Aplikace matematiky

Jiří Michálek

Linear transformations of locally stationary processes

Aplikace matematiky, Vol. 34 (1989), No. 1, 57–66

Persistent URL: http://dml.cz/dmlcz/104334

LINEAR TRANSFORMATIONS OF LOCALLY STATIONARY PROCESSES

JIŘÍ MICHÁLEK

(Received November 19, 1987)

Summary. The paper deals with linear transformations of harmonizable locally stationary random processes. Necessary and sufficient conditions under which a linear transformation defines again a locally stationary process are given.

Keywords: harmonizable process, locally stationary process, covariance function.
AMS subject classification: 60G.

The notion of a weakly locally stationary process was introduced by Silverman in [1]. Let $\{x(t), t \in \mathbb{R}_1\}$ be a second order random process with a vanishing expected value and with a covariance function $R(\cdot,\cdot)$ defined on $\mathbb{R}_1 \times \mathbb{R}_1$. If for every pair s, t of reals one can write

$$R_x(s, t) = R_x^{(1)}\left(\frac{s+t}{2}\right) R_x^{(2)}(s-t),$$

where $R_x^{(1)} \geqq 0$ and $R_x^{(2)}$ is a stationary covariance, then, in accordance with [1], $R_x(\cdot, \cdot)$ is a locally stationary covariance function. A process possessing such a covariance function is called weakly locally stationary, too. Further, we shall need some facts about the harmonic analysis of nonstationary random processes. Following [2] we say that a random process $\{x(t), t \in \mathbb{R}_1\}$ is harmonizable if it can be written in the form of a stochastic integral understood in the quadratic mean sense

$$x(t) = \int_{-\infty}^{+\infty} e^{it\lambda} \, d\xi(\lambda)$$

where $\{\xi(\lambda), \lambda \in \mathbb{R}_1\}$ is a second order random process with zero mean and a covariance function $\gamma(\cdot,\cdot)$ of bounded variation on $\mathbb{R}_1 \times \mathbb{R}_1$. A random process is harmonizable if and only if its covariance function $R_x(\cdot, \cdot)$ is harmonizable, i.e.

$$R_x(s, t) = \iint_{-\infty}^{+\infty} e^{i(s\lambda - t\mu)} \, dd\gamma(\lambda, \mu).$$

Let us suppose that the process $\{x(t), t \in \mathbb{R}_1\}$ is locally stationary and harmonizable. In the theory of weakly stationary processes linear transformations of these processes play a very important role. If $\{x(t), t \in \mathbb{R}_1\}$ is a weakly stationary process having a spectral decomposition $x(t) = \int_{-\infty}^{+\infty} e^{it\lambda} \, d\xi(\lambda)$, and $\phi(\cdot) \in \mathscr{L}_2(\mathbb{R}_1, \gamma(\cdot))$ where $\gamma(\cdot)$ is the corresponding spectral measure, then the process

$$(1) \qquad y(t) = \int_{-\infty}^{+\infty} e^{it\lambda} \phi(\lambda) \, d\xi(\lambda) \,, \quad t \in \mathbb{R}_1$$

is weakly stationary, too. In the case of a locally stationary process the situation is not clear. We shall formulate the following problem: if $\{x(t), t \in \mathbb{R}_1\}$ is locally stationary and harmonizable, under which conditions put on a function $\phi(\cdot)$ the process (1) will be locally stationary as well.

