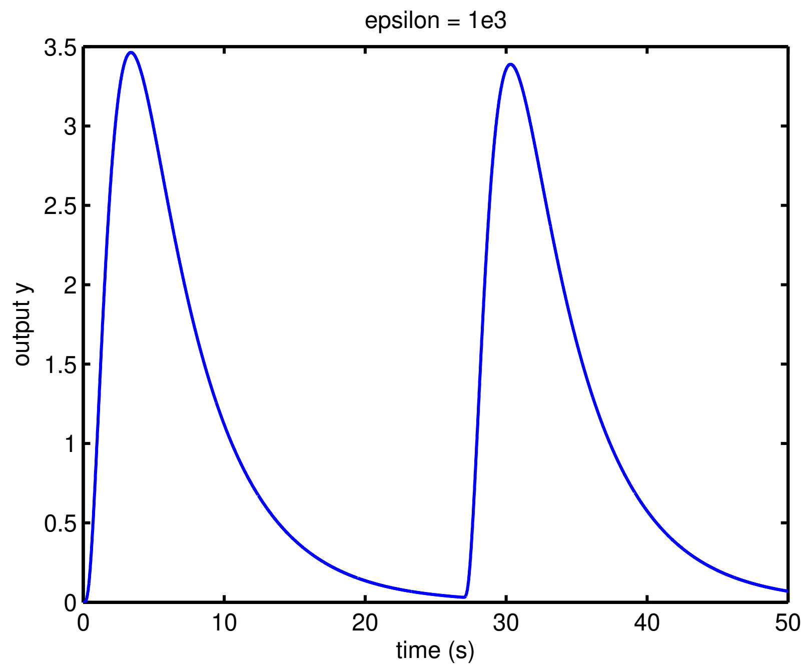


Figure 5.7: Output response for $\epsilon = 1e3$.

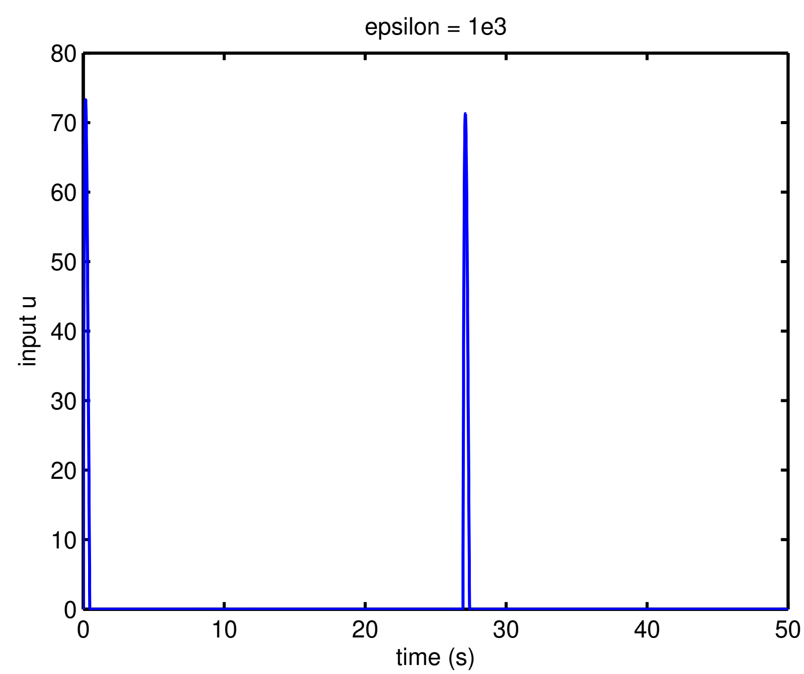


Figure 5.8: Input response for $\epsilon = 1e3$.

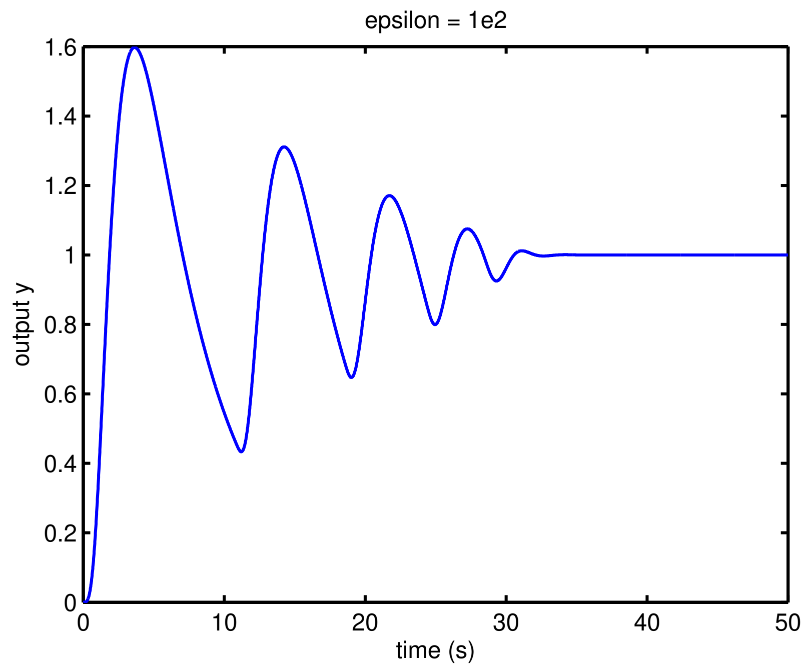


Figure 5.9: Output response for $\epsilon = 1e2$.

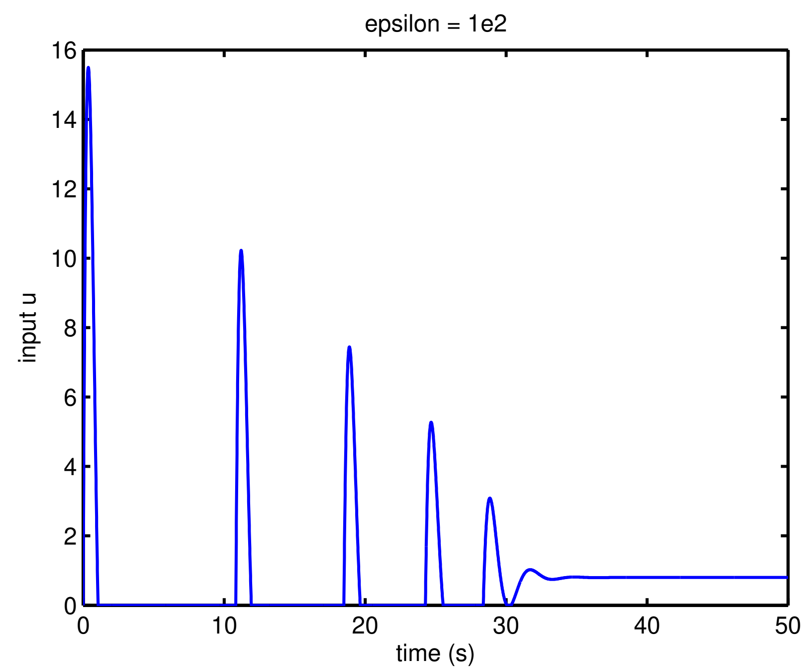


Figure 5.10: Input response for $\epsilon = 1e2$.

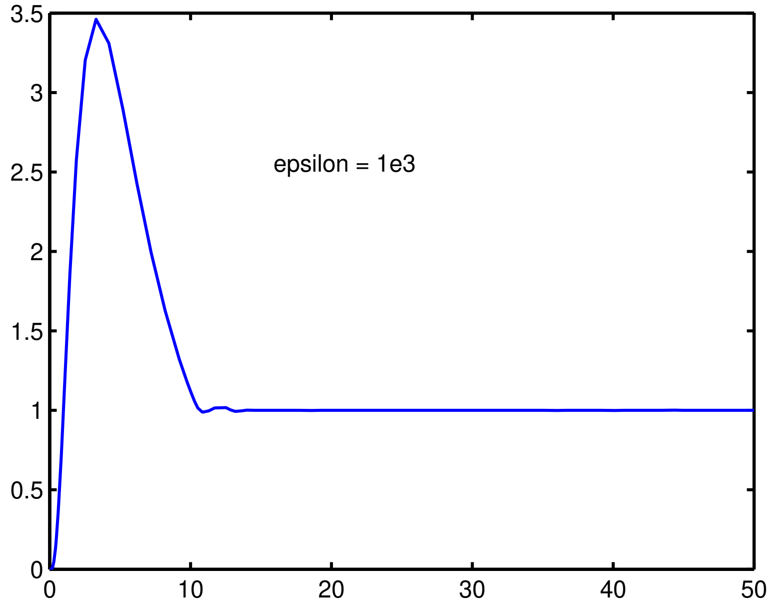


Figure 5.11: Output response with RLQcR control.

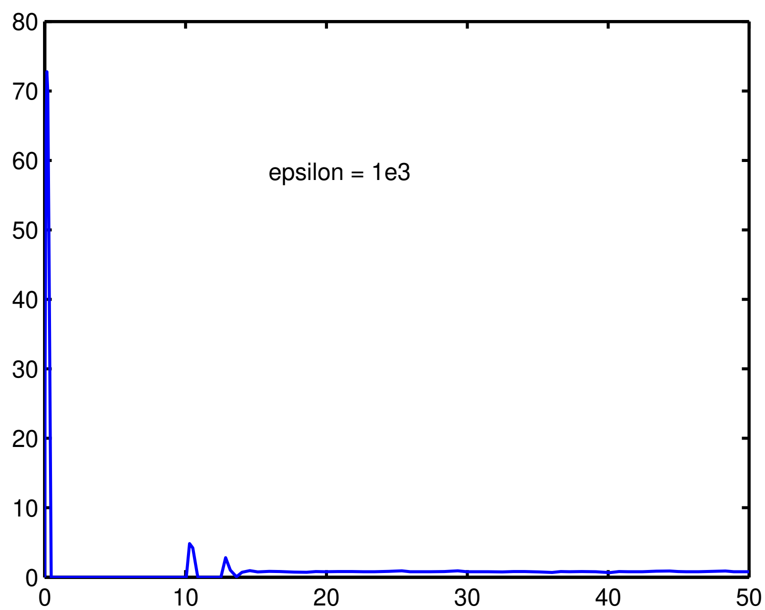


Figure 5.12: Input response with RLQcR control.

Notice the huge benefit of using the above RLTQcR controller over that of the LTQcR (5.12). Additionally, we point out that the response of the RLTQcR controller is far superior to that of the RTcR. The result for the RTcR, which yields the best settling time for the same example, are provided in Figure 5.13. The overshoot is much lower, but the settling time is six times slower than that of the RLQcR.

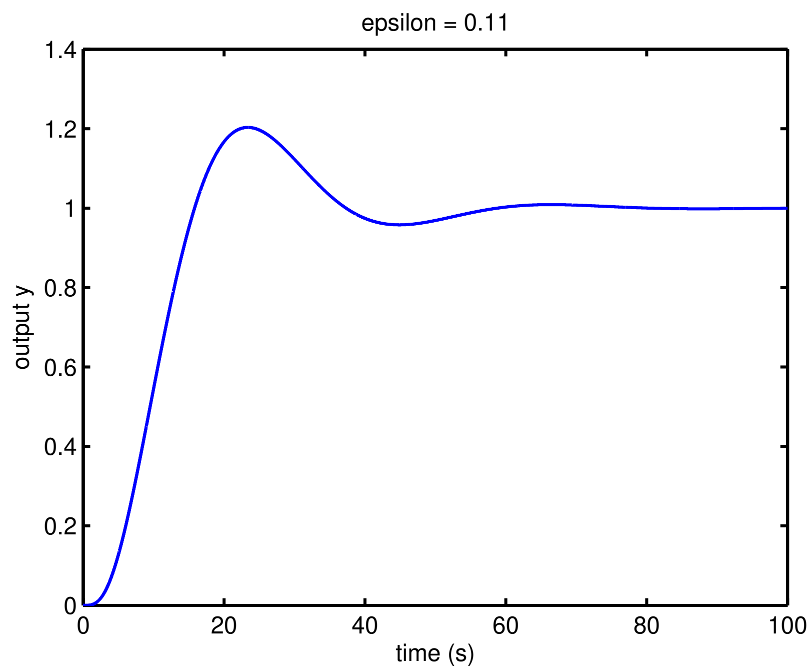


Figure 5.13: Output response with reset anti-windup for TcR $\epsilon = 0.11$.

△

Before we depart this section a remark regarding saturating signals is given.

Remark 5.3.1. If control signal saturation from above is to be considered then we can replace the RLQcR controller with

$$\begin{array}{ll} & \text{if } ((0 < u < \overline{u}) \text{ or} \\ \dot{\eta} = y - y_{ref} & (u = 0 \text{ and } e < 0) \text{ or} \\ & (u = \overline{u} \text{ and } e > 0)) \\ & \\ \dot{\eta} = 0 & \text{else,} \end{array} \tag{5.15}$$

with

$$u = \begin{cases} 0 & [K_x \;\; K_\eta][x \;\; \eta]^T \leq 0 \\ [K_x \;\; K_\eta]\,[x \;\; \eta]^T & 0 < [K_x \;\; K_\eta][x \;\; \eta]^T < \overline{u} \\ \overline{u} & [K_x \;\; K_\eta][x \;\; \eta]^T \geq \overline{u} \end{cases},$$

with $0 < u(0) < \overline{u}$ and fixed.

This control strategy will play a crucial role in the experimental results presented at the end of this thesis.

5.4 Appendix

In this section, we discuss some results that were used in this chapter without justification.

Some preliminary results are initially given. The ultimate goal of this introductory material is to show that the optimal gain matrix K_ϵ has the property that

$$K_\epsilon = [K_x \; K_\eta] = \epsilon \overline{K}, \tag{5.16}$$

where $K_\eta = -\epsilon$ and where the components of $\overline{K}$ are

$$\overline{K}_{1r} = \left(\frac{\zeta_0 + \zeta_1 \epsilon + \zeta_2 \epsilon^2 + ...}{1 + \gamma_1 \epsilon + \gamma_2 \epsilon^2 + ...} \right)$$

with $r \in \{1, ..., n+1\}$, $\zeta_i \in \mathbb{R}$, $\gamma_j \in \mathbb{R}$ $\forall i, j \in \{0, 1, 2, ...\}$. Note that $\overline{K}$ is continuously differentiable with respect to ϵ near and at the origin!

The first introductory result is provided next.

Lemma 5.4.1. Let $M \in \mathbb{R}^{n\times n}$ and $N(\epsilon) \in \mathbb{R}^{n\times n}$ with each element of $N(\epsilon)$ defined by

$$N_{ij}(\epsilon) = \frac{\alpha_1^{ij}\epsilon + \alpha_2^{ij}\epsilon^2 + ...}{1 + \beta_1^{ij}\epsilon + \beta_2^{ij}\epsilon^2 + ...}$$

where $\alpha_s^{ij} \in \mathbb{R}$, $\beta_r^{ij} \in \mathbb{R}$, and $\epsilon \in (0,1)$, for all $i, j, s, r \in \{1, 2, ...\}$. Then the determinant of $M + N(\epsilon)$ is

$$|M + N(\epsilon)| = \frac{\tilde{\alpha}_0^{ij} + \tilde{\alpha}_1^{ij}\epsilon + \tilde{\alpha}_2^{ij}\epsilon^2 + ...}{1 + \tilde{\beta}_1^{ij}\epsilon + \tilde{\beta}_2^{ij}\epsilon^2 + ...}$$

with all constants defined as above, i.e. real, and $\tilde{\alpha}_0^{ij} \in \mathbb{R}$.

Moreover, if $\epsilon \to 0$ and if the determinant of M is nonzero, then $\tilde{\alpha}_0^{ij} \neq 0$.

Proof. We will prove the result via induction.

1. For $n = 1$ we have $M \in \mathbb{R}^{1\times 1}$, $N(\epsilon) \in \mathbb{R}^{1\times 1}$ and $|M + N(\epsilon)|$

$$\begin{aligned} &= M + \frac{\alpha_1^{11}\epsilon + \alpha_2^{11}\epsilon^2 + ...}{1 + \beta_1^{11}\epsilon + \beta_2^{11}\epsilon^2 + ...} \\ &= \frac{M(1 + \beta_1^{11}\epsilon + \beta_2^{11}\epsilon^2 + ...) + (\alpha_1^{11}\epsilon + \alpha_2^{11}\epsilon^2 + ...)}{1 + \beta_1^{11}\epsilon + \beta_2^{11}\epsilon^2 + ...} \\ &= \frac{M + (\alpha_1^{11} + M\beta_1^{11})\epsilon + (\alpha_2^{11} + M\beta_2^{11})\epsilon^2 + ...}{1 + \beta_1^{11}\epsilon + \beta_2^{11}\epsilon^2 + ...}. \end{aligned}$$

   This completes the case for $n = 1$.

2. Next, assume $n = k$ holds true.

3. Let's now show that the result also holds for $n = k + 1$. Define

$$M + N(\epsilon) := \begin{bmatrix} \overline{a}_{11} & \overline{a}_{12} & \dots & \overline{a}_{1(k+1)} \\ \overline{a}_{21} & \overline{a}_{22} & \dots & \overline{a}_{2(k+1)} \\ \vdots & & \ddots & \vdots \\ \overline{a}_{(k+1)1} & \overline{a}_{(k+1)2} & \dots & \overline{a}_{(k+1)(k+1)} \end{bmatrix}.$$

The determinant becomes:

$$|M + N(\epsilon)| = \overline{a}_{11} \begin{vmatrix} \overline{a}_{22} & \dots & \overline{a}_{2(k+1)} \\ \vdots & \ddots & \vdots \\ \overline{a}_{(k+1)2} & \dots & \overline{a}_{(k+1)(k+1)} \end{vmatrix} + \dots$$
$$+(-1)^{1+n}\overline{a}_{1(k+1)} \begin{vmatrix} \overline{a}_{21} & \dots & \overline{a}_{2k} \\ \vdots & \ddots & \vdots \\ \overline{a}_{(k+1)2} & \dots & \overline{a}_{(k+1)k} \end{vmatrix}.$$

It is now easy to deduce that all determinants above are of size k, and since addition and multiplication are the only operations involved, the result follows.

The second statement: if $\epsilon \longrightarrow 0$ and the determinant of M is nonzero, then $\tilde{\alpha}_0^{ij} \neq 0$, is a direct result of the above proof and matrix perturbation theory [81]. □

Remark 5.4.1. Notice from the above result, under the assumption that the determinant of M is nonzero, that the inverse of the matrix $(M+N(\epsilon))^{-1}$, where $\epsilon \longrightarrow 0$, always exists; and moreover, all components will be of the form given by

$$\overline{a}_{ij}^{\epsilon} = \frac{\alpha_0^{ij} + \alpha_1^{ij}\epsilon + \alpha_2^{ij}\epsilon^2 + \dots}{1 + \beta_1^{ij}\epsilon + \beta_2^{ij}\epsilon^2 + \dots}.$$

We now come back to showing that

$$K_\epsilon = [K_x \ K_\eta] = \epsilon[\overline{K}_x \ \overline{K}_\eta].$$

where $lim_{\epsilon \to 0}[\overline{K}_x \ \overline{K}_\eta]$ exists. First, let's show that

$$K_\eta = -\epsilon.$$

Recall that in order to obtain the gain matrix K_ϵ and consequently K_η we need to solve the continuous algebraic Riccati equation for (5.8):

$$\tilde{A}^T P + P\tilde{A} - P\tilde{B}R^{-1}\tilde{B}^T P + Q = 0 \tag{5.17}$$

with $R^{-1} = 1$,

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & \epsilon^2 \end{bmatrix},$$

$$\tilde{A} = \begin{bmatrix} A & 0 \\ c & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} b \\ 0 \end{bmatrix}.$$

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix},$$

with appropriate dimensions. Moreover,

$$K_\epsilon = [K_x \;\; K_\eta] = -\tilde{B}P \Rightarrow \;\; K_x = -b^T P_{11} \;\;\; K_\eta = -b^T P_{12}.$$

By manipulation of (5.17) we obtain

$$P_{12}^T b b^T P_{12} \;\; = \;\; \epsilon^2,$$

but $P_{12}^T b = b^T P_{12}$ (note, they are scalars) so we have

$$\begin{aligned} (b^T P_{12})^2 &= \epsilon^2 \\ (K_\eta)^2 &= \epsilon^2 \\ K_\eta &= \pm\epsilon \end{aligned}$$

Let us show that K_η must be negative (i.e. $K_\eta = -\epsilon$). In order to show that $K_\eta < 0$, we will use one key result for stable Metzler matrices, which is justified by Lemma 3.2.1. Namely, every principal submatrix of a stable Metzler matrix is itself stable. First, without loss of generality let's consider the case when $\epsilon \to 0$; then the determinant of the closed-loop matrix is

$$det \begin{bmatrix} A + bK_x & bK_\eta \\ c + dK_x & dK_\eta \end{bmatrix} \begin{cases} > 0, & n \text{ odd} \\ < 0, & n \text{ even} \end{cases},$$

since $K_\epsilon = [K_x \quad K_\eta] \to 0$ for $\epsilon \to 0$ and the closed loop matrix must be stable by lqr design. Now, notice that if

$$det \begin{bmatrix} A & b \\ c & d \end{bmatrix} \begin{cases} < 0, & n \text{ odd} \\ > 0, & n \text{ even} \end{cases}, \tag{5.18}$$

then

$$det \begin{bmatrix} A + bK_x & b \\ c + dK_x & d \end{bmatrix} \begin{cases} < 0, & n \text{ odd} \\ > 0, & n \text{ even} \end{cases}$$

for sufficiently small K_x. However, if the above is true then since

$$det \begin{bmatrix} A + bK_x & bK_\eta \\ c + dK_x & dK_\eta \end{bmatrix} = K_\eta det \begin{bmatrix} A + bK_x & b \\ c + cK_x & d \end{bmatrix}$$

it follows that $K_\eta < 0$. Therefore, it suffices to show that (5.18) holds true. We will show this by induction.

(1) Let's show that (5.18) holds true for $k = 1, 2$. In this case (5.18) claims that the result should be negative for $k = 1$ and positive for $k = 2$. Indeed the determinant for $k = 1$ is equal to $Ad - bc < 0$, since A is a negative scalar and all other variables are positive. Note that $bc \neq 0$ because $(d - cA^{-1}b) \neq 0$.

In the case of $k = 2$, we have (along the last row)

$$\begin{aligned} det \begin{bmatrix} A & b \\ c & d \end{bmatrix} &= c_1(a_{12}b_2 - a_{22}b_1) - c_2(a_{11}b_2 - a_{21}b_1) + d \times det(A) \\ &\geq 0. \end{aligned}$$

The summation along the last row is nonnegative by the assumption that A is stable, and all other matrices (b, c, d) are nonnegative. The result that the determinant is positive follows from the assumption that $(d - cA^{-1}b) \neq 0$. This concludes both $k = 1$ and $k = 2$.

(2) Assume that (5.18) holds true for $k = n - 1$ for both n odd and even.

(3) Let's now show that (5.18) holds true for $k = n$. Let's compute the determinant by going along the last row of the matrix

$$\begin{bmatrix} A & b \\ c & d \end{bmatrix}.$$

Now notice that

$$det \begin{bmatrix} A & b \\ c & d \end{bmatrix} =$$

$$\sum_{i=1}^{n}(-1)^{n+1+i}c_i \times det(A_i) + d \times det(A),$$

where $A_i =$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1(i-1)} & a_{1(i+1)} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2(i-1)} & a_{2(i+1)} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \\ a_{i1} & a_{i2} & \dots & a_{i(i-1)} & a_{i(i+1)} & \dots & a_{in} & b_i \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \\ a_{n1} & a_{n2} & \dots & a_{n(i-1)} & a_{n(i+1)} & \dots & a_{nn} & b_n \end{bmatrix}$$

and where, by rearrangement, $det(A_i) = (-1)^{n-i}$ multiplied by the determinant of

$$\begin{bmatrix} a_{11} & \dots & a_{1(i-1)} & a_{1(i+1)} & \dots & a_{1n} & b_1 \\ a_{21} & \dots & a_{2(i-1)} & a_{2(i+1)} & \dots & a_{2n} & b_2 \\ \vdots & & \vdots & \vdots & & \vdots & \\ a_{(i-1)1} & \dots & \dots & \dots & \dots & a_{(i-1)n} & b_{i-1} \\ a_{(i+1)1} & \dots & \dots & \dots & \dots & a_{(i+1)n} & b_{i+1} \\ \vdots & & \vdots & \vdots & & \vdots & \\ a_{n1} & \dots & a_{n(i-1)} & a_{n(i+1)} & \dots & a_{nn} & b_n \\ a_{i1} & \dots & a_{i(i-1)} & a_{i(i+1)} & \dots & a_{in} & b_i \end{bmatrix}$$

but the latter rearrangement (which we will refer to as A_i^*) is just in the form of $k = n-1$ which by assumption holds true or is equal to zero. Thus, we have

$$det\begin{bmatrix} A & b \\ c & d \end{bmatrix} = \sum_{i=1}^{n} -c_i \times det(A_i^*) + d \times det(A),$$

which is positive if n is even or negative if n is odd. Hence, indeed $K_\eta < 0$ and by the results presented thus far

$$K_\eta = -\epsilon.$$

With the knowledge above we now move to study the result for K_x. For convenience, but without loss of generality, we assume $d = 0$. Here we use the help of [87]. In [87] it has been pointed out how one can recursively find the nonnegative symmetric and unique P that solves (5.17) by following the steps:

1. Choose K_1 so that $\tilde{A} + \tilde{B}K_1$ is stable.

2. Having chosen $K_1, ..., K_k$ obtain P_k from

$$(\tilde{A} + \tilde{B}K_k)^T P_k + P_k(\tilde{A} + \tilde{B}K_k) + Q + K_k^T K_k = 0 \tag{5.19}$$

3. Define $K_{k+1} := \tilde{B}^T P_k$.

We will show the result for $K_x = \epsilon\overline{K_x}$ holds true by induction with

$$K_1 = [0 \quad -\epsilon].$$

By the discussion of Chapter 4 (and [17]) we know that there exists an ϵ^* such that for all $\epsilon \in (0, \epsilon^*]$ K_1 stabilizes $\tilde{A}$. Also, K_1 clearly satisfies (5.16). Therefore, we have a starting point for our algorithm above. In order to show the desired result we must show that the desired result holds for P_k.

For $K_{k+1} = [K_{k+1}^x \quad K_{k+1}^\eta]$ we have (5.19):

$$(\tilde{A} + \tilde{B}K_k)^T P_k + P_k(\tilde{A} + \tilde{B}K_k) = -\left(K_k^T K_k + \begin{bmatrix} 0 & 0 \\ 0 & \epsilon \end{bmatrix}\right)$$

For convenience, we let

$$P_k = \left[\begin{array}{c|c} P1 & P0 \\ \hline P0' & P2 \end{array}\right],$$

where $P1 \in \mathbb{R}^{n\times n}$, $P0 \in \mathbb{R}^{n\times 1}$ and $P2 \in \mathbb{R}^{1\times 1}$. Notice that $K_{k+1}^x = -b^T P1$ and $K_{k+1}^\eta =$

$-b^T P0$. Here we are solely interested in K^x_{k+1} since we already know the value of K^η_{k+1} for all k.

By simplification we arrive at the result that

$$\begin{gathered}(A+bK^x_k)^T P1 + P1(A+bK^x_k)+ \\ c^T P0' + P0c + (K^x_k)^T(K^x_k) = 0,\end{gathered} \tag{5.20}$$

with

$$P0 = (A+bK^x_k)^{-T}(-c^T P2 + \epsilon P1b + \epsilon(K^x_k)^T) \tag{5.21}$$

and

$$P2 = \frac{\epsilon - b^T(A+bK^x_k)^{-T}[\epsilon P1b + \epsilon(K^x_k)^T]}{-c(A+bK^x_k)^{-1}b}. \tag{5.22}$$

Next, by induction we are interested in showing that $P1$ is in the form of $\epsilon(M+N(\epsilon))$ (defined in Lemma 5.4.1) for all k.

