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LINEAR TRANSFORMATIONS THAT PRESERVE THE NILPOTENT MATRICES

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Let \mathfrak{sl}_n be the algebra of $n \times n$ matrices with zero trace and entries in a field with at least n elements. Let \mathcal{N} be the set of nilpotent matrices. The main result in this paper is that the group of nonsingular linear transformations L on \mathfrak{sl}_n such that $L(\mathcal{N}) = \mathcal{N}$ is generated by the inner automorphisms: $X \rightarrow S^{-1}XS$; the maps: $X \rightarrow aX$, for $a \neq 0$; and the map: $X \rightarrow X'$ that sends a matrix X to its transpose.

Introduction. Let M_n be the algebra of $n \times n$ matrices over a field K and let S be an algebraic set in M_n . There are a number of theorems characterizing the linear maps L on M_n that preserve S , i.e. $L(S) \subseteq S$. For example there are results for $\{X: \det X = 0\}$ by Dieudonné [4], $\{X: \text{rank } X \leq 1\}$ by Jacob [8] and Marcus and Moyls [10], the orthogonal group by Pierce and Botta [2] and other linear groups by Dixon [5]. In every instance the transformations L that preserve these various algebraic sets have one of these two forms:

$$(1) \quad L(X) = PXQ, \quad \text{for all } X$$

or

$$(2) \quad L(X) = PX'Q, \quad \text{for all } X$$

where P and Q are in M_n . There are conditions on P and Q which depend on the algebraic set S . For example if $S = \{X: \det X = 0\}$ and L is nonsingular then P and Q are nonsingular; if S is the orthogonal group then $PQ = I$ and P must be a scalar multiple of a matrix in the orthogonal group over the algebraic closure of K . For a good survey of further results of this type see Marcus [9].

In this paper we characterize the nonsingular linear transformations L that preserve the set \mathcal{N} of nilpotent matrices. Since the linear span of \mathcal{N} is the space \mathfrak{sl}_n of matrices with trace zero, we may as well assume that L is a transformation on \mathfrak{sl}_n . (In order to see that \mathcal{N} spans \mathfrak{sl}_n , let E_{ij} be the matrix whose only nonzero entry is a 1 in position (i, j) . The nilpotent matrices E_{ij} and $E_{ii} + E_{ij} - E_{ji} - E_{jj}$ for $i \neq j$ span \mathfrak{sl}_n .)

Actually we characterize all nonsingular semilinear mappings that preserve nilpotence. The main theorem can be extended to matrices with entries from an integral domain. The extension follows from a modification of a result of Chevalley [3, p. 104, Théorème 3].

THEOREM. *Let $n \geq 3$, K be a field with at least n elements and suppose that L is a nonsingular linear transformation on \mathfrak{sl}_n such that $L(\mathfrak{N}) \subseteq \mathfrak{N}$. Then L either has form (1) or (2), where PQ is a non-zero scalar matrix.*

Without the assumption that L is nonsingular the theorem is false. Any map whose image is contained in the algebra \mathfrak{U} of the strictly upper triangular matrices preserves nilpotence. The proof of the theorem depends on a result of Gerstenhaber about maximal spaces of nilpotent matrices. We also use some elementary algebraic geometry and