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Linear Transformations on Matrices*

Marvin Marcus**

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Let K be a field and let $M_n(K)$ denote the vector space of all $n \times n$ matrices over K. Suppose $I(X)$ is an invariant defined on a subset $\mathfrak{A}$ of $M_n(K)$. This paper surveys certain results concerning the following problem. Describe the set $\mathscr{L}(I, \mathfrak{A})$ of all linear transformations $T: \mathfrak{A} \to \mathfrak{A}$ that hold the invariant I fixed:

$$I(T(X)) = I(X), \qquad X \in \mathfrak{A}.$$

Key words: Matrices; invariants; determinant; generalized matrix function; rank.

1. Introduction

Let K be a field and let $M_n(K)$ denote the vector space of all $n \times n$ matrices over K. Over the last 80 years, a great deal of effort has been devoted to the following question. Suppose $I(X)$ is an invariant defined on a subset $\mathfrak{A}$ of $M_n(K)$. Describe the set $\mathscr{L}(I, \mathfrak{A})$ of all linear transformations $T: \mathfrak{A} \to \mathfrak{A}$ that hold the invariant I fixed:

$$I(T(X)) = I(X), \qquad X \in \mathfrak{A}. \tag{1}$$

Even in this generality, it is clear that $\mathscr{L}(I, \mathfrak{A})$ is a multiplicative semigroup with an identity. The invariant I can be a scalar valued function, e.g., $I(X) = \det(X)$; or for that matter it can describe a property, e.g., $\mathfrak{A}$ can equal $M_n(C)$ and $I(X)$ can mean that X is unitary, so that we are simply asking for the structure of all linear transformations T that map the unitary group into itself.

Much of a beginning course in linear algebra is devoted to the study of one aspect of this question for certain choices of I; for example, if $I(X) = \rho(X)$, the rank of X, then it is well known that the three standard linear operations on the rows and columns of a matrix leave ρ fixed and this fact permits us to compute $\rho(X)$ by reducing X to some normal form. The similarity theory is another example of this problem. In this case take $I(X)$ to be the set of all elementary divisors of the characteristic matrix of X, and then the linear operators T that we wish to study are precisely those for which $I(X) = I(T(X))$.

In the survey paper [18, 1962] [1] some of the aspects of this general problem are discussed. But since the time that paper was written there have been a number of developments. The purpose of this paper is to describe some of these.

2. Survey of Results

The scalar invariants are functions I for which $I(X)$ is either an element of K (we will assume that char $K = 0$, so that integer-valued functions are included) or a p-tuple of elements of K. Prob-

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** Present address: Department of Mathematics, University of California, Santa Barbara, California 93106.

[1] Figures in brackets indicate the literature references at the end of this paper.

ably the first three problems of this kind were considered by Frobenius [11, 1897]:

(i) $\mathfrak{A}=M_n(C)$, $\quad I(X)=\det(X)$;

(ii) $\mathfrak{A}$ is the space of real-symmetric (or odd order skew-symmetric) matrices and $I(X)=\det(X)$;

(iii) $\mathfrak{A}=\{X \mid \mathrm{tr}(X)=0\} \subset M_n(C)$ and $I(X)=\det(X)$.

Frobenius proved what one might expect, namely that $\mathscr{L}(\det, M_n(C))$ consists of linear transformations of the form

$$T(X)=UXV, \qquad X \in M_n(C), \tag{2}$$

or

$$T(X)=UX^TV, \qquad X \in M_n(C). \tag{3}$$

In problem (i) $\det(UV)=1$; in problem (ii) $U=\xi A$, $V=A^T$ where ξ is an appropriately chosen constant and $\det A=1$; in (iii) $U=\xi A$, $V=A^{-1}$. I. Schur [34, 1925] extended and improved the result (i) as follows: Take $\delta(X)$ to be the $\binom{n}{k}^2$-tuple of k^{th} order subdeterminants of X in some order where $k \geqslant 3$ is fixed; then Schur proved that if $\delta(T(X))=S(\delta(X))$ for S a fixed nonsingular matrix, then T has one of the two forms (2) or (3) (without the restriction $\det(UV)=1$). Dieudonné [8, 1949] showed that if T is a semi-linear transformation of $M_n(K)$ onto itself which holds the cone $\det(X)=0$ invariant, then T is of the form

$$T(X)=U(\sigma(x_{ij}))\,V, \qquad X \in M_n(K),$$

or

$$T(X)=U(\sigma(x_{ij}))^TV, \qquad X \in M_n(K),$$

where σ is an automorphism of the field. In the paper [9, 1957] Dynkin states that the Frobenius theorems can be obtained using some results on the structure of maximal subgroups of the classical groups.