By induction, let $i=1$. From (5.20)-(5.22) and simplification we have:

$$(A^T P1 + P1A) + c^T P0^T + P0c = 0$$

with

$$P0 = \epsilon\left(A^{-T}P1b + \frac{A^{-T}c^T}{cA^{-1}b} - \frac{A^{-T}c^T b^T A^{-T} P1b}{cA^{-1}b}\right).$$

With the use of Lemma 5.4.1 and existence of $P1$ the result follows for $i=1$ since we have to solve an equation of the form:

$$\left(\underbrace{\begin{bmatrix} & & \\ & matrix & \\ & & \end{bmatrix}}_{\text{constant}} + \epsilon \underbrace{\begin{bmatrix} & & \\ & matrix & \\ & & \end{bmatrix}}_{\text{constant}}\right)\begin{bmatrix} p_{11} \\ \vdots \\ p_{nn} \end{bmatrix} = \epsilon \underbrace{\begin{bmatrix} & & \\ & vector & \\ & & \end{bmatrix}}_{\text{constant}},$$

where $P1$ has been represented as a vector $[p_{11} \; ... \; p_{nn}]^T$.

Next, continuing by induction we assume the result holds for $i = k-1$ and show that it holds for $i = k+1$. However, we have already simplified the results for (5.20)-(5.22) with $i = k$ and we can deduce that indeed $P1$ is of the form represented by $\epsilon(M + N(\epsilon))$ and hence

$$K_x = \epsilon \overline{K}_x$$

for some $\overline{K}_x$ with appropriate dimensions, with the property that $lim_{\epsilon \to 0} \overline{K}_x(\epsilon)$ exists.

5.5 Conclusion

In this chapter, we have presented a new type of LQ approach that includes a clamping controller and an anti-reset windup clamping control, which may highly improve on the response of the controllers presented in Chapter 4. In order to facilitate the improvement in transient response, we made the extra assumption that the model of the system is known a priori, which was not done in Chapter 4.

Chapter 6

Servomechanism Problem: MIMO tuning regulators

6.1 Introduction

In this chapter we study the servomechanism problem for MIMO positive LTI systems. In particular, this chapter considers the tracking problem of nonnegative constant reference signals for stable known and unknown MIMO positive LTI systems under measurable and unmeasurable disturbances. The chapter extends the results of the SISO case of Chapter 4 to MIMO positive systems.

This chapter is organized as follows. The problem statement is redefined first and a very interesting existence condition is then outlined. The main discussion on the servomechanism problem for unknown MIMO positive LTI systems under nonnegative control and measurable disturbances is then considered; thereafter, we tackle the same problem under nonnegative control and unmeasurable disturbances. The chapter is finalized with an extension to the discrete-time positive LTI systems case, and several interesting comments.

6.2 Preliminaries

The plant of interest is given first:

$$\begin{aligned} \dot{x} &= Ax + Bu + E\omega \\ y &= Cx + Du + F\omega \\ e &:= y_{ref} - y \end{aligned} \tag{6.1}$$

where A is an $n \times n$ Metzler stable matrix, $B \in \mathbb{R}_+^{n\times m}$, $C \in \mathbb{R}_+^{r\times n}$, $D \in \mathbb{R}_+^{r\times m}$, $E \in \mathbb{R}_+^{n\times q}$, $F \in \mathbb{R}_+^{r\times q}$; the signal $y_{ref} \in \mathbb{R}_+^r$ is a constant, as is $\omega \in \mathbb{R}_+^q$. Assume $m = r$, i.e., the number of inputs is equal to the number of outputs.

First, we extend the result of Lemma 4.1.2 for $d - cA^{-1}b$ and $f - cA^{-1}e_\omega$ to the multi-variable case.

Corollary 6.2.1. Consider the system matrices of the plant (6.1). Then,

$$D - CA^{-1}B \in \mathbb{R}_+^{r\times m}$$

and

$$F - CA^{-1}E \in \mathbb{R}_+^{r\times q}.$$

Proof.

$$\begin{aligned} -A^{-1} &\in \mathbb{R}_+^{n\times n} && \text{by Lemma 4.1.1} \\ -CA^{-1}B &\in \mathbb{R}_+^{r\times m} && \text{since } C \in \mathbb{R}_+^{r\times n},\ B \in \mathbb{R}_+^{n\times m} \\ D - CA^{-1}B &\in \mathbb{R}_+^{r\times m} && \text{since } D \in \mathbb{R}_+^{r\times m} \end{aligned}$$

The result for $F - CA^{-1}E$ follows in the same fashion. □

Next, we provide an important assumption which will be commonly used throughout this chapter and the next. The assumption is needed in order to ensure that the steady state value of the input exists and that the transmission zeros of the plant exclude the origin.

Assumption 6.2.1. Given the plant (6.1) assume that $rank(D - CA^{-1}B) = r$.

The latter assumption implies that the inverse

$$(D - CA^{-1}B)^{-1}$$

exists.

Next we redefine the *steady state* u_{ss}, x_{ss} and y_{ss} for the MIMO case.

Definition 6.2.1. Consider the plant (6.1) under Assumption 6.2.1. Define

$$u_{ss} := (D - CA^{-1}B)^{-1}y_{ref} - (D - CA^{-1}B)^{-1}(F - CA^{-1}E)\omega \tag{6.2}$$

$$K_r := (D - CA^{-1}B)^{-1} \tag{6.3}$$

$$K_d := -(D - CA^{-1}B)^{-1}(F - CA^{-1}E) \tag{6.4}$$

$$x_{ss} := -A^{-1}(Bu_{ss} + E\omega) \tag{6.5}$$

and

$$y_{ss} := Cx_{ss} + Du_{ss} + F\omega. \tag{6.6}$$

From the above definitions, we can conclude that

$$u_{ss} = K_r y_{ref} + K_d\omega.$$

If we assume that $u_{ss} \in \mathbb{R}^m_+$, then we can make the following claim. The result is stated as a corollary to Lemma 4.1.3.

Corollary 6.2.2. Consider the plant (6.1). If $u_{ss} \in \mathbb{R}^m_+$, then

$$x_{ss} \geq 0 \text{ and } y_{ss} \geq 0.$$

Proof. It follows that

$$\begin{aligned} -A^{-1} &\in \mathbb{R}^{n\times n}_+ && \text{by Lemma 4.1.1} \\ -A^{-1}(Bu_{ss} + E\omega) &\in \mathbb{R}^n_+ && \text{since } B \in \mathbb{R}^{n\times m}_+, u_{ss} \in \mathbb{R}^m_+, E \in \mathbb{R}^{n\times q}_+, \omega \in \mathbb{R}^q_+. \end{aligned}$$

Similarly for y_{ss}. □

Let us now shift to a discussion of the servomechanism problem for MIMO positive LTI systems.

6.3 Servomechanism Problem: measurable disturbances and nonnegative control

In this section the main ideas behind the servomechanism problem under measurable disturbances are considered. First, two problems of interest associated with measurable disturbances are introduced. Second, a crucial outcome is obtained, which was not as evident for the SISO case, i.e. we point out that for positive systems one, in general, cannot solve the tracking problem with nonnegative control inputs for all tracking and disturbance signals, i.e., for all

$$\begin{aligned} y^i_{ref} &\in Y^i_{ref} := [0, \overline{y}^i_{ref}],\ i = 1, ..., r \quad \text{where } \overline{y}^i_{ref} > 0 \\ \omega_i &\in \Omega^i := [0, \overline{\omega}_i],\ i = 1, ..., \overline{\Omega} \quad \text{where } \overline{\omega}_i > 0. \end{aligned} \tag{6.7}$$

Until otherwise stated, in this chapter, assume Y_{ref} and Ω are defined as above.

With the latter result in mind, we then restrict the tracking and disturbance signals to a feasible set and solve the servomechanism problem for unknown stable MIMO positive LTI systems.

Next, we introduce two problems of interest. The first problem considers unknown plants[1], which do not experience any perturbations. The second problem considers unknown plants that may experience perturbations.

The new non-robust servomechanism problem for MIMO is outlined below.

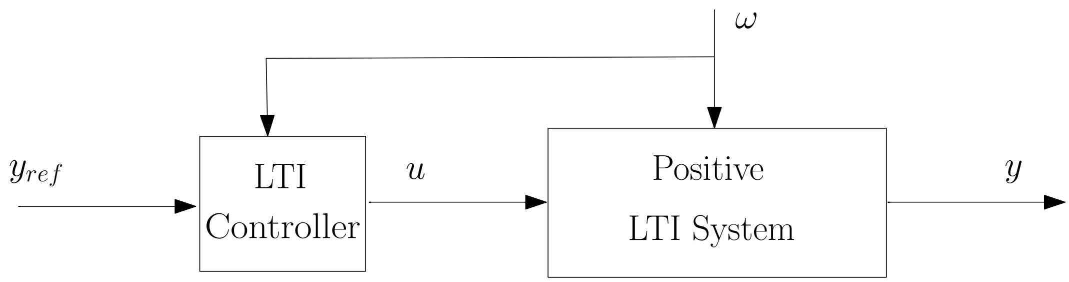


Figure 6.1: Open-loop MIMO LTI control.

Problem 6.3.1. Consider the plant (6.1) where the disturbance ω is measurable, the tracking signal y_{ref} is measurable, and the initial condition $x_0 \in \mathbb{R}^n_+$. Assume that Assumption 6.2.1 holds true.

Find an LTI controller connected as in the diagram (Figure 6.1) such that the controlled system for every $y_{ref} \in Y_{ref}$, and every $\omega \in \Omega$ has the property:

(a) the states $x(t) \geq 0$, output $y(t) \geq 0$, and input $u(t) \geq 0 \; \forall \; t$; and

(b) tracking of the reference signal occurs, i.e. $e(t) = y_{ref} - y(t) \to 0$, as $t \to \infty$.

The second problem considers plants that may undergo some perturbations.

[1]once again by unknown we mean that the numerical knowledge of the matrices is not known, but we do assume that the plant is a stable positive LTI system.

Problem 6.3.2. Consider the Problem 6.3.1 with the additional requirement that if an LTI controller has been found so that conditions (a), (b) are satisfied, then for all perturbations of the nominal plant model which maintain properties (a) and (b) of Problem 6.3.1, it is desired that the controller can still achieve closed-loop stability and asymptotic tracking and regulation.

We now illustrate that both Problem 6.3.1 and Problem 6.3.2 are in general unattainable under the restriction of nonnegative control of the input. The first step in showing the latter is the presentation of one key result from matrix theory and Lemma 4.1.1 (which is restated for quick reference).

Lemma 6.3.1 ([72]). If a nonnegative matrix $\mathcal{A}$ is square and nonsingular, then its inverse $\mathcal{A}^{-1}$ is also nonnegative if and only if $\mathcal{A}$ is a monomial matrix.

Lemma 6.3.2 ([55]). Let $\mathcal{A}$ be a Metzler matrix; then $-A^{-1}$ exists and is a nonnegative matrix if and only if A is Hurwitz.

We are now ready to state the first major result. The following observation with respect to Problem 6.3.1 and Problem 6.3.2 is obtained.

Theorem 6.3.1. Assume that the disturbances ω are measurable and that the plant model (6.1) is completely known; then:

[i] A necessary condition for a solution to Problem 6.3.1 and Problem 6.3.2 to exist is that $rank(D - CA^{-1}B) = r$ (Assumption 6.2.1).

[ii] Assume that $m = r = 1$ and that condition [i] above holds; then if $K_d\omega = 0$ there exists a solution to Problem 6.3.1 and Problem 6.3.2, and if $K_d\omega \neq 0$ with $y_{ref} = 0$, there exists no solution to Problem 6.3.1 or Problem 6.3.2 (consistent with Assumption 4.1.1).

[iii] Otherwise generically, for almost all plant parameters of model (6.1) there exists no solution to Problem 6.3.1 or Problem 6.3.2.

Proof. The necessary condition follows from LTI systems (i.e. no transmission zeros at the origin see Chapter 2), so we will concentrate on [ii] and [iii]. A necessary result for

Problem 6.3.1 and Problem 6.3.2 to hold is the need for the steady-state value of the input $u_{ss} := u(\infty)$ to be nonnegative. We now show that this in general does not hold. Recall that from (6.1) the control steady-state u_{ss} is given by

$$u_{ss} = K_r y_{ref} + K_d \omega. \tag{6.8}$$

From Corollary 6.2.1, we can conclude that the matrices $(D - CA^{-1}B)$ and $(F - CA^{-1}E)$ are nonnegative, which implies that if $m = r = 1$ and $K_d\omega = 0$ that $u_{ss} = K_r y_{ref} > 0$ for $y_{ref} > 0$, and that the feedforward controller $u = K_r y_{ref}$ (which is simply the feedforward controller of Chapter 4) solves Problem 6.3.1. Next, assume that $r > 1$. In order to have a nonnegative u_{ss} we need the inverse of $(D - CA^{-1}B)$ (i.e. K_r) to be nonnegative if $K_d\omega = 0$, since y_{ref} is nonnegative. Notice that by Lemma 6.3.1 this holds if and only if $(D - CA^{-1}B)$ is a monomial matrix, which generically is not the case, i.e. $(D - CA^{-1}B)$ is a monomial matrix if and only if $(D - CA^{-1}B)^{-1}$ is a monomial matrix which is true if and only if

$$[0 \;\; I] \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} [0 \;\; I]^T \text{ or } [0 \;\; I] \frac{adj \begin{pmatrix} A & B \\ C & D \end{pmatrix}}{det \begin{pmatrix} A & B \\ C & D \end{pmatrix}} [0 \;\; I]^T \tag{6.9}$$

is monomial, where $det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \neq 0$ follows from the assumption that $rank(D - CA^{-1}B) = r$. It directly follows now that the class of parameters of (C, A, B, D) which results in (6.9) being monomial is a hypersurface in the parameter space of (A, B, C, D) [22].

Also, notice that if $y_{ref} = 0$, then $u_{ss} = K_d\omega$ and there never exists a case when all entries of K_d are positive, by a similar argument as above.

Assume now that $r = 1$, and that $y_{ref} = 0$. In this case, unless $K_d\omega = 0$, $u_{ss} = K_d\omega$ is negative for all $\omega > 0$, which implies that there exists no solution to Problem 6.3.1 (as we have already commented on in the SISO case of Chapter 4). $\square$

Theorem 6.3.1 is an important result, as it tells us that if tracking/disturbance rejection is considered for positive systems (Definition 2.2.1) then in general no solution exists for the class of signals considered in (6.7). Although it may appear that Problem 6.3.1 (and Problem 6.3.2) is restrictive, there are still many subclasses of tracking and disturbance signals that can be considered. We will concentrate on these subclasses for the remainder of the chapter. In particular, the above result leads us to two new restricted problems in which we want to find the largest subclass of signals $y_{ref} \in Y_{ref} \subset \mathbb{R}^r_+$ and $\omega \in \Omega \subset \mathbb{R}^q_+$ such that the two problems, given above, can be solved.

Problem 6.3.3. Obtain the largest subclass of tracking signals $y_{ref} \in Y_{ref} \subset \mathbb{R}^r_+$ and disturbance signals $\omega \in \Omega \subset \mathbb{R}^q_+$ such that Problem 6.3.1 is solvable.

Next, we show that Problem 6.3.3 can be solved by using a feedforward compensator.

Theorem 6.3.2. Problem 6.3.3 is solvable if and only if

$$(y_{ref}, \omega) \in Y_{ref} \times \Omega := \{(\overline{y}_{ref}, \overline{\omega}) \in \mathbb{R}^r_+ \times \mathbb{R}^q_+ \mid K_r\overline{y}_{ref} \geq -K_d\overline{\omega} \text{ component-wise}\}. \tag{6.10}$$

Moreover, it suffices to use the following feedforward compensator in Figure 6.1

$$u = K_r y_{ref} + K_d\omega \tag{6.11}$$

as the control input.

Proof.

($\Rightarrow$) Since Problem 6.3.3 is solvable, then

$$\begin{aligned} u_{ss} &= K_r y_{ref} + K_d \omega \text{ which implies component-wise:} \\ 0 &\leq K_r y_{ref} + K_d \omega \text{ or that:} \\ K_r y_{ref} &\geq -K_d \omega \end{aligned}$$

($\Leftarrow$) We will show that using the feedforward controller all conditions of Problem 6.3.3 hold. Condition (a) and (c) hold since u is set to u_{ss}. Condition (b) can be guaranteed if the control input $u \geq 0$ component-wise for all time. However, the feedforward compensator (2.9) has the property that:

$$u = u_{ss} \geq 0,$$

which will guarantee nonnegativity of the states and outputs, thus solving Problem 6.3.3 with $Y_{ref} \times \Omega$ defined by (6.10). $\square$

Feedforward compensators are an effective theoretical tool to solve the tracking and regulation problem. However in practice, due to possible changes to the parameters of the plant, feedforward controllers in general may lead to unsatisfactory tracking/regulation. Thus, just as in the case of SISO systems, we will consider perturbations to the plant model with the use of a tuning regulator in the next section.

Before departing this section we make the following assumption, which parallels Assumption 4.1.1 of Chapter 4 and considers the results of this section.

Assumption 6.3.1. Given the plant (6.1). Assume Assumption 6.2.1 holds true. Also, assume the sets Ω and Y_{ref} are chosen such that

$$(y_{ref}, \omega) \in Y_{ref} \times \Omega := \{(\overline{y}_{ref}, \overline{\omega}) \in \mathbb{R}^r_+ \times \mathbb{R}^{\overline{\Omega}}_+ \mid K_r \overline{y}_{ref} > -K_d \overline{\omega} \text{ component-wise}\}. \quad (6.12)$$

The latter assumption is nothing else but a positivity constraint on the steady state input u_{ss}.

6.4 Servomechanism Problem: unmeasurable disturbances and nonnegative control

In this section, the focus shifts toward solving the servomechanism problem under unmeasurable disturbances and nonnegative control inputs. In particular, we return to Problem 6.3.1 under the more realistic case of unmeasurable disturbances. The motivation for assuming that the model of the plant is unknown is that: (i) often this in fact is the case, particularly for industrial systems, and (ii) since there are no assumptions made on the plant model other than it be open-loop stable, the resultant controller obtained, which uses on-line-tuning of a single parameter, is highly robust just like in the case of SISO systems of Chapter 4.

The main problem of interest in this section is provided next.

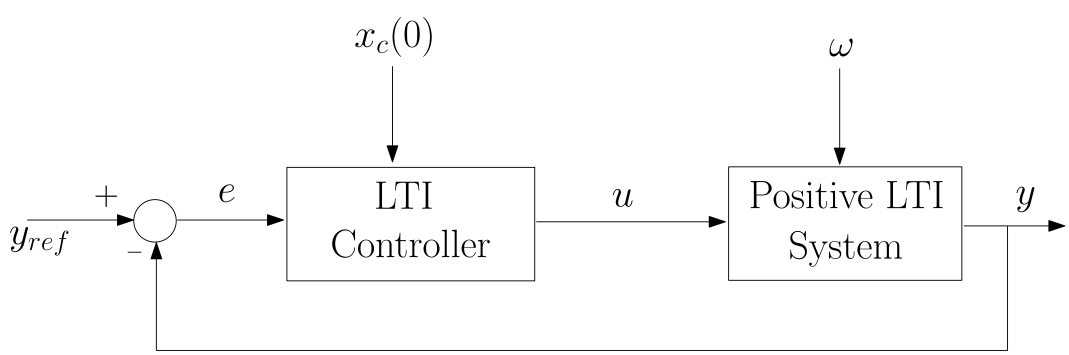


Figure 6.2: Closed-loop LTI system.

Problem 6.4.1. Consider the plant (6.1) where the disturbance ω is unmeasurable, the tracking signal y_{ref} is measurable, and the initial condition $x_0 \in \mathbb{R}^n_+$. Assume that Assumption 6.3.1 holds true.

Find an LTI controller connected as in the diagram (Figure 6.2) such that the closed-loop system satisfies

(a) asymptotic stability in the sense of Lyapunov with respect to the origin, and for every $y_{ref} \in Y_{ref}$, $\omega \in \Omega$, the initial condition of the controller $x_c(0)$ chosen such that $u(0) \in \mathbb{R}_+$

(b) the states $x(t) \geq 0$, output $y(t) \geq 0$, and input $u(t) > 0 \; \forall \; t$; and

(c) tracking of the reference signal occurs, i.e. $e(t) = y_{ref} - y(t) \to 0$, as $t \to \infty$. In addition,

(d) assume that a controller has been found so that conditions (a), (b), (c) are satisfied; then for all perturbations of the nominal plant model which maintain properties (a) and (b), it is desired that the controller can still achieve asymptotic tracking and regulation, i.e. the controller is *robust* and property (c) still holds.

Next, for quick reference, the MIMO Tuning Regulator (TR) of interest in this section is provided.

$$\begin{array}{rcl} \dot{\eta} & = & \epsilon(y_{ref} - y) \\ u & = & K_r \eta, \;\; u(0) > 0 \text{ and fixed} \end{array} , \tag{6.13}$$

where $K_r = (D - CA^{-1}B)^{-1}$with $\epsilon \in (0, \epsilon^*]$, $\epsilon^* > 0$ to be shown to exist.

We now solve Problem 6.4.1 with the use of the TR controller.

Theorem 6.4.1. Consider system (6.1). Then for all $x(0) \in \mathbb{R}^n_+$ there exists an $\epsilon^*(x(0), u(0)) > 0$ such that for all $\epsilon \in (0, \epsilon^*(x(0), u(0))]$ the controller (6.13) solves Problem 6.4.1.

Through the remainder of this section $\epsilon^*(x(0), u(0))$ will be denoted by simply ϵ^*.

The proof is given next.

Proof. The closed-loop system of the plant and the controller is given in Figure 6.3. The

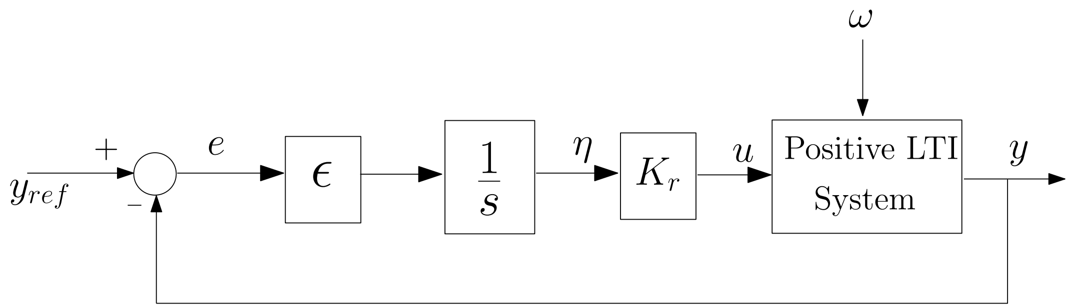


Figure 6.3: Closed-loop LTI system.

closed-loop system is LTI and by [17] there always exists an $\epsilon^+ > 0$ such that $\forall\, \epsilon \in (0, \epsilon^+]$ the closed-loop matrix is stable. This implies that all the conditions of Problem 6.4.1 will hold true if the nonnegativity condition (b) holds true. Through the remainder of the proof we assume $\epsilon \in (0, \epsilon^+]$.

Let us recall the two key assumptions:

1. $u(0) > 0$ (by the definition of the TR controller (6.13));

2. $u_{ss} > 0$ (by Assumption 6.3.1).

First, recall that by the definition of positive LTI systems we know that if $u(t) \geq 0$ for all t, then the states $x(t)$ and the outputs $y(t)$ also remain nonnegative for all t. Let us show now that there exists an $\epsilon^* \leq \epsilon^+$ such that for all $\epsilon \in (0, \epsilon^*]$, $u(t) > 0$ for all t under the two assumptions listed above. However, since

$$u(t) = K_r \eta(t),\ \forall\ t$$

then

$$\dot{u} = K_r \dot{\eta} = \epsilon K_r (y - y_{ref}),\ \forall\ t.$$

In order to prove the above, we use the results of singular perturbation (SP). The closed-loop system with the tuning regulator (6.13) is of the form:

$$\begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & B \\ -\epsilon K_r C & -\epsilon K_r D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} E & 0 \\ -\epsilon K_r F & \epsilon K_r \end{bmatrix} \begin{bmatrix} \omega \\ y_{ref} \end{bmatrix}. \tag{6.14}$$

Now, since the equilibrium (x_{ss}, u_{ss}), which depends on ω, y_{ref}, is independent of ϵ, we can transform the system as needed, i.e. let $z = x - x_{ss}$ and $q = u - u_{ss}$ in (6.14), resulting in the new system

$$\begin{bmatrix} \dot{q} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\epsilon K_r D & -\epsilon K_r C \\ B & A \end{bmatrix} \begin{bmatrix} q \\ z \end{bmatrix}. \tag{6.15}$$

Notice that if $q(t, \epsilon) + u_{ss} > 0$ for all t, then $u(t) > 0$ for all t.

Next, let us scale the derivatives by $\epsilon dt = d\tau$ (i.e. scaling of time $\epsilon t = \tau$) resulting in the transformed system

$$\begin{bmatrix} \overset{\odot}{q} \\ \epsilon \overset{\odot}{z} \end{bmatrix} = \begin{bmatrix} -K_r D & -K_r C \\ B & A \end{bmatrix} \begin{bmatrix} q \\ z \end{bmatrix}, \tag{6.16}$$

with $\epsilon \overset{\odot}{q} = \dot{q}$ and $\epsilon \overset{\odot}{z} = \dot{z}$.

Notice that if $q(\tau, \epsilon) + u_{ss} > 0$ for all τ, then $q(t, \epsilon) + u_{ss} > 0$ for all t and consequently $u(t) > 0$ for all t. Therefore, it remains to show that indeed $q(\tau, \epsilon) + u_{ss} > 0$ for all τ.

Our model (6.16) now satisfies the singular perturbation model. In order to use the singular perturbation (SP) results, we must show that all assumptions of SP hold true. However, as (6.16) is linear and time invariant, and the boundry-layer model is

$$\dot{p} = Ap$$

with A stable, it suffices to show that the reduced model (given below) yields exponential stability. Now by setting $\epsilon = 0$ we obtain $z = h(q) = -A^{-1}Bq$, and since A is Hurwitz, $h(q)$ exists and is unique. Next, by substituting $h(q)$ into $\overset{\odot}{q}$ we obtain the reduced model:

$$\overset{\odot}{q} \;=\; -K_r Dq - K_r(CA^{-1}B)q = -q.$$

Denote the solution of $\overset{\odot}{q} = -q$ by $\overline{q}(\tau)$, which is clearly exponentially stable (as needed by SP) and monotonic. Thus, by SP we have:

$$q(\tau,\epsilon) - \overline{q}(\tau) = O(\epsilon)\ \forall \tau$$

uniformly in τ, where

$$\begin{aligned} \overline{q}(\tau) &= e^{-\tau}\overline{q}(0) \text{ and} \\ \overline{q}(\tau) + u_{ss} &= u_{ss} + e^{-\tau}\overline{q}(0) \end{aligned}$$

with $\overline{q}(0) = q(0) = u(0) - u_{ss}$ by definition. Now, since $u(0) > 0$, then there exists an $\epsilon^* \leq \epsilon^+$ such that $q(\tau,\epsilon) + u_{ss} > 0$ for all $\epsilon \in (0,\epsilon^*]$ since $\overline{q}(\tau) + u_{ss}$ is *monotonically* approaching u_{ss}.

This completes the proof. □

Before completing this section, we consider a Corollary to Theorem 6.4.1 with the extra assumption that $0 \leq u(t) \leq \overline{u}$, $\overline{u} \in$ *interior*$(\mathbb{R}^n_+)$ fixed, for all $t \in [0,\infty)$ and where $u(0)$ in (6.13) is fixed such that $0 \leq u(0) \leq \overline{u}$.

Corollary 6.4.1. Consider system (6.1) and controller (6.13) where $0 < u(0) < \overline{u}$, $0 < u_{ss} < \overline{u}$, $\overline{u} \in$ *interior*$(\mathbb{R}^n_+)$ fixed. Then for all $x(0) \in \mathbb{R}^n_+$ there exists an $\epsilon^*(x(0),u(0)) > 0$ such that for all $\epsilon \in (0,\epsilon^*(x(0),u(0))]$ the controller (6.13) solves Problem 6.4.1 with condition (b) replaced by

(b') the states $x(t) \geq 0$, output $y(t) \geq 0$ and input $u(t)$ has the property $0 < u(t) < \overline{u}$.

