Linear Time-Variable Systems: Balancing and Model Reduction

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Abstract—**A "uniformly balanced" realization for linear time-variable systems is defined. This representation is characterized by the fact that its controllability and observability Gramians are equal and diagonal. Existence and uniqueness of the uniformly balanced realization is studied. Such a framework has many remarkable properties and leads to a novel method for approximating time-variable systems, where the subsystems of the balanced realization can be taken as a reduced model. The reduced model is examined from the point of view of stability, controllability, and observability.**

I. INTRODUCTION

MULLIS and Roberts [25] and Moore [23] have recently introduced a novel coordinate system for realizing finite dimensional linear time-invariant systems. The realizations so obtained exhibit certain symmetries between the input and output maps of the realization and are called "balanced realizations." In the balanced coordinate system, the controllability and observability Gramians are equal and diagonal. By ordering these diagonal elements we are able to measure the "degree of controllability *and* observability" of different components of the state vector. The states corresponding to small diagonal elements are "nearly uncontrollable" *and* "nearly unobservable," and thus "nearly redundant," so that the most controllable and observable part can be retained as a reduced model. This amounts to taking a subsystem of the balanced realization as an approximation to the original system. For constant systems obtained in this way, reduced models are almost always stable if the original system is stable [4], [23]. Pernebo and Silverman [5], [24] showed, moreover, that the stability of the reduced model is guaranteed if the diagonal elements of the controllability and observability Gramians are distinct.

In this paper, we consider a generalization of balancing to time-variable systems, and study the properties of such a realization. This framework leads to what is possibly the first systematic procedure for lower order approximation of time-variable systems. In Section II some preliminary results and definitions are given. The earlier work of Silverman [2], [3] in which a class of system representations termed "uniform" were introduced are particularly useful. The class of uniform realizations for time-variable systems behave in many ways like that of minimal realizations (controllable and observable) for time-invariant systems. In Section III a characterization of systems which are equivalent to a balanced one is given. Also, a set of reasonable properties of the system (A, B, C) is proposed, which ensures the existence of balanced realizations. We then investigate the uniqueness of such a realization. Applications of balanced realizations to periodic systems are also considered. In Section IV we further study the properties of balanced realizations, which leads to a natural setting for model reduction of time-variable systems. We justify that the reduced model is in fact a good one, if the diagonal elements of the controllability and observability Gramians can be separated into "large" and "small" sets. Stability of the subsystems (reduced models) is of prime importance and it turns out that once the stability of subsystems is guaranteed, then the subsystems preserve many of the properties of the original system including balancedness.

This paper is an extended and more complete version of the conference papers [6] and [7] where some preliminary results without the proofs were given. Verriest and Kailath [22] have also considered balancing for the special class of analytic systems subsequent to our initial work [6]. It turns out that there are fundamental differences between discrete and continuous balanced realizations. Time-invariant discrete balanced systems are considered in [5], [24]–[27], while discrete time-variable systems are studied in [28].

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II. BACKGROUND

We shall be concerned throughout this paper with linear continuous-time system representations of the type

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1a}$$

$$y(t) = C(t)x(t) \tag{1b}$$

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where $x(t) \in \Re^n$, $u(t) \in \Re^r$, $y(t) \in \Re^m$ are the state, input, and output vectors, respectively, at time $t \in (-\infty, \infty)$. $A(t)$, $B(t)$, and $C(t)$ are matrices of order compatible with $x(t)$, $u(t)$, and $y(t)$ and are assumed to be continuous functions of time. A system representation of this type will be denoted by the triplet (A, B, C). The impulse response corresponding to this triplet is given by

$$H(t,\tau) = C(t)\Phi(t,\tau)B(\tau) \qquad t \geqslant \tau \tag{2}$$

where $\Phi(t,\tau)$ is the transition matrix associated with the homogeneous part of (1a). A realization of an impulse response $H(t,\tau)$ is any triplet (A,B,C) such that (2) holds. It is well known [18] that an impulse response can be realized by a system of type (1) iff it is separable in the form

$$H(t,\tau) = \Psi(t)\theta(\tau) \qquad t \geqslant \tau$$

where $\Psi(t)$ and $\theta(t)$ are continuous matrices of finite dimensions. The system $(0, \theta, \psi)$ then trivially realizes $H(t,\tau)$. However, such a realization is unstable and therefore undesirable practically. In general, there is no unique solution to the realization problem and different realizations of the same impulse response have quite distinct characteristics. It is necessary, therefore, to examine the properties of "equivalent" representations.

Definition 1: The representation $(\bar{A}, \bar{B}, \bar{C})$ is said to be *algebraically equivalent* to (A, B, C), if and only if there exists a continuously differentiable matrix $T(t)$, nonsingular for all $t \in \Re$, such that

$$\bar{A}(t) = T^{-1}(t)\left[A(t)T(t) - \dot{T}(t)\right] \tag{3a}$$

$$\bar{B}(t) = T^{-1}(t)B(t) \tag{3b}$$

$$\bar{C}(t) = C(t)T(t). \tag{3c}$$

■

The above type of equivalence will be denoted symbolically as follows:

$$(A, B, C) \overset{T}{\rightarrow} (\bar{A}, \bar{B}, \bar{C}). \tag{4}$$

It may be readily verified that if (4) holds, then

$$\bar{\Phi}(t,\tau) = T^{-1}(t)\Phi(t,\tau)T(\tau) \tag{5}$$

which, together with (3b) and (3c), implies that $\bar{H}(t,\tau) = H(t,\tau)$. Hence, input–output properties of a system are invariant under algebraic equivalence. However, the internal properties of a system may change under such a transformation. For example, the internal stability (Lyapunov stability or exponential stability) and boundedness[1] of the coefficient matrices A, B, C are not preserved under algebraic equivalence. Therefore, the following type of equivalence will be more important for our purpose.

Definition 2: The representation $(\bar{A}, \bar{B}, \bar{C})$ is said to be *topologically equivalent* to (A, B, C), if $(A, B, C) \overset{T}{\rightarrow} (\bar{A}, \bar{B}, \bar{C})$ where T is Lyapunov transformation, i.e., T, T^{-1}, and $\dot{T}$ are continuous and bounded. ■

It is a routine matter to show [11] that if T is a Lyapunov transformation, then the boundedness and internal stability of (A, B, C) is invariant under such a transformation. Other system properties of interest in this paper are controllability and observability. For bounded realizations, the following definitions of uniform complete controllability/observability are equivalent to the ones introduced by Kalman [1].

Definition 3 [2]: A bounded realization (A, B, C) is said to be *uniformly completely controllable* if $\exists \delta > 0$ such that

$$G_c(t-\delta, t) \geqslant \alpha_1(\delta)I > 0 \qquad \forall t \in \Re \tag{6}$$ [2]

where

$$G_c(t-\delta, t) \triangleq \int_{t-\delta}^{t} \Phi(t,\tau)B(\tau)B'(\tau)\Phi'(t,\tau)\,d\tau. \tag{7}$$

■

Definition 4: A bounded realization (A, B, C) is said to be *uniformly completely observable* if $\exists \delta > 0$ such that

$$G_0(t, t+\delta) \geqslant \alpha_2(\delta)I > 0 \qquad \forall t \in \Re \tag{8}$$

where

$$G_0(t, t+\delta) \triangleq \int_{t}^{t+\delta} \Phi'(\tau,t)C'(\tau)C(\tau)\Phi(\tau,t)\,d\tau. \tag{9}$$

■

If $(A, B, C) \overset{T}{\rightarrow} (\bar{A}, \bar{B}, \bar{C})$, the controllability and observability Gramians of the transformed system have the following forms:

$$\bar{G}_c(t) = T^{-1}(t)G_c(t)T^{-1\prime}(t) \tag{10}$$

$$\bar{G}_0(t) = T'(t)G_0(t)T(t) \tag{11}$$

and

$$\bar{G}_0(t)\bar{G}_c(t) = T'(t)G_0(t)G_c(t)T^{-1\prime}(t). \tag{12}$$

Equation (10) implies that $\bar{G}_c$ is congruent [19] to G_c, so that they have the same number of positive, negative, and zero eigenvalues. Equation (11) implies the same about $\bar{G}_0$ and G_0. Equation (12) implies that the eigenvalues of the product G_0G_c are invariant under algebraic equivalence. This set of invariants plays a crucial role in the theory to be presented.

Definition 5 [3]: A system representation (A, B, C) is said to be *uniform* if

1) $A(\cdot), B(\cdot), C(\cdot)$ are continuous and bounded

2) (A, B, C) is uniformly completely controllable and observable. ■

If the impulse response $H(t,\tau)$ is separable, i.e., $H(t,\tau)$

[1] A matrix $M(t)$ is said to be *bounded* if $\exists$ a constant $K \ni \|M(t)\| \leqslant K$ $\forall t \in \Re$, where $\|\cdot\|$ is the Euclidean norm.

[2] For symmetric matrices $A > B$ ($A \geqslant B$) means $A - B$ is positive (semi-) definite.

$=\Psi(t)\theta(\tau)$, $t \geqslant \tau$, then corresponding to any such separation there is a realization $(0, \theta, \Psi)$ for which we can define the controllability and observability matrices

$$M(t-\delta, t)=\int_{t-\delta}^{t} \theta(\tau)\theta'(\tau)\, d\tau \tag{13}$$

$$N(t, t+\delta)=\int_{t}^{t+\delta} \Psi'(\tau)\Psi(\tau)\, d\tau. \tag{14}$$

Silverman [3] gave the necessary and sufficient conditions for existence of uniform realization of a given impulse response $H(t,\tau)$.