In an old result, Pòlya [32] restricted T to be a linear transformation which affixes in a prescribed way + and − signs to the elements of X, and asked whether such a T exists which satisfies per $(T(X))=\det(X)$ for $n>2$. Pòlya answered the question negatively and many years later in [23, 1961] the question was answered negatively for arbitrary linear transformations T. Along the lines of the Frobenius and Schur results, the structure of $\mathscr{L}(E_r, M_n(C))$ is determined where $E_r(X)$ is the rth elementary symmetric function of the eigenvalues of X, i.e., the sum of all r-square principle subdeterminants of X. It was proved in [28, 1959] that for $4 \leqslant r < n$ any $T \in \mathscr{L}(E_r, M_n(C))$ is of the form

$$T(X)=UXV, \qquad X \in M_n(C),$$

or

$$T(X)=UX^TV, \qquad X \in M_n(C),$$

where $UV=e^{i\varphi}I_n$ and $r\varphi \equiv 0\,(2\pi)$. Just recently Beasley [1, 1970] completed the argument by showing that for $r=3$, precisely the same result holds. E. P. Botta, in several papers considers the choice $I(X)=d(X)$ where d is a generalized matrix function in the sense of Schur, i.e.,

$$d(X)=\sum_{\sigma \in H} \lambda(\sigma) \prod_{i=1}^{n} x_{i,\sigma(i)},$$

where λ is a nonzero function defined on a subgroup H of S_n. In [3, 1967] Botta determined the structure of $\mathscr{L}$ $(d, M_n(K))$ when H is a transitive cyclic subgroup of S_n. In [4, 1968] H is taken to be a doubly transitive or regular proper subgroup of S_n and λ is a character of H of degree 1. In [5, 1967] Botta reproves an earlier result of Marcus and May [22, 1962] showing that $\mathscr{L}$ (per, $M_n(K)$) consists of precisely those T of the form

$$T(X) = DPXQL, \qquad X \epsilon M_n(K),$$

or

$$T(X) = DPX^T Q\, L, \qquad X \epsilon M_n(K),$$

where P and Q are permutation matrices, D and L are diagonal matrices and per $(DL) = 1$.

Many of the results concerning the structure of $\mathscr{L}(I, \mathfrak{A})$ can be reduced to the problem of determining linear maps on $M_n(K)$ which map the set of rank 1 matrices into itself. W. L. Chow [6, 1949], L. K. Hua [12, 1951] and Jacobson and Rickart [13, 1950] considered $1-1$ onto maps T of $M_n(K)$ which have the property that both T and T^{-1} preserve coherence. In the present context, this amounts to assuming that T and T^{-1} both have the property that whenever X and Y differ by a matrix of rank 1, then $T(X)$ and $T(Y)$ differ by a matrix of rank 1. For linear maps this means $\rho(T(X)) = \rho(X)$ for all X. In [26, 1959] Marcus and Moyls proved that if $T: M_{m,n}(K) \rightarrow M_{m,n}(K)$ ($M_{m,n}(K)$ is the space of all $m \times n$ matrices over K) is linear and $\rho(T(X)) = 1$ whenever $\rho(X) = 1$, then T has the form