The latter Corollary simply states that we would like to bound our input signal both from below by zero and from above by some constant $\overline{u}$.

Proof. The proof follows a similar argument as the proof of Theorem 6.4.1. □

We note that in the proof we let $\epsilon^* \to 0$. Clearly, we do not actually require $\epsilon^* \to 0$ to obtain the needed result. Note, the interested reader can refer to [17] to observe how "on-line tuning" can be used to find the ideal ϵ^*; although our clamping controller is different, the procedure is the same.

In the next section we take the results of this section into consideration and provide a clamping controller that parallels the SISO results of Chapter 4.

6.5 Implementation

Here we present a clamping controller that utilizes the discussion of Section 4.5. All results of Section 4.5 carry over to the MIMO case with the TR controller (6.13) of this chapter. Thus, under Section 4.5 and the TR (6.13) we provide a MIMO tuning clamping regulator (TcR) and an accompanying example next.

$$\begin{array}{rcl} \dot{\eta} & = & \epsilon(y_{ref} - y), \quad \eta_0 = 0 \\ u & = & \alpha(\eta) K_r \eta \end{array} , \tag{6.17}$$

where

$$\alpha(\eta) = \begin{cases} 0 & \text{if } \exists\ i \in \{1, ..., r\} \text{ such that } (K_r\eta)_i \le 0, \\ 1 & \qquad\qquad\qquad otherwise \end{cases}$$

We now provide an example.

Example 6.5.1. The following plant is a stable compartmental system, which has been used during the SISO case. However, here in order to illustrate the MIMO case, we insert an additional input and output into the system. Thus, consider the reservoir network of Figure 6.4; recall that each reservoir is identified by a number $(1, 2, ..., 6)$ where the water storage level $(x_1, x_2, ..., x_6)$ is a state of the system. Also γ and ϕ are the splitting coefficients of the flows at the branching points. The system is of order 6 and the input into the reservoir is in (L/s).

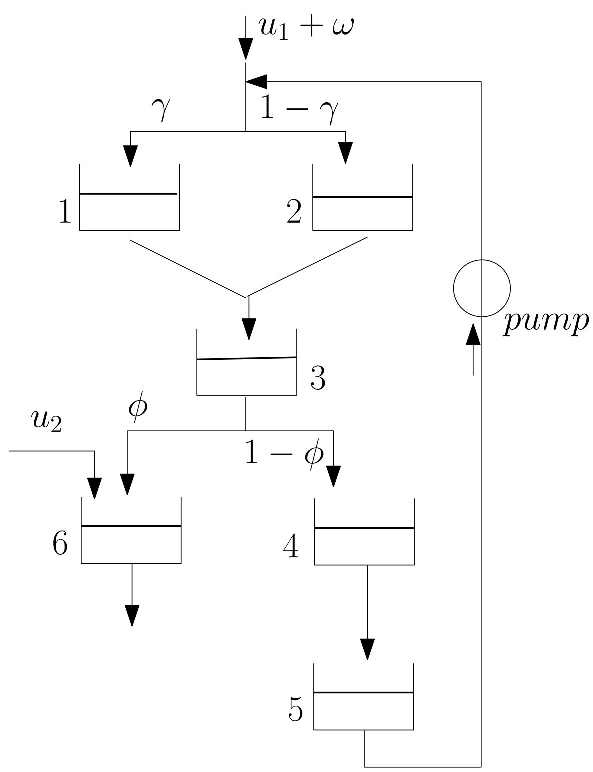


Figure 6.4: System set up for Example 6.5.1.

Consider the case where $\gamma = 0.5$, $\phi = 0.7$, $\alpha_1, ..., \alpha_6 = 0.8,\ 0.7,\ 0.5,\ 1,\ 2,\ 0.8.$ Note that all the rates are also measured in L/s. This results in the following MIMO system:

$$\dot{x} = \begin{bmatrix} -0.8 & 0 & 0 & 0 & 2 & 0 \\ 0 & -0.7 & 0 & 0 & 0 & 0 \\ 0.8 & 0.7 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.15 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0.35 & 0 & 0 & -0.8 \end{bmatrix} x + \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u + \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 \end{bmatrix}^T \omega$$

Also, assume the output y is of the form

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x$$

Next, assume that the initial condition $x_{i0} = 0\ (L)\ \forall i = 1, ..., 6$, i.e. initially there is no water in the tanks, $\omega = 0.5\ (L/s)$, and that the tracking signal is $y_{ref} = [5\ 5]^T\ (L)$.

We now proceed to find the controller (6.11), which will solve the problem under the assumption that the disturbance ω is measurable. First we obtain K_r by Procedure 2.3.1. By applying $\overline{u}_1 = 1$ and $\overline{u}_2 = 1$ in steady state, we obtain:

$$\begin{aligned} K_r &= \begin{bmatrix} 2.8571 & 0 \\ 2.4107 & 1.2500 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 0.3500 & 0 \\ -0.6750 & 0.8000 \end{bmatrix}. \end{aligned}$$

In similar fashion, we can obtain the gain matrix

$$K_d = \begin{bmatrix} -0.5 \\ 0 \end{bmatrix}.$$

Now, it is easy to see that

$$u_{ss} = K_r y_{ref} + K_d \omega = [1.5 \; 0.625]^T > 0,$$

and therefore by Theorem 6.3.2 we can proceed to use the feedforward controller to solve Problem 6.3.3. Figure 6.5 illustrates the simulated input response, while Figure 6.6 shows the output y.

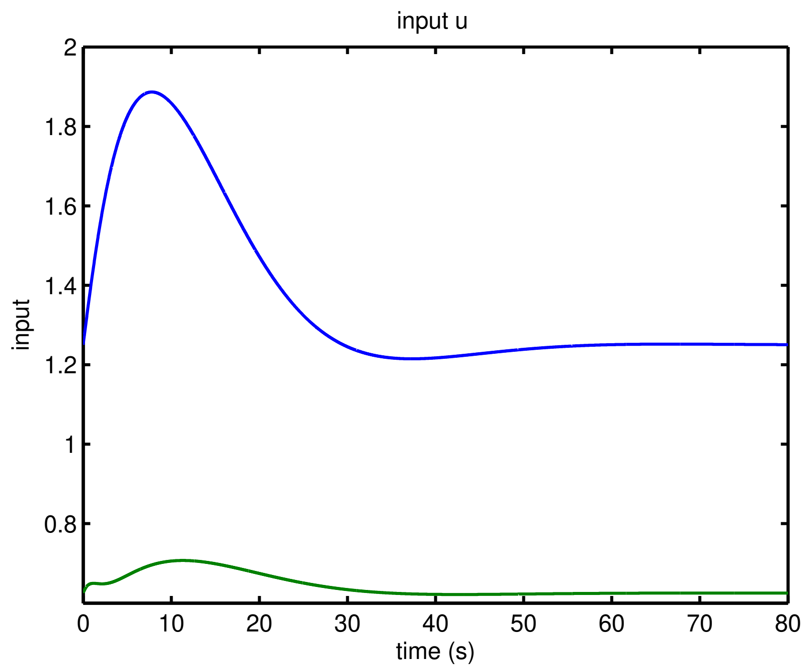


Figure 6.5: Input response for Example 6.5.1.

$\triangle$

Next, we revisit the latter example under unmeasurable disturbances. In this case only the MIMO TcR controller will be used.

Example 6.5.2. Consider the plant of Example 6.5.1 with initial conditions $x_0 = [2\ 2\ 2\ 2\ 2\ 2]$ and under the use of the TcR controller. In particular, we set $\epsilon = 0.1$, and obtain the following results via Figure 6.7 and Figure 6.8.

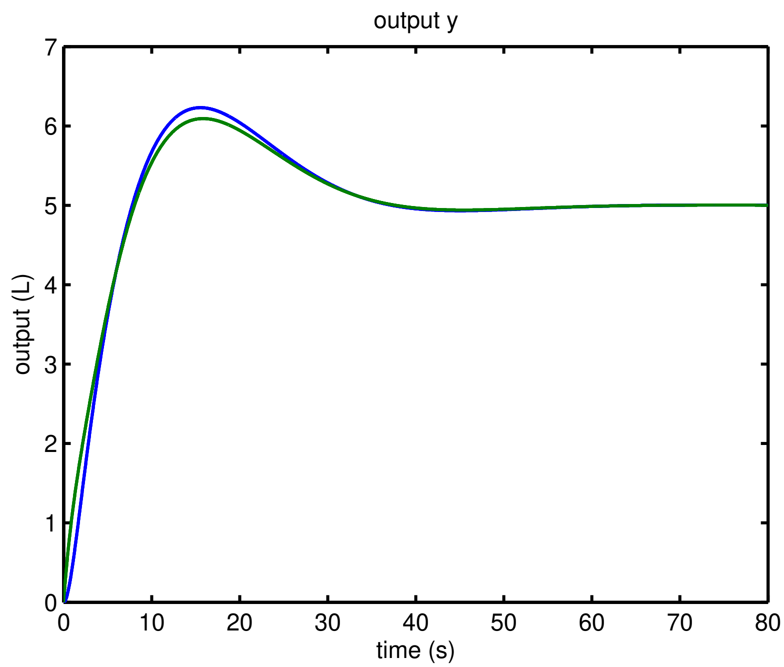


Figure 6.6: Output response for Example 6.5.1.

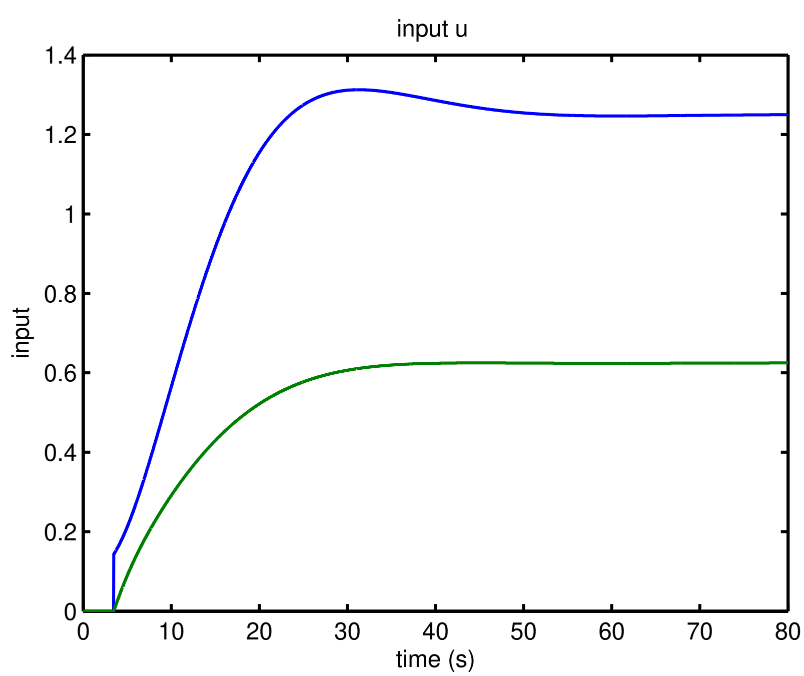


Figure 6.7: Input response for Example 6.5.2.

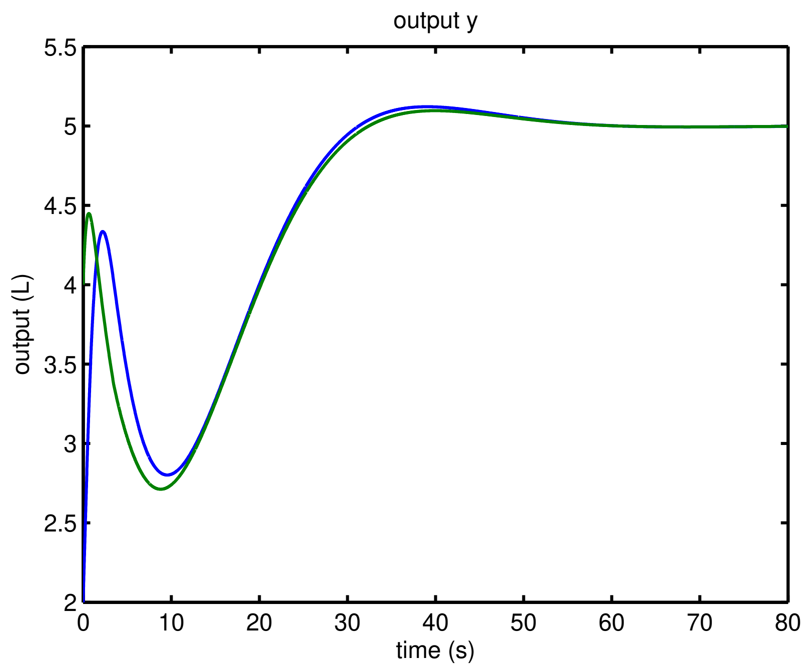


Figure 6.8: Output response for Example 6.5.2.

$\triangle$

The TcR controller will be used on experimental results discussed in Chapter 9.

6.6 Discrete Time Control of Positive Systems

In this section, we would like to point out that all of our results for continuous TR control can be transferred to the discrete-time case. The main distinction of course being that the system plant (6.1) has to be discretized $(A, B, C, D, E, F) \rightarrow (A_d, B_d, C_d, D_d, E_d, F_d)$ (note that for positive LTI discrete systems A_d is a nonnegative matrix and not a Metzler matrix like in the case of continuous systems) and instead of the continuous control strategy, discrete control must be implemented. This translates to the following replacements:

K_r replaced by $K_r^d = (D_d + C_d(I - A_d)^{-1}B_d)^{-1}$

K_d replaced by $K_d^d = -(D_d + C_d(I - A_d)^{-1}B_d)^{-1}(F + C(I - A_d)^{-1}E_d)$;

note that although the structure of K_r^d and K_d^d may seem different, the result is actually the same, i.e. $K_r = K_r^d$ and $K_d = K_d^d$. The second replacement is with respect to the controller given by (6.13), which is now replaced by

$$Eq.(6.13) \Rightarrow \eta_{k+1} = \eta_k + \epsilon(y_{ref} - y_k), \tag{6.18}$$

$$u_k = K_r^d \eta_k, \quad u_0 > 0 \tag{6.19}$$

Similar exchanges can also be made in the SISO case of Chapter 4. Once again we note that the proof of the TR controller involved the use of singular perturbation results, and the proof of the corresponding discrete results can likewise be captured in the discrete-time case as well, by using the two-time scaling results of discrete-time singular perturbation, with the use of [67], [54], i.e. all continuous-time case results can be extended to the discrete-time case.

6.7 Conclusion

In this chapter we have discussed a variation of the servomechanism problem for stable unknown MIMO positive linear systems. In particular, we have fully extended the results of the SISO case from Chapter 4 to the multi-variable case. The shift now turns to the extension of the SISO LTQcR controller to the MIMO case, which we tackle in the next chapter.

Chapter 7

Servomechanism Problem: MIMO LTQcR

In the single-input single-output (SISO) case both the tuning regulator (TR) and the linear tuning quadratic regulator (LTQR) were introduced to deal with the servomechanism problem. In Chapter 6 the results of the tuning regulator for SISO systems were extended to the MIMO case; here in this chapter we successfully extend the results on the SISO LTQR control of Chapter 5 to the MIMO case. The focus of this chapter, just like in the SISO LTQR case, will be on control strategies which incorporate the LTQR controller under nonnegative inputs.

This chapter is organized into the following sections. The first section presents preliminary results, where the MIMO plant of interest is reintroduced along with the servomechanism problem for MIMO systems. The MIMO linear tuning quadratic regulator (LTQR) is then defined and the main theoretical results behind the LTQR control law for MIMO positive LTI systems are presented. The chapter concludes with an application of the optimal control strategy to one of the examples originally presented in the previous chapter, when the MIMO tuning clamping regulator was used.

7.1 Preliminaries

Throughout this section we revisit the plant (6.1), which is reintroduced next.

The plant of interest is given first:

$$\begin{aligned} \dot{x} &= Ax + Bu + E\omega \\ y &= Cx + Du + F\omega \\ e &:= y - y_{ref} \end{aligned} \tag{7.1}$$

where A is an $n \times n$ Metzler stable matrix, $B \in \mathbb{R}_+^{n\times m}$, $C \in \mathbb{R}_+^{r\times n}$, $D = 0$, $E \in \mathbb{R}_+^{n\times q}$, $F \in \mathbb{R}_+^{r\times q}$; the signal $y_{ref} \in Y_{ref}$ is a constant, as is $\omega \in \Omega$, where Y_{ref} and Ω have been defined in Assumption 6.3.1. Assume $m = r$, i.e., the number of inputs is equal to the number of outputs.

The robust servomechanism problem of interest in this chapter is outlined below and deals strictly with nonnegative inputs. The problem, just like the plant above, has been originally presented in Chapter 6 (see Problem 6.4.1) and is presented here for quick reference.

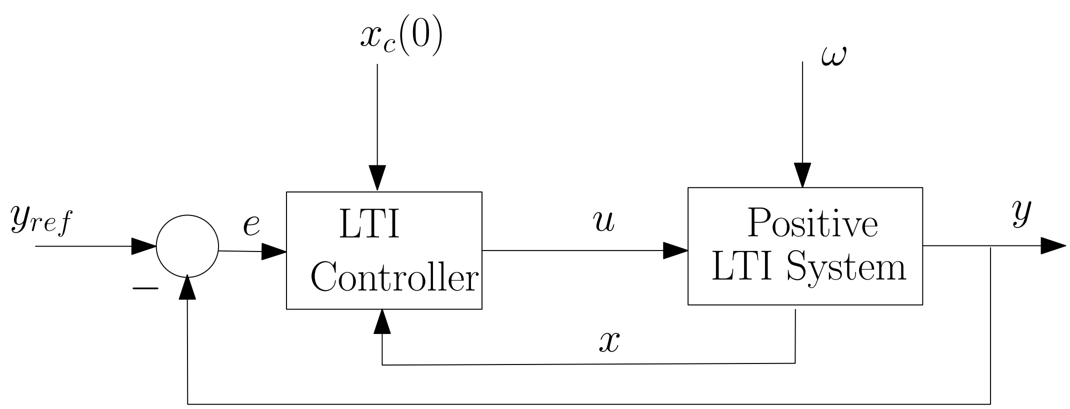


Figure 7.1: Closed-loop LTI system.

Problem 7.1.1. Consider the plant (7.1) where the disturbance ω is unmeasurable, the tracking signal y_{ref} and the states x are measurable, and the initial condition $x_0 \in \mathbb{R}_+^n$.

Assume that Assumption 6.2.1 holds true.

Find an LTI controller connected as in the diagram (Figure 7.1) such that the closed-loop system satisfies

(a) asymptotic stability in the sense of Lyapunov with respect to the origin, and for every $y_{ref} \in Y_{ref}$, $\omega \in \Omega$, with the initial condition of the controller $x_c(0)$ such that $u(0) \in \mathbb{R}_+$

(b) the states $x(t) \geq 0$, output $y(t) \geq 0$, and input $u(t) > 0 \; \forall \; t$; and

(c) ensures tracking of the reference signal, i.e. $e(t) = y(t) - y_{ref} \to 0$, as $t \to \infty$. In addition,

(d) assume that a controller has been found so that conditions (a), (b), (c) are satisfied; then for all perturbations of the nominal plant model which maintain properties (a) and (b), it is desired that the controller can still achieve asymptotic tracking and regulation, i.e. the controller is *robust* and property (c) still holds.

The main distinction between Chapter 6 and the present chapter is the fact that here we assume the system model is *known*, i.e. the matrices (A, B, C, D) are given to the designer. This assumption will allow us to discuss less restrictive control strategies in the hope of improving the transient response and settling time of those given in the previous chapter.

7.2 Servomechanism Problem: MIMO LTQR approach

In this section, the solution to Problem 7.1.1 under the MIMO Linear Tuning Quadratic Regulator (LTQR) controller, which is presented below, is outlined.

The MIMO LTQR controller is given by:

$$\begin{aligned} \dot{\eta} &= y - y_{ref} \\ u &= [K_x \;\; K_\eta] \begin{bmatrix} x \\ \eta \end{bmatrix}, \; u(0) > 0 \text{ and fixed} \end{aligned} , \tag{7.2}$$

where $K_x \in \mathbb{R}^{m \times n}$ and $K_\eta \in \mathbb{R}^{m \times r}$ ($K_\epsilon = [K_x \; K_\epsilon]$) are found by solving the quadratic control problem:

$$\int_0^\infty \epsilon^2 \eta^T Q \eta + u^T u d\tau \tag{7.3}$$

where $\epsilon > 0$ and $Q = ((-CA^{-1}B)^{-1})^T((-CA^{-1}B)^{-1})$, for the stabilizable and detectible system:

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} &= \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix} u \\ \eta &= [0 \;\; I_r] \begin{bmatrix} x \\ \eta \end{bmatrix} \end{aligned}$$

The latter control law can also be presented in a slightly different fashion, which yields the same gain matrix as that of the control strategy above, i.e. we can replace (7.3) with

$$\int_0^\infty \epsilon^2 e^T Q e + \dot{u}^T \dot{u} d\tau, \tag{7.4}$$

where $\epsilon > 0$ for the system:

$$\begin{aligned} \begin{bmatrix} \ddot{x} \\ \dot{e} \end{bmatrix} &= \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ e \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix} \dot{u} \\ e &= [0 \;\; I_r] \begin{bmatrix} \dot{x} \\ e \end{bmatrix} \end{aligned}$$

and minimizing (7.4), we obtain the optimal controller

$$\dot{u} = K_\epsilon [\dot{x}^T \ e^T]^T,$$

or by

$$u = K_x x + K_\eta \eta.$$

For a detailed description on (7.3) and its use in LTI systems under constant tracking and constant disturbances see [16].

The main difference between the SISO LQTR and the MIMO LQTR control is the choice of Q in the optimization function. The choice of $Q = ((-CA^{-1}B)^{-1})^T((-CA^{-1}B)^{-1}) > 0$ will play a major role in the proof of the results for the MIMO LTQcR extension.

Remark 7.2.1. Notice that clearly, by definition of the optimization integral, as

$$\epsilon \longrightarrow 0 \qquad K_\epsilon = [K_x \ K_\eta] \longrightarrow 0.$$

The main result under the MIMO LTQR is presented next.

Theorem 7.2.1. Consider system (7.1). Then for all $x(0) \in \mathbb{R}^n_+$ there exists an $\epsilon^*(x(0), u(0)) > 0$ such that for all $\epsilon \in (0, \epsilon^*(x(0), u(0))]$ the controller (7.2) solves Problem 7.1.1.

Through the remainder of this section, as was done in the previous Chapter, $\epsilon^*(x(0), \eta(0))$ will be denoted by simply ϵ^*.

The proof is given next.

Proof. The closed-loop system of the plant and the controller is given in Figure 7.2. The closed-loop system is LTI and by the existence of a unique linear quadratic gain K_ϵ there always exists an $\epsilon^+ > 0$ such that $\forall\ \epsilon \in (0, \epsilon^+]$ the closed-loop matrix is stable. Thus, once again if the nonnegativity condition (b) of Problem 7.1.1 holds true we can satisfy all other conditions of Problem 7.1.1. Through the remainder of the proof we assume

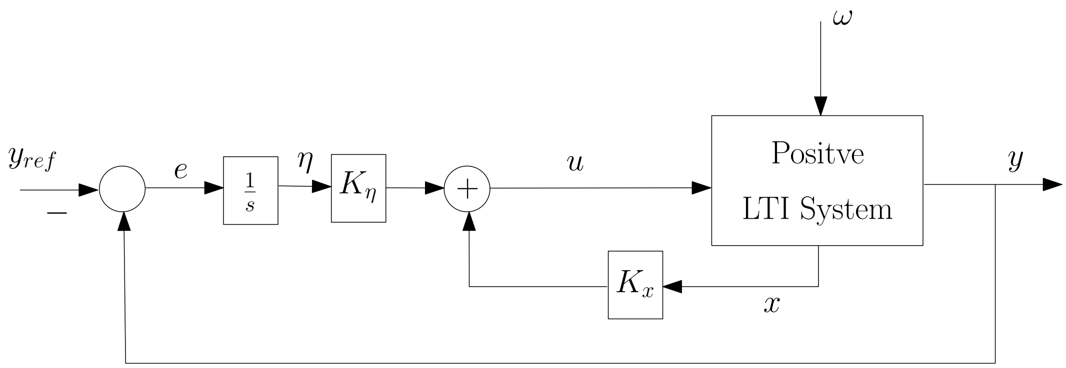


Figure 7.2: Closed-loop LTI system.

$\epsilon \in (0, \epsilon^+]$; however, since we are considering a linear quadratic problem, ϵ^+ can be chosen arbitrarily. Without loss of generality let us assume ϵ^+ is fixed.

Let us recall the two key assumptions:

1. $u(0) > 0$ (by the definition of the MIMO LTQR controller);

2. $u_{ss} > 0$ (by the choice of Y_{ref} and Ω).

First, by the definition of positive LTI systems we know that if $u(t) \geq 0$ for all t, then the states $x(t)$ and the outputs $y(t)$ also remain nonnegative for all t. Let us show now that there exists an $\epsilon^* \leq \epsilon^+$ such that for all $\epsilon \in (0, \epsilon^*]$, $u(t) > 0$ for all t under the two assumptions listed above.

Under the set up of the closed-loop system the steady states x_{ss}, u_{ss} are independent of ϵ therefore we can shift the closed loop system by setting $z = x - x_{ss}$ and $q = u - u_{ss}$, which results in:

$$\begin{bmatrix} \dot{z} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} A & B \\ K_x A + K_\eta C & K_x B \end{bmatrix} \begin{bmatrix} z \\ q \end{bmatrix} \tag{7.5}$$

$$= \begin{bmatrix} A & B \\ \epsilon(\overline{K}_x A + \overline{K}_\eta C) & \epsilon \overline{K}_x B \end{bmatrix} \begin{bmatrix} z \\ q \end{bmatrix}. \tag{7.6}$$

Note we assume that $K_\epsilon = [K_x \ K_\eta] = \epsilon[\overline{K}_x \ \ \overline{K}_\eta]$, where $lim_{\epsilon \to 0} \overline{K}_x(\epsilon)$ is finite (comment re K_η is made below); this assumption is justified at the end of the proof in Remark 7.2.2.

For convenience, rewrite

$$\begin{bmatrix} \dot{q} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \epsilon \overline{K}_x B & \epsilon(\overline{K}_x A + \overline{K}_\eta C) \\ B & A \end{bmatrix} \begin{bmatrix} q \\ z \end{bmatrix}. \tag{7.7}$$

Notice that if $q(t,\epsilon) + u_{ss} > 0$ for all t, then $u(t) > 0$ for all t.

Next, let us scale the derivatives by $\epsilon dt = d\tau$ (i.e. scaling of time $\epsilon t = \tau$) resulting in the transformed system

$$\begin{bmatrix} \overset{\odot}{q} \\ \epsilon \overset{\odot}{z} \end{bmatrix} = \begin{bmatrix} \overline{K}_x B & \overline{K}_x A + \overline{K}_\eta C \\ B & A \end{bmatrix} \begin{bmatrix} q \\ z \end{bmatrix}, \tag{7.8}$$

with $\epsilon \overset{\odot}{q} = \dot{q}$ and $\epsilon \overset{\odot}{z} = \dot{z}$.

Notice that if $q(\tau,\epsilon)+u_{ss} > 0$ for all τ, then $q(t,\epsilon)+u_{ss} > 0$ for all t and consequently $u(t) > 0$ for all t. Therefore, it remains to show that indeed $q(\tau,\epsilon) + u_{ss} > 0$ for all τ.

We have now transformed our model (7.8) into that of the singular perturbation model (SP). As the result parallels that of the SISO LQTR case we shift our attention to the reduced model only:

$$\overset{\odot}{q} \;=\; -K_r(-CA^{-1}B)q = -q,$$

where $\overline{K}_\eta = -K_r$ (see Remark 7.2.2). Denote the solution of $\overset{\odot}{q} = -q$ by $\overline{q}(\tau)$, which is clearly exponentially stable (as needed by SP) and monotonic. Thus, by SP we have:

$$q(\tau,\epsilon) - \overline{q}(\tau) = O(\epsilon)\ \forall \tau$$

uniformly in τ, where

$$\begin{aligned} \overline{q}(\tau) &= e^{-\tau}\overline{q}(0) \text{ and} \\ \overline{q}(\tau) + u_{ss} &= u_{ss} + e^{-\tau}\overline{q}(0) \end{aligned}$$

with $\overline{q}(0) = q(0) = u(0) - u_{ss}$ by definition. Now, since $u(0) > 0$, then there exists an $\epsilon^* \leq \epsilon^+$ such that $q(\tau, \epsilon) + u_{ss} > 0$ for all $\epsilon \in (0, \epsilon^*]$ since $\overline{q}(\tau) + u_{ss}$ is *monotonically* approaching u_{ss}.