Theorem 1: $H(t,\tau)$ is a uniformly realizable impulse response if and only if $H(t,\tau)=\Psi(t)\theta(\tau)$ where Ψ and θ are continuous matrices of finite order, and $\exists\delta>0 \ni$

i) $\lambda_m\{N(t,t+\delta)M(t-\delta,t)\} \geqslant \beta_1(\delta)>0$[3] $\quad \forall t$

ii) $\lambda_M\{\theta'(t)M^{-1}(t-\delta,t)\theta(t)\} \leqslant \beta_2(\delta)<\infty \quad \forall t$

iii) $\lambda_M\{\Psi(t)M(t-\delta,t)\Psi'(t)\} \leqslant \beta_3(\delta)<\infty \quad \forall t$

iv) $\lambda_M\left\{M^{-1}(t-\delta,t)\frac{\partial}{\partial t}M(t-\delta,t)\right\} \leqslant \beta_4(\delta)<\infty \quad \forall t.$ ■

The importance of uniform realizations is that this class plays a role similar to that of minimal (completely controllable and observable) realizations for time-invariant systems, as is shown by the following Theorems 2 and 3.

Theorem 2 [3]: Let (A,B,C) be a uniform realization of the impulse response $H(t,\tau)$. Then $(\bar{A},\bar{B},\bar{C})$ is also a uniform realization of $H(t,\tau)$ iff (A,B,C) and $(\bar{A},\bar{B},\bar{C})$ are topologically equivalent. ■

This shows that the class of uniform realizations of an impulse response matrix is closed under topological equivalence. Of course, a similar closure property holds for minimal time-invariant realizations. More importantly, for uniform realizations, "input–output" and "internal stability" are equivalent, which for time-invariant systems is equivalent to saying there are no pole-zero cancellations.

Definition 6:

1) A system with impulse response $H(t,\tau)$ is (zero state) *BIBO stable* if and only if $\exists k_1>0 \ni$

$$\int_{-\infty}^{t} \|H(t,\tau)\|\, d\tau \leqslant k_1 \qquad \forall t.$$

2) A realization (A,B,C) is *exponentially stable* if and only if $\exists k_2>0$ and $k_3>0 \ni$

$$\|\Phi(t,t_0)\| \leqslant k_2 e^{-k_3(t-t_0)} \qquad \forall t \geqslant t_0. \quad ■$$

We note that BIBO stability is independent of the particular realization. In contrast, exponential stability is a characteristic of the internal structure of the system. The following theorem shows that equivalence for uniform realizations.

[3] $\lambda_m\langle A\rangle$ and $\lambda_M\langle A\rangle$ denote the minimum and maximum eigenvalues of a matrix A.

Theorem 3 [2]: If the realization (A,B,C) is uniform, then it is BIBO stable iff it is exponentially stable. ■

Our analysis in the next section relies heavily on the six technical lemmas which for the continuity of discussion are contained in Appendix A. We close this section with the following useful Theorem.

Theorem 4: Suppose $G(t)$ is symmetric, uniformly positive definite, and Lyapunov. If there exists an eigenvalue decomposition

$$G(t)=U(t)\Sigma^2(t)U'(t)$$

where $U(t)$ is unitary and differentiable, then $\Sigma(t)$ is Lyapunov.

Proof: If we show $\Sigma^2(t)$ is Lyapunov, then its square root $\Sigma(t)$ is also Lyapunov (Lemma A2). Continuity of $\Sigma^2(t)$, $\Sigma^{-2}(t)$, and $(d/dt)\Sigma^2(t)$ follows from Lemma A5. Boundedness of $\Sigma^2(t)$ and $\Sigma^{-2}(t)$ is trivial and we need only to show that $(d/dt)\Sigma^2(t)$ is bounded.

$$\dot{G}(t)=\dot{U}(t)\Sigma^2(t)U'(t)+U(t)\frac{d}{dt}(\Sigma^2(t))U'(t) + U(t)\Sigma^2(t)\dot{U}'(t).$$

Pre- and postmultiplying the above equation by $U'(t)$ and $U(t)$, respectively, and using the fact that $U(t)$ is unitary we obtain

$$\frac{d}{dt}\Sigma^2(t)=U'(t)\dot{G}(t)U(t)-U'(t)\dot{U}(t)\Sigma^2(t) + \Sigma^2(t)U'(t)\dot{U}(t).$$

Defining $D(t)\triangleq U'(t)\dot{U}(t)$, we have

$$\frac{d}{dt}\Sigma^2(t)=U'(t)\dot{G}(t)U(t)-D(t)\Sigma^2(t)+\Sigma^2(t)D(t). \tag{15}$$

Since $U(t)$ is a unitary matrix, we have

$$D(t)+D'(t)=0$$

which means $D(t)$ is skew-symmetric (in particular, its diagonal elements are zero). Considering that $\Sigma^2(t)$ is diagonal, then $D(t)\Sigma^2(t)$ and $\Sigma^2(t)D(t)$ also have zero diagonal elements. Taking the diagonal elements of (15) we therefore have

$$\frac{d}{dt}\sigma_i^2(t)=i\text{th diagonal element of } (U'\dot{G}U)$$

but diagonal elements of $(U'\dot{G}U)$ are bounded since G is Lyapunov and U is unitary. Thus we have $(d/dt)\Sigma^2(t) \leqslant M$. ■

III. Uniformly Balanced Realizations

In this section we introduce the notion of a "uniformly balanced" realization for time-variable systems. We then deal with the existence, uniqueness, and other properties of such a realization. We start with the following definition.

Definition 7: A system representation (A,B,C) is said to be *uniformly balanced* if

1) (A,B,C) is uniform

2) $G_c(t-\delta,t)=G_0(t,t+\delta)=\Sigma(t)$, where $\Sigma(t)$ is a diagonal matrix. ■

Since uniformly balanced realizations form a subclass of the class of uniform realizations, we will always assume in this section that the systems we are dealing with can be uniformly realized (for necessary and sufficient conditions, see Theorem 1) and moreover, that a uniform realization (A,B,C) is given. The following theorem and its corollary characterizes the existence of a uniformly balanced realization within the class of uniform realizations.

Theorem 5: Let the impulse response $H(t,\tau)$ have a uniform realization (A,B,C). Then $H(t,\tau)$ has a uniformly balanced realization. $\Leftrightarrow$ The product G_0G_c has an eigenvalue decomposition of the form

$$G_0(t)G_c(t)=T(t)\Sigma^2(t)T^{-1}(t) \tag{16}$$

where $T(t)$ is Lyapunov.

Proof: ($\Leftarrow$) Applying the Lyapunov transformation $T^{-1'}(t)$ we obtain

$$(A,B,C)\overset{T^{-1'}(t)}{\to}(\bar{A},\bar{B},\bar{C})$$

where $(\bar{A},\bar{B},\bar{C})$ is uniform (Theorem 2) and from (12), the product of its Gramians is given by

$$\bar{G}_0\bar{G}_c=T^{-1}(t)G_0(t)G_c(t)T(t)=\Sigma^2(t).$$

By Lemma 3, $\Sigma^2(t)$ is diagonal (i.e., has no Jordan blocks), so that $\bar{G}_0\bar{G}_c=\Sigma^2(t)=\Sigma^{2'}(t)=\bar{G}_c\bar{G}_0$. By Lemma A4, $\exists$ a unitary matrix U such that

$$\bar{G}_0(t)=UD_0U', \qquad \bar{G}_c(t)=UD_cU' \tag{17}$$

where D_0 and D_c are diagonal, and

$$\bar{G}_0\bar{G}_c=UD_0D_cU'=\Sigma^2(t). \tag{18}$$

Since $\bar{G}_0$ and $\bar{G}_c$ are Lyapunov (Lemma A1) and the product of two Lyapunov transformations is again Lyapunov [11], then using Lemma A2, we know that $\bar{G}_c^{1/2}\Sigma^{-1/2}(t)$ is also Lyapunov. Applying this transformation we obtain the following:

$$(\bar{A},\bar{B},\bar{C})\overset{\bar{G}_c^{1/2}\Sigma^{-1/2}}{\to}(\tilde{A},\tilde{B},\tilde{C}).$$

Using (10) and (11) we have

$$\tilde{G}_c(t)=\Sigma^{1/2}\bar{G}_c^{-1/2}\bar{G}_c\bar{G}_c^{-1/2}\Sigma^{1/2}=\Sigma(t)$$
$$\tilde{G}_0(t)=\Sigma^{-1/2}\bar{G}_c^{1/2}\bar{G}_0\bar{G}_c^{1/2}\Sigma^{-1/2}. \tag{19}$$

Using (17) we obtain

$$\tilde{G}_0(t)=\Sigma^{-1/2}UD_c^{1/2}\underbrace{U'U}_{I}D_0\underbrace{U'U}_{I}D_c^{1/2}U'\Sigma^{-1/2}$$
$$=\Sigma^{-1/2}UD_c^{1/2}D_0D_c^{1/2}U'\Sigma^{-1/2}.$$

Since D_0 and D_c are diagonal, they commute, and using (18) we have

$$\tilde{G}_0(t)=\Sigma^{-1/2}UD_0D_cU'\Sigma^{-1/2}=\Sigma^{-1/2}\Sigma^2\Sigma^{-1/2}=\Sigma(t). \tag{20}$$

Equations (19) and (20) show that realization $(\tilde{A},\tilde{B},\tilde{C})$ is indeed uniformly balanced.