First, we immediately see that the process $\{y(t), t \in \mathbb{R}_1\}$ must be of the second order, i.e. for every $s, t \in \mathbb{R}_1$ the integral

$$R_y(s, t) = \iint_{-\infty}^{+\infty} e^{i(s\lambda - t\mu)} \phi(\lambda) \bar{\phi}(\mu) \, dd\gamma(\lambda, \mu)$$

must exist. The process $\{y(t), t \in \mathbb{R}_1\}$ will be locally stationary if its covariance function $R_y(\cdot,\cdot)$ is a product of $R_y^{(1)}$ and $R_y^{(2)}$,

$$R_y(s, t) = R_y^{(1)} \left(\frac{s + t}{2}\right) R_y^{(2)}(s - t)$$

with $R_y^{(1)} \geqq 0$ and $R_y^{(2)}(\cdot)$ being a stationary covariance function. Let us consider the transformation

$$T: \frac{\lambda + \mu}{2} = u \,, \quad \lambda - \mu = v$$

which, under the local stationarity of $\{x(t), t \in \mathbb{R}_1\}$, makes it possible to express $R_y(\cdot,\cdot)$ in the form

$$R_y(s, t) = \iint_{-\infty}^{+\infty} e^{iu(s-t)} \, e^{iv[(s+t)/2]} \, \phi\left(u + \frac{v}{2}\right) \bar{\phi}\left(u - \frac{v}{2}\right) dF_1(u) \, dF_2(v)$$

where

$$(2) \qquad \iint_{E \times F} dF_1(u) \, dF_2(v) = \iint_{T^{-1}(E \times F)} dd\gamma(\lambda, \mu)$$

($E \times F$ is a measurable rectangle in $\mathbb{R}_1 \times \mathbb{R}_1$). This relation is in more detail explained in [3]. Because $R_x^{(2)}(y) = \int_{-\infty}^{+\infty} e^{iyu} \, dF_1(u)$ is a stationary covariance function, $F_1(u)$ must be a distribution function of a nonnegative measure of finite variation; because $R_x^{(1)}(\cdot) \geqq 0$, the Fourier image of $F_2(\cdot)$ must be nonnegative.

Now, if the following *separation* of the variables u, v

$$\phi\left(u + \frac{v}{2}\right) \bar{\phi}\left(u - \frac{v}{2}\right) = f(u) \, g(v)$$

is possible then

$$R_y(s, t) = \int_{-\infty}^{+\infty} e^{iu(s-t)} f(u) \, dF_1(u) \int_{-\infty}^{+\infty} e^{iv\,(s+t)/2} \, g(v) \, dF_2(v) \,.$$

Further, if $\int_{-\infty}^{+\infty} e^{iu(s-t)} f(u) \, dF_1(u)$ is a stationary covariance function and, simultaneously, if

$$\int_{-\infty}^{+\infty} e^{iv[(s+t)/2]}\, g(v)\, dF_2(v) \geqq 0$$

for every $s,\ t \in \mathbb{R}_1$ then $R_y(\cdot,\cdot)$ will be locally stationary. The following theorem gives necessary and sufficient conditions on $\phi(\cdot)$ in order that the process $\{y(t), t \in \mathbb{R}_1\}$ may be locally stationary.

Theorem 1. *Let $\{x(t), t \in \mathbb{R}_1\}$ be a harmonizable locally stationary random process,*

$$x(t) = \int_{-\infty}^{+\infty} e^{it\lambda}\, d\xi(\lambda)\,.$$

Then the process $\{y(t), t \in \mathbb{R}_1\}$ where $y(t) = \int_{-\infty}^{+\infty} e^{it\lambda}\, \phi(\lambda)\, d\xi(\lambda)$ is locally stationary if and only if there exist functions $f(\cdot)$, $g(\cdot)$ such that

1° $\quad \phi\left(u + \frac{v}{2}\right) \bar{\phi}\left(u - \frac{v}{2}\right) = f(u)\, g(v)$ a.e. $[F_1 \times F_2]$,

2° $\quad \int_{-\infty}^{+\infty} e^{itu} f(u)\, dF_1(u)$ is a stationary covariance function,

3° $\quad \int_{-\infty}^{+\infty} e^{isv} g(v)\, dF_2(v) \geqq 0$ for every $s \in \mathbb{R}_1$,

where $F_1(\cdot)$, $F_2(\cdot)$ are induced by the transformation T described above under the local stationary of $\{x(t), t \in \mathbb{R}_1\}$.