We now justify our result for $\overline{K}_x$ and $\overline{K}_\eta$.

Remark 7.2.2. Next, we show that as $\epsilon \to 0$

$$K_\eta = \epsilon \overline{K}_\eta = -\epsilon K_r,$$

with $\overline{K}_\eta = -K_r = (-CA^{-1}B)^{-1}$ (recall in this chapter we assume $D = 0$).

The gain matrix $[K_x \ K_\eta]$ can of course be obtained from the solution of the ARE

$$\tilde{A}^T P + P\tilde{A} - P\tilde{B}R^{-1}\tilde{B}^T P + \tilde{Q} = 0, \tag{7.9}$$

which is related to (7.3), where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad R^{-1} = I,$$

$$\tilde{Q} = \begin{bmatrix} 0 & 0 \\ 0 & \epsilon^2((-CA^{-1}B)^{-1})^T((-CA^{-1}B)^{-1}) \end{bmatrix},$$

and

$$P = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{12}^T & P_{22} \end{array}\right].$$

Therefore, the gain matrix can be expressed as

$$\begin{aligned} K_\epsilon = [K_x \quad K_\eta] &= -\tilde{B}^T P & (7.10)\\ &= -[B^T \quad 0] \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} & (7.11)\\ &= -[B^T P_{11} \quad B^T P_{12}] \end{aligned}$$

and by definition of the control problem is *unique*.

Notice that the closed loop matrix with the control in place is:

$$\begin{bmatrix} A + BK_x & BK_\eta \\ C & 0 \end{bmatrix}. \tag{7.12}$$

Additionally, by manipulating (7.9) we can obtain

$$\begin{aligned} P0^T BB^T P0 &= \epsilon^2 Q \\ (K_\eta)^T (K_\eta) &= [\epsilon(-CA^{-1}B)^{-1}]^T [\epsilon(-CA^{-1}B)^{-1}]. \end{aligned}$$

Now, one solution to the above equation is

$$K_\eta = -\epsilon(-CA^{-1}B)^{-1}. \tag{7.13}$$

However, from Chapter 6 and [17] (with $D = 0$) we know that as $\epsilon \to 0$ the control gain (7.13) will stabilize (7.12) and since the LQ problem has a unique solution, then (7.13) must be the only possible solution for K_η. Therefore, indeed as $\epsilon \to 0$ we have $\overline{K}_\eta = -K_r = (-CA^{-1}B)^{-1}$.

It now remains to show that

$$K_x = \epsilon \overline{K}_x, \tag{7.14}$$

where the components of $\overline{K}_x$ are of the form

$$\overline{k}_{rs} = \left(\frac{\zeta_0 + \zeta_1 \epsilon + \zeta_2 \epsilon^2 + ...}{1 + \gamma_1 \epsilon + \gamma_2 \epsilon^2 + ...} \right) \quad r, s \in \{1, ..., m(n)\}$$

with $\zeta_i \in \mathbb{R}$, $\gamma_j \in \mathbb{R}\ \forall i, j \in \{0, 1, 2, ...\}$. Note that $\overline{K}_x$ is continuously differentiable and well defined with respect to ϵ, near and at the origin.

The process for proving the result is identical to the case of SISO LQTR control, except for the recursive algorithm of [87], given below

1. Choose K_1 so that $\tilde{A} + \tilde{B}K_1$ is stable.

2. Having chosen $K_1, ..., K_k$ obtain P_k from

$$(\tilde{A} + \tilde{B}K_k)^T P_k + P_k(\tilde{A} + \tilde{B}K_k) + Q + K_k^T K_k = 0 \tag{7.15}$$

3. Define $K_{k+1} := \tilde{B}^T P_k$,

instead of

$$K_1 = [0 \quad -\epsilon]$$

for the SISO case, we start with

$$K_1 = [0 \quad -\epsilon(-CA^{-1}B)^{-1}].$$

for the MIMO case. The remainder of the result for showing that

$$K_x \rightarrow \epsilon \overline{K}_x, \text{ as } \epsilon \rightarrow 0.$$

is identical to the SISO LQTR case of Chapter 5.

This completes the proof. □

In the case of SISO LQTR control a Corollary to Theorem 5.2.1 with the extra assumption that $0 \leq u(t) \leq \overline{u}$, $\overline{u} > 0$ fixed, for all $t \in [0, \infty)$ was introduced. The same result of bounding the input signals can be extended to the results of this chapter with the new MIMO LQTR controller. In similar fashion the results of clamping signals at zero and at some upper value of $\overline{u} > 0$ can also be done. The details of the extension of the Corollary and the results on bounding the signals are omitted.

Next, the linear tuning quadratic clamping regulator (LTQcR) for MIMO plants is defined. This control law will be used in the example presented in the sequel and in the experimentation of Chapter 9.

$$\dot{\eta} = y - y_{ref}; \qquad u = \alpha[K_x \;\; K_\eta][x \; \eta]^T, \; u(0) > 0 \tag{7.16}$$

$$\alpha = \begin{cases} 0 & \text{if } \exists \, i \in \{1, ..., r\} \text{ s.t } ([K_x \;\; K_\eta][x \; \eta])_i \leq 0, \\ 1 & \qquad\qquad otherwise \end{cases}$$

where $K_x \in \mathbb{R}^{m \times n}$ and $K_\eta \in \mathbb{R}^{m \times r}$ are found in the same fashion as the LTQR control approach.

7.3 LTQcR MIMO example

In this section we illustrate the results of this chapter via an example. In particular, we consider an example that was tackled in Chapter 6 and show the improvement of the current control strategy.

Example 7.3.1. Consider the system of Example 6.5.2, i.e. consider the system of reservoirs of Figure 7.3, note that each reservoir is identified by a number $(1, 2, ..., 6)$

where the water storage level $(x_1, x_2, ..., x_6)$ is a state of the system. Also γ and ϕ are the splitting coefficients of the flows at the branching points. The system is of order 6, as we assume the pump dynamics can be neglected. As pointed out in [30], the dynamics of each reservoir can be captured by a single differential equation:

$$\dot{x}_i = -\alpha_i x_i + v + e_i \omega, \quad z = \alpha_i x_i$$

for all $i = 1, ..., 6$, where x_i is the water storage (in L) and $\alpha > 0$ is the ratio between outflow rate z and storage, with $e_i \omega$ being the disturbance rate into the storage. The input into the reservoir is designated by v and is in (L/s).

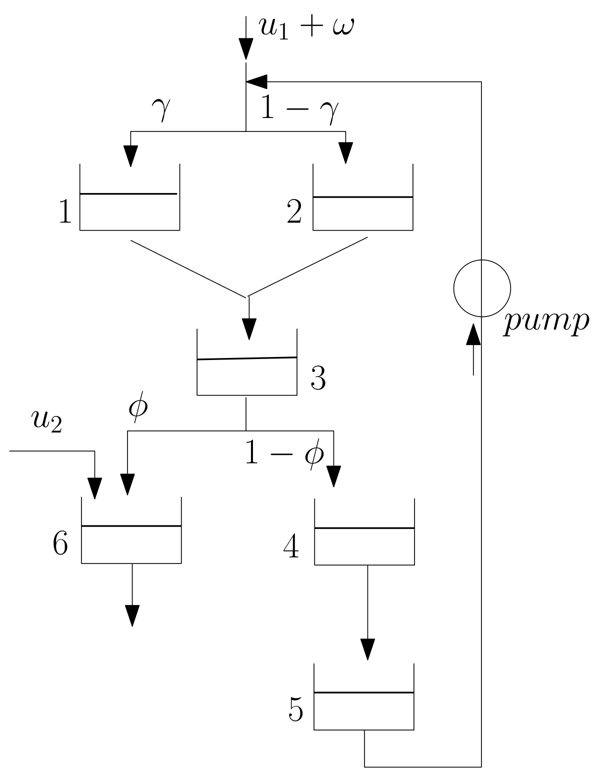


Figure 7.3: System set up for Example 7.3.1.

Consider the case where $\gamma = 0.5$, $\phi = 0.7$, $\alpha_1, ..., \alpha_6 = 0.8,\ 0.7,\ 0.5,\ 1,\ 2,\ 0.8$. Note that all the rates are measured in L/s. This results in the following system:

$$\dot{x} = \begin{bmatrix} -0.8 & 0 & 0 & 0 & 2 & 0 \\ 0 & -0.7 & 0 & 0 & 0 & 0 \\ 0.8 & 0.7 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.15 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0.35 & 0 & 0 & -0.8 \end{bmatrix} x$$

$$+ \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u + \quad [0.5 \;\; 0.5 \;\; 0 \;\; 0 \;\; 0 \;\; 0]^T \omega \tag{7.17}$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x \tag{7.18}$$

It is now desired to solve Problem 7.1.1 for this system, where unlike Chapter 6, we assume that the knowledge of the reservoir model (7.17)-(7.18) is now available.

Assume the initial condition and the disturbance, just like in the previous chapter, is $x_0 = [2\ 2\ 2\ 2\ 2\ 2]$ and $\omega = 0.5$, respectively. Additionally, assume that we would like to track the reference input $y_{ref} = [5\ 5]^T$. With the choice of $\epsilon = 10$, the desired result is obtained. Figure 7.4 illustrates the simulated input response; Figure 7.5 shows the output y response; and Figure 7.6 shows the states response. Notice that all inputs, outputs, and states are nonnegative.

Notice that the clamping on the control input occurs initially (0-3 seconds). Thereafter, the LTQR takes over and tracking occurs rather quickly in comparison to the

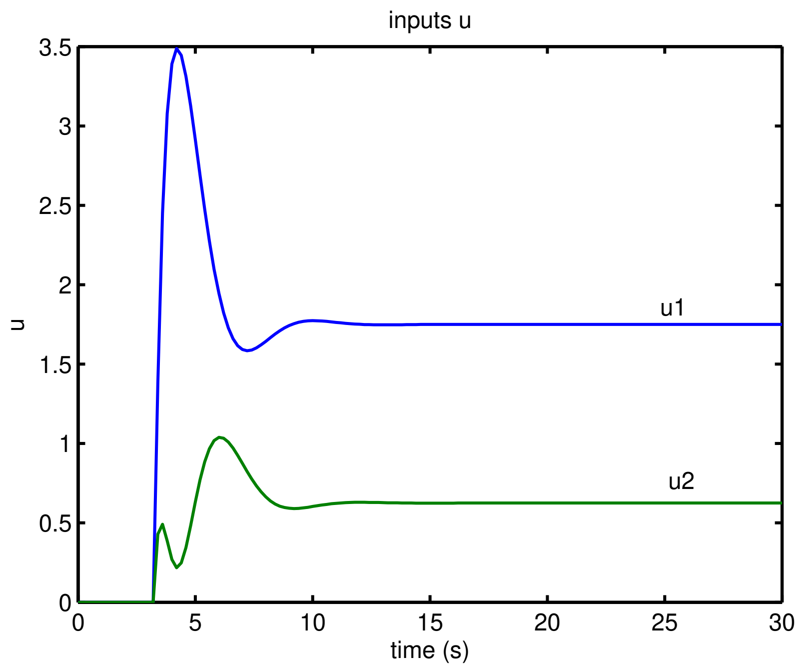


Figure 7.4: Input response for Example 7.3.1.

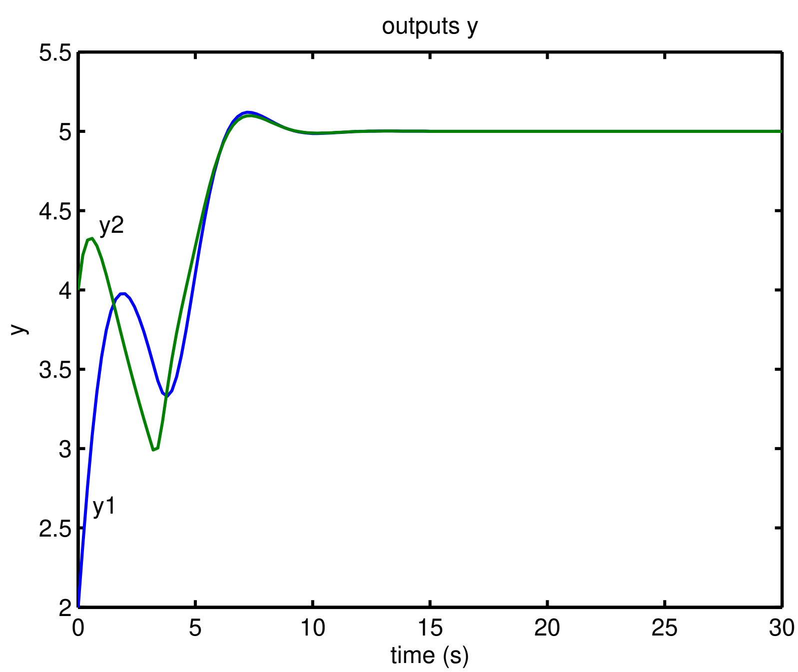


Figure 7.5: Output response for Example 7.3.1.

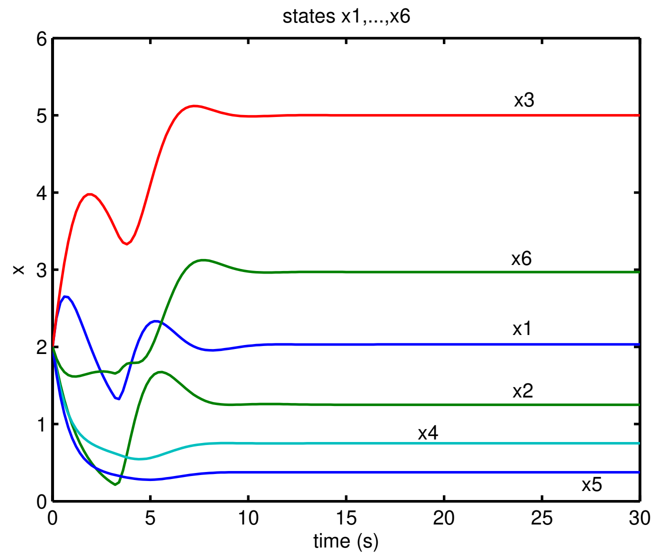


Figure 7.6: State response for Example 7.3.1.

example presented in Chapter 6. The simulation diagrams from Chapter 6 for both the input and output have been reproduced for quick reference in Figure 7.7 and Figure 7.8, respectively.

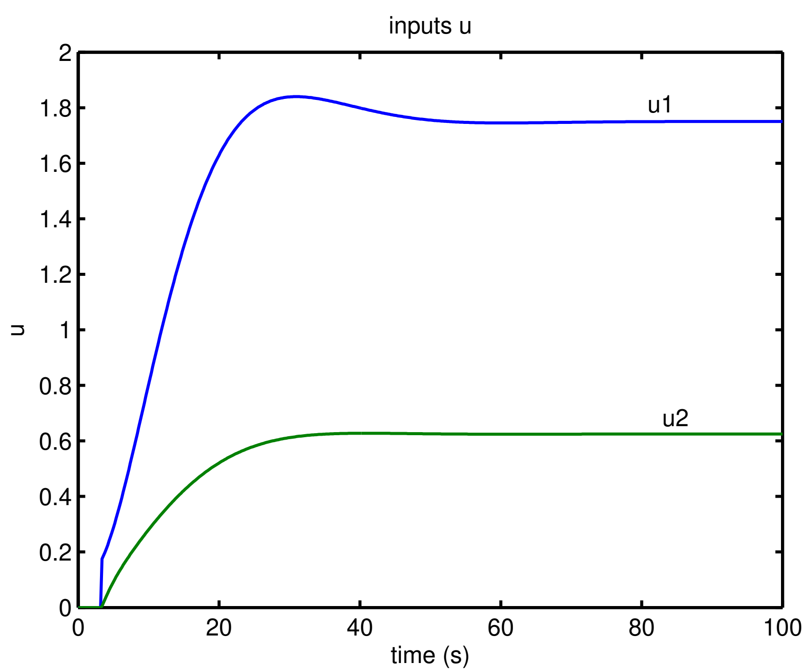


Figure 7.7: Input response for Example 6.5.2.

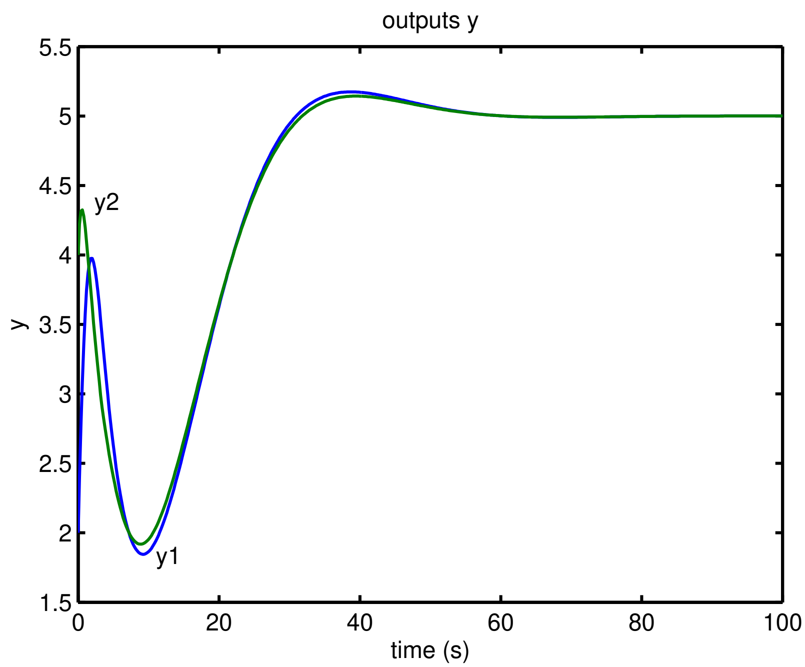


Figure 7.8: Output response for Example 6.5.2.

△

7.4 Conclusion

In this chapter, the results of LTQR for SISO were extended to the MIMO case. It was shown that through a new choice of an optimizing matrix Q one was able to implement the LQTR to solve the servomechanism problem under nonnegative control.

Chapter 8

Model Predictive Control (MPC)

In this dissertation we have considered the positive servomechanism problem for both SISO and MIMO positive LTI plants. In the approach of the solution to the servomechanism problem we have implemented clamping type of controllers, as pointed out via Chapters 4 - Chapter 7. This nonlinear constraint on the input to the system can also naturally be considered via the use of Model Predictive Control (MPC), which we touch upon in this chapter.

The interest in Model Predictive Control is quite overwhelming and one can simply look at one of the recent conferences [12] and find numerous citations. For a very thorough background into MPC and the literature associated with it, the interested reader is referred to [62]. It is not our interest here to develop and overview what previous MPC researchers have done, but rather to point out that via our results of the TR and LTQR controllers, we will be able to provide a link between MPC control and TR (or LTQR) control of the positive LTI servomechanism problem.

In short, Model Predictive Control (MPC) is based on optimal cost minimization of a sampled data discrete-time system with a finite-horizon window and subject to linear constraints which arise in the case of nonnegative inputs, e.g. $u \geq 0$ component-wise, as in this thesis. The MPC cost minimization is then treated via quadratic programming

(QP) [62]. The potential problem that MPC carries is that this constrained QP problem can be computationally intensive, and therefore for "fast" systems may result in no real-time solution.

Before we leap into the control strategy for MPC, let us first recall several facts from Chapter 6 about the transformation of the continuous TR control case to the discrete-time case. The main distinction of course being that the system plant (6.1) has to be discretized $(A, B, C, D, E, F) \rightarrow (A_d, B_d, C_d, D_d, E_d, F_d)$ and instead of the continuous control strategy, discrete control must be implemented. This translates to the following replacements:
K_r replaced by $K_r^d = (D_d + C_d(I - A_d)^{-1}B_d)^{-1}$
K_d replaced by $K_d^d = -(D_d + C_d(I - A_d)^{-1}B_d)^{-1}(F + C(I - A_d)^{-1}E_d)$;
note that although the structure of K_r^d and K_d^d may seem different, the result is actually the same, i.e. $K_r = K_r^d$ and $K_d = K_d^d$. The second replacement is with respect to the controller given by (6.13), which is now replaced by

$$Eq.(6.13) \quad \Rightarrow \quad \eta_{k+1} = \eta_k + \epsilon(y_{ref} - y_k), \tag{8.1}$$
$$u_k = K_r^d \eta_k, \; u_0 > 0$$

Similar exchanges can be made in the SISO case of Chapter 4. Once again we note that the proof of the TR controller involved the use of singular perturbation results, and the proof of the corresponding discrete results can likewise be captured in the discrete-time case as well, by using the two-time scaling results of discrete-time singular perturbation, with the use of [67], [54], i.e. all continuous-time case results can be extended to the discrete-time case.

Next, the control strategy for MPC, under linear discrete-time conditions, is outlined. Consider the discretized system (6.1) constrained by $u_i \geq 0$ for all $i = 1, ..., m$; then, minimize the following performance index and obtain the control u under MPC control

[62], [19]:

$$\begin{aligned} J_{MPC} &= \sum_{k=1}^{N+1} \rho\eta(k-1)^T\eta(k-1) + [u(k)-u(k-1)]^T[u(k)-u(k-1)]d\tau \\ u(k) &= K_x x(k) + K_\eta \eta(k) \quad \text{where} \\ \eta(k+1) &= \eta(k) + e(k-1) \end{aligned} \tag{8.2}$$

for a given $\rho > 0$, and under the nonnegative constraint $u_i \geq 0$ for all $i = 1, ..., m$, and a window size $N > 0$.

Recall that we can always solve the positive robust servomechanism problem (Problem 6.3.2) using TR (or LTQR) control. However, if we consider Problem 6.3.2 under MPC control with nonnegative inputs for known systems, then in general we will obtain an improved transient response compared to the TR (or LTQR) control. This improvement should not be surprising, since MPC gives the optimal controller subject to any input constraints - unlike the results of TR (or LTQR). However, how can we justify that the MPC approach may actually work?

Let us attempt to answer this question next. First, there are three necessary conditions that must be satisfied for the MPC controller to work:

(i) there must exist a solution to the robust servomechanism problem for the system;

(ii) the steady-state feasibility conditions $u_{ss} > 0$, component-wise, must hold;

(iii) the window size N must be "large enough".

To satisfy (i)-(iii) we apply the same conditions as we assumed in the case of TR control, i.e. open loop plant stability, no transmission zeros present at one (or simply $(D_d + C_d(I - A_d)^{-1}B_d)$ being full rank), and the corresponding steady-state feasibility conditions of nonnegativity. So if we assume that these conditions all hold for the MPC problem just as they held for the TR control case, then:

- if the window size of the performance index is "large enough" it is known, from the MPC literature [60], that the MPC closed loop system will be stable, and that for a given constant tracking and disturbance signal that the control signal has the property that

$$\lim_{k\to\infty} u(k) = \overline{u}$$

  is a constant.

Thus, the only remaining item to show is that the MPC controller obtained, will have the property that error regulation actually occurs. However, the MPC controller obtained finds a discrete control signal u (8.2) which minimizes the performance index subject to control input constraints (nonnegative). We know that there exists a control signal corresponding to the controller TR (8.1) which satisfies the linear constraints and results in error regulation, and so we have an existence result which states that there exists a control signal which provides error regulation and stability for Problem 6.3.2. Thus, if we start the MPC problem out with a discrete-time version of TR (8.1), then there always exists a solution to the MPC problem since we can set our performance index in such a way (specifically $\rho \to 0$ in (8.2)) that will result in arbitrarily small gains for the u_{MPC} (8.2), knowing that when the MPC gain is zero, then the TR is the solution.

In practice, for sufficiently small $\rho > 0$, the MPC controller obtained indeed has the property that error regulation actually occurs, but not necessarily for large ρ.

The existence result that MPC control can be used for sufficiently small ρ is important for the study of positive LTI systems.

Let us now illustrate a comparison of the TR controller and the MPC controller.

Example 8.0.1. Let us revisit the system of Example 6.5.2. The system is depicted in Figure 8.1.

Consider the case where $\gamma = 0.5$, $\phi = 0.7$, $\alpha_1, ..., \alpha_6 = 0.8,\ 0.7,\ 0.5,\ 1,\ 2,\ 0.8$. This

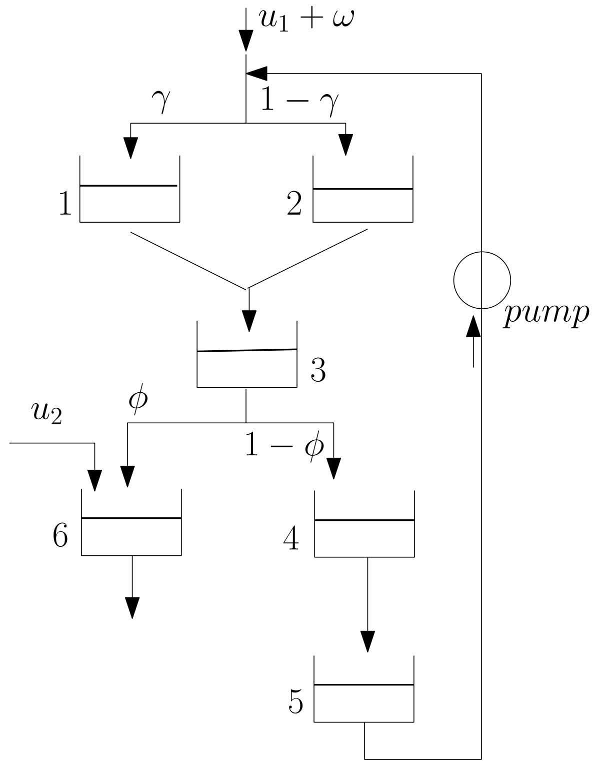


Figure 8.1: System set up for Example 8.0.1.

results in the following system:

$$\dot{x} = \begin{bmatrix} -0.8 & 0 & 0 & 0 & 2 & 0 \\ 0 & -0.7 & 0 & 0 & 0 & 0 \\ 0.8 & 0.7 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.15 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0.35 & 0 & 0 & -0.8 \end{bmatrix} x + \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u + \quad [0.5 \;\; 0.5 \;\; 0 \;\; 0 \;\; 0 \;\; 0]^T \omega$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x \tag{8.3}$$

It is easy to verify that the above compartmental system is stable, as $\sigma(A) = \{-0.8, \;\; -$

$0.21, \; -0.99 \pm 0.52i, \; -2.10, \; -0.70\}$ and that

$$rank(D - CA^{-1}B) = rank \begin{bmatrix} 2.86 & 0 \\ 2.41 & 1.25 \end{bmatrix} = 2 \tag{8.4}$$

$$\Rightarrow K_r = \begin{bmatrix} 0.35 & 0 \\ -0.675 & 0.8 \end{bmatrix}. \tag{8.5}$$

It is now desired to solve Problem 6.3.2 for this system, where it is assumed that knowledge of the reservoir model (7.17)-(7.18) is available.

Assume that the initial condition $x_0 = [1.2 \; 0.6 \; 1.8 \; 0.9 \; 0 \; 0.6]$ and $\omega = 0.1$. Additionally, assume that the tracking reference is $y_{ref} = [1 \; 2]^T$. Now setting $\epsilon = 0.1$ for our (discretized) TR controller (8.1) with an employed clamp, the following result is obtained. Figure 8.2 illustrates the simulated input response, while Figure 8.3 shows the output y.