$$T(X) = UXV, \qquad X \epsilon M_{m,n}(C),$$

or

$$T(X) = UX^T V, \qquad X \epsilon M_{m,n}(C),$$

where U and V are fixed nonsingular matrices in $M_m(K)$ and $M_n(K)$ respectively. This result is fairly easy to apply because it does not require the a priori existence of T^{-1}. R. Westwick in [36, 1967] extended the result in [26, 1959] to linear maps on the space of n-contravariant tensors which hold the nonzero decomposable elements set-wise fixed. In another paper [35, 1964] Westwick, using techniques in [6, 1949] determined the structure of linear maps on the space $\wedge^m V$ of skew-symmetric tensors into itself which hold the set of nonzero decomposable elements set-wise fixed. In a thesis at the University of British Columbia [7, 1967] L. Cummings proved that if T maps the symmetric power $V^{(m)}$ into itself and holds the set of non-zero decomposable elements set-wise fixed then T is induced by a linear map of V. Cummings' result requires that the underlying field be algebraically closed of characteristic either 0 or exceeding m. In another thesis [14, 1971] M. H. Lim reconsiders this problem and relaxes the conditions on the field. Beasley [2, 1970] considered the problem of determining all linear transformations $T: M_n(K) \rightarrow M_n(K)$, K algebraically closed, which hold the set of rank k matrices set-wise fixed. Beasley required additional hypotheses on T in order to prove that T has the form (2) or (3). Djoković [10, 1969] proved that if T maps the set of rank k matrices into itself and is nonsingular, then in fact T maps the set of rank 1 matrices into itself and the result in [26, 1959] applies. Much earlier [30, 1941] Morita proved that if T maps the set of rank 1 matrices into itself and maps the set of rank 2 matrices into the set of matrices of rank at least 2, then T has the form (2) or (3). He then used this to prove a result of Schur to the effect that if $I(X) = \alpha_1(X)$ is the Hilbert norm of X, i.e., the square root of the largest eigenvalue of X^*X, and $T \epsilon \mathscr{L}(I, M_{m,n}(C))$, then T has the form (2) or (3) in which U and V are unitary. In a later paper [31, 1944], Morita shows that if $\mathfrak{A}$ is the set of n-square complex skew-symmetric matrices and $I(X)$ is again the Hilbert norm of X, then for $n \neq 4$ and $T \epsilon \mathscr{L}(I, \mathfrak{A})$,

$$T(X) = U^T XU, \qquad X \epsilon M_n(C),$$

where U is a fixed unitary matrix; or if $n=4$, then $T(X)$ can also have the alternative form

$$T(X)=U^T X^+ U, \qquad X \in M_n(C), \tag{4}$$

where X^+ is the matrix obtained from X by interchanging the $(1, 4)$ and the $(2, 3)$ entry. A result similar to this was obtained by Westwick in his thesis [35, 1964]. In [29, 1960] Marcus and Westwick proved a theorem somewhat along the lines of the Morita theorem as follows. Let k be a fixed integer satisfying $4 \leqslant 2k \leqslant n$. Let $\mathfrak{A}$ be the set of skew-symmetric matrices over the field R of real numbers and let $T \in \mathscr{L}(E_{2k}, \mathfrak{A})$. If $n \geqslant 5$, then there exists a real matrix P such that

$$T(X)=\alpha P X P^T, \qquad X \in \mathfrak{A}, \tag{5}$$

where $\alpha P P^T = I_n$ is $2k<n$ and $\alpha P P^T$ is unimodular if $2k=n$. If $2k=n=4$, then either T has the form (5) or

$$T(X)=\alpha P \begin{bmatrix} 0 & x_{34} & x_{24} & x_{23} \\ -x_{12} & 0 & x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{bmatrix} P^T,$$

where

$$X=\begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} \\ -x_{12} & 0 & x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{bmatrix},$$

and $\alpha P P^T$ is unimodular. Later on Marcus and Minc [24, 1962] proved that if $T \in \mathscr{L}(E_r', M_{m,n}(C))$ where $1<r \leqslant n$ and $E_r'(X)$ is the value of E_r at the squares of the singular values of X, i.e., $E_r'(X)$ is just the value of E_r at the eigenvalues of X^*X, then T has the form (2) if $m \neq n$ and either (2) or (3) if $m=n$, where $U \in M_n(C)$ and $V \in M_n(C)$ are unitary.