Proof. Let us suppose that both $\{x(t), t \in \mathbb{R}_1\}$ and $\{y(t), t \in \mathbb{R}_1\}$ are locally stationary. Then the covariance function $R_y(\cdot,\cdot)$ of $\{y(t), t \in \mathbb{R}_1\}$ can be written as the product

$$R_y(s, t) = \iint_{-\infty}^{+\infty} e^{i(s\lambda - t\mu)}\, \phi(\lambda)\, \bar{\phi}(\mu)\, dd\gamma(\lambda, \mu) = R_y^{(1)}\left(\frac{s+t}{2}\right) R_y^{(2)}(s - t)$$

where $R_y^{(1)}(\cdot) \geqq 0$ and $R_y^{(2)}(\cdot)$ is a stationary covariance. By means of transformation T (described above) we can express

$$R_y(s, t) = \iint_{-\infty}^{+\infty} e^{iv[(s+t)/2]}\, e^{iu(s-t)}\, \phi\left(u + \frac{v}{2}\right) \bar{\phi}\left(u - \frac{v}{2}\right) dF_1(u)\, dF_2(v)$$

where $F_1(\cdot)$ is a probability distribution function (without loss of generality we can put $R_x(0, 0) = 1$) and the Fourier image of $F_2(\cdot)$ is nonnegative. We immediately see that

$$R_y(s, s) = R_y^{(1)}(s)\, R_y^{(2)}(0)\,, \quad R_y\left(\frac{t}{2}, -\frac{t}{2}\right) = R_y^{(2)}(t)\, R_y^{(1)}(0)$$

and hence

$$R_y^{(2)}(0)\, R_y^{(1)}(s) = \iint_{-\infty}^{+\infty} e^{isv}\, \phi\left(u + \frac{v}{2}\right) \bar{\phi}\left(u - \frac{v}{2}\right) dF_1(u)\, dF_2(v)\,,$$

$$R_y^{(1)}(0)\, R_y^{(2)}(t) = \iint_{-\infty}^{+\infty} e^{itu}\, \phi\left(u + \frac{v}{2}\right) \bar{\phi}\left(u - \frac{v}{2}\right) dF_1(u)\, dF_2(v)\,.$$

In this way we obtain the relation

$$R_y(0,0)\iint_{-\infty}^{+\infty} e^{isv}\, e^{itu}\, \phi\left(u+\frac{v}{2}\right)\bar{\phi}\left(u-\frac{v}{2}\right) dF_1(u)\, dF_2(v) =$$

$$= \iint_{-\infty}^{+\infty} e^{isv}\, \phi\left(u+\frac{v}{2}\right)\bar{\phi}\left(u-\frac{v}{2}\right) dF_1(u)\, dF_2(v) \times$$

$$\times \iint_{-\infty}^{+\infty} e^{itu}\, \phi\left(u+\frac{v}{2}\right)\bar{\phi}\left(u-\frac{v}{2}\right) dF_1(u)\, dF_2(v)$$

holding for every pair $(s, t) \in R_2$. Properties of the two-dimensional Fourier transform imply that

$$R_y(0,0)\iint_{-\infty}^{uv} \phi\left(x+\frac{y}{2}\right)\bar{\phi}\left(x-\frac{y}{2}\right) dF_1(x)\, dF_2(y) =$$

$$= \int_{-\infty}^{u}\int_{-\infty}^{+\infty} \phi\left(x+\frac{y}{2}\right)\bar{\phi}\left(x-\frac{y}{2}\right) dF_2(y)\, dF_1(x) \times$$

$$\times \int_{-\infty}^{v}\int_{-\infty}^{+\infty} \phi\left(x+\frac{y}{2}\right)\bar{\phi}\left(x-\frac{y}{2}\right) dF_1(x)\, dF_2(y)$$