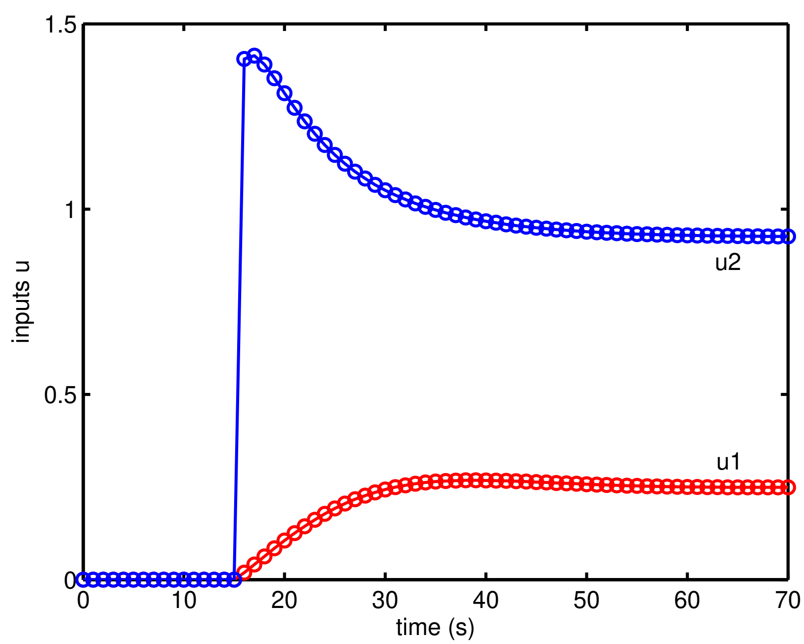


Figure 8.2: Input response for Example 8.0.1 under TcR (6.17).

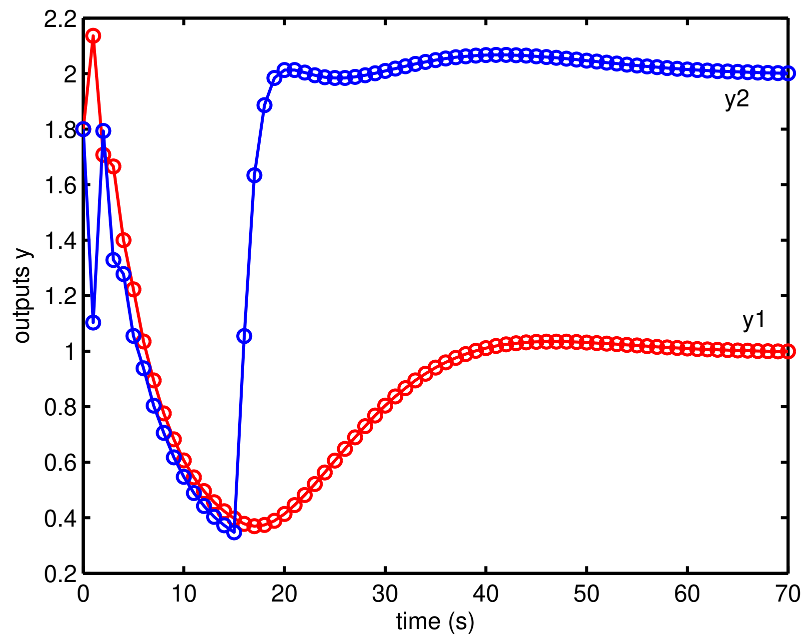


Figure 8.3: Output response for Example 8.0.1 under TcR (6.17).

Next, we consider the same tracking system, but under MPC control (8.2) with sampling interval $h = 1s$ and window size of $N = 15$. The initial conditions, disturbances, and tracking signals are identical to the TR example above. We illustrate the results of MPC with $\rho = 10^5$ (8.2) via Figure 8.4 and Figure 8.5 for both inputs and outputs, respectively.

$\triangle$

Notice that both controllers, TR (with a clamp) and MPC, employ a clamping for the first $\approx 17s$. This clamp is a direct result of the nonnegativity constraint placed on the control input. It is observed on comparing Figure 8.5 with Figure 8.3, that the MPC controller is some four-times faster than the TcR controller, which shows the advantage of having a system's model.

Next, we shift our attention to experimental results.

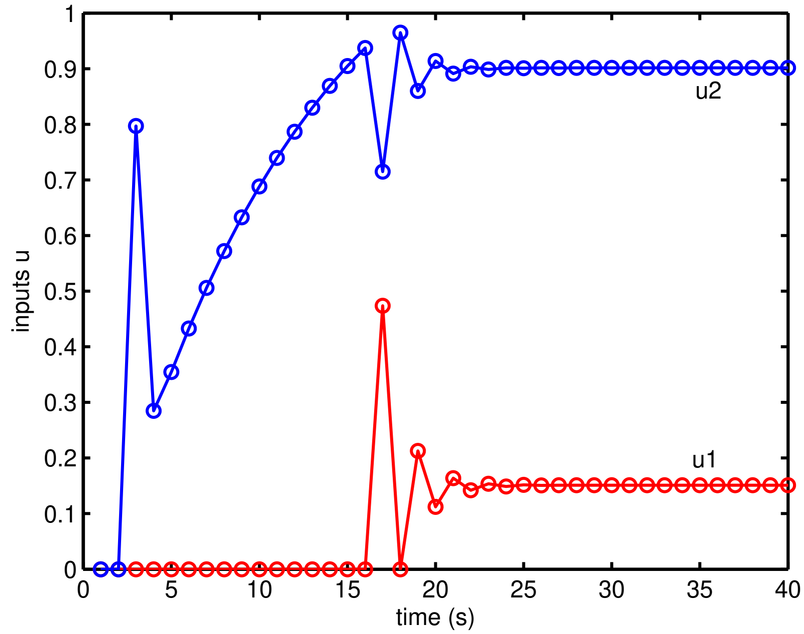


Figure 8.4: Input response for Example 8.0.1 under MPC (8.2).

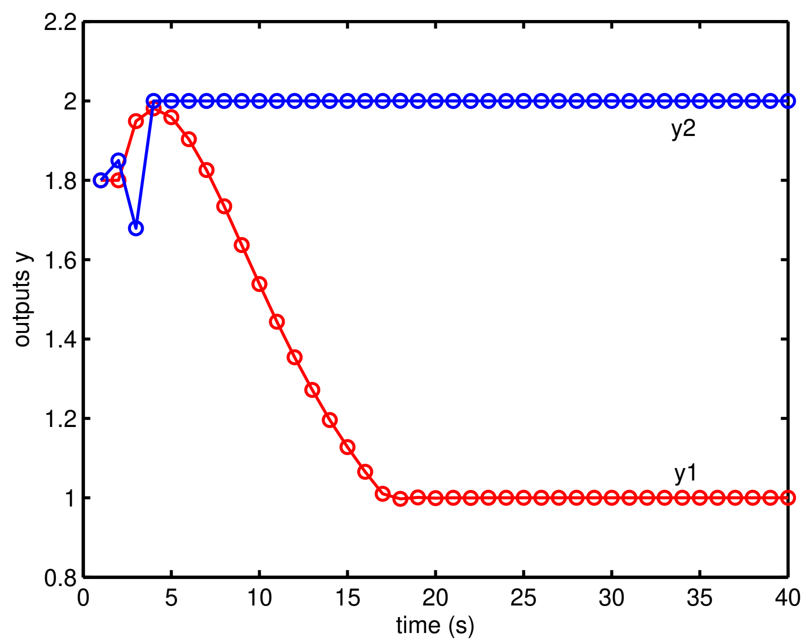


Figure 8.5: Output response for Example 8.0.1 under MPC (8.2).

Chapter 9

Servomechanism Problem: Experimental Results

This chapter illustrates, via an experimental study of an experimental industrial hydraulic system (MARTS) [18], the effectiveness, ease and robustness of the set of controllers presented in Chapter 4 - Chapter 7.

The main purpose of this study is to show that with no mathematical model of a system, or with a very crude approximation, the tuning clamping regulator (TcR) and the linear tuning quadratic clamping regulator (LTQcR) can *robustly* solve the problem of tracking a reference signal under disturbances and system changes, i.e. for large perturbations[1] of the nominal plant model the TcR and the LTQcR can achieve asymptotic reference tracking regulation and disturbance rejection as defined by the servomechanism problem of this thesis.

The interest of this chapter will not be focused on the technicalities of the experimental setup or identification of the model, but rather we concentrate on how well the TcR and the LTQcR perform and abide to robustness issues. All the control laws used within this chapter are definitely tested, as the "waterworks experiment" incorporates nonlinear

[1]that do not destabilize the system

effects of the plant, actuator valve dynamics, time-delays, as well as sizing effects due to actuator valve constraints [18].

This chapter is organized as follows. The experimental setup is described first, followed by the main experimental results, simulations, and discussion on the:

- SISO TcR results
- SISO LTQcR results
- MIMO TcR and LTQcR results.

Concluding remarks finalize the chapter.

9.1 Experiment: waterworks

The purpose of this section is to introduce an experimental waterworks setup for which the control strategies of the previous chapters will be tested on. All details of the setup and industrial components used within the experiment (see Figure 9.1) are provided next. A more in-depth summary has also been given in [18].

The entire waterworks apparatus has been assembled from *industrial components*; this includes the actuators, sensors, valves, piping, and all digital communication. We note that the actuators (valves) are controlled by compressed air, and all signal communication between the actuators/sensors to the digital computer are obtained by commercial current variation ($4ma$ to $20ma$) techniques [18], and controlled within the loop by voltages ($0V$ to $10V$). Although all components used are industrial, we have chosen to incorporate a standard personal computer running MATLAB Version 7.2.0.232 (R2006a) to carry out all real-time control. Below is a full list of components used within the experiment.

- 1 personal computer with an AMD Athlon(tm) 64 Processor 3200+ and 896 RAM

- 2 PCI-DAS6014 Analog and Digital I/O Boards

- 2 Foxboro Model V4A 1/2 inch Body H Needle Diaphram Control valve C/W I/P Transducer Model E69-BIIQ-R-S
  The I/P transducer is of the equal % type (3psi = 4 ma and 15 psi = 20 ma)

- 2 Taylor Model B 3401T 1/2 inch Differential Pressure Transmitter

- 1 Foxboro (Canada) Model E13-DL-I KAL2 Differential Transmitter and Model IFO-F2-S1 Integral Orifice Manifold Assembly in-lin type. This flow meter has a range of O to 2 (US) gallons/ rein

- 2 ASCO solenoid valves. These solenoids are 100 V(AC) on/off 1/4 inch size and are activated by a voltage 3 to 32 V DC to the solid state relay

- 1 Compressed Air Regulator Model 2515346 (from Canox Toronto)

- 2 Magnetic Drive Pumps Model 13-874-11 (from Fisher Scientific). The pumps are 1/12 HP, 1/2 in in/out and can deliver 32 l/rein at 10 head

- 4 Solid state relay model EOM1DE42 (5VDC) (from Electrosonic)

- 1 24V power supply model HPFSO24O1O (from Electrosonic)

- 1 Disk Drive power supply model CP206-A (from Active Components, Toronto)

- Hammond power supply HPFT 00512015 (from Electrosonic).

The apparatus (Figure 9.1) consists of four water tanks, interconnected via numerous piping and valves, where the water circulates between the tanks and can be controlled via two digitally controlled valves (we will only be interested in using one valve for the SISO constraint of our theory and both valves for the MIMO case) that provide water inflow into two upper tanks. An overview diagram of the system is provided, see Figure 9.2. The experimental apparatus has numerous valves which can be opened/closed to increase/decrease the water flow between respective tanks during the experimentation; thus, allowing for major perturbation of the system model. Moreover, unmeasurable disturbances are present within the apparatus; in particular a water inflow disturbance is available via a digital on/off control input (both are not measured during experimentation). The only measurements taken during the experimentation are that of the height of the water in Tank 1 (for the SISO case) and Tank 1/Tank 2 (for the MIMO case) - see Figure 9.2 - via a sensor which provides a voltage level (varying from $1V$ corresponding to near empty, to $5V$ near full, with a $1V$ increase/decrease representing approximately $2.4L$ of water rise/drop) and the valve control voltage ($0V$ corresponding to nearly closed and $10V$ corresponding to fully open) of the input into Tank 1/Tank 2 (Valve A_1 and A_2, in Figure 9.2).

Note: by inspection since the system is compartmental and stable the setup yields a positive system, as desired.

The focus now turns to experimental results.

9.2 SISO TcR experimental results

Throughout this section we refer to Figure 9.2, under the assumption that u_2 is turned off, and with the valve settings as indicated in Table 9.1. The goal will be to illustrate the theory behind the tuning clamping regulator via the use of various perturbations and disturbances on the nominal plant, which is represented by Case 1 from Table 9.1. In all cases, the initial level of Tank 1 is equal to $x_0 = 4.4V$, and the initial condition of the servo compensator is $\eta_0 \approx 0$ (unless stated otherwise). In Table 9.1 "on" ("off") represents that a valve is open (closed).

Table 9.1: Experimental Cases

Case	A_1	A_2	B	C	D	E	F	G	ω_1	ω_2
1	on	off	off	off	on	off	off	on	small	none
2	on	off	off	on	on	on	off	on	small	none
3	on	off	off	on	on	on	on	on	small	small
4	on	off	on	on	on	on	on	on	small	small
5	on	off	on	on	on	on	on	on	large	large

9.2.1 Experiment I: Case 1 of Table 9.1

In the first experiment we consider Case 1 under various initial conditions and ϵ values, i.e. various water levels of Tank 1 and for different types of tuning parameters. In particular, we consider the cases of the

(a) clamping controller (4.20) with $\epsilon = 0.07$, $y_{ref} = 2V$ (Figure 9.3), which results in a settling time of ≈ 16.5 minutes (991 sec).

(b) tuning clamping regulator (4.21) with $\overline{u} = 10$, $\epsilon = 0.07$, $y_{ref} = 2V$ (Figure 9.4), which results in a settling time of ≈ 15 minutes (897 sec); and

(c) tuning clamping regulator (4.21) under an extra feedforward term $u_{ff} = ky_{ref}$ where $k = 2.5$, with $\overline{u} = 10$, $\epsilon = 0.07$, $y_{ref} = 2V$ (Figure 9.5), which results in a settling time of ≈ 12 minutes (720 sec).

We note that in the case of initial condition $x_0 = 0$ the two cases (a) and (b) are identical, as no clamping occurs, while case (c) outperforms both (a) and (b) cases. In general, we found that for various initial conditions the tuning clamping regulator with the addition of the extra feedforward works best, i.e. has best $\%OS$ and settling times.

9.2.2 Experiment II: Case 1,2,3,4 of Table 9.1

Next, we put the above tuning clamping regulator with the addition of the extra feedforward term to the test by considering numerous perturbations into the system, as described in Table 9.2. In particular, we consider the case of

$$\begin{aligned} \overline{u} &= 10 \\ x_0 &= 3.53V \\ \epsilon &= 0.07 \\ y_{ref} &= 2V, \end{aligned}$$

under the transitions of Table 9.2, where Case 1 occurs during the time period $0 < t < 26min$, Case 2 occurs during the time period $26 < t < 53min$, Case 3 occurs during the time period $53 < t < 67min$, and Case 4 occurs during the time period $t > 67min$. Figure 9.6 illustrates the results.

It is to be noted that no adjustments to the controller are made during experimentation and no model is ever used to obtain the control law. Note that saturation has played a key role in this set up.

Table 9.2: Purturbation experiment

Time (min)	Current Case
0	Case 1
≈ 26	Case 2
≈ 53	Case 3
≈ 67	Case 4.

9.2.3 Experiment III: Case 1,2,3,4,5 of Table 9.1

In this experiment we repeat Experiment II of the previous subsection and at approximately 75 minutes add a large disturbance, described by the addition of Case 5 of Table 9.1 to Table 9.2, coming into both Tank 1 and Tank 2. The point of this experiment is to show that if the steady state u_{ss} does not abide to Assumption 4.1.1, and no solution exists, then the proposed TcR controller will turn itself off. Figure 9.7 illustrates the results.

In these experiments, we have used the tuning clamping regulator to illustrate experimentally the servomechanism problem for unknown stable positive systems, i.e. a positive system whose mathematical model is unknown. The main contribution of the tuning clamping regulator is its ability to provide a solution to the servomechanism problem for stable positive systems provided a solution exists; if no solution exists under the saturation constraints then the tuning clamping regulator will shut itself off whenever the disturbances are too large. The use of the regulator is straight forward and very practical from the perspective of cost and validation of a solution. Once a solution exists under the tuning clamping regulator, then more in-depth studies can be carried out to improve the results via better controllers, e.g. faster settling times. In the next section we test the SISO LTQcR approach and carry out a comparison to the results of this section.

9.3 SISO LTQcR experimental results

The main purpose of this section is to show that with a *very crude mathematical model of the system*, the LTQcR control strategy, presented in Chapter 5, can *robustly* solve the problem of tracking a reference signal under disturbances and system changes, i.e. for large perturbations that do not destabilize the system the LTQcR regulator can achieve asymptotic reference tracking regulation. More importantly, we also illustrate the latter results via the experimental setup of Section 9.1 where it is shown that the LTQcR outperforms the tuning clamping regulator that was tested in the previous section. However, note that unlike the tuning clamping regulator, the LTQcR does use a crude approximation of the system model. It is shown in this section that the extra information used within the design of the LTQcR regulator leads to improvements within settling time (in some situations of $\approx$ 10-fold) and overshoots (in some situations $\approx$ 3-fold) of the experimental results, while maintaining all robustness properties. We note that the main purpose of this section, just like the previous, is to check out the theory in an experimental setting.

Here we test the RLTQcR and compare the results to the tuning clamping regulator of the previous section.

9.3.1 Experimental results

Throughout this subsection we refer to Figure 9.2 and Table 9.3. Our goal will be to illustrate the theory behind the LTQcR via the use of various perturbations and disturbances on the nominal plant, which is represented by Case 1 from Table 9.3 (or Table 9.1 in the previous section).

Before we begin the discussion on the experimental results we present a crude model approximation of the waterworks. This approximation (1D model) has been obtained via experimental measurements of the tuning clamping regulator of the previous section.

In Section 9.2 the waterworks setup was controlled via the use of a tuning clamping regulator, which saturated at zero, thus the control law was off for a certain amount of time allowing us to measure the dominant eigenvalue of the system: approximately -0.00891. The resultant system modeled by Case I in Table 9.3, can thus be represented by:

$$\begin{aligned} \dot{x} &= -0.00891x + u \\ y &= x. \end{aligned} \tag{9.1}$$

The model is clearly a very crude approximation and if simulated will not exactly resemble that of the plant. Note that not even steady-state experiments have been performed for the latter approximation. However, our goal here is not to identify the mathematical model, but rather to show how powerful Problem 5.1.1 is (especially the importance of Problem 5.1.1 (d)), and to illustrate the use of the LTQcR designing controllers for positive systems. We illustrate the effectiveness of the RLTQcR controller next.

First, we set the initial level of Tank 1 equal to $x_0 = 4.4V$ (this is done in order to replicate the initial conditions of the TcR approach of the previous section) and then run the experiment with the RLTQcR control under $\overline{u} = 10$, $\epsilon = 50$, $y_{ref} = 2V$ (see Figure 9.8). Notice that the settling time of approximately 2.5 minutes (148 sec) for the RLTQcR, clearly outperforms the results of the TcR or the RTcR.

We note that in the case of initial condition $x_0 = 0$ (1V reading) the tuning clamping regulator cases (a) and (b) are identical, and turn out to have a settling time of approximately 12 minutes, while the LTQcR control results in a settling time of approximately 2 minutes (121 sec); see Figure 9.9.

In general, we found that for various initial conditions the RLTQcR works best, i.e. has best $\%OS$ and settling times, however it does incorporate a crude mathematical model into the design, while the tuning clamping regulator, although conservative, uses

Table 9.3: Experimental Cases

Case	A	B	C	D	E	F	G	ω_1	ω_2
1	on	off	off	on	off	off	on	small	none
2	on	off	on	on	on	off	on	small	none
3	on	off	on	on	on	on	on	small	small
4	on	on	on	on	on	on	on	small	small
5	on	on	on	on	on	off	on	small	small
6	on	off	off	on	on	off	on	small	small
7	on	on	on	on	on	on	on	large	large

no mathematical model for the control design.

9.3.2 Experiment II

Next, we put the RLTQcR controller to the test by considering numerous perturbations into the system, just as we did for the tuning case. We consider the case of

$$\begin{aligned} \overline{u} &= 10 \\ x_0 &= 3.60V \\ \epsilon &= 50 \\ y_{ref} &= 2V, \end{aligned}$$

under the transitions of Table 9.4.

Figure 9.10 illustrates the results of the RLTQcR controller, while Figure 9.11 shows the results of the tuning clamping regulator (4.21) for a similar perturbation configuration (see Table 9.2). Notice that the RLTQcR outperforms the tuning clamping regulator. Also, it is worth pointing out that saturation has played a key role in this set up.

Table 9.4: Purturbation experiment

Time (min)	Current Case
0	Case 1
≈ 8.5	Case 2
≈ 15	Case 3
≈ 19	Case 4
≈ 28	Case 5
≈ 32	Case 6

9.3.3 Experiment III

In this experiment we repeat Experiment II of the previous subsection and at approximately 38 minutes add a large disturbance coming into both Tank 1 and Tank 2, i.e. we add Case 7 of Table 9.3 into Table 9.4. The point of this experiment is to show that if the steady state u_{ss} does not abide to Assumption 4.1.1, then the controller shuts itself off; we have also seen this effect in the TcR case of the previous section. Figure 9.12 illustrates the results of the LTQcR while Figure 9.13 illustrates the tuning regulator control law.

In conclusion, we note that in general if the disturbance of the system is too large, then the tuning clamping regulator (and the linear tuning quadratic clamping regulator) will shut itself off.

9.4 MIMO TcR and LTQcR: experimental results

The main purpose of this section is to extend the experimental results of Section 9.2 and Section 9.3 to the MIMO TcR and the MIMO LTQcR case. The interest is to solve the positive servomechanism problem under MIMO control strategies of the TcR and the LTQcR. Here, we once again refer to Figure 9.2, under the assumption that both u_1 and u_2 are at our disposal and the water levels to control are those of tank 1 and tank

2. Table 9.5 outlines the experiments that have been performed. The goal will be to illustrate the theoretical results of the MIMO tuning clamping regulator and the MIMO linear tuning quadratic clamping regulator (Chapter 6 and Chapter 7). In all cases, the initial condition of the servo compensator is $\eta(0) \approx 0$.

Table 9.5: Experimental Cases

Case	A_1	A_2	B	C	D	E	F	G	ω_1	ω_2
1	on	on	off	on	on	on	off	on	small	small
2	on	on	on	on	on	on	off	on	small	small
3	on	on	on	on	on	on	on	on	small	small
4	on	on	on	on	on	on	on	on	large	large

First the TcR control approach is considered.

The MIMO TcR was defined in Chapter 6 and is given below for convenience.

Controller 9.4.1. Due to unmeasurable disturbances on the system the controller used in this section will be:

$$\begin{aligned} \dot{\eta} &= \epsilon(y_{ref} - y), \quad \eta_0 = 0 \\ u &= \alpha K \eta \end{aligned} , \tag{9.2}$$

where $K = (D - CA^{-1}B)^{-1}$ and

$$\alpha = \begin{cases} 0 & \text{if } \exists\, i \in \{1, ..., r\} \text{ such that } (K\eta)_i \leq 0, \\ 1 & \textit{otherwise} \end{cases}$$

In this case, to implement the controller we have to determine $K = (D - CA^{-1}B)^{-1}$; this can be done using Algorithm 2.3.1, in which case, we perform two steady state experiments on the plant to obtain this gain (K).

Using Algorithm 2.3.1, we first take

$$u = [7\ 0]^T,$$

where $\overline{u}_1 = 7$. Second, we take

$$u = [0\ 6]^T,$$

where $\overline{u}_2 = 6$. The resultant steady-state response with $\overline{u}_1$ and $\overline{u}_2$ as above was

$$\overline{y}_1 = [3.08 \ \ 1.71]^T$$

and

$$\overline{y}_1 = [1.06 \ \ 4.13]^T.$$

Two of the steady state responses are captured in Figures 9.14-9.15.

Next, by Algorithm 2.3.1, we obtain the gain matrix K_1:

$$\begin{aligned} K_1 &= \begin{bmatrix} 7 & 0 \\ 0 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 3.08 & 1.06 \\ 1.71 & 4.13 \end{bmatrix} \\ &= \begin{bmatrix} 0.440 & 0.177 \\ 0.244 & 0.688 \end{bmatrix}, \end{aligned}$$

which results in

$$K = K_1^{-1} = \begin{bmatrix} 2.650 & -0.680 \\ -0.941 & 1.694 \end{bmatrix}.$$

From the above result we can estimate our system A matrix as:

$$A = -K = \begin{bmatrix} -2.650 & 0.680 \\ 0.941 & -1.694 \end{bmatrix}$$

since

$$K = (-CA^{-1}B)^{-1} = -A,$$

where $C = B = I_{2\times 2}$. As a result an approximate model of this (stable) system is:

$$\dot{x} = \begin{bmatrix} -2.650 & 0.680 \\ 0.941 & -1.694 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{9.3}$$

$$y = x \tag{9.4}$$

with $x = [x_1 \ x_2]^T$ being the water levels of tanks one and two, respectively. This model will be used when the MIMO LTQcR control implementation will be considered.

First, we illustrate, via experimental results, the MIMO TcR control on the water-works setup.

9.4.1 MIMO TcR: Experiments

In the first set of experiments we consider Case 1 of Table 9.5 and illustrate the results obtained for different values of ϵ in the control law (9.2). Throughout these first set of experiments

$$y^1_{ref} = 2V$$

and

$$y^2_{ref} = 3V,$$

and the initial condition of the servo compensator is zero ($\eta(0) = 0$).

The results are described next:

(a) The first experiment uses the MIMO TcR controller (9.2) with $\epsilon = 0.05$, $y_{ref} = [2\ 3]^T$, and with initial conditions of the tanks being: tank 1 ($1.372V$) and tank 2 ($1.716V$). Figure 9.16 shows that with the choice of $\epsilon = 0.05$ a resultant settling time of approximately 1445 seconds and $\%OS = 37\%$ is attained. Figure 9.17 gives the corresponding input response.

(b) The second experiment uses the MIMO TcR controller (9.2) with $\epsilon = 0.02$, $y_{ref} = [2\ 3]^T$, and with initial conditions of the tanks being: tank 1 ($1.06V$) and tank 2 ($1.70V$). Figure 9.18 shows that with the choice of $\epsilon = 0.02$ the resultant settling time of approximately 1190 seconds is attained, and Figure 9.19 gives the input response.

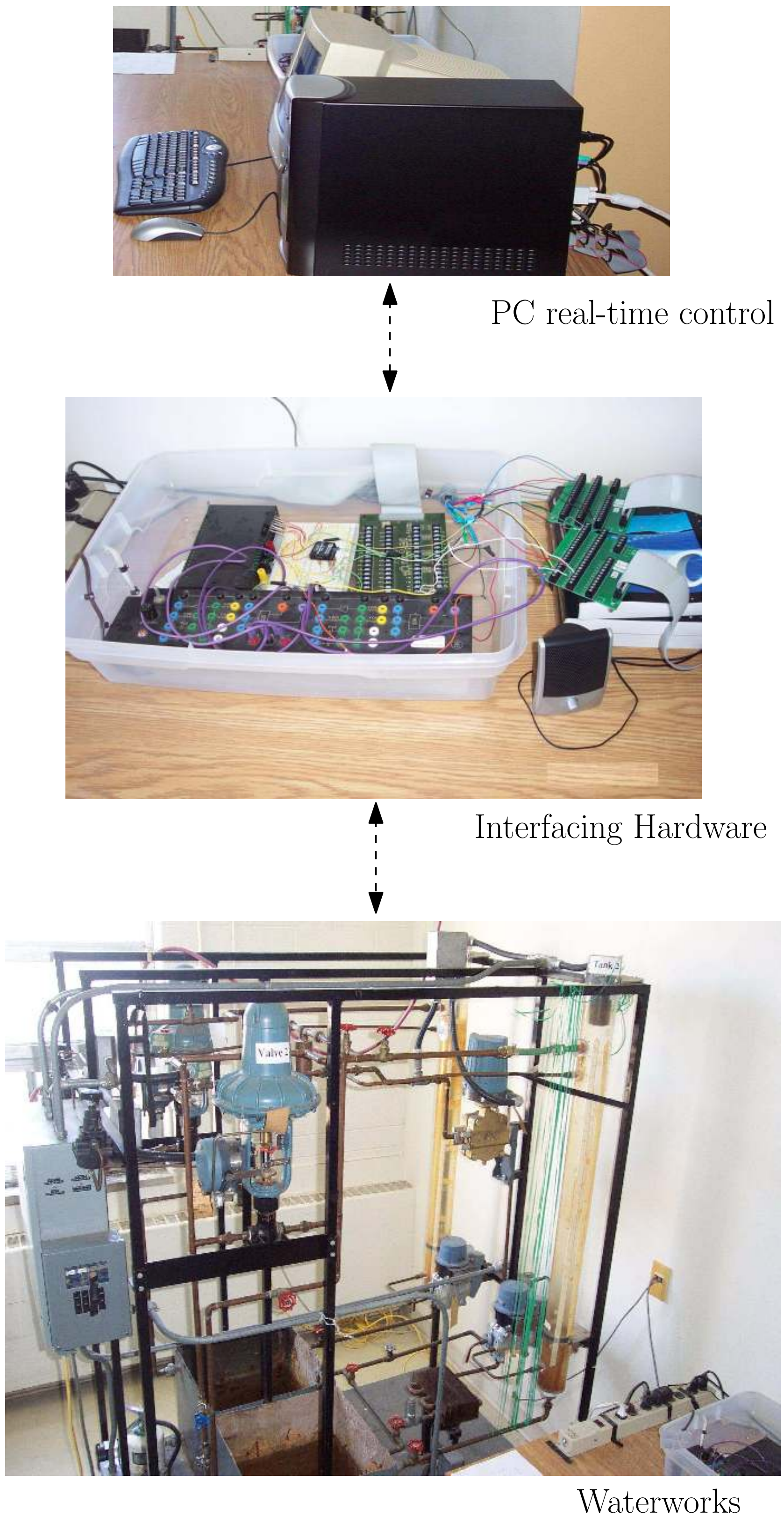


Figure 9.1: Experimental setup of the waterworks.