In [17, 1959] it is proved that if T is a linear map of $M_n(C)$ into itself such that $T(X)$ is unitary whenever X is unitary, then T is of the form (2) or (3) where U and V are unitary. B. Russo [33, 1969] recently used this result to prove the following interesting theorem. If $I(X)$ is the sum of the singular values of X and if T maps the identity matrix into itself, then $T \in \mathscr{L}(I, M_n(C))$ has the form (2) or (3), where U and V are unitary. Marcus and Gordon [20, 1970] recently proved the following result. Let $f(t)=f(t_1, \ldots, t_n)$ be a continuous, real-valued function defined for all $t_j \geqslant 0$, $1 \leqslant j \leqslant n$, and for $X \in M_{m,n}(C)$, let

$$I(X)=f(\alpha_1(X), \alpha_2(X), \ldots, \alpha_n(X))$$

where $\alpha_1(X) \geqslant \alpha_2(X) \geqslant \ldots \geqslant \alpha_n(X)$ are the singular values of X. If $f(t_1, \ldots, t_n)$ is concave, symmetric, strictly increasing in each t_j, and $f(0)=0$, then $T \in \mathscr{L}(I, M_{m,n}(C))$ has the form (2) if $m \neq n$ and either (2) or (3) if $m=n$ where $U \in M_m(C)$ and $V \in M_n(C)$ are unitary. By specializing f to

$$f(t)=\sum_{j=1}^{n} t_j^{\sigma}$$

where $0<\sigma \leqslant 1$, the above theorem reduces to the following result. If $T: M_{m,n}(C) \to M_{m,n}(C)$ satisfies

$$\sum_{j=1}^{n} \alpha_j(T(X))^{\sigma}=\sum_{j=1}^{n} \alpha_j(X)^{\sigma}$$

for all $X \in M_{m,n}(C)$, then (2) or (3) holds with unitary U and V. In the paper [27, 1970] the following result is proven using representation theory techniques. Let $f(t)=f(t_1, \ldots, t_n)$ satisfy the conditions

(i) $f(t)=0$ if and only if $t=0$;

(ii) f is positively homogeneous of degree $\rho \neq 0$; i.e., $f(ct)=c^\rho f(t)$, all $c \geq 0$, $t \geq 0$ (i.e., $t_j \geq 0, j=1, \ldots, n$).

If $I(X)=f(\alpha_1(X), \ldots, \alpha_n(X))$ as before, then $\mathscr{L}(I, M_{m,n}(C))$ is a subgroup of the group of $mn \times mn$ unitary matrices $U(mn, C)$ where we associate each $T \in \mathscr{L}(I, M_{m,n}(C))$ with its matrix representation with respect to the lexicographically ordered basis $\{E_{st}=(\delta_{is}\delta_{tj}), i, j=1, \ldots, n\}$.

3. Current Work and Some Questions

M. J. S. Lim, in work closely related to that of Marcus and Westwick [29, 1960], has recently published [15, 1970] the following result. Let T map the space of skew-symmetric matrices over an algebraically closed field K into itself. Assume that T maps the set of rank 4 matrices into itself. Then for $n \neq 4$, T is of the form

$$T(X)=\alpha PXP^T$$

or

$$T(X)=\alpha PX^TP^T.$$

In case $n=4$, T is one of the above forms, or else

$$T(X)=\alpha PX^+P^T$$

where X^+ is defined in (4).

Just recently Marcus and Holmes [21, 1971] have proved the following results. Let $X \in M_n(C)$. For any subgroup H of the symmetric group S_m of degree m and character χ of degree 1 on H, let $K(X): P \to P$ be the induced transformation [19, 1967] on the symmetry class of tensors (P, ν) associated with H and χ. Define $I(X)=tr\, K(X), X \in M_n(C)$.

(i) Let $m \leq n$ or $\chi \equiv 1$. $\mathscr{L}(I, M_n(C))$ is a group if and only if $H \neq \{e\}$.

(ii) Let $H=S_m$, $\chi \equiv 1$ and $\mathfrak{A} \subset M_n(C)$ an algebra with the property that $\mathfrak{A}^*=\{X^*: X \in \mathfrak{A}\}=\mathfrak{A}$, when X^* denotes the conjugate transpose of X. Then $\mathscr{L}(I, \mathfrak{A})$ is a group.