($\Rightarrow$) Suppose that $(\tilde{A},\tilde{B},\tilde{C})$ is a uniformly balanced realization with $\tilde{G}_0(t)=\tilde{G}_c(t)=\Sigma(t)$, then by Theorem 2, $(\tilde{A},\tilde{B},\tilde{C})$ and (A,B,C) are topologically equivalent, and there exists a Lyapunov transformation $L(t)$ such that $(\tilde{A},\tilde{B},\tilde{C})\overset{L(t)}{\to}(A,B,C)$. Therefore,

$$G_0(t)G_c(t)=L'(t)\tilde{G}_0(t)\tilde{G}_c(t)L^{-1'}(t)$$
$$=L'(t)\Sigma^2(t)L^{-1'}(t).$$

Letting $T(t)\triangleq L'(t)$, we have the required decomposition (16). ■

Notice that any uniform realization (A,B,C) could be used in the previous theorem without affecting the results. Since $G_c^{1/2}(t)$ is a Lyapunov transformation,

$$(A,B,C)\overset{G_c^{1/2}(t)}{\to}(\bar{A},\bar{B},\bar{C}) \tag{21}$$

where the realization $(\bar{A},\bar{B},\bar{C})$ is uniform with

$$\bar{G}_c(t)=G_c^{-1/2}G_cG_c^{-1/2}=I \tag{22}$$
$$\bar{G}_0(t)=G_c^{1/2}G_0G_c^{1/2}\triangleq S(t). \tag{23}$$

Using this uniform realization $(\bar{A},\bar{B},\bar{C})$ we then have the following corollary to the above theorem.

Corollary 1: Let $H(t,\tau)$ have a uniform realization $(\bar{A},\bar{B},\bar{C})$ as in (21). Then $H(t,\tau)$ has a uniformly balanced realization $\Leftrightarrow$ $G_0(t)\bar{G}_c(t)=S(t)$ has an eigenvalue decomposition of the form

$$\bar{G}_0(t)\bar{G}_c(t)=U(t)\Sigma^2(t)U'(t) \tag{24}$$

where $U(t)$ is unitary and $\dot{U}(t)$ is continuous and bounded.

Proof: ($\Leftarrow$) Since $U(t)$ is unitary, then boundedness and continuity of $\dot{U}(t)$ implies that $U(t)$ is Lyapunov and the proof follows from Theorem 5.

($\Rightarrow$) We only have to prove that $T(t)$ in Theorem 5 can be chosen unitary. Let $(\tilde{A},\tilde{B},\tilde{C})$ be a uniformly balanced realization with $\tilde{G}_0=\tilde{G}_c=\Sigma$, then $\exists$ Lyapunov transformation T such that

$$(\tilde{A},\tilde{B},\tilde{C})\overset{T}{\to}(\bar{A},\bar{B},\bar{C})$$

where $(\bar{A},\bar{B},\bar{C})$ is the realization in (21) with

$$\bar{G}_0(t)=T'(t)\Sigma(t)T(t)=I \tag{25}$$
$$\bar{G}_c(t)=T^{-1}(t)\Sigma(t)T^{-1'}(t)=S(t). \tag{26}$$

Taking $T(t)=\Sigma^{-1/2}(t)V(t)$, then $V(t)$ is Lyapunov since $T(t)$ and $\Sigma(t)$ are Lyapunov also. Equation (25) implies that $V'(t)V(t)=I$, i.e., $V(t)$ is unitary and (26) implies that

$$\bar{G}_0(t)\bar{G}_c(t)=S(t)=V'(t)\Sigma^2(t)V(t). \tag{27}$$

Hence, $V(t)$ is the required unitary transformation. ■

Remark: Given any uniform realization (A, B, C), we can obtain a topologically equivalent uniformly balanced realization $(\tilde{A}, \tilde{B}, \tilde{C})$ via a transformation $T(t) = U(t)\Sigma^{-1/2}(t)G_c^{1/2}(t)$

$$(A, B, C) \overset{G_c^{1/2}}{\rightarrow} (\bar{A}, \bar{B}, \bar{C}) \overset{U\Sigma^{-1/2}}{\rightarrow} (\tilde{A}, \tilde{B}, \tilde{C})$$

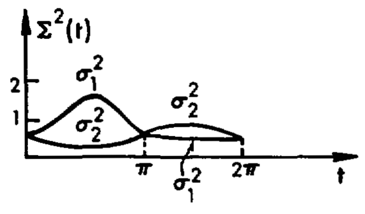
provided that $\dot{U}(t)$ is continuous and bounded in decomposition (24).

Since we can always construct a uniform realization $(\bar{A}, \bar{B}, \bar{C})$ as in (21), with properties given in (22) and (23), then we can use Corollary 1 for analyzing the existence of uniformly balanced realizations. The problem is then simplified since only boundedness and continuity of $\dot{U}$ are required (U is then Lyapunov, since it is unitary). Moreover, the symmetric decomposition (24) has been studied more extensively in the literature [13] than the nonsymmetric decomposition (16). For example, Lemma 6A holds only for symmetric operators. Since the product $\bar{G}_0\bar{G}_c$ in (24) is Lyapunov, then it follows from Theorem 4 that $\Sigma^2(t)$ is also Lyapunov in that decomposition. However, $T(t)$ in (16) or $U(t)$ in (24) cannot necessarily be chosen Lyapunov as can be seen from the following example.

Example 1: Let $\bar{G}_0\bar{G}_c = S(t)$ equal

$$\begin{cases} \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 1-\frac{1}{2}\cos(2t) & 0 \\ 0 & \frac{1}{2}-\frac{1}{4}\sin(t) \end{bmatrix} \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} & t \in [2k\pi, (2k+1)\pi) \\ \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2}-\frac{1}{4}\sin(t) \end{bmatrix} \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} & t \in [(2k+1)\pi, (2k+2)\pi). \end{cases} \quad k \in Z$$

Notice that the above decomposition is the only one (up to a permutation) guaranteeing the continuous differentiability of $\Sigma^2(t)$



$\Sigma^2(t)$ is Lyapunov, $S(t)$ is also Lyapunov (check the continuity of $\dot{S}$), however, $U(t)$ is not, since

$$\dot{U}(t) = \begin{cases} \begin{bmatrix} -\sin(t) & \cos(t) \\ -\cos(t) & -\sin(t) \end{bmatrix} & t \in [2k\pi, (2k+1)\pi) \\ \begin{bmatrix} -\sin(t) & -\cos(t) \\ \cos(t) & -\sin(t) \end{bmatrix} & t \in [(2k+1)\pi, (2k+2)\pi) \end{cases} \qquad k \in Z \quad (28)$$

and $\dot{U}(t)$ is discontinuous at $(k\pi)$, for example,

$$\dot{U}(\pi^-) = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} \quad \dot{U}(\pi^+) = \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix}.$$

■

We now define some "reasonable" properties of uniform systems (A, B, C) or $(\bar{A}, \bar{B}, \bar{C})$ that will ensure the transformations $T(t)$ in (16) or $U(t)$ in (24) to be Lyapunov, thus guaranteeing the existence of a uniformly balanced realization. In the sequel, whenever we use the term eigenvalues of G_0G_c, we will refer to the diagonal elements of $\Sigma^2(t)$ in (16) or (24), i.e., to a choice of decomposition that makes $\Sigma(t)$ continuously differentiable.

Property I: The product G_0G_c of the uniform system (A, B, C) has eigenvalues $\sigma_i^2(t)$ that only cross at isolated points, constituting the set Ω. (A point $t_1 \in \Re$ is called an *isolated* point if $\exists$ neighborhood U of t_1 such that $U \cap \Re = \{t_1\}$, i.e., t_1 is not a limit point.)

Property II: Two eigenvalues of $\sigma_i^2(t)$ and $\sigma_j^2(t)$, $i \neq j$, do not have common derivatives at their crossing points.

Property III: G_0G_c has a continuous second derivative in a neighborhood of each $t \in \Omega$.

The first two properties could be called "generic" since they will be satisfied for "random" uniform systems (A, B, C). The third property requires local smoothness (first derivative) of the triplet (A, B, C). We can then establish the following result.

Theorem 6: If a uniform system (A, B, C) satisfies Properties I–III, then there exists a uniformly balanced realization $(\tilde{A}, \tilde{B}, \tilde{C})$ which is topologically equivalent to (A, B, C).

Proof: See Appendix B. ■

Notice that the conditions in the above theorem are not necessary. Full necessary and sufficient conditions are still an open question. Uniqueness of uniformly balanced realization will now be studied.

Theorem 7: Suppose (A, B, C) is a uniformly balanced realization with $G_0 = G_c = \Sigma(t)$. If Property I holds, then this representation is unique up to a constant permutation.