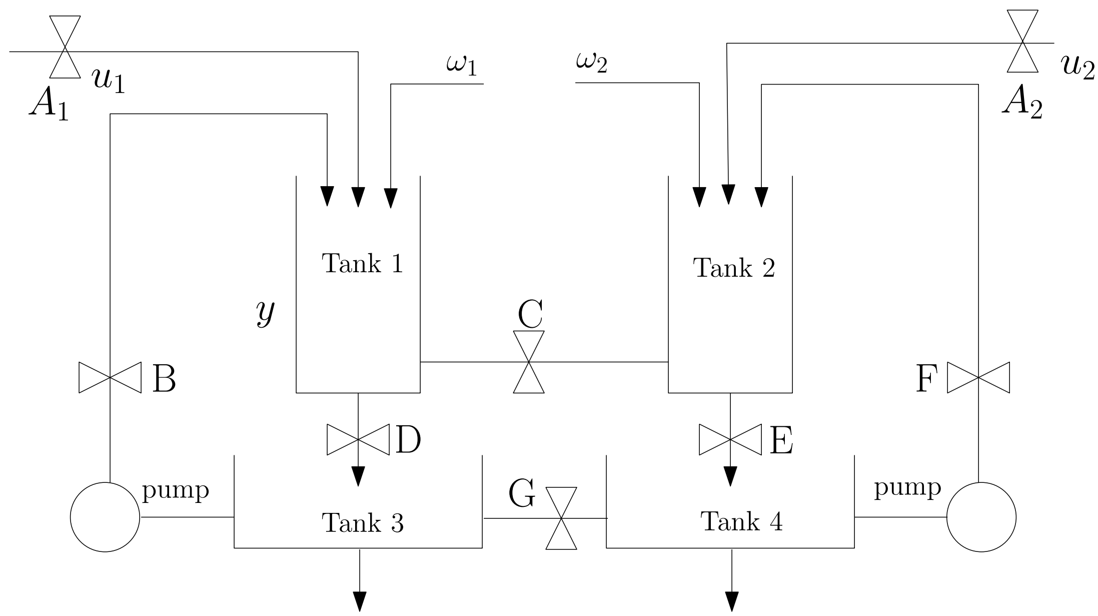


Figure 9.2: Diagram of the waterworks.

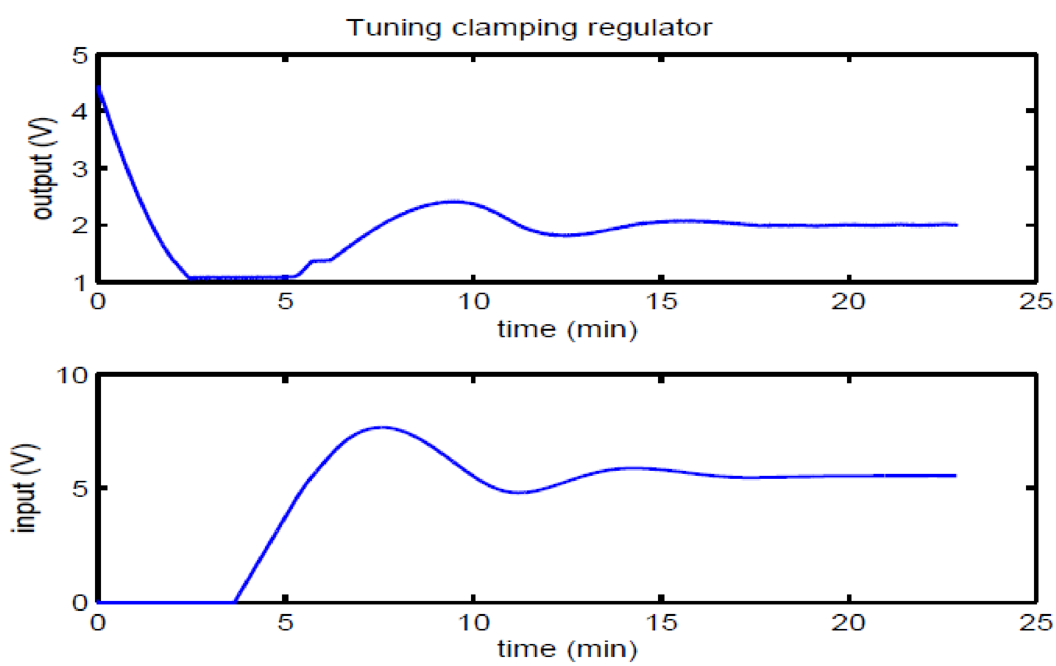


Figure 9.3: Experiment 1: Clamping controller (4.20).

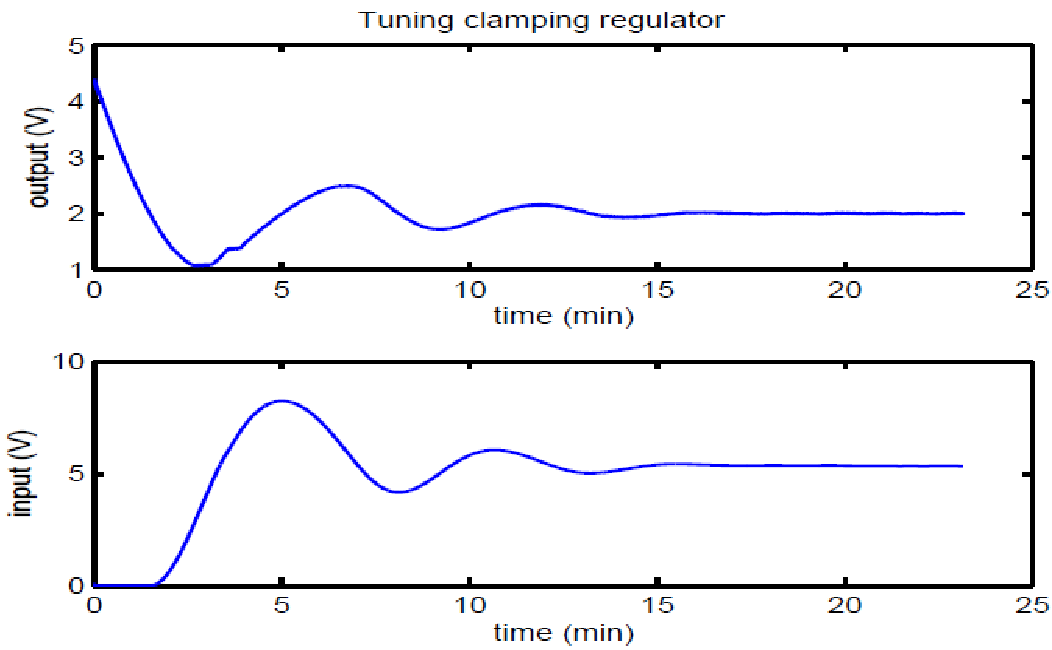


Figure 9.4: Experiment 1: Tuning clamping regulator (4.21).

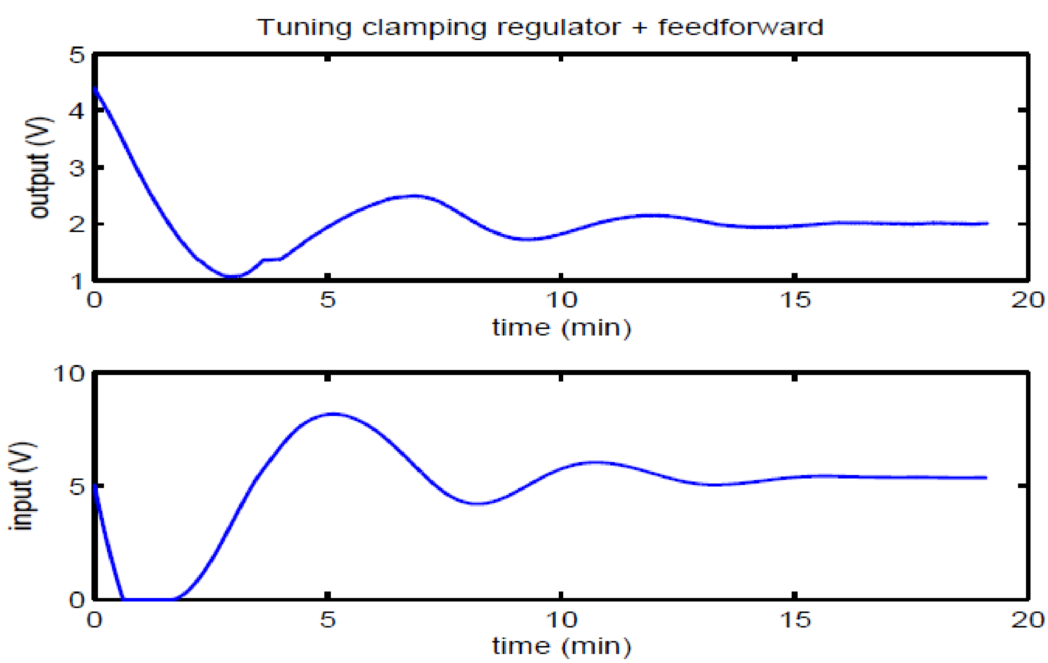


Figure 9.5: Experiment 1: Tuning clamping regulator (4.21) + u_{ff}.

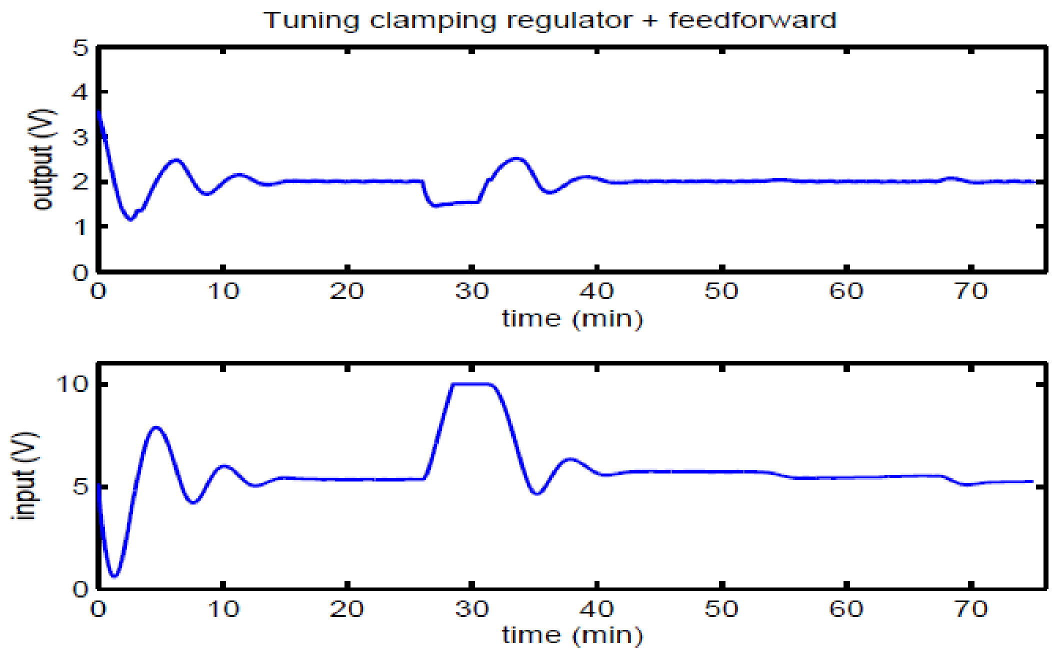


Figure 9.6: Experiment 2: Tuning clamping regulator (4.21) + u_{ff} under perturbations.

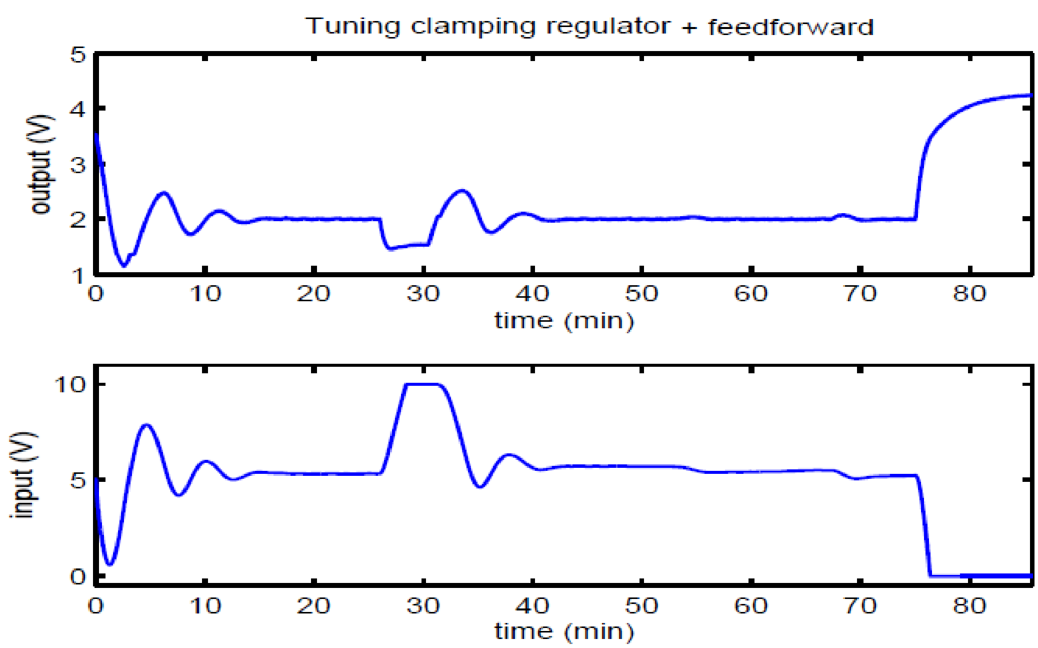


Figure 9.7: Experiment 3: Tuning clamping regulator (4.21) + u_{ff} under large disturbances.

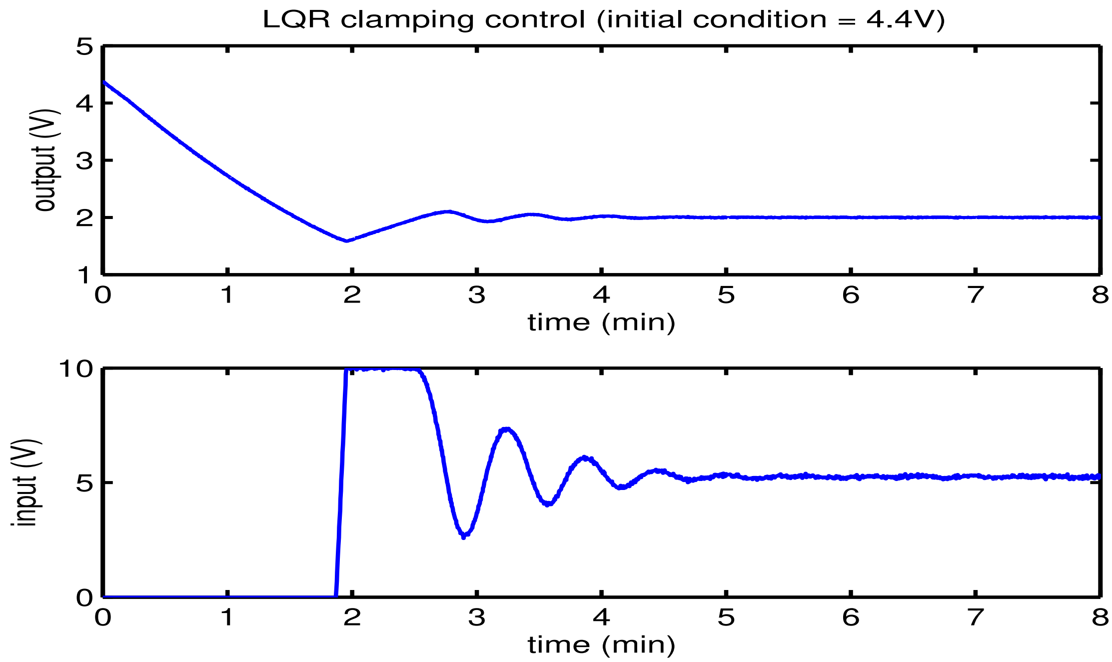


Figure 9.8: Experiment 1: LTQcR (5.15).

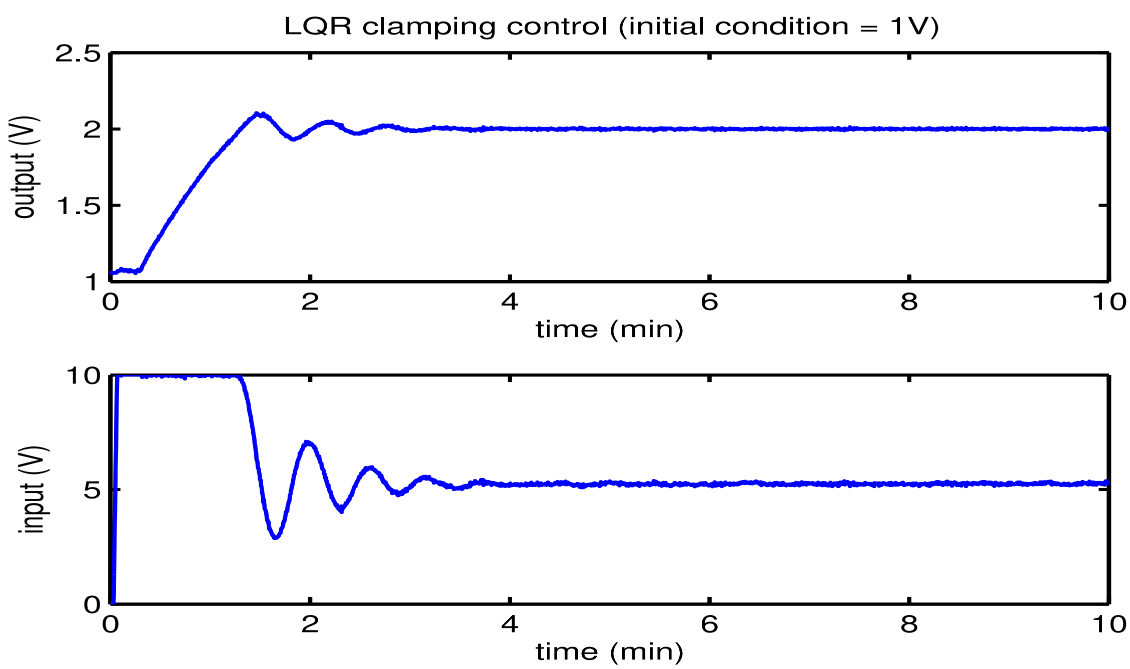


Figure 9.9: Experiment 1: LTQcR (5.15) with $V = 1.1$ (empty tank).

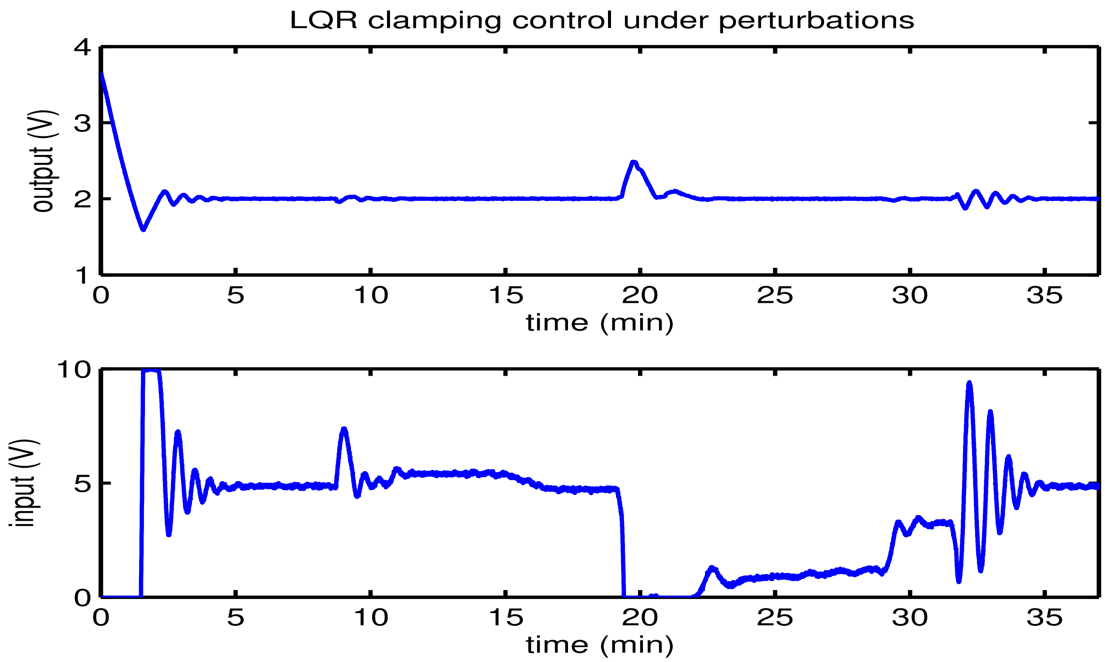


Figure 9.10: Experiment 2: LTQcR(5.15) under perturbations.

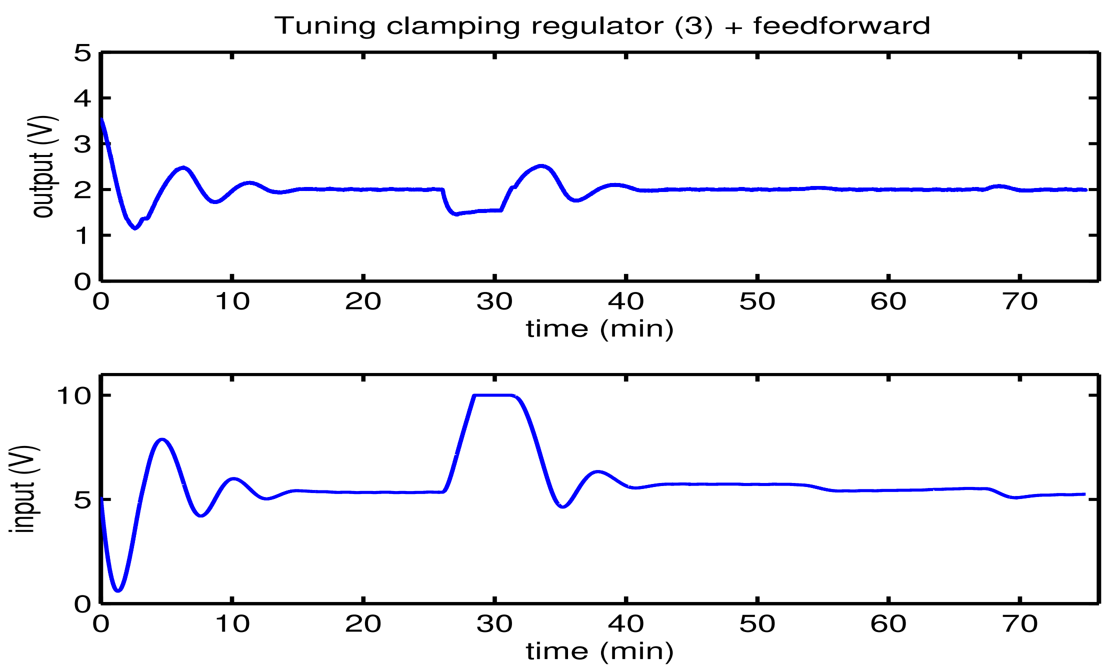


Figure 9.11: Experiment 2: Tuning clamping regulator (4.21) with feedforward term under perturbations.

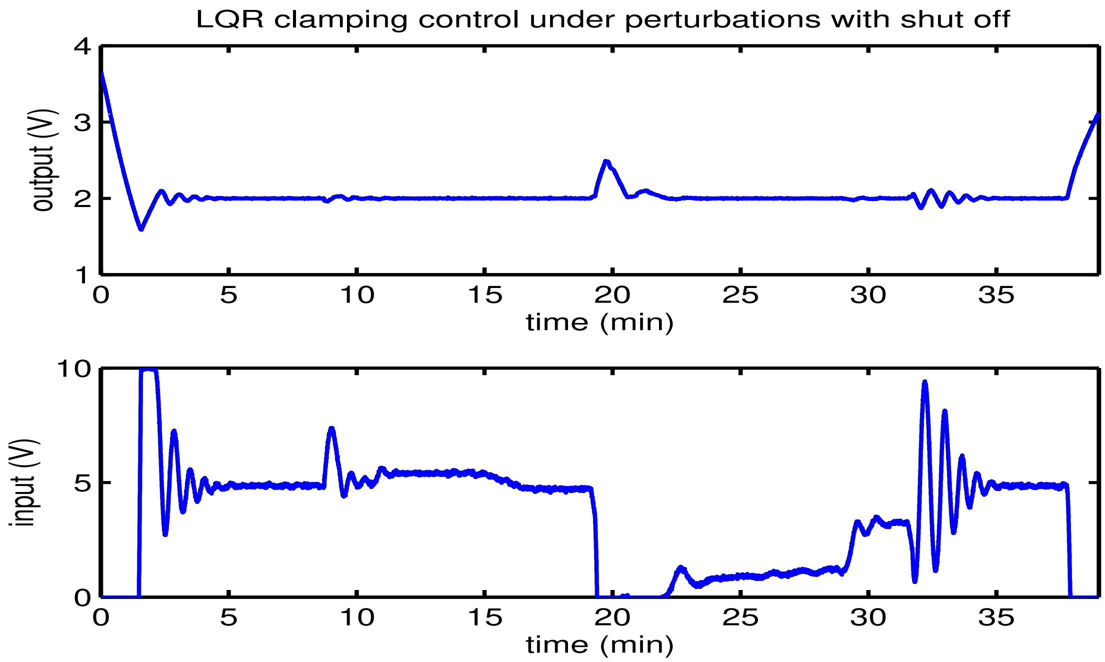


Figure 9.12: Experiment 3: LTQcR (5.15) under large disturbances and perturbations.