for every $u, v \in R_1$. This fact proves that

$$(3)\qquad \phi\left(x+\frac{y}{2}\right)\bar{\phi}\left(x-\frac{y}{2}\right) = \frac{1}{R_y(0,0)}\int_{-\infty}^{+\infty} \phi\left(x+\frac{v}{2}\right)\bar{\phi}\left(x-\frac{v}{2}\right) dF_2(v) \times$$

$$\times \int_{-\infty}^{+\infty} \phi\left(u+\frac{y}{2}\right)\bar{\phi}\left(u-\frac{y}{2}\right) dF_1(u) = f(x)\, g(y)$$

a.e. $[F_1 \times F_2]$.

As $R_y^{(1)}(\cdot) \geqq 0$ then

$$\int_{-\infty}^{+\infty} e^{ivs}\left\{\int_{-\infty}^{+\infty} \phi\left(u+\frac{v}{2}\right)\bar{\phi}\left(u-\frac{v}{2}\right) dF_1(u)\right\} dF_2(v) \geqq 0$$

must be nonnegative for every $s \in R_1$. Similarly, as $R_y^{(2)}(\cdot)$ is a stationary covariance function then

$$\int_{-\infty}^{+\infty} e^{iut}\left\{\int_{-\infty}^{+\infty} \phi\left(u+\frac{v}{2}\right)\bar{\phi}\left(u-\frac{v}{2}\right) dF_2(v)\right\} dF_1(u)$$

must be a stationary covariance function, too. Since $F_1(\cdot)$ is a probability distribution function $R_y^{(2)}(\cdot)$ will be a covariance function if and on if

$$\int_{-\infty}^{+\infty} \phi\left(u+\frac{v}{2}\right)\bar{\phi}\left(u-\frac{v}{2}\right) dF_2(v) \geqq 0 \quad \text{a.e.}\quad [F_1].$$

On the contrary, let the conditions 1°, 2°, 3° of Theorem 1 hold. The covariance function $R_y(\cdot, \cdot)$ can be expressed as

$$R_y(s, t) = \iint_{-\infty}^{+\infty} e^{iv[(s+t)/2]} e^{iu(s-t)} \phi\left(u + \frac{v}{2}\right) \bar{\phi}\left(u - \frac{v}{2}\right) dF_1(u) \, dF_2(v)$$

because $\{x(t), t \in \mathbb{R}_1\}$ is locally stationary. As $\phi(u + v/2) \, \bar{\phi}(u - v/2) = f(u) \, g(v)$ a.e. $[F_1 \times F_2]$ then

$$\begin{aligned} R_y(s, t) &= \iint_{-\infty}^{+\infty} e^{iv[(s+t)/2]} g(v) \, e^{iu(s-t)} f(u) \, dF_1(u) \, dF_2(v) = \\ &= \int_{-\infty}^{+\infty} e^{iv[(s+t)/2]} g(v) \, dF_2(v) \int_{-\infty}^{+\infty} e^{iu(s-t)} f(u) \, dF_1(u) = \\ &= R_y^{(1)}\left(\frac{s+t}{2}\right) R_y^{(2)}(s - t) \end{aligned}$$

where $R_y^{(1)}(\cdot) \geqq 0$ and $R_y^{(2)}(\cdot)$ is a stationary covariance. We have proved that the process $\{y(t), t \in \mathbb{R}_1\}$ is locally stationary. Q.E.D.