Proof: Suppose that (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ are two equivalent uniformly balanced realizations satisfying Property I with

$$G_c = G_0 = \Sigma(t) \quad \text{and} \quad \bar{G}_c = \bar{G}_0 = \bar{\Sigma}(t).$$

Then there exists a Lyapunov transformation T such that $(A, B, C) \overset{T}{\rightarrow} (\bar{A}, \bar{B}, \bar{C})$ and

$$\bar{\Sigma}(t) = T'(t)\Sigma(t)T(t),$$
$$\bar{\Sigma}(t) = T^{-1}(t)\Sigma(t)T^{-1\prime}(t)$$

so that

$$\overline{\Sigma}^2(t) = T^{-1}(t)\Sigma^2(t)T(t). \qquad (29)$$

On Ω^c the eigenvalues of Σ and $\overline{\Sigma}$ are distinct so that the only solution to (29) is a permutation matrix. Since a permutation matrix can only vary discontinuously, $T(t)$ must be a *constant* permutation matrix (since T is Lyapunov). ■

Two types of systems that can always be uniformly realized (see [2]) are minimal periodic and time-invariant systems (i.e., systems that possess periodic and time-invariant representations, respectively). They can be characterized as follows.

Lemma 1[3]: A necessary and sufficient condition for a separable impulse response $H(t,\tau)$ to possess a periodic (time-invariant) realization is that

$$H(t+\Delta, \tau+\Delta) = H(t,\tau) \qquad \forall t \geqslant \tau$$

for some (all) constant Δ. ■

The following necessary conditions for a system to be periodic (time-invariant) are easily derived.

Lemma 2: If a system (A, B, C) is periodic (time-invariant) then $G_c(t-\delta, t)$ and $G_o(t, t+\delta)$ are periodic (time-invariant) for all δ.

Proof: This easily follows from the periodicity (time-invariance) of $B(t)$, $C(t)$, and $\Phi(t, t+\delta)$, the latter being equal to

$$\Phi(t, t+\delta) = e^{-A\delta}$$

for time-invariant systems and equal to (for some $P(t)$ and R)

$$\delta(t, t+\delta) = P(t)e^{-R\delta}P^{-1}(t+\delta)$$

for periodic systems, where $P(t)$ is periodic (Floquet theory [20]). ■

Our interest in the above lemma lies in its significance for uniformly balanced realizations. We close this section with the following theorem.

Theorem 8: Suppose that $H(t,\tau)$ has a minimal periodic (time-invariant) realization, and (A, B, C) is an arbitrary uniform realization for it, i.e., not necessarily periodic (time-invariant), then

1) the eigenvalues of G_0G_c are periodic (time-invariant),

2) the uniformly balanced realization $(\tilde{A}, \tilde{B}, \tilde{C})$, if it exists, is periodic (time-invariant) if Property I holds.

Proof:

1) Since (A, B, C) is topologically equivalent to a periodic (time-invariant) representation $(\overline{A}, \overline{B}, \overline{C})$, then there exists a Lyapunov transformation T such that

$$G_0G_c = T\overline{G}_0\overline{G}_cT^{-1}. \qquad (30)$$

By Lemma 2, the eigenvalues of $\overline{G}_0\overline{G}_c$ are periodic (time-invariant) and by (30) those of G_0G_c also are.

2) If (A, B, C) can be uniformly balanced, then the topologically equivalent, periodic system $(\overline{A}, \overline{B}, \overline{C})$ can also be, according to Theorem 5. But in the eigenvalue decomposition

$$\overline{G}_0\overline{G}_c = \overline{T}\overline{\Sigma}^2\overline{T}^{-1}.$$

$\overline{T}$ and $\overline{\Sigma}^2$ can obviously be chosen periodic (time-invariant). If one constructs a balanced realization $(\tilde{A}, \tilde{B}, \tilde{C})$ from $(\overline{A}, \overline{B}, \overline{C})$ along the lines of (16)–(19) in Theorem 5, then all the decompositions there can also be chosen periodic (time-invariant) so that $(\tilde{A}, \tilde{B}, \tilde{C})$ will be periodic (time-invariant). The uniqueness in Theorem 7 then completes the proof. ■

IV. Model Reduction

In this section we show how the uniformly balanced realization leads to a method for model reduction which is a generalization of that introduced by Moore [4], [23]. The basic idea is to eliminate that part of the system corresponding to relatively small singular values, i.e., the weakly controllable and observable part. So far, we have considered balancing over a finite interval δ (see Definition 7), and have not assumed anything about stability of the original systems. However, since we are interested in using the uniformly balanced coordinate system for model reduction, we now assume that we start with a uniform realization $(\tilde{A}, \tilde{B}, \tilde{C})$ which is asymptotically stable. Uniformity of the realization implies that there exists a $\delta > 0$ such that

$$\begin{aligned} \tilde{G}_c(t-\delta, t) &\geqslant \alpha_1(\delta) > 0 \\ \tilde{G}_0(t, t+\delta) &\geqslant \alpha_2(\delta) > 0. \end{aligned} \qquad (31)$$

We note that if (31) holds for some δ, then it will hold for any $\delta' \geqslant \delta$, and therefore uniformity is preserved for any $\delta' \geqslant \delta$. In particular, as $\delta' \to \infty$, we have

$$\tilde{G}_c(-\infty, t) = \int_{-\infty}^{t} \tilde{\Phi}(t,\tau)\tilde{B}(\tau)\tilde{B}'(\tau)\tilde{\Phi}'(t,\tau)\,d\tau$$

$$\tilde{G}_0(t,\infty) = \int_{t}^{\infty} \tilde{\Phi}'(\tau,t)\tilde{C}'(\tau)\tilde{C}(\tau)\tilde{\Phi}(\tau,t)\,d\tau$$

where asymptotic stability and boundedness of $(\tilde{A}, \tilde{B}, \tilde{C})$ guarantee that $\tilde{G}_c(-\infty, t)$ and $\tilde{G}_o(t,\infty)$ are uniformly bounded. Under the assumption that $\tilde{G}_o(t,\infty)\tilde{G}_c(-\infty,t)$ satisfies the properties of Theorem 6, we can uniformly balance $(\tilde{A}, \tilde{B}, \tilde{C})$, giving

$$(\tilde{A}, \tilde{B}, \tilde{C}) \xrightarrow{T} (A, B, C)$$

where now

$$G_0(t,\infty) = G_c(-\infty, t) = \Sigma(t)$$

$$= \begin{bmatrix} \sigma_1(t) & & & \\ & \sigma_2(t) & & 0 \\ 0 & & \ddots & \\ & & & \sigma_n(t) \end{bmatrix}.$$

The following lemma is important for model reduction.

Lemma 3: Suppose that $(\tilde{A}, \tilde{B}, \tilde{C})$ is a uniform realization which is also asymptotically stable. Let $(\hat{A}, \hat{B}, \hat{C})$ be a topologically equivalent uniformly balanced realization with $\hat{G}_0(t,\infty)=\hat{G}_c(-\infty,t)=\hat{\Sigma}(t)$, and let (A,B,C) be *any* realization obtained by simple reordering of the state variables of $(\hat{A},\hat{B},\hat{C})$. If (A,B,C) is partitioned as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\left[\begin{array}{c|c}A_{11} & A_{12}\\ \hline A_{21} & A_{22}\end{array}\right]\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\left[\begin{array}{c}B_1\\ \hline B_2\end{array}\right]u \tag{32}$$

$$y=\left[\begin{array}{c|c}C_1 & C_2\end{array}\right]\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{33}$$

then the subsystem (A_{11},B_1,C_1) satisfies the equations

$$\dot{\Sigma}_1(t)=A_{11}(t)\Sigma_1(t)+\Sigma_1(t)A'_{11}(t)+B_1(t)B'_1(t) \tag{34a}$$

$$-\dot{\Sigma}_1(t)=A'_{11}(t)\Sigma_1(t)+\Sigma_1(t)A_{11}(t)+C'_1(t)C_1(t) \tag{34b}$$

where the diagonal matrix $\Sigma_1(t)$ is a submatrix of $\Sigma(t)=G_0(t,\infty)=G_c(-\infty,t)$, to conform with the partitioning in (32) and (33), and there exists $\beta_i>0$, $i=1,2$, such that

$$0<\beta_1 I\leqslant\Sigma_1(t)\leqslant\beta_2 I<\infty \qquad \forall t. \tag{35}$$

Proof: $(\hat{A},\hat{B},\hat{C})$ is stable and uniformly balanced with

$$\hat{\Sigma}(t)=\hat{G}_0(t,\infty)=\hat{G}_c(-\infty,t)=\int_{-\infty}^{t}\hat{\Phi}(t,\tau)\hat{B}(\tau)\hat{B}'(\tau)\hat{\Phi}'(t,\tau)\,d\tau.$$

Taking the derivative, we obtain the following:

$$\dot{\hat{\Sigma}}(t)=\hat{A}(t)\hat{\Sigma}(t)+\hat{\Sigma}(t)\hat{A}'(t)+\hat{B}(t)\hat{B}'(t).$$

Now let: $(\hat{A},\hat{B},\hat{C})\xrightarrow{P}(A,B,C)$, where P is a permutation matrix. Permutation P in effect reorders the state variables corresponding to $(\hat{A},\hat{B},\hat{C})$, and since $\hat{\Sigma}(t)$ is diagonal, this transformation simply interchanges the diagonal elements of $\hat{\Sigma}(t)$, and we obtain

$$\dot{\Sigma}(t)=A(t)\Sigma(t)+\Sigma(t)A'(t)+B(t)B'(t)$$

$$\left[\begin{array}{c|c}\dot{\Sigma}_1 & 0\\ \hline 0 & \dot{\Sigma}_2\end{array}\right]=\left[\begin{array}{c|c}A_{11} & A_{12}\\ \hline A_{21} & A_{22}\end{array}\right]\left[\begin{array}{c|c}\Sigma_1 & 0\\ \hline 0 & \Sigma_2\end{array}\right]+\left[\begin{array}{c|c}\Sigma_1 & 0\\ \hline 0 & \Sigma_2\end{array}\right]\left[\begin{array}{c|c}A'_{11} & A'_{21}\\ \hline A'_{12} & A'_{22}\end{array}\right]+\left[\begin{array}{c}B_1\\ \hline B_2\end{array}\right]\left[\begin{array}{c|c}B'_1 & B'_2\end{array}\right]$$

$$\Rightarrow\ \dot{\Sigma}_1=A_{11}\Sigma_1+\Sigma_1A'_{11}+B_1B'_1.$$

Equation (34b) follows similarly. ■

From the above lemma, we notice that *any* subsystem (A_{11},B_1,C_1) of a uniformly balanced asymptotically stable (UBAS) realization (A,B,C) satisfies two Lyapunov equations of the type (34a), (34b). This immediately implies some preliminary information about the stability (not asymptotic stability) of the subsystems (A_{11},B_1,C_1). We summarize these properties in the following theorem. Notice that every property holding for a subsystem (A_{11},B_1,C_1) also holds for the system (A,B,C), since it is a trivial subsystem of itself.