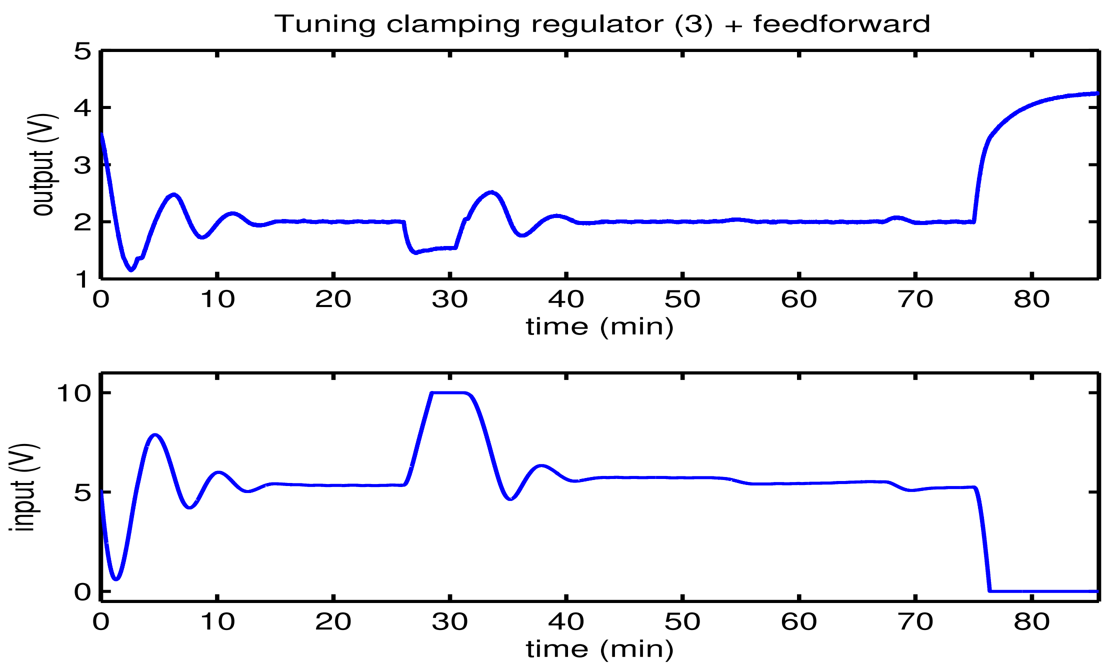


Figure 9.13: Experiment 3: Tuning clamping regulator (4.21) + u_{ff} under perturbations.

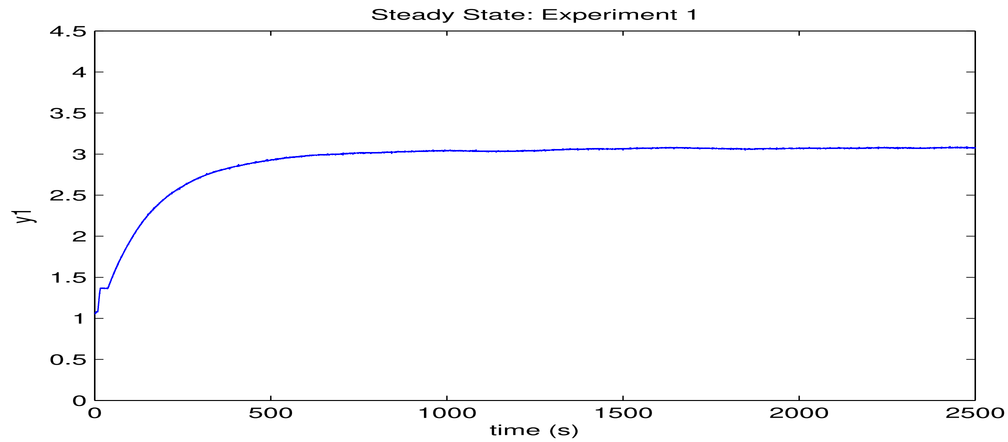


Figure 9.14: Steady state experiment 1: response of tank 1 (y_1).

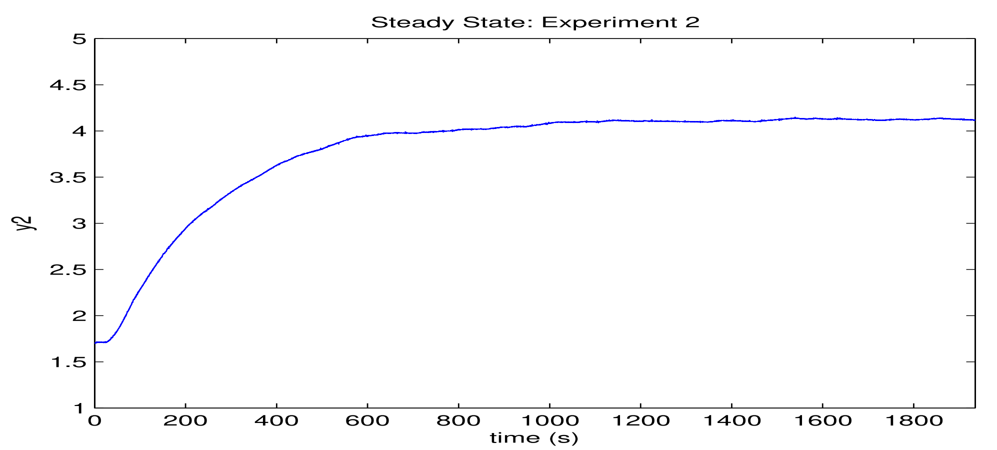
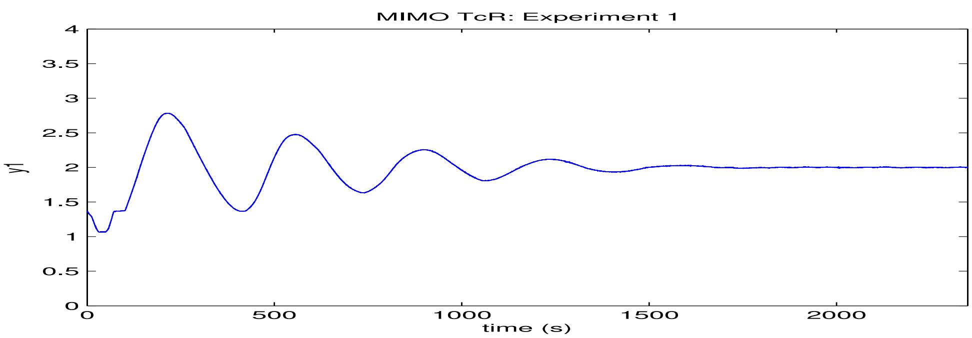
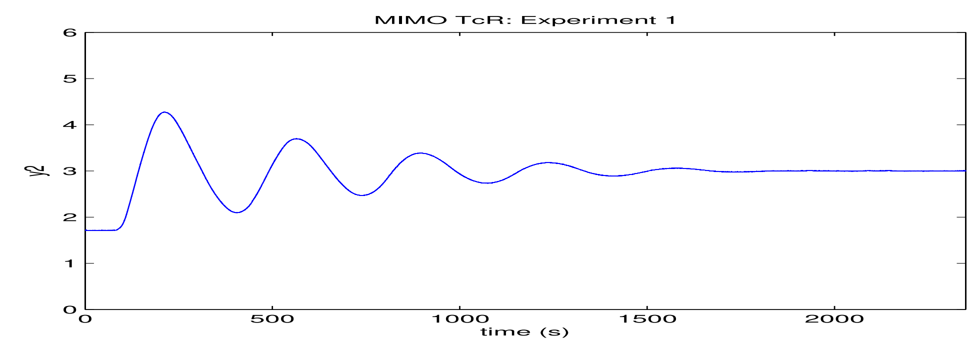


Figure 9.15: Steady state experiment 2: response of tank 2 (y_2).

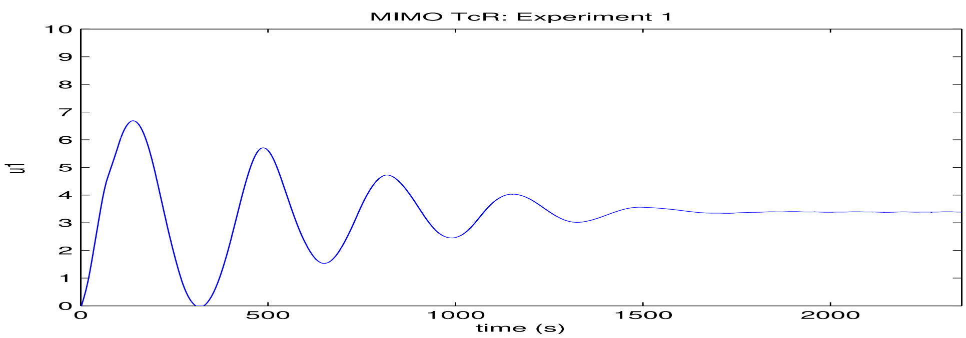


(a) Experiment 1: tank 1 (y_1) response

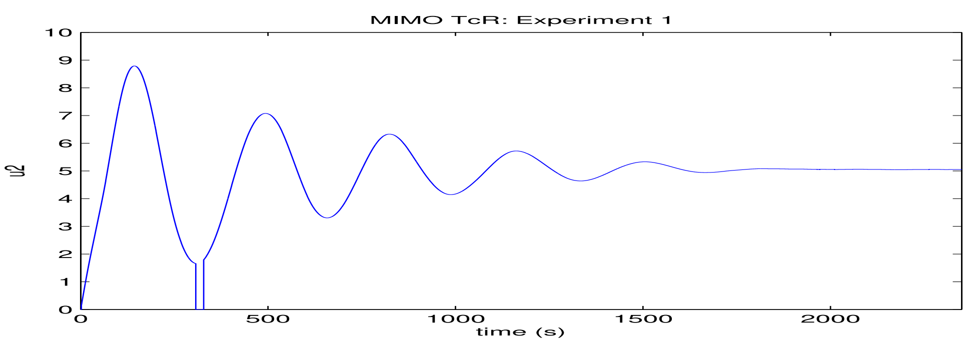


(b) Experiment 1: tank 2 (y_2) response

Figure 9.16: MIMO TcR Experiment 1: Output response with $\epsilon = 0.05$ and initial conditions of tanks: tank 1 ($1.372V$) and tank 2 ($1.716V$)

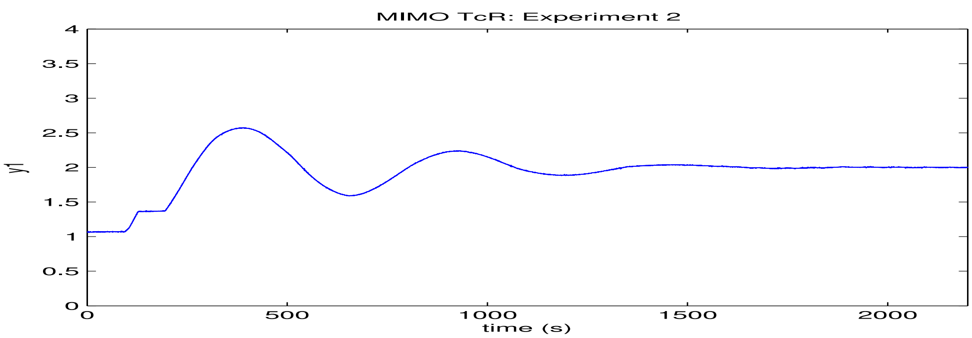


(a) Experiment 1: input (u_1) response

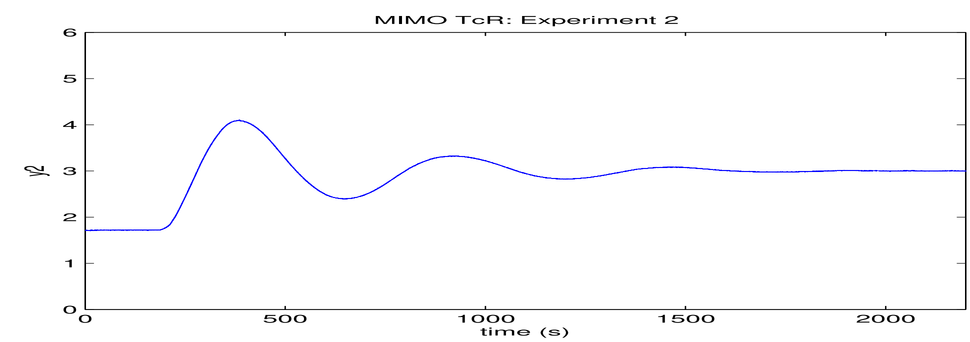


(b) Experiment 1: input (u_2) response

Figure 9.17: MIMO TcR Experiment 1: Input response with $\epsilon = 0.05$ and initial conditions of tanks: tank 1 ($1.372V$) and tank 2 ($1.716V$)

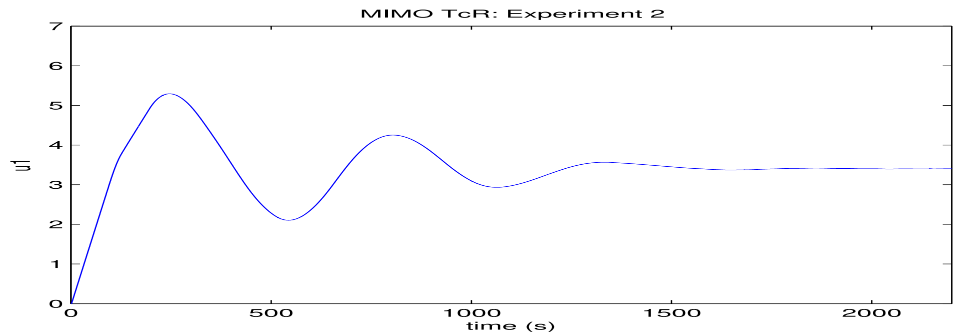


(a) Experiment 2: tank 1 (y_1) response

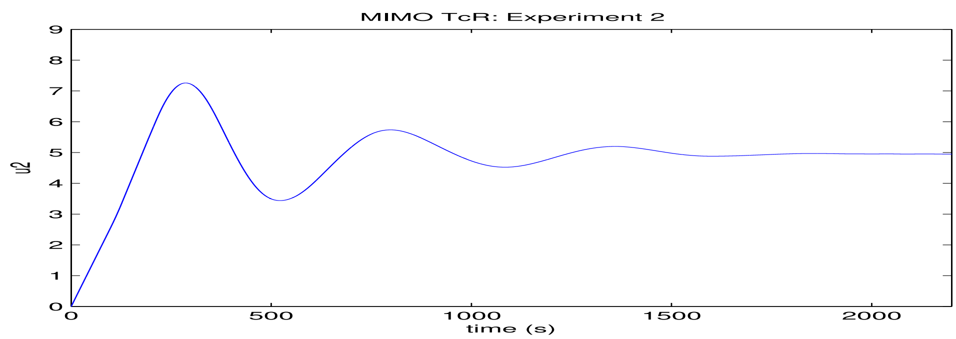


(b) Experiment 2: tank 2 (y_2) response

Figure 9.18: MIMO TcR Experiment 2: Output response with $\epsilon = 0.02$ and initial conditions of tanks: tank 1 ($1.06V$) and tank 2 ($1.70V$)



(a) Experiment 2: input (u_1) response



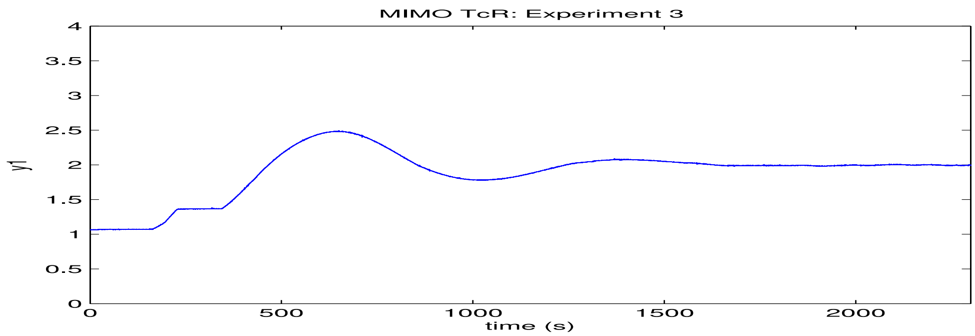
(b) Experiment 2: input (u_2) response

Figure 9.19: MIMO TcR Experiment 2: Input response with $\epsilon = 0.02$ and initial conditions of tanks: tank 1 ($1.06V$) and tank 2 ($1.70V$)

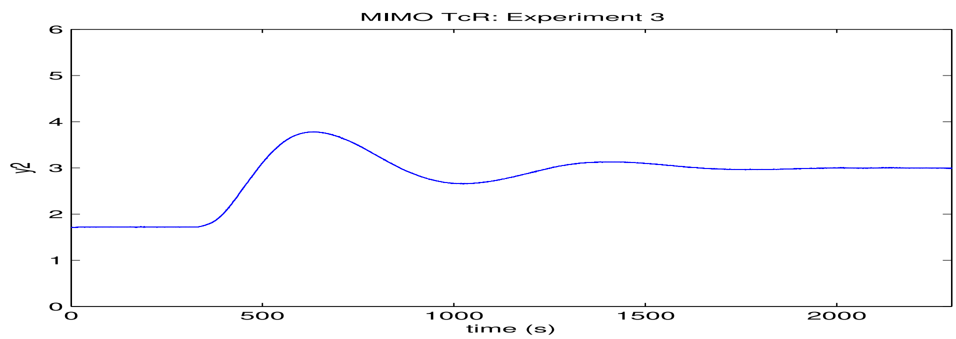
(c) The third experiment uses the MIMO TcR controller (9.2) with $\epsilon = 0.01$, $y_{ref} = [2\ 3]^T$, and with initial conditions of the tanks being: tank 1 ($1.06V$) and tank 2 ($1.70V$). Figure 9.20 shows that with the choice of $\epsilon = 0.01$ the resultant settling time of approximately 1251 seconds is attained. Figure 9.21 gives the input response.

(d) The fourth experiment uses the MIMO TcR controller (9.2) with $\epsilon = 0.02$, $y_{ref} = [3.5\ 1.5]^T$, and with initial conditions of the tanks being: tank 1 ($3.02V$) and tank 2 ($3.34V$). In this case since the y_{ref} chosen does not abide to the steady-state conditions of the input:

$$u(\infty) = K_r y_{ref} = \begin{bmatrix} 2.650 & -0.680 \\ -0.941 & 1.694 \end{bmatrix} [3.5\ 1.5]^T = [8.255\ \ -0.753]^T$$



(a) Experiment 3: tank 1 (y_1) response



(b) Experiment 3: tank 2 (y_2) response

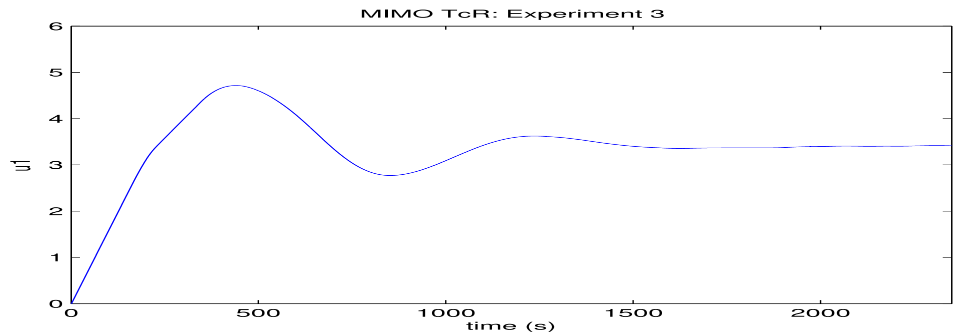
Figure 9.20: MIMO TcR Experiment 3: Output response with $\epsilon = 0.01$ and initial conditions of tanks: tank 1 ($1.06V$) and tank 2 ($1.70V$)

the controller should turn off and remain turned off, as no solution to the problem of the servomechanism exists. This is illustrated via Figure 9.25(a) (for $t \in [0, 120]$) for the input response into the system (clearly it remains shut off) and Figure 9.24(a) for the output, which slowly gets emptied. This experiment was run for 120 seconds after which the reference was switched back to $y_{ref} = [2\ 3]^T$.

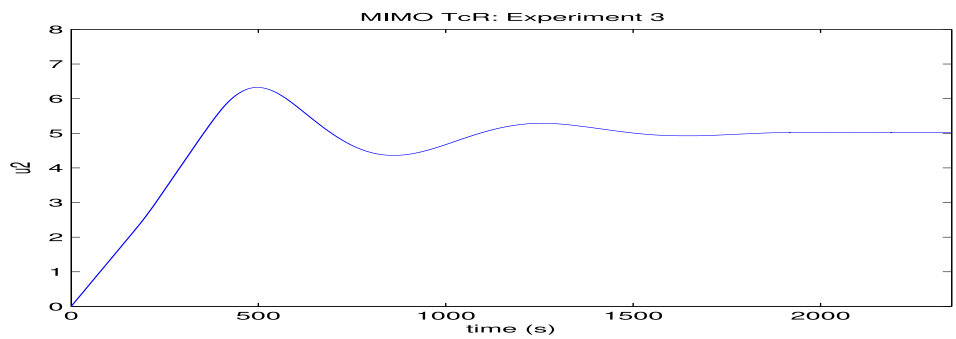
Note that our K_r has been found by experimentation and thus is merely an approximation, so one must be careful in examining the steady-state values before implementation, e.g. if we consider $y_{ref} = [3\ 2]$, the steady state result for u is

$$u_{ss} = [6.59\ 0.57]^T,$$

and since 0.57 is "small" and "close" to zero, one cannot be certain that the system



(a) Experiment 3: input (u_1) response



(b) Experiment 3: input (u_2) response

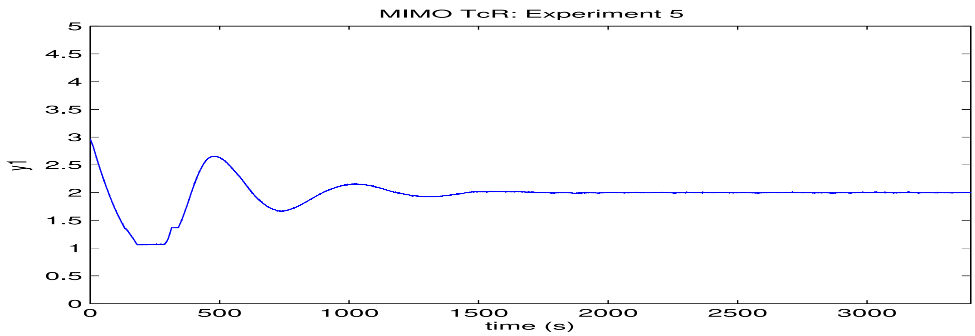
Figure 9.21: MIMO TcR Experiment 3: Input response with $\epsilon = 0.01$ and initial conditions of tanks: tank 1 ($1.06V$) and tank 2 ($1.70V$)

will actually track the designated reference.

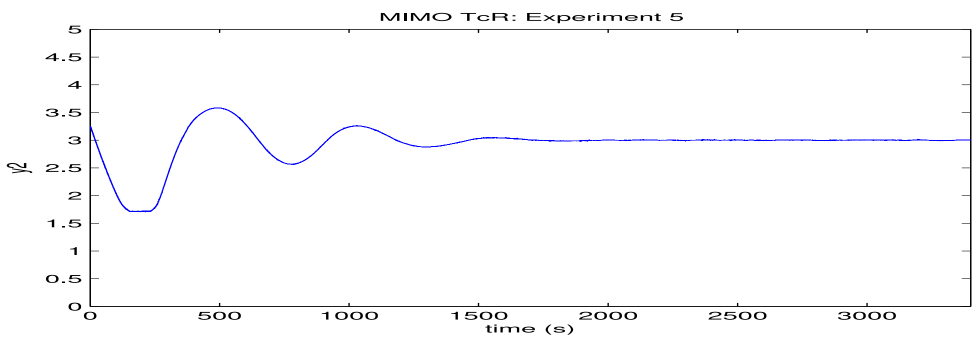
(e) The fifth experiment takes the MIMO TcR (9.2) with $\epsilon = 0.02$, $y_{ref} = [2\ 3]^T$, and initial conditions of tanks: tank 1 ($3.02V$) and tank 2 ($3.34V$). Figure 9.22 shows that with the latter choice of $\epsilon = 0.02$ the resultant settling time of approximately 1427 seconds is attained. Figure 9.23 shows the input response.

(f) The final experiment with respect to the MIMO TcR (9.2) case considers multiple perturbations. In particular, this experiment begins at $t = 120s$ (see point (f) above) with Case 1 of Table 9.5 with and then transitions to Case 2 and onward. The complete list of transitions is given below:

- Case 1: time $\approx 120 - 2550s$
- Case 2: time $\approx 2551 - 3600s$



(a) Experiment 5: tank 1 (y_1) response



(b) Experiment 5: tank 2 (y_2) response

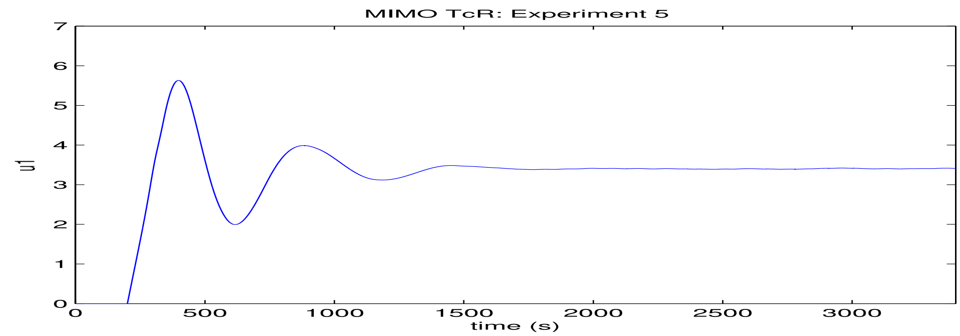
Figure 9.22: MIMO TcR Experiment 5: Output response with $\epsilon = 0.02$ and initial conditions of tanks: tank 1 ($3.02V$) and tank 2 ($3.34V$)

- Case 3: time $\approx 3601 - 5000s$
- Case 4: time $\approx 5000 - 5150s$

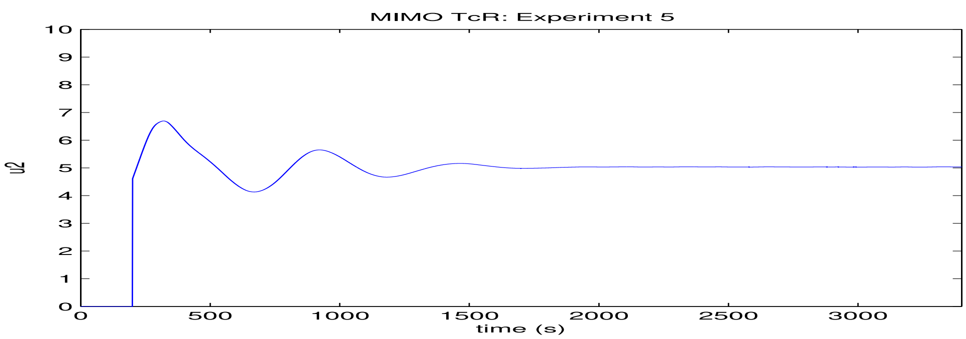
In this experiment we chose $\epsilon = 0.02$, $y_{ref} = [2\ 3]^T$, and initial conditions of tanks: tank 1 ($3.02V$) and tank 2 ($3.34V$).

Figure 9.24 shows the output response and Figure 9.25 shows the input response. Notice that for Case 4, the inputs get turned off as the disturbance gets too large and no solution exists to the servomechanism problem.