In Theorem 1 we met an interesting relation concerning the function $\phi(\cdot)$, namely

$$\phi\left(u + \frac{v}{2}\right) \bar{\phi}\left(u - \frac{v}{2}\right) = f(u) \, g(v) \, [F_1 \times F_2] \quad \text{a.s.}$$

Let us now suppose a somewhat stronger condition, namely

$$\phi\left(u + \frac{v}{2}\right) \bar{\phi}\left(u - \frac{v}{2}\right) = f(u) \, g(v)$$

for every $u, v \in \mathbb{R}_1$. Then for $v = 0$ we get

$$|\phi(u)|^2 = f(u) \, g(0) \geqq 0 \tag{4}$$

and similarly for $u = 0$

$$\phi\left(\frac{v}{2}\right) \bar{\phi}\left(-\frac{v}{2}\right) = f(0) \, g(v) \, .$$

Both the relations together give that (provided $f(0) \neq 0$, $g(0) \neq 0$)

$$f(u) \, g(v) = \frac{|\phi(u)|^2 \, \phi\left(\frac{v}{2}\right) \bar{\phi}\left(-\frac{v}{2}\right)}{f(0) \, g(0)}$$

and hence

$$\phi(\lambda) \, \bar{\phi}(\mu) = K \, . \left|\phi\left(\frac{\lambda + \mu}{2}\right)\right|^2 \phi\left(\frac{\lambda - \mu}{2}\right) \bar{\phi}\left(-\frac{\lambda - \mu}{2}\right)$$

where $K = f(0) \, g(0)$, $u = (\lambda + \mu)/2$, $v = \lambda - \mu$.

As $g(v) = \int_{-\infty}^{+\infty} \phi(u + v/2) \, \bar{\phi}(u - v/2) \, dF_1(u)$ (see Theorem 1), thus $g(0) = \int_{-\infty}^{+\infty} |\phi(u)|^2 \, dF_1(u) \geqq 0$ and hence the assumption $g(0) > 0$ is quite natural.

This fact together with (4) yields that $f(u) \geqq 0$ for every $u \in R_1$, hence also $K > 0$. In the sequel, for simplicity, we will assume $K = 1$. In this way we have obtained the following functional equation for the function $\phi(\cdot)$

$$(5) \qquad \phi(\lambda)\,\bar{\phi}(\mu) = \left|\phi\left(\frac{\lambda+\mu}{2}\right)\right|^2 \phi\left(\frac{\lambda-\mu}{2}\right)\bar{\phi}\left(-\frac{\lambda-\mu}{2}\right), \quad \lambda, \mu \in R_1,$$
$$\phi(0) = 1$$

which is very close to the local stationarity. If the function $g(v) = \phi(v/2)\,\bar{\phi}(-v/2)$ is a characteristic function then the covariance function $\phi(\cdot)\,\bar{\phi}(\cdot)$ will be locally stationary because

$$\left|\phi\left(\frac{\lambda+\mu}{2}\right)\right|^2 \geqq 0$$

and

$$\phi\left(\frac{\lambda-\mu}{2}\right)\bar{\phi}\left(-\frac{\lambda-\mu}{2}\right)$$

is a stationary covariance. We see that the linear transformation between two locally stationary random processes determined by the function $\phi(\cdot)$ is closely connected with the question which covariances of the type $\phi(\cdot)\,\bar{\phi}(\cdot)$ are locally stationary.

Let us try to solve the functional equation (5). At the first sight it is evident that $\phi(\cdot) = 1$ is a solution of (5) and thus the set of solutions is nonempty. Similarly, the function $\phi(\cdot)$ equal to 1 at 0 and vanishing otherwise also solves this equation. Hence, there is a discontinuous solution of (5). It is evident as well that the product $\phi_1\phi_2(\cdot)$ solves (5) if $\phi_1(\cdot)$ and $\phi_2(\cdot)$ are solutions of (5). The equation can be easily expressed in an equivalent form

$$\phi(u+v)\,\bar{\phi}(u-v) = |\phi(u)|^2\,\phi(v)\,\bar{\phi}(-v)\,, \quad u, v \in R_1\,,$$