Theorem 9: Let (A,B,C) be a UBAS realization. Then *any* subsystem (A_{11},B_1,C_1) has the following properties:

i) $\lambda_{\max}(A_{11}(t)+A'_{11}(t))\leqslant 0 \qquad \forall t$

ii) $\operatorname{Re}\lambda(A_{11}(t))\leqslant 0 \qquad \forall t$

iii) $\|\Phi_{A_{11}}(t,t_0)\|\leqslant 1 \qquad \forall t\geqslant t_0$

iv) $\|x_1(t)\|\leqslant\|x_1(\tau)\| \qquad \forall t\geqslant\tau$

where $\operatorname{Re}\lambda(A_{11})$ denotes the real part of eigenvalues of A_{11}.

Proof:

i) Adding (34a) and (34b) we obtain the following:

$$(A_{11}(t)+A'_{11}(t))\Sigma_1(t)+\Sigma_1(t)(A_{11}(t)+A'_{11}(t))=-(B_1(t)B'_1(t)+C'_1(t)C_1(t)). \tag{36}$$

Let $v(t)$ be a vector satisfying (pointwise) $(A_{11}(t)+A'_{11}(t))v(t)=\lambda_{\max}(t)v(t)$, then pre- and postmultiplying (36) with $v'(t)$ and $v(t)$, respectively, gives

$$2\lambda_{\max}(t)\underbrace{v'(t)\Sigma(t)v(t)}_{>0}=-\underbrace{v'(t)(B_1(t)B'_1(t)+C'_1(t)C_1(t))v(t)}_{\leqslant 0}.$$

Therefore, $\lambda_{\max}(t)\leqslant 0\ \forall t$.

ii) Using the Bendixon inequality [17]

$$\lambda_{\min}\left(\frac{M+M'}{2}\right)\leqslant\operatorname{Re}\lambda(M)\leqslant\lambda_{\max}\left(\frac{M+M'}{2}\right) \tag{37}$$

for any matrix M, the result follows immediately from property i).

iii) Let $\Phi_{A_{11}}(t,t_0)$ be the transition matrix corresponding to $\dot{x}_1(t)=A_{11}(t)x_1(t)$; then using the Wazewski's inequality [11]

$$\exp\int_{t_0}^{t}\lambda_{\min}\left(\frac{A_{11}(\tau)+A'_{11}(\tau)}{2}\right)d\tau\leqslant\|\Phi_{A_{11}}(t,t_0)\|\leqslant\exp\int_{t_0}^{t}\lambda_{\max}\left(\frac{A_{11}(\tau)+A'_{11}(\tau)}{2}\right)d\tau \tag{38}$$

and the results follow from property i).

iv) From property iii) it follows that

$$\|x_1(t)\|=\|\Phi_{A_{11}}(t,\tau)x_1(\tau)\|\leqslant\|\Phi_{A_{11}}(t,\tau)\|\cdot\|x_1(\tau)\|\leqslant\|x_1(\tau)\|.$$ ■

Remark: Property iv) of the last theorem implies that $(d/dt)\|x_1(t)\|^2\leqslant 0$. Therefore we have

$$\langle A_{11}(t)x_1(t),x_1(t)\rangle+\langle x_1(t),A_{11}(t)x_1(t)\rangle=x'_1(t)(A_{11}(t)+A'_{11}(t))x_1(t)=\frac{d}{dt}\|x_1(t)\|^2\leqslant 0. \tag{39}$$

In circuit theory a realization (A_{11}, B_1, C_1) satisfying (39) is called *dissipative* [14].

If the uniformly balanced framework is to be used for model reduction, then the asymptotic stability of subsystem (reduced models) is of prime importance. Theorem 10 below characterizes the asymptotic stability (AS) of the subsystems. We first need the following lemma.

Lemma 4: Suppose that $\dot{x}(t) = A(t)x(t)$ is exponentially stable and $A(t), C(t)$ are bounded. Then

$$P(t) = \int_t^\infty \Phi'(\tau, t)C'(\tau)C(\tau)\Phi(\tau, t)\, d\tau \tag{40}$$

exists as a bounded nonnegative definite symmetric matrix, and it satisfies

$$-\dot{P}(t) = A'(t)P(t) + P(t)A(t) + C'(t)C(t). \tag{41}$$

Moreover, $P(t)$ is the *unique* bounded matrix satisfying (41).

Proof: All claims are clear except perhaps for the last. Let $Q(t)$ be a second bounded matrix satisfying (41). Set $R(t) = P(t) - Q(t)$, then

$$-\dot{R}(\tau) = A'(\tau)R(\tau) + R(\tau)A(\tau).$$

Pre- and postmultiplying the above equation by $\Phi'(\tau, t)$ and $\Phi(\tau, t)$, respectively, and integrating we obtain

$$\int_{t_1}^t \frac{d}{d\tau}(\Phi'(\tau, t)R(\tau)\Phi(\tau, t))\, d\tau = 0.$$

This implies that

$$R(t) - \Phi'(t_1, t)R(t_1)\Phi(t_1, t) = 0.$$

Letting $t_1 \to \infty$, using the asymptotical decay of $\Phi(t_1, t)$ and boundedness of $R(t_1)$ gives $R(t) = 0$. ■

Theorem 10: Let (A, B, C) be a UBAS realization. Then for *any* subsystem (A_{11}, B_1, C_1) the following are equivalent statements.

i) $A_{11}(t)$ is AS.

ii) $(A_{11}(t), B_1(t))$ is uniformly controllable.

iii) $(A_{11}(t), C_1(t))$ is uniformly observable.

iv) $(A_{11}(t), B_1(t), C_1(t))$ is uniformly balanced with

$$\Sigma_1(t) = \int_t^\infty \Phi'_{A_{11}}(\tau, t)C_1'(\tau)C_1(\tau)\Phi_{A_{11}}(\tau, t)\, d\tau = \int_{-\infty}^t \Phi_{A_{11}}(t, \tau)B_1(\tau)B_1'(\tau)\Phi_{A_{11}}(t, \tau)\, d\tau.$$

Proof: i) ⇒ iv). Using the last lemma, we let

$$G_1(t) = \int_t^\infty \Phi'_{A_{11}}(\tau, t)C'(\tau)C(\tau)\Phi_{A_{11}}(\tau, t)\, d\tau$$

where $G_1(t)$ satisfies

$$-\dot{G}_1(t) = A_{11}'(t)G_1(t) + G_1(t)A_{11}(t) + C_1'(t)C_1(t).$$

However, according to (34b) $\Sigma_1(t)$ also satisfies the above equation. By uniqueness proved in the last lemma, we have $G_1(t) = \Sigma_1(t)$. We can similarly show that

$$\Sigma_1(t) = \int_{-\infty}^t \Phi_{A_{11}}(t, \tau)B_1(\tau)B_1'(\tau)\Phi_{A_{11}}(t, \tau)\, d\tau,$$

hence (A_{11}, B_1, C_1) is uniformly balanced.

iv) ⇒ iii)

$$\Sigma_1(t) = \lim_{\delta\to\infty} G_1(t, t+\delta) = \lim_{\delta\to\infty} \int_t^{t+\delta} \Phi'_{A_{11}}(\tau, t)C'(\tau)C(\tau)\Phi_{A_{11}}(\tau, t)\, d\tau.$$

Then using the definition of the limit, we can find δ large enough such that $G_1(t, t+\delta)$ is uniformly bounded away from zero, and hence (A_{11}, C_1) is uniformly observable.

iii) ⇒ i). Take

$$V(x_1(t), t) \triangleq x_1'(t)\Sigma_1(t)x_1(t) \geqslant \alpha_1 x_1'(t)x_1(t) > 0$$

as a Lyapunov function, with its derivative given by

$$\frac{dV(x_1(t), t)}{dt} = x_1'(t)\left[A_{11}'(t)\Sigma_1(t) + \Sigma_1(t)A_{11}(t) + \dot{\Sigma}_1(t)\right]x_1(t) = -x_1'(t)C_1'(t)C_1(t)x_1(t).$$

To establish asymptotic stability, compute the change in V along a length δ of the trajectory. Thus,

$$\int_t^{t+\delta} \frac{dV(x_1(\tau), \tau)}{d\tau}\, d\tau = -\int_t^{t+\delta} x_1'(\tau)C_1'(\tau)C_1(\tau)x_1(\tau)\, d\tau$$

$$V(x_1(\tau), \tau)\Big|_t^{t+\delta} = -x_1'(t)\int_t^{t+\delta}\Phi'(\tau, t)C_1'(\tau)C_1(\tau)\Phi(\tau, t)\, d\tau\, x_1(t) = -x_1'(t)G_0(t)x_1(t) \leqslant -\alpha_2 x_1'(t)x_1(t) < 0.$$

Therefore, $V(x_1(t+\delta), t+\delta) - V(x_1(t), t)$ is uniformly strictly less than zero. This establishes the asymptotic stability of $A_{11}(t)$ (see also [29]). Equivalence with ii) can similarly be proved. ■

This theorem shows that any of the conditions i), ii), and iii) imply that the subsystem (A_{11}, B_1, C_1) is UBAS. Silverman and Anderson [2] have shown that several broad classes of systems have the uniform complete observability property. For example, the class of all the nth order linear differential equations with bounded coefficients are uniformly observable. They have also shown that for periodic systems uniform complete observability is equivalent to complete observability (in some interval), and since complete observability of any pair (A_{11}, C_1) is generic, then we have the following corollary to Theorem 10.