Next, in order to validate the theoretical results of the MIMO LTQcR control strategy two experimental results have been included. Recall, the MIMO LTQcR controller has been defined in Chapter 7 by:



(a) Experiment 5: input (u_1) response



(b) Experiment 5: input (u_2) response

Figure 9.23: MIMO TcR Experiment 5: Input response with $\epsilon = 0.02$ and initial conditions of tanks: tank 1 ($3.02V$) and tank 2 ($3.34V$)

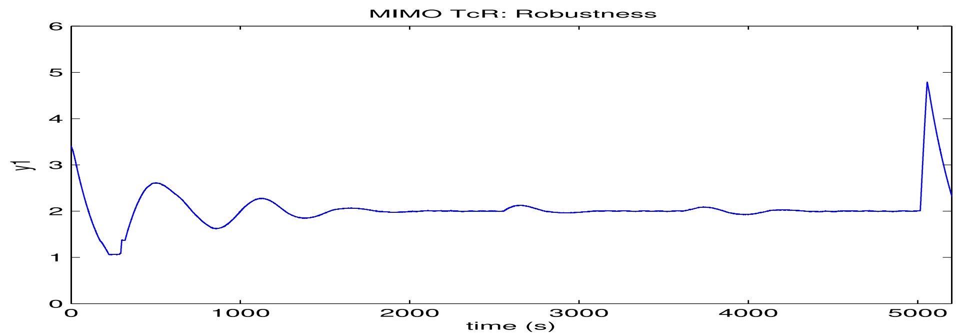
Controller 9.4.2. Assume $\text{rank}(D - CA^{-1}B) = r$. Given $\epsilon > 0$, the controller is described by:

$$\dot{\eta} = y - y_{ref}; \qquad u = \alpha[K_x \quad K_\eta][x\ \eta]^T,\ u(0) > 0 \tag{9.5}$$

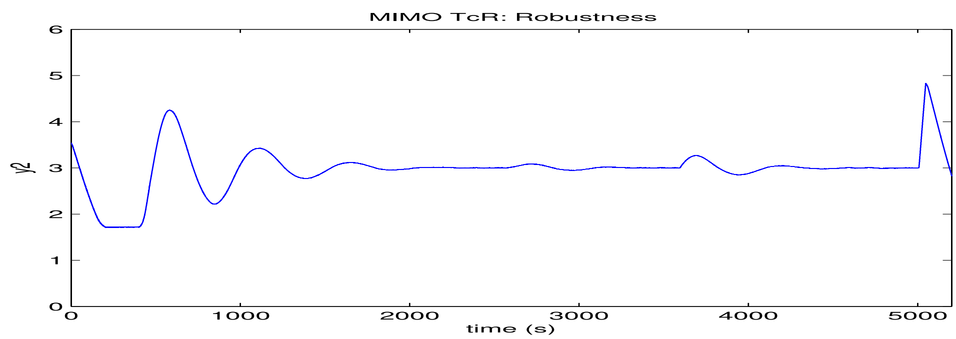
$$\alpha = \begin{cases} 0 & \text{if } \exists\ i \in \{1, ..., r\} \ \text{ s.t } \ ([K_x \quad K_\eta][x\ \eta])_i \leq 0, \\ 1 & \qquad\qquad\qquad otherwise \end{cases}$$

where $K_x \in \mathbb{R}^{m \times n}$ and $K_\eta \in \mathbb{R}^{m \times r}$ are found by solving the LQ control problem:

$$\int_0^\infty \epsilon^2 e^T Q e + \dot{u}^T \dot{u} d\tau \tag{9.6}$$



(a) Robustness Experiment: tank 1 (y_1) response

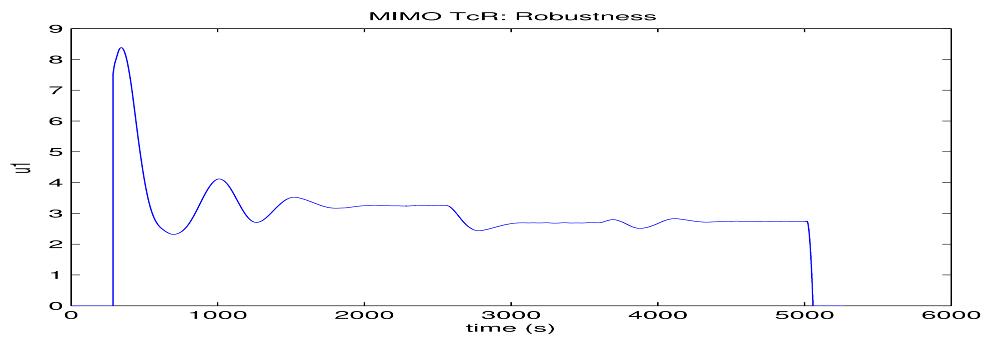


(b) Robustness Experiment: tank 2 (y_2) response

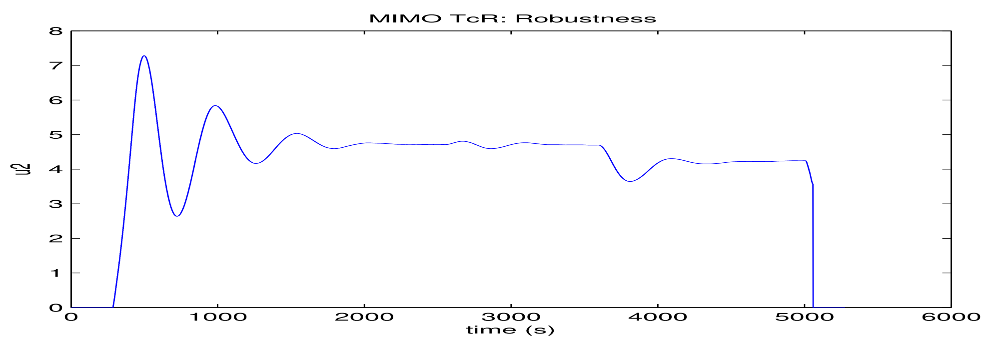
Figure 9.24: MIMO TcR Robustness Experiment: Output response with $\epsilon = 0.02$ and initial conditions of tanks: tank 1 ($3.44V$) and tank 2 ($3.51V$)

with $\rho > 0$ and $Q = ((D - CA^{-1}B)^{-1})^T (D - CA^{-1}B)^{-1})$. $\square$

Next, two experimental simulations are presented under different values of ϵ (the ϵ has been used for consistency with the other Chapters and is clear from the context of discussion) for (9.5); see Figure 9.26 and Figure 9.28 for the output response and Figure 9.27 and Figure 9.29 for the corresponding inputs. Notice that the LTQcR results in a faster settling time of approximately 990 seconds, but is nowhere close to the results of the SISO RLTQcR control.

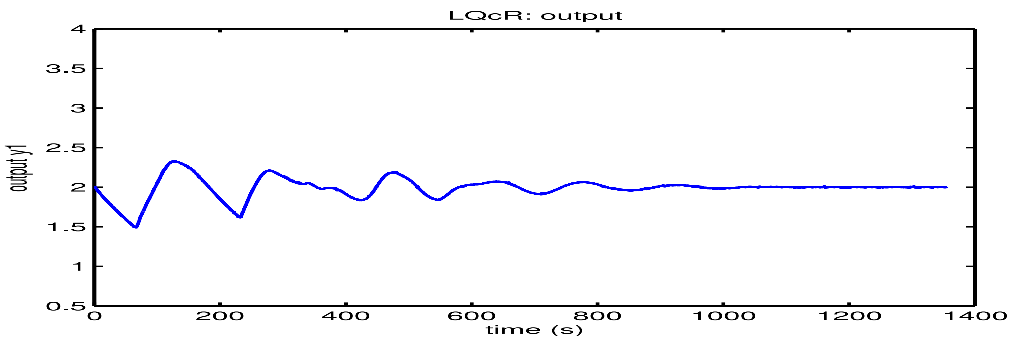


(a) Robustness Experiment: input 1 (u_1) response

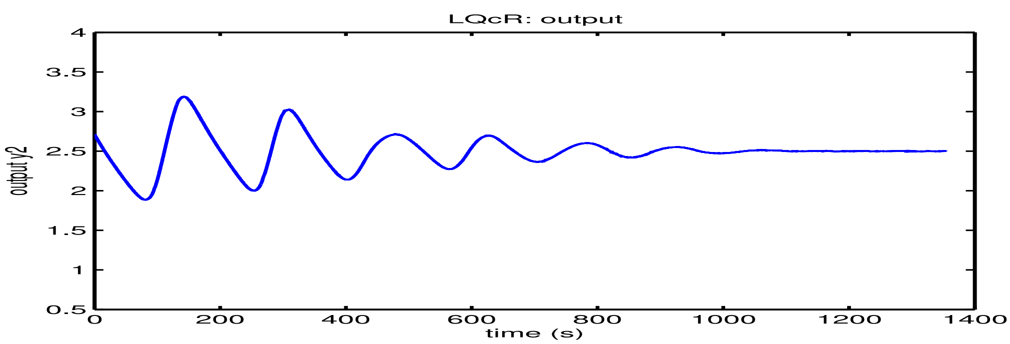


(b) Robustness Experiment: input 2 (u_2) response

Figure 9.25: MIMO TcR Robustness Experiment: Input response with $\epsilon = 0.02$ and initial conditions of tanks: tank 1 ($3.44V$) and tank 2 ($3.51V$)

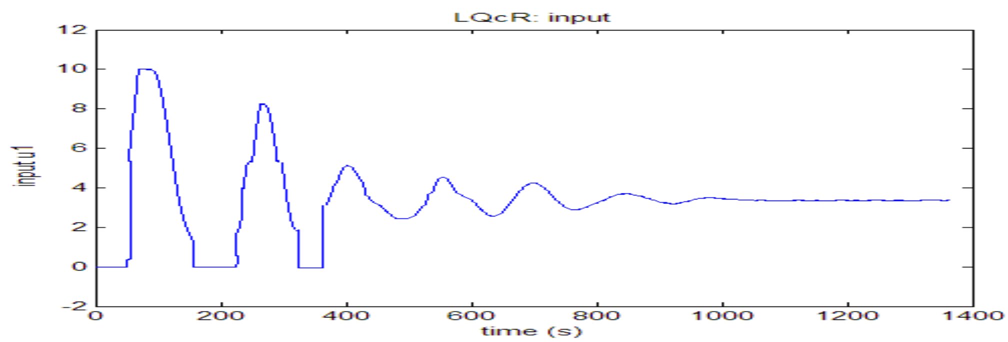


(a) LTQcR Experiment: output 1 (y_1) response

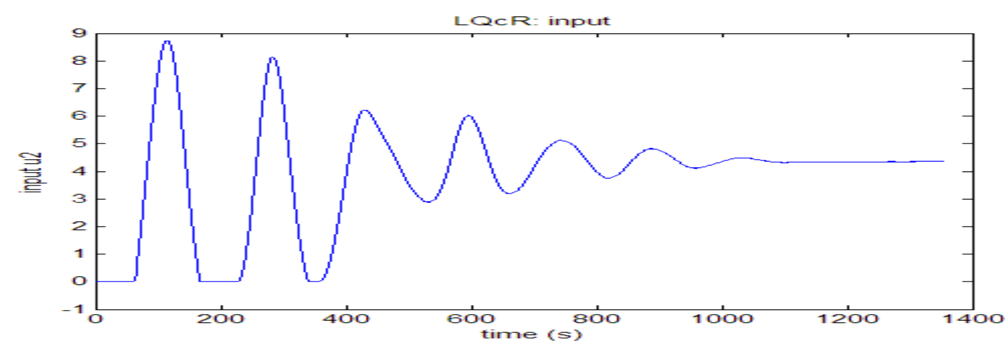


(b) LTQcR Experiment: output 2 (y_2) response

Figure 9.26: MIMO LTQcR Experiment: Output response with $\epsilon = 2$ and initial conditions of tanks: tank 1 ($1.95V$) and tank 2 ($2.71V$)

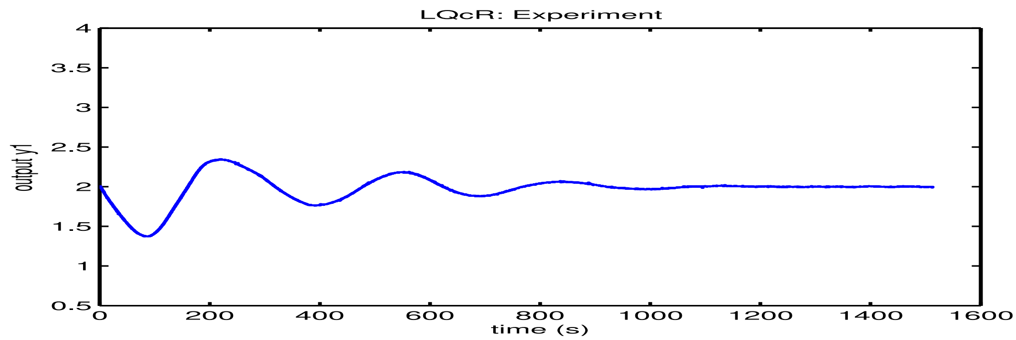


(a) LTQcR Experiment: input 1 (u_1) response

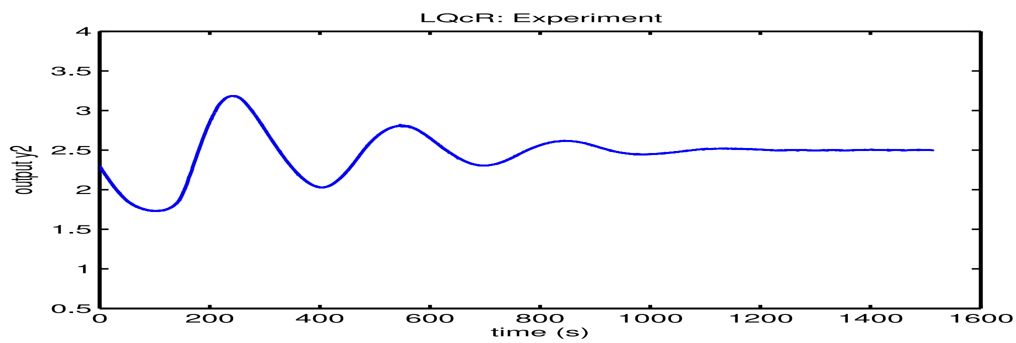


(b) LTQcR Experiment: input 2 (u_2) response

Figure 9.27: MIMO LTQcR Experiment: Input response with $\epsilon = 2$ and initial conditions of tanks: tank 1 ($1.95V$) and tank 2 ($2.71V$)



(a) LTQcR Experiment: output 1 (y_1) response

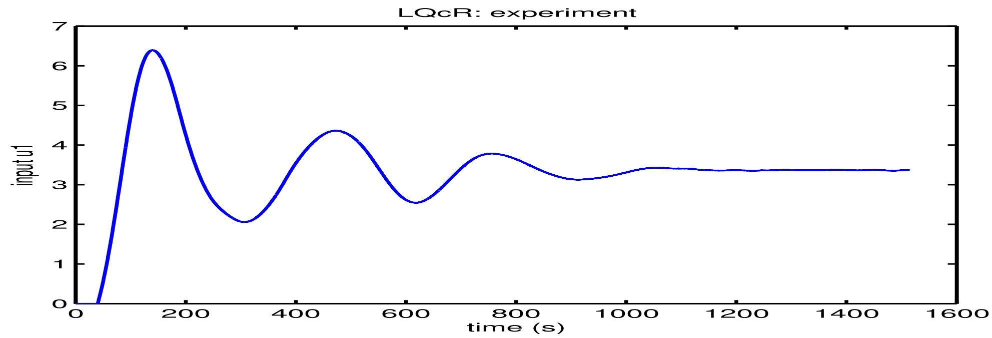


(b) LTQcR Experiment: output 2 (y_2) response

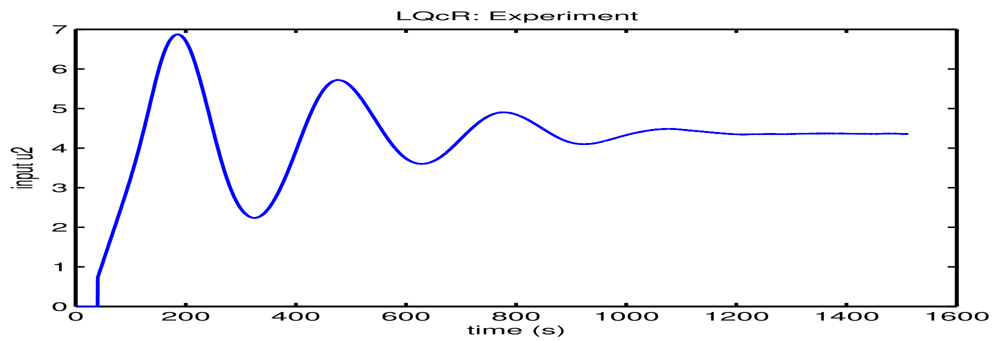
Figure 9.28: MIMO LTQcR Experiment: Output response with $\epsilon = 0.5$ and initial conditions of tanks: tank 1 ($2.08V$) and tank 2 ($2.29V$)

9.5 Conclusion

In this chapter we have used both the TcR and the LTQcR to illustrate experimentally the servomechanism problem for stable positive SISO and MIMO systems. In conclusion, the tuning clamping regulator and the linear tuning quadratic clamping regulator have justified the theoretical results of this dissertation; moreover, we can also conclude that the TcR and the LTQcR can be easily implemented via Matlab real-time experimentation and are very practical to use from the perspective of cost and validation of a solution. The clear underlying problem with the current approach is the sluggishness of the response, especially for the MIMO case. The SISO RLTQcR has generated extremely robust and good response with the restrictions of saturation effects which were present on the system.



(a) LTQcR Experiment: input 1 (u_1) response



(b) LTQcR Experiment: input 2 (u_2) response

Figure 9.29: MIMO LTQcR Experiment: Input response with $\epsilon = 0.5$ and initial conditions of tanks: tank 1 ($2.08V$) and tank 2 ($2.29V$)

The next set of steps clearly point to trying to improve the performance of the MIMO case, where possible anti-reset type behavior with some form of online self-tuning control could be implemented. This, however, we leave for future considerations.

Chapter 10

Conclusions and Future Research

This dissertation has studied positive linear time-invariant (LTI) systems. In particular the problems of state and output feedback stabilization, and the positive servomechanism problem, for positive LTI systems have been discussed. The problem of stabilization has also been extended to include results on the positive separation principle of LTI systems, i.e. the design of a state feedback stabilizing gain in conjunction with an observer feedback. Following the problem of positive stabilization, the focal point of the dissertation shifted toward finding necessary and sufficient conditions for reference tracking and disturbance rejection of stable positive LTI systems via robust control strategies, i.e. the *servomechanism problem* for positive LTI systems. Once the necessary and sufficient conditions were established, results on finding adequate control methodologies that solve the servomechanism problem were outlined. Finally, the theoretical results were verified via experimentation on a waterworks positive system composed of industrialized components.

In the next two sections we summarize the results and contributions of the thesis and discuss several extensions that may be of interest.

10.1 Summary and Contributions

Chapter 2 has provided primarily the building blocks for the thesis. In this chapter Section 2.1 defined all common terms and symbols used throughout the thesis. The chapter then outlined positive systems and compartmental systems and any associated results that were later needed in the dissertation. Next, a discussion of tuning regulators and feedforward control was reviewed and finally Section 2.4 discussed singular perturbation theory [45].

The contributions of Chapter 3 can be summarized as follows. The chapter was divided into three sections. The first section illustrated introductory results by presenting all definitions needed within the chapter. The second section of Chapter 3 outlined feedback stabilization, observer design, and the separation principle for positive single-input single-output (SISO) systems. The final section considered the latter problems of feedback stabilization, observer design, and the separation principle for the case of positive multi-input multi-output (MIMO) systems. In addition to the SISO and MIMO results, Chapter 3 illustrated numerous examples that outlined key differences between the single-input single-output case and the multi-input multi-output case.

Next, the focus shifted to the study of the tuning clamping regulator problem for stable *(un)known* SISO positive LTI systems under unmeasurable and measurable disturbances. In particular, existence conditions were provided, along with the actual control law, which solve the servomechanism problem for constant tracking and (un)measurable disturbance signals for unknown positive LTI systems. Chapter 4 not only considered the problem from the viewpoint of tuning clamping regulators (TcR), but also from the perspective of reset anti-windup tuning clamping regulators (RTCR). Finally, the chapter concluded with results on bidirectional control inputs.

Chapter 5 was an extension to the tuning clamping regulator results of Chapter 4, except unlike in the case of TcR control, in Chapter 5 it was assumed that the plant model was known; in this case, it was shown that the results for tracking and disturbance rejection can be significantly improved over those presented for unknown models of Chapter 4. The new control law that was utilized with the extra information was the linear quadratic clamping regulator (LQcR) and the reset linear quadratic clamping regulator (RLQcR). In Chapter 5 we have also noted that arbitrarily fast response, as for example in the case of perfect control type behaviour [20] for minimum phase LTI systems, may not be attainable for positive LTI systems under nonnegative control.

The next two chapters, Chapter 6 and Chapter 7, studied the servomechanism problem for MIMO positive LTI systems. In particular, Chapter 6 considered the tracking problem of nonnegative constant reference signals for stable known and unknown MIMO positive LTI systems under measurable and unmeasurable disturbances under a new tuning clamping regulator (TcR), and Chapter 7 extended the results of the SISO LQcR case to the MIMO positive systems case.

In Chapter 8 a short discussion of an existence condition for MPC controllers constrained to positive LTI systems under the servomechanism problem was outlined. It was shown via TcR or LQcR, that we can use MPC controllers to solve the servomechanism problem under nonnegative inputs.

The final chapter under clamping type controllers consisted of an experimental study of a waterworks setup [18], illustrating the effectiveness, ease and robustness of the set of controllers presented in Chapter 4 - Chapter 7. The main purpose of this study was to show that with no mathematical model of a system, or with only a very crude model approximation, the tuning clamping regulator (TcR) and the linear quadratic clamping regulator (LQcR) can *robustly* solve the problem of tracking a reference signals under disturbances and system changes.

10.2 Future Work

The evolution of the current research stemming from this dissertation can take on many facets. The first, of many extensions, can be in the study of the positive decentralized servomechanism problem. Namely, this dissertation provides answers only to the tracking and disturbance rejection problem using centralized control methods, which uses the notion of one overall controller for the system; however, in many practical applications centralized control may not always be considered as a feasible solution, and hence decentralized control must be considered. Decentralized control is applied in large-scale applications where the notion of several controllers is used to manage an overall task.

Another important extension is that of anti-reset and self-tuning regulators for MIMO positive systems. Throughout the experimental results it has become clear that the current approach of TcR and LQcR, although feasible, lacks various performance properties, e.g. settling time properties which are associated with reset windup. Therefore, it would be of great benefit to consider not only anti-reset controllers, as in the SISO case, but also self-tuning algorithms that would greatly improve various performance specifications.

Although this dissertation has already touched upon optimal control, the study is far from complete. There are numerous open questions regarding optimal control and its role within positive systems; and thus, a continuous study of robust constraint optimal-type controllers for positive systems, for example a more in depth study of Model Predictive Control (MPC), is needed. The importance of constrained optimal control in positive systems is vital for future applications, especially in the biomedical field, which relies heavily on constraints and changing environments. For example, an application of constrained optimal control can be considered in the study of automated drug infusion. This problem, as already described in the introduction and covered in Chapter 4, still poses numerous practical challenges; e.g. how will the controller be incorporated into a surgical procedure and how will it fair in a real-life setup? The problem of infusing anesthesia is not the only biomedical problem associated with positive systems; in fact, the control of insulin in diabetics, and many other drug related procedures can also be considered. Currently the study of drug infusion is gathering interest within the biological systems community, and since many of the questions are dependent on systems control theory and positive systems, this area will continue to grow and pose interesting theoretical and practical questions.

Appendix A

Counter Examples

This Appendix presents counter examples to three important papers in the field of stabilization of positive linear systems. Each section of this Appendix is entitled by the title of the paper under consideration.

A.1 Positive Linear Observers for Positive Linear Systems [15]

In [15], the author claims that there exists a convergent positive linear observer (or by duality stabilizing matrix) for a given LTI positive system if and only if Theorem 2 of [15] holds true. The Theorem is repeated for completeness below.

Theorem A.1.1 ([15]). Given a Metzler matrix $A \in \mathbb{R}^{n\times n}$ and a nonnegative matrix $C \in \mathbb{R}_+^{p\times n}$, define a nonnegative matrix $E_o \in \mathbb{R}_+^{n\times p}$ such that the elements of the i^{th} line of E_o are solution of the following linear programming problem:

$$\max_{e_{ij}} \sum_{j=1}^{p} e_{ij}c_{ji},$$

subject to

$$\sum_{j=1}^{p} e_{ij}c_{jk} \leq a_{ik}, \quad \forall\ k \neq i.$$

In case this optimization problem is unbounded ($c_{jk} = 0,\ \forall\ k \neq i$), then any value of e_{ij} such that

$$\sum_{j=1}^{p} e_{ij}c_{ji} > a_{ii}$$

is admissible.

Then there exists a convergent positive linear observer for the given system if and only if $\lambda_{max}(A - E_oC) < 0$.

The constructed example below shows that the Theorem in [15] is incomplete. Let

$$A = \begin{bmatrix} 0.5 & 9 & 4 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix},\ C = \begin{bmatrix} 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}.$$

By the algorithm presented in [15], we obtain the observer gain matrix:

$$L = \begin{bmatrix} 0 & 0.5 & 0 \\ 4 & 0 & 0 \end{bmatrix}^T,$$

which results in the unstable closed loop matrix which is Metzler and is stable:

$$A_c = \begin{bmatrix} 0.5 & 9 & 4 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 4 \\ 0.5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3.5 & 9 & 0 \\ 1 & -2.5 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

with eigenvalues $\{-1,\ -6.0414,\ 0.0414\}$. However, via the results of Chapter 3, we obtain a new stabilizing matrix:

$$L = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 1 & 0 \end{bmatrix}^T,$$

resulting in the new closed loop matrix which is Metzler and is stable.

$$\begin{bmatrix} -3.5 & 9 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Another problem of Theorem A.1.1 of [15] is the fact that it confines the stabilizing gain matrices to be nonnegative which, as pointed out in Chapter 3, is a result that must be relaxed.

A.2 Stabilization of positive linear systems [23]

This paper presents results on stabilization of SISO positive linear systems with the maximum eigenvalue of the linear system being at the origin. A significant statement of the paper, which the authors prove incorrectly, is summarized below (Proposition 8 on page 265 of the paper).

Proposition A.2.1 ([23]). Suppose that $\tilde{A}$ is a singular compartmental and irreducible matrix and that $\tilde{g} \in \mathbb{R}_+^n$. Then the pair $(\tilde{A},\ \tilde{g})$ is stabilizable if and only if

$$\tilde{g} \neq 0 \text{ and there exists at least one } i \in 1, ..., n \text{ such that } \forall j \neq i \text{ with } \tilde{g}_j \neq 0, \text{ also } \tilde{a}_{ji} \neq 0 \tag{A.1}$$

Above, the word stabilization means the standard LTI stabilization and not the stabilization we defined in Chapter 3, i.e. Proposition A.2.1 is stating that for the special class of singular compartmental and irreducible matrices positive stabilization is equivalent to the standard stabilization of linear systems. We show that this is false by an example below.

Consider the following two matrices

$$\tilde{A} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

and

$$\tilde{g} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

where $\tilde{A}$ is an irreducible compartmental matrix with maximum eigenvalue at the origin and $\tilde{g}$ is a nonnegative vector. Notice that

$$rank[\tilde{A} \quad \tilde{g}] = 3,$$

i.e. the system is stabilizable in the linear system sense, yet no k_s exists in order to make

$$\tilde{A} + \tilde{g}k_s$$

Metzler stable, i.e. condition (12) in the paper (equation (A.1) above) does not hold.

This shows that Proposition 8 (Proposition A.2.1 above) is in fact false.

A.3 Positive Observers for Linear Compartmental Systems [25]

This paper presents results for constructing positive observers for LTI compartmental systems. Once again by duality the construction of observers can be carried over to positive stabilization for compartmental systems.

Here we present an example that shows the incorrectness of Theorem 3.13 in [25]. Due to the overwhelming build up to the Theorem, we omit the Theorem itself and details associated with it. The interested reader can refer back to [25] for completeness.

Theorem 3.13 on page 599 can be disproved by the following example. Take the continuous LTI system with matrices[1]:

$$F = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$C = [1 \quad 1],$$

now this system has one trap[2] and $(F_{22}, C_2) = (0, 1)$, which satisfies modifiability[3] as per assumption of Theorem 3.13, yet there is no K that solves the problem, i.e. by the stabilization results of Chapter 3 (see Theorem 3.2.1 with $A = F^T$ and $b = C^T$).

The following set of examples above has illustrated several important observations on major papers on the subject of observer design for positive and compartmental systems. In this thesis we have answered all these questions and more. For complete statements of all the papers of this Appendix see [15], [23], and [25].

[1] note that the paper uses F instead of A due to the assumption that A is reduced to a special structure form, which is represented with the matrix F

[2] see [25] for clarification

[3] see [25] for clarification

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