$\phi(0) = 1$. First we shall be interested in continuous solutions of the equation (5). Let $\phi(\cdot)$ be a solution of (5) continuous at zero with $\phi(\lambda_0) = 0$, $\lambda_0 \neq 0$. Then

$$0 = \phi(\lambda_0)\,\bar{\phi}(\mu) = \left|\phi\left(\frac{\lambda_0+\mu}{2}\right)\right|^2 \phi\left(\frac{\lambda_0-\mu}{2}\right)\bar{\phi}\left(\frac{\mu-\lambda_0}{2}\right)$$

for every real μ. For $\mu = 0$ we have

$$0 = \left|\phi\left(\frac{\lambda_0}{2}\right)\right|^2 \phi\left(\frac{\lambda_0}{2}\right)\bar{\phi}\left(-\frac{\lambda_0}{2}\right)$$

and hence either $\phi(\lambda_0/2) = 0$ or $\phi(-\lambda_0/2) = 0$. In the case of $\phi(\lambda_0/2) = 0$ we again obtain either $\phi(\lambda_0/4) = 0$ or $\phi(-\lambda_0/4) = 0$. In this way we can construct a sequence $\{\lambda_n\}_{n=1}^{\infty}$, $\lambda_n \to 0$ for $n \to \infty$ with $\phi(\lambda_n) = 0$. This conclusion contradicts the assumption that $\phi(0) = 1$. We can summarize; if there exists a continuous at zero solution $\phi(\cdot)$ of (5) then $\phi(\lambda) \neq 0$ for every $\lambda \in R_1$. Thus $1/\phi(\cdot)$ is a solution of (5) as well.

We see that all solutions of (5) continuous at zero form a group with respect to multiplication. Let us describe this group explicitly. At the beginning we must realize that if $\phi(\cdot)$ is a solution of (5) then the absolute value $|\phi(\cdot)|$ solves the same equation, hence $\phi(\cdot)/|\phi(\cdot)|$ is a solution of (5) as well. As $|\phi(\cdot)/|\phi(\cdot)|| = 1$ the equation (5) in this case has the form

$$\phi(\lambda)\,\bar{\phi}(\mu) = \phi\left(\frac{\lambda-\mu}{2}\right)\bar{\phi}\left(\frac{\mu-\lambda}{2}\right), \quad \lambda, \mu \in \mathbb{R}_1\,, \quad \phi(0) = 1 \tag{6}$$

and $|\phi(\lambda)| = 1$ for every $\lambda \in \mathbb{R}_1$. Then one can write $\phi(\lambda) = \mathrm{e}^{\mathrm{i}\alpha(\lambda)}$ where $\alpha(\cdot)$ is a real function, and we have obtained an equivalent transcription of (6)

$$\alpha(\lambda) - \alpha(\mu) = \alpha\left(\frac{\lambda-\mu}{2}\right) - \alpha\left(\frac{\mu-\lambda}{2}\right)$$

or

$$\alpha(u+v) - \alpha(u-v) = \alpha(u) - \alpha(v)\,.$$

We see that $\Delta_h\alpha(\lambda) = \Delta_h\alpha(0)$ for every λ, $h \in \mathbb{R}_1$. This implies that

$$\alpha(\lambda) = C_0 + C_1\lambda$$

and hence $\phi(\lambda) = \mathrm{e}^{\mathrm{i}(C_0 + C_1\lambda)}$. As we demand $\phi(0) = 1$, we have $C_0 = 0$.

The equation (5) for the absolute value $A(\cdot) = |\phi(\cdot)|$ has the form

$$A(\lambda)\,A(\mu) = A^2\left(\frac{\lambda+\mu}{2}\right)A\left(\frac{\lambda-\mu}{2}\right)A\left(\frac{\mu-\lambda}{2}\right), \quad \lambda, \mu \in \mathbb{R}_1\,,$$

$A(0) = 1$ and $A(\lambda) > 0$.