(iii) In (i) take $H=S_m$, $m \geq 3$ and $\chi \equiv 1$. If $\mathscr{L}_1(I, M_n(C))$ denotes the subgroup of $\mathscr{L}(I, M_n(C))$ of those $T: M_n(C) \to M_n(C)$ satisfying $T(I_n)=\xi I_n$, then for any $T \in \mathscr{L}_1(I, M_n(C))$ there exists a fixed nonsingular matrix $P \in M_n(C)$ such that

$$T(X)=\xi P^{-1}XP, \qquad X \in M_n(C), \tag{6}$$

or

$$T(X)=\xi P^{-1}X^TP, \qquad X \in M_n(C). \tag{7}$$

In this case $tr\, K(X)$ is the completely symmetric function of the eigenvalues of X, denoted here by $h_m(X)$. Thus this result states that if $T(I_n)=\xi I_n$ and $h_m(T(X))=h_m(X)$ for all $X \in M_n(C)$ then T has the form (6) or (7).

(iv) In (i) take $A_m \subset S_m$ to be the alternating group, $m \geq 3$ and $\chi \equiv 1$. Then the group $\mathscr{L}_1(I, M_n(C))$ consists precisely of those linear transformations T of the form (6) or (7).

There are a number of questions which remain unanswered. For example, a more direct proof of the result in [27, 1970] might be based on the following.

CONJECTURE 1: *Let* T *be an mn-square complex matrix, and assume that for arbitrary unitary matrices* $U \epsilon M_n(C)$, $V \epsilon M_n(C)$ *the matrix* $(U \otimes V)T$ *has eigenvalues of modulus 1. (The matrix* $U \otimes V$ *is the usual Kronecker product of* U *and* V.) *Then* T *is unitary.*

CONJECTURE 2: *If* $T\colon M_n(C) \rightarrow M_n(C)$ *is a linear map and if* $h_m(T(X)) = h_m(X)$, $X \epsilon M_n(C)$ *(recall that* $h_m(X)$ *is the completely symmetric function of the eigenvalues of* X), *then in fact* $T(I_n) = \xi I_n$ *and hence from* [21, 1971] T *has the form (6) or (7).*

CONJECTURE 3: *Let* $P_m(X)$ *denote the* mth *induced power matrix of* X [25] *and suppose that* $T\colon M_n(C) \rightarrow M_n(C)$ *satisfies* $P_m(T(X)) = S(P_m(X))$, $X \epsilon M_n(C)$ *where* $S\colon M_N(C) \rightarrow M_N(C)$ *is a fixed non-singular linear map,* $N = \binom{m+m-1}{m}$. *Then* T *has the form (2) or (3).*

CONJECTURE 4: *Suppose* K(X) *is an invariant matrix* [16, Chapter X] *defined by means of a Young tableau. If* $T\colon M_n(C) \rightarrow M_n(C)$ *and* $\operatorname{tr} K(T(X)) = \operatorname{tr} K(X)$ *for all* $X \epsilon M_n(C)$, *then* T *must have the form (2) or (3) with some appropriate conditions on the* U *and* V. *Of course, the result in* [28, 1959] *and* [21, 1971] *are special cases of this.*

As a possible extension of Schur's theorem [30, 1941] *consider*

CONJECTURE 5: *Let* $T\colon M_n(C) \rightarrow M_n(C)$ *and* $h_m(T(X)^*T(X)) = h_m(X^*X)$ *(see Conjecture 2 in which* h_m *is defined), then* T *has the form (2) or (3).*

As a variant of the result in [17, 1959], *let* G *be any of the following classical groups: the real orthogonal group, the rotation group, the symplectic group.*

CONJECTURE 6: *Let* $T\colon M_n(R) \rightarrow M_n(R)$ *map* G *into itself. Then* T *must have the form (2) or (3) in which* U *and* V *belong to* G. *(In the case of the rotation group,* U *and* V *could be simply real orthogonal with* $\det(UV) = 1$.)

CONJECTURE 7: *Suppose* T *is a mapping of the space of 2-contravariant tensors into itself; and suppose moreover that for each decomposable element* $x \otimes y$ *we have* $\| T(x \otimes y) \| = \| x \otimes y \|$ *(Euclidean norm). Then* T *is unitary. This can be restated in terms of linear maps* $T\colon M_n(C) \rightarrow M_n(C)$. *Thus suppose for* $\rho(X) = 1$, $\| T(X) \| = \| X \|$ *where* $\| X \|$ *is just the Euclidean norm. Show that* T *is unitary.*

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