Corollary 2: Suppose (A, B, C) is a UBAS realization. If the pair (A_{11}, C_1) *or* (A_{11}, B_1) is periodic, then the following properties for the subsystem (A_{11}, B_1, C_1) hold generically.

i) $A_{11}(t)$ is AS.

ii) (A_{11}, B_1, C_1) is uniformly balanced with $\Sigma_1 = G_0 = G_c$.

iii) (A_{11}, B_1, C_1) is UBAS. ■

For a linear differential equation of the form (1) which is asymptotically stable, in general, nothing can be said about the asymptotic stability of the subsystems. However, for uniformly balanced realizations, asymptotic stability of A_{11} in many cases will follow. The authors have extensively studied such a question in [8]. Here we quote a theorem and its corollary which gives a sufficient condition which guarantees the asymptotic stability of *any* subsystem. In view of property i) of Theorem 9, the properties of this theorem are "generically" satisfied.

Theorem 11 [8]: Let (A, B, C) be a UBAS realization. If

$$\lambda_{\max}\left(\frac{A(t)+A'(t)}{2}\right) \notin \mathfrak{L}_1[t_0,\infty)$$

then *every* subsystem (A_{11}, B_1, C_1) is UBAS.[4] ■

Corollary 3 [8]: Let (A, B, C) be a UBAS realization with $A(t)$ periodic. If

$$\exists t_1 \ni \lambda_{\max}\left(\frac{A(t_1)+A'(t_1)}{2}\right) < 0,$$

then *every* subsystem (A_{11}, B_1, C_1) is UBAS. ■

Notice that both Corollaries 2 and 3, although different in nature, imply that for periodic uniformly balanced realizations, the subsystems (A_{11}, B_1, C_1) are generically UBAS. We also emphasize that the condition on $\lambda_{\max}(t)$ is only sufficient as can be verified by the following example.

Example 2: Let (A, B, C) be given by

$$\left(\begin{bmatrix} -1 & 0 \\ -2\sqrt{2} & -2 \end{bmatrix}, \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}, [\sqrt{2}, 2]\right).$$

Then (A, B, C) is UBAS with $\Sigma = I_{2\times 2}$. We note that $\lambda_{\max}(A + A') = 0 \in \mathfrak{L}_1$, yet every subsystem is UBAS. ■

Justification for Reduced Model: We now give two different arguments to justify that the subsystem (A_{11}, B_1, C_1) is in fact a "good" reduced model. The first argument is based on input and output energy, and the second one is based on an inequality that we derive. A rigorous treatment of this subject should be of interest for further research.

For model reduction to be meaningful we assume that the eigenvalues of $\Sigma(t)$ can be divided into two groups of "small" and "large," and via a constant permutation we can always rearrange the balanced realization (A, B, C) such that

$$\Sigma(t) = \begin{bmatrix} \Sigma_1(t) & 0 \\ 0 & \Sigma_2(t) \end{bmatrix}$$

where $\Sigma_1(t) > \Sigma_2(t) \geqslant \alpha I > 0 \ \forall t$. If $\Sigma_1(t) \gg {}_2(t)$, i.e., there is a "gap" between the eigenvalues of $\Sigma(t)$, then in the balanced framework, the states corresponding to $\Sigma_1(t)$ are very controllable and observable. On the other hand, since $\Sigma_2(t)$ is small, the states corresponding to $\Sigma_2(t)$ are "nearly uncontrollable" and "nearly unobservable," and therefore "nearly redundant." Upon deleting those states, we obtain the subsystem (A_{11}, B_1, C_1) corresponding to Σ_1, which we claim is a reasonable reduced model. We can further justify this by the following energy argument: Kalman [1] showed that the minimum control energy required to get to state $e_i' = [0 \ 0 \ \cdots \ 1 \ 0 \ \cdots \ 0]$ at time τ is given by

$$U(\tau) = B'(\tau)\Phi'(t,\tau)\Sigma^{-1}(t)e_i.$$

Therefore, the input power needed to drive the state in the e_i direction is given by

$$\begin{aligned}\int_{-\infty}^{t} &\|U(\tau)\|^2 d\tau \\ &= e_i'\Sigma^{-1}(t)\int_{-\infty}^{t}\Phi(t,\tau)B(\tau)B'(\tau)\Phi'(t,\tau)\,d\tau\,\Sigma^{-1}(t)e_i \\ &= e_i'\Sigma^{-1}(t)e_i = \frac{1}{\sigma_i(t)}.\end{aligned} \tag{42}$$

And the power we can obtain at the output from state e_i is

$$\begin{aligned}\int_t^{\infty}\|y(\tau)\|^2 d\tau &= \int_t^{\infty} e_i'\Phi'(\tau,t)C'(\tau)C(\tau)\Phi(\tau,t)e_i\,d\tau \\ &= e_i'\Sigma(t)e_i = \sigma_i(t).\end{aligned} \tag{43}$$

If $\sigma_i(t)$ is small, then (42) implies that it takes a relatively large amount of input energy to drive the state in the direction e_i. Therefore, the states in the e_i direction are very weakly coupled to the input (hard to control). Similarly, (43) shows that very little output energy can be observed from state e_i. Therefore, the states in the e_i direction are very weakly coupled to the output (hard to observe). In conclusion, if σ_i is small, then the states in the e_i direction are nearly uncontrollable and nearly unobservable, and therefore nearly redundant, and can be deleted.

We now derive an inequality that will give further insight to justify that the subsystem (A_{11}, B_1, C_1) is a reasonable reduced mode. Let

$$\left(\frac{A_{11}+A_{11}'}{2}\right)\mathcal{V}_1(t) = \lambda_1^{\min}(t)\mathcal{V}_1(t), \qquad \|\mathcal{V}_1(t)\| = 1 \quad \forall t$$

where $\lambda_1^{\min}(t)$ is the pointwise minimum eigenvalue of $((A_{11} + A_{11}')/2)$. Pre- and postmultiplying (36) by $\mathcal{V}_1'(t)$ and $\mathcal{V}_1(t)$, respectively, gives

$$2\lambda_1^{\min}(t)\mathcal{V}_1'(t)\Sigma_1(t)\mathcal{V}_1(t) = -\tfrac{1}{2}\mathcal{V}_1'(t)(B_1B_1' + C_1'C_1)\mathcal{V}_1(t).$$

Therefore,

$$\sigma_{\min}(\Sigma_1(t)) \leqslant \frac{\|B_1'\mathcal{V}_1\|^2 + \|C_1\mathcal{V}_1\|^2}{4(-\lambda_1^{\min}(t))}. \tag{44}$$

Similarly

$$\sigma_{\max}(\Sigma_2(t)) \geqslant \frac{\|B_2'\mathcal{V}_2\|^2 + \|C_2\mathcal{V}_2\|^2}{4(-\lambda_2^{\max}(t))} \tag{45}$$

[4] A real valued measurable function $f\colon [t_0,\infty) \to \mathfrak{R}$ is said to be *integrable* or *summable* if $f \in \mathfrak{L}_1[t_0,\infty)$, where

$$\mathfrak{L}_1[t_0,\infty) \triangleq \left\{ f\colon [t_0,\infty) \to \mathfrak{R} \ni \int_{t_0}^{\infty} |f(t)|\,dt < \infty \right\}.$$

where $\lambda_2^{\max}(t)$ is the pointwise maximum eigenvalue, and $\mathcal{V}_2(t)$ is the normalized eigenvector of $((A_{22}+A'_{22})/2)$. Subtracting (45) from (44) we obtain

$$[\sigma_{\min}(\Sigma_1)-\sigma_{\max}(\Sigma_2)] \leqslant \frac{\|B_1'\mathcal{V}_1\|^2+\|C_1\mathcal{V}_1\|^2}{4(-\lambda_1^{\min}(t))}-\frac{\|B_2'\mathcal{V}_2\|^2+\|C_2\mathcal{V}_2\|^2}{4(-\lambda_2^{\max}(t))}. \tag{46}$$

Also by Wazewski's inequality [11] we have

$$\exp\int_{t_0}^{t}\lambda_1^{\min}\left(\frac{A_{11}(\tau)+A'_{11}(\tau)}{2}\right)d\tau \leqslant \|\Phi_{A_{11}}(t,t_0)\| \leqslant \exp\int_{t_0}^{t}\lambda_1^{\max}\left(\frac{A_{11}(\tau)+A'_{11}(\tau)}{2}\right)d\tau$$

$$\exp\int_{t_0}^{t}\lambda_2^{\min}\left(\frac{A_{22}(\tau)+A'_{22}(\tau)}{2}\right)d\tau \leqslant \|\Phi_{A_{22}}(t,t_0)\| \leqslant \exp\int_{t_0}^{t}\lambda_2^{\max}\left(\frac{A_{22}(\tau)+A'_{22}(\tau)}{2}\right)d\tau.$$

Interpretation: If $\sigma_{\min}(\Sigma_1)\gg\sigma_{\max}(\Sigma_2)$, then the difference on the right-hand side of inequality (46) is also large. This means the following.