We can write $A(\lambda) = \mathrm{e}^{a(\lambda)}$ and arrive at the equation

$$a(\lambda) + a(\mu) = 2a\left(\frac{\lambda+\mu}{2}\right) + a\left(\frac{\lambda-\mu}{2}\right) + a\left(\frac{\mu-\lambda}{2}\right), \quad \lambda, \mu \in \mathbb{R}_1\,.$$

We immediately obtain that $a(0) = 0$ and the latter relation can be rewritten as

$$\Delta_h^2 a(\lambda) = \Delta_h^2 a(0)\,.$$

Solving the difference equation $\Delta_h^3 a(\lambda) = 0$ we obtain that

$$a(\lambda) = K_0 + K_1\lambda + K_2\lambda^2\,.$$

As we need $a(0) = 0$, we have $K_0 = 0$. In this way we have proved that every continuous at zero solution of (5) has the form

$$\phi(\lambda) = \mathrm{e}^{K\lambda^2}\,.\,\mathrm{e}^{Q\lambda}$$

where $K \in \mathbb{R}_1$, $Q \in \mathbb{C}$.

Corollary 1. *Let* $\{x(t), t \in \mathbb{R}_1\}$ *be a harmonizable locally stationary process*

$$x(t) = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}t\lambda}\,\mathrm{d}\xi(\lambda)\,.$$

Then the process $\{y(t), t \in \mathbb{R}_1\}$, $y(t) = \int_{-\infty}^{+\infty} e^{it\lambda} e^{K\lambda^2} e^{Q\lambda} d\xi(\lambda)$ *with* $K \leqq 0$, $Q \in \mathbb{C}$ *is locally stationary, too.*

Proof. It is evident that

$$R_y(s, t) = \iint_{-\infty}^{+\infty} e^{i(s\lambda - t\mu)} e^{K(\lambda^2 + \mu^2)} e^{Q\lambda} e^{\bar{Q}\lambda} dd\gamma(\lambda, \mu).$$

By means of the transformation $T: (\lambda + \mu)/2 = u$, $\lambda - \mu = v$ and the local stationarity of $\{x(t), t \in \mathbb{R}_1\}$ we get

$$R_y(s, t) =$$
$$= \iint_{-\infty}^{+\infty} e^{iu(s-t)} e^{iv(s+t)/2} e^{2Ku^2} e^{(Q+\bar{Q})u} e^{K(v^2/2)} e^{(Q-\bar{Q})v/2} dF_1(u) dF_2(v).$$

As $Q + \bar{Q} = 2 \operatorname{Re} Q$ and $e^{2Ku} . e^{2\operatorname{Re}Qu} > 0$,

$$\int_{-\infty}^{+\infty} e^{iu(s-t)} e^{2Ku^2} e^{2\operatorname{Re}Qu} dF_1(u)$$

is a stationary covariance function. Similarly $(Q - \bar{Q})/2 = i \operatorname{Im} Q$ and hence

$$\int_{-\infty}^{+\infty} e^{iv(s+t)/2} e^{i\operatorname{Im}Qv} e^{K(v/2)} dF_2(v) =$$
$$= \int_{-\infty}^{+\infty} e^{iv(s+t)/2} \left(\int_{-\infty}^{+\infty} e^{ivx} \frac{1}{\sqrt{(-2\pi K)}} e^{(x-\operatorname{Im}Q)^2/2K} dx\right) dF_2(v) =$$
$$= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} e^{iv[(s+t)/2+x]} dF_2(v)\right) \frac{1}{(-2\pi K)} e^{(x-\operatorname{Im}Q)^2/2K} dx \geqq 0$$

because under the local stationarity of $\{x(t), t \in \mathbb{R}_1\}$ we have

$$\int_{-\infty}^{+\infty} e^{ivy} dF_2(v) \geqq 0$$

for every $y \in \mathbb{R}_1$. Q.E.D.

Corollary 2. *Every continuous locally stationary covariance function* $R(\cdot,\cdot)$ *of the type*

$$R(s, t) = \phi(s) \bar{\phi}(t), \quad R(0, 0) = 1$$

has the form

$$R(s, t) = e^{-a(s^2+t^2)} . e^{bs+\bar{b}t},$$

where $a \geqq 0$, $b \in \mathbb{C}$.