1) $[\|B_1\mathcal{V}_1\|^2+\|C_1'\mathcal{V}_1\|^2]$ is large *or* $|\lambda_1^{\min}((A_{11}+A'_{11})/2)|$ is small which by Wazewski's inequality it implies that $\|\Phi_{A_{11}}(t,t_o)\|$ is slowly decaying.

2) $[\|B_2'\mathcal{V}_2\|^2+\|C_2\mathcal{V}_2\|^2]$ is small *or* $|\lambda_2^{\max}((A_{22}+A'_{22})/2)|$ is large which again by Wazewski's inequality means that $\|\Phi_{A_{22}}(t,t_o)\|$ is decaying fast.

Statements 1) and 2) together imply that (A_{11},B_1,C_1) is in some sense the dominant subsystem of (A,B,C) and can be used as a reduced model. This appears to be the first such notion of dominance for time-variable systems. We finally note that stability and uniform balancedness of the original system is preserved in the reduced model according to Theorems 10, 11 and Corollaries 2, 3. We close this section with the following example.

Example 3: Consider the stable time-variable system

$$A(t)=\begin{bmatrix}\frac{1}{2}\sin^2(t)-\frac{20}{21}\sin(2t)-\frac{5}{2} & \frac{40}{21}\sin^2(t)+\frac{1}{4}\sin(2t)+\frac{1}{21}\\ \frac{40}{21}\sin^2(t)+\frac{1}{4}\sin(2t)-\frac{41}{21} & -\frac{1}{2}\sin^2(t)-\sin(2t)-2\end{bmatrix}$$

$$B(t)=C'(t)=\begin{bmatrix}10\cos(t)+2\sin(t)\\ -10\sin(t)+2\cos(t)\end{bmatrix}.$$

The product of the controllability and observability Gramians is given by

$$G_0(t)G_c(t)=\begin{bmatrix}400-399\sin^2(t) & -399\sin(t)\cos(t)\\ -399\sin(t)\cos(t) & 1-399\sin^2(t)\end{bmatrix}.$$

This product has the eigenvalue decomposition

$$G_0(t)G_c(t)=T(t)\Sigma^2(t)T^{-1}(t) = \begin{bmatrix}\cos(t) & \sin(t)\\ -\sin(t) & \cos(t)\end{bmatrix}\begin{bmatrix}400 & 0\\ 0 & 1\end{bmatrix} \cdot\begin{bmatrix}\cos(t) & -\sin(t)\\ \sin(t) & \cos(t)\end{bmatrix}.$$

Since $T(t)$ is Lyapunov, we can then obtain a uniformly balanced realization (Theorem 5, Corollary 1). The uniformly balanced realization $(\bar{A},\bar{B},\bar{C})$ (which in this case is time-invariant!) is given by

$$\bar{A}=\begin{bmatrix}-\frac{5}{2} & -\frac{20}{21}\\ -\frac{20}{21} & -2\end{bmatrix}\qquad \bar{B}=\bar{C}'=\begin{bmatrix}10\\ 2\end{bmatrix}$$

with

$$\bar{G}_0=\bar{G}_c=\Sigma=\begin{bmatrix}20 & 0\\ 0 & 1\end{bmatrix}.$$

The impulse response for this system is given by

$$h(t)=20.95e^{-1.25t}+83.048e^{-3.25t}. \tag{47}$$

Since in the matrix Σ, $20\gg1$, then as justified above, the subsystem $(A_{11},B_1,C_1)=(-2.5,10,10)$ can be taken as a reduced model. Its impulse response is given by

$$h_R(t)=100e^{-2.5t}. \tag{48}$$

A comparison of $h(t)$ and $h_R(t)$ is given in Table I.

The DC gain of the original system is (42.31) while for the reduced model it is (40), which is in good agreement. The graph of the two impulse responses is given in Fig 1.

V. Conclusions

It is obvious that not all the realizations are equally useful for practical implementation or for answering various theoretical questions. We have attempted here to introduce a "good" realization for time-variable systems. Such a framework has many interesting properties and leads to a natural setting for performing model reduction. The idea of taking a subsystem as a reduced model is very appealing and attractive. However, much work needs to be done to further understand balanced and related realizations and their implications in linear system theory. In particular, more research is desirable to find some measure of close-

TABLE I

t	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	1.5	2	3
h(t)	104	73.23	51.84	27.67	15.46	9.398	3.85	1.842	.5
$h_R(t)$	100	73.16	54	29	15.34	8	2.35	.674	.055
$\|h(t)-h_R(t)\|$	4	.07	2.16	1.33	.12	1.39	1.5	1.168	.44

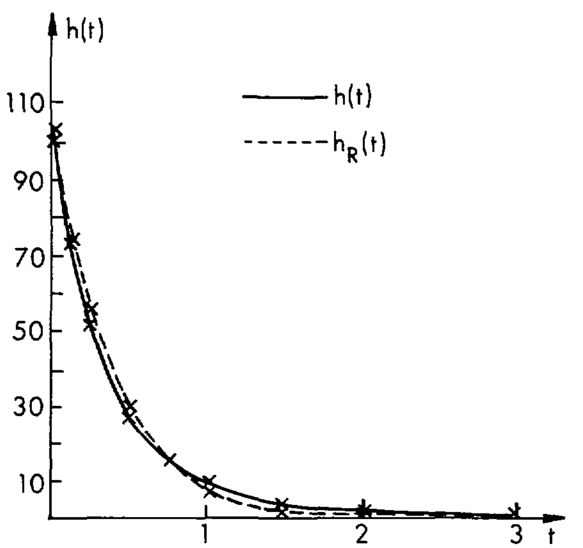


Fig. 1.

ness between the original system and its reduced model. At this point, however, balanced realizations yield a new perspective and view of linear systems and model reduction.

APPENDIX A

This Appendix contains some technical Lemmas which are useful throughout the paper.

Lemma A1: The controllability and observability Gramians G_c and G_0 of a uniform realization are Lyapunov transformation.

Proof: Considering the observability Gramian [9] and differentiating, we have

$$\dot{G}_0(t) = \Phi'(t+\delta,t)C'(t+\delta)C(t+\delta)\Phi(t+\delta,t) - C'(t)C(t) - A'(t)G_0(t) - G_0(t)A(t).$$

Since the realization (A, B, C) is uniform, then the continuity and boundedness of G_0, G_0^{-1}, and $\dot{G}_0$ can easily be deduced. ■

Lemma A2 [3]: If G is a symmetric positive definite Lyapunov transformation, then so is $G^{1/2}$. ■

Lemma A3: Let G_0 and G_c be two symmetric positive definite matrices, then their product G_0G_c is semisimple (i.e., has no Jordan blocks) and has positive eigenvalues.

Proof: It is well known [19] that a symmetric positive definite matrix G_0 always has a decomposition of the type

$$G_0 = S_0 S_0$$

where $S_0 = G_0^{1/2}$ is symmetric and positive definite. Clearly

$$S_0^{-1}(G_0G_c)S_0 = S_0G_cS_0$$

is symmetric and positive definite since S_0 is a congruence transformation. Hence,

$$\lambda(G_0G_c) = \lambda(S_0^{-1}G_0G_cS_0) = \lambda(S_0G_cS_0).$$

Therefore, the product G_0G_c is semisimple with positive eigenvalues. ■

Lemma A4 [12]: Two Hermitian matrices G_1 and G_2 can be simultaneously diagonalized by a unitary matrix U iff $G_1G_2 = G_2G_1$ (i.e., G_1 and G_2 commute). ■

Finally, we are interested in the smoothness of an eigenvalue decomposition of a time-variable symmetric matrix. The strongest results known in this context are given in Kato [13]. Rephrased in our terminology, the following results are shown there.

Lemma A5 [13]: Eigenvalues of a continuously differentiable (analytic) self-adjoint operator can be chosen to be continuously differentiable (analytic) on the real axis. ■

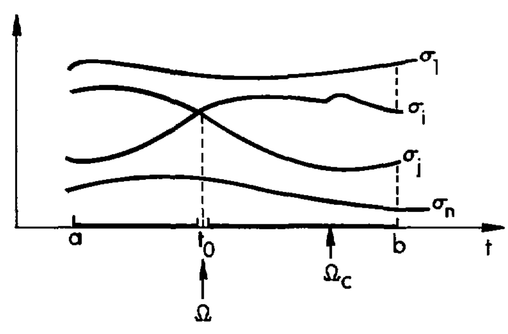
Unfortunately, similar results, with the same generality do not hold for eigenvectors. As a general rule, we can state that eigenvectors behave worse, more singularly, and with much less continuity than the eigenvalues even for self-adjoint operators. However, one of the most remarkable results of analytic function theory for self-adjoint operator is the existence of an orthonormal basis depending smoothly on t.

Lemma A6 [13]: If the self-adjoint operator $G(t)$ is analytic, then the orthonormal eigenvectors $\{\nu_i(t)\}_{i=1}^{n}$ can be chosen as analytic function of t. ■

Note that in the above theorem analyticity of the operator $G(t)$ is essential. Kato [13, Example 5.3, p. 111] shows that smoothness of eigenvectors can be lost completely if the analyticity of $G(t)$ is replaced by infinite differentiability. However, for eigenvalues (Lemma A5), the assumption of analyticity can be removed, and they still behave nicely.

APPENDIX B

Before we prove Theorem 6, we need some preliminary results. According to Assumptions I–III we have the following situation: $S(t)$ is a symmetric real matrix for real t, and therefore has real eigenvalues. $S(t)$ has continuous derivative, thus eigenvalues $\{\sigma_i(t) | i = 1, \cdots, n\}$ can be chosen to have continuous derivative (Lemma A5)



On Ω_c the complement of Ω no eigenvalues cross, on Ω two eigenvalues $\sigma_i(t)$ and $\sigma_j(t)$ cross with values $\sigma = \sigma_i(t_0) = \sigma_j(t_0)$, but have different derivatives (Propositions I, II). Moreover, in an open neighborhood of t_0, i.e., $(t_0 - \delta, t_0 + \delta)$, $S(t)$ has continuous second derivative (Proposition III). For simplicity we assume $t_0 = 0$! For an interval $I(0) = (a, b)$ we can show that the projector $P(t)$ on the invariant subspace of the *pair* $\sigma_i(t)$ and $\sigma_j(t)$ has *continuous derivative* since over the whole interval $I(0) = (a, b)$, this pair is separated from the other eigenvalues [13, Remark 5.10, p. 115] and $\tilde{S}^{(1)}(t) \triangleq t^{-1}[S(t) - \sigma]P(t)$ is *continuous.*

We now show that in the interval $0(o) \triangleq (-\delta, +\delta)$ where $S(t)$ has continuous second derivative, there $P(t)$ has also continuous *second derivative* and $\tilde{S}^{(1)}(t)$ continuous *first derivative.*

Lemma B1: If $S(t)$ is symmetric, real, and has continuous second derivative in $0(o)$ and if $P(t)$ is the projector on the σ group, then

1) $P(t)$ has continuous second derivative in $0(o)$
2) $\tilde{S}^{(1)}(t)$ has continuous first derivative in $0(o)$.

Proof:

1) The projection operator $P(t)$ is given by [13, p. 67]

$$P(t) = -\frac{1}{2\pi i}\int_\Gamma R(\xi, t)\, d\xi$$

where resolvent $R(\xi, t) \triangleq (S(t) - \xi)^{-1}$. $R(\xi, t)$ is twice continuously differentiable since

$$\frac{\partial}{\partial t}R(\xi, t) = R(\xi, t)S'(t)R(\xi, t)$$

$$\frac{\partial^2}{\partial t^2}R(\xi, t) = -2R(\xi, t)S'(t)R(\xi, t)S'(t)R(\xi, t) - R(\xi, t)S''(t)R(\xi, t).$$

We then have that [13, Formula (2.8), p. 76] $P(t)$ is twice continuously differential in $0(o)$.

2) We have $\tilde{S}^{(1)}(t) = t^{-1}(S(t) - \sigma)P(t) \triangleq t^{-1}\tilde{S}(t)$ with $\tilde{S}(0) = (S(0) - \lambda)P(0) = 0$ [13, pp. 112–113]. Clearly $\tilde{S}(t)$ is twice continuously differentiable in $0(o)$ since $S(t)$ and $P(t)$ are so in $0(o)$. Now define the following:

$$\frac{\partial}{\partial t}\tilde{S}^{(1)}(t) \triangleq \begin{cases} t^{-1}\tilde{S}'(t) - t^{-2}\tilde{S}(t) & \text{on } 0(o)/\{0\} \quad (1a) \\ \dfrac{\tilde{S}''(0)}{2} & \text{on } \{0\}. \quad (1b) \end{cases}$$

Now, (1a) is continuous on $0(o)/\{0\}$ since it is even differentiable there. Moreover, using L'Hospital's rule we have

$$\lim_{t\to 0}\frac{\partial}{\partial t}\tilde{S}^{(1)}(t) = \lim_{t\to 0}\frac{t\tilde{S}'(t) - \tilde{S}(t)}{t^2} = \lim_{t\to 0}\frac{\tilde{S}'(t) + t\tilde{S}''(t) - \tilde{S}'(t)}{2t} = \lim_{t\to 0}\frac{\tilde{S}''(t)}{2} = \frac{\tilde{S}''(0)}{2}.$$

Hence (1b) makes (1a) also continuous at $\{0\}$. Therefore, the derivative $(\partial/\partial t)\tilde{S}^{(1)}(t)$ is continuous! ∎

Now we use this lemma to complete Theorem 6.8 [13, p. 123] and obtain the desired results, namely the following.

Theorem B1: Let $S(t)$ satisfy the conditions of Lemma B1 and let Property II hold. Then the projectors $P_i(t)$ and $P_j(t)$, which project on the eigenvectors $u_i(t)$ and $u_j(t)$ of $\sigma_i(t)$ and $\sigma_j(t)$, respectively, have continuous derivative.

Proof: Follow [13, Theorem 6.8, p. 123] until line 14. There it says that $P_i(t)$ and $P_j(t)$ are also the projectors of $\tilde{S}^{(1)}(t)$ and $\tilde{\sigma}_i^{(1)}(t)$ and $\tilde{\sigma}_j^{(1)}$ which are defined as $\tilde{\sigma}_{i,j}^{(1)}(t) = t^{-1}(\sigma_{i,j}(t) - \sigma)$. But, by assumption, $\sigma_i^{(1)}(t) \neq \sigma_j^{(1)}(t)$ on $0(o)$. Hence, these projectors have continuous derivative on $0(o)$, since $\tilde{S}^{(1)}(t)$ has continuous derivative there [13, Remark 5.10, p. 115]. Once we know that $P_j(t)$ and $P_i(t)$ have continuous derivative, the eigenvectors $u_j(t)$ and $u_i(t)$ satisfy the same condition because of Dolezal. ∎

We now use the above theorem to prove Theorem 6.

Proof of Theorem 6: As discussed in Corollary 1, we have to prove this theorem in the case $G_0G_c = S$ is a symmetric matrix to be decomposed as $S = U\Sigma^2U'$. We first prove the continuity of $\dot{U}$.

The matrix $S(t) - \sigma_i^2(t)I$ has continuous derivative everywhere and constant rank $(n-1)$ on Ω_c, the complement of Ω. Its nullspace is the eigenvector $u_i(t)$ and this vector thus has continuous derivative on Ω_c (by Dolezal theorem, [15]). For $t_0 \in \Omega$ it is proven that under assumption of Property II, $U(t)$ is continuous at t_0 since $\dot{S}(t)$ is continuous at t_0 [13, p. 123]. Continuity of $\dot{U}$ follows from Theorem B1. The boundedness of $\dot{U}$ is then proven as follows. Writing the following:

$$\dot{S}(t) = \dot{U}\Sigma^2U' + U\dot{\Sigma}^2U' + U\Sigma^2\dot{U}' = U(U'\dot{U}\Sigma^2 + \dot{\Sigma}^2 + \Sigma^2\dot{U}'U)U' = UFU'$$

we have that F is bounded (since U and $\dot{S}$ are also bounded). Also from $U'U = I$, it follows that $\dot{U}'U + U'\dot{U} = 0$, and thus that $\dot{U}'U$ is antisymmetric, or zero on diagonal. Therefore:

$$F = \begin{cases} \dot{\Sigma}^2 & \text{on diagonal} \\ U'\dot{U}\Sigma^2 + \Sigma^2\dot{U}'U = U'\dot{U}\Sigma^2 - \Sigma^2U'\dot{U} & \text{off diagonal.} \end{cases}$$

From the latter equation it follows that

$$f_{ij} = u_i'\dot{U}_j\left(\sigma_j^2 - \sigma_i^2\right) \qquad i \neq j$$

and thus

$$\begin{cases} u_i'\dot{U}_j = f_{ij}/\left(\sigma_j^2 - \sigma_i^2\right) & i \neq j \\ u_j'\dot{U}_j = 0 \end{cases} \tag{2}$$

yields that $U'\dot{U}$ is bounded on Ω. This implies that $\dot{U}$ is bounded on Ω. For finite $t_0 \in \Omega_c$ the continuity of $\dot{U}$ implies that $\dot{U}$ is also bounded on $\Omega_c/\{\infty\}$. For the points at infinity we may reason as follows. By Property II, $\lim_{t\to\infty}(\sigma_i^2(t) - \sigma_j^2(t)) \neq 0$, and thus there exists a sequence $t_k \to \infty$ such that

$$\sigma_i^2(t_k)-\sigma_j^2(t_k)\geqslant\delta>0$$

on this sequence. Hence, by (2) $u_i'\dot{U}_j$ is bounded on $\{t_k\}$. By continuity $u_i'(\infty)u_j(\infty)$ is also bounded. This proves the *uniform* boundedness of $\dot{U}$. ■

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