Proof. In order to be locally stationary the covariance function $\phi(\cdot) \bar{\phi}(\cdot)$ must satisfy

$$\phi(s) \bar{\phi}(t) = R_1\left(\frac{s+t}{2}\right) R_2(s-t)$$

where $R_1(\cdot) \geqq 0$ and $R_2(\cdot)$ is a stationary covariance. One immediately sees that

$$R_1(x) = |\phi(x)|^2, \quad R_2(y) = \phi\left(\frac{y}{2}\right) \bar{\phi}\left(-\frac{y}{2}\right)$$

and thus the function $\phi(\cdot)$ must be a solution of the equation

$$\phi(s)\,\bar{\phi}(t) = \left|\phi\left(\frac{s+t}{2}\right)\right|^2 \phi\left(\frac{s-t}{2}\right)\bar{\phi}\left(\frac{t-s}{2}\right), \quad \phi(0) = 1\,.$$

As was proved above the continuous solution of this functional equation is

$$\phi(\lambda) = e^{a\lambda^2 + b\lambda}$$

where $a \in R_1$, $b \in C$.

Thus $R_1(x) = |e^{ax^2+bx}|^2 = e^{2ax^2} \,.\, e^{(b+\bar{b})x}$ and

$$R_2(y) = e^{a(y/2)^2 + by/2} \,.\, e^{a(-y/2)^2 + \bar{b}(-y/2)} = e^{ay^2/2} \,.\, e^{(b-\bar{b})y/2}\,.$$

Indeed, we obtain that $R_1(\cdot) \geqq 0$; $R_2(\cdot)$ must be a stationary covariance. As $R_2(\cdot)$ is continuous it will be a stationary covariance if and only if $R_2(\cdot)$ is a characteritic function. It means that the coefficient a must be less or equal to zero because the inequality

$$|R_2(y)| = e^{ay^2/2} \leqq 1$$

must hold for every $y \in R_1$. Then

$$R_2(y) = \int_{-\infty}^{+\infty} e^{iyv} \frac{1}{2\pi(-a)} e^{(v-(b-\bar{b})/2)^2/2a} \, dv$$

in the case $a < 0$ and

$$R_2(y) = \int_{-\infty}^{+\infty} e^{iyv} \, dF_0\left(v - \frac{b-\bar{b}}{2}\right)$$

for $a = 0$ where $F_0(v) = 0$ for $v \leqq 0$, $F_0(v) = 1$ otherwise. Q.E.D.

References

[1] *R. A. Silverman:* Locally stationary random processes. IRE Transactions of Information Theory IT-3 (1957), 3, 182—187.

[2] *M. Loève:* Probability Theory. D. van Nostrand, Toronto—New York—London, 1955.

[3] *J. Michálek:* Spectral decomposition of locally stationary random processes. Kybernetika 22 (1986), 3, 244—255.

Souhrn

LINEÁRNÍ TRANSFORMACE LOKÁLNĚ STACIONÁRNÍCH PROCESŮ

JIŘÍ MICHÁLEK

V článku je řešena otázka, za jakých podmínek je lineární transformace harmonizovatelného slabě lokálně stacionárního procesu opět lokálně stacionární proces. Jsou nalezeny nutné a postačující podmínky pro funkci, kterou je tato lineární transformace určena.

Резюме

JIŘÍ MICHÁLEK

ЛИНЕЙНЫЕ ПРЕОБРАЗОВАНИЯ ЛОКАЛЬНО СТАЦИОНАРНЫХ ПРОЦЕССОВ

В статье решен вопрос, при каких условиях линейное преобразование гармонизуемого в широком смысле локально стационарного процесса опять является локально стационарным. Найдены необходимые и достаточные условия для функции, определяющей такое линейное преобразование.

Author's address: Dr. *Jiří Michálek*, CSc., ÚTIA ČSAV, Pod vodárenskou věží 4, 182 08 Praha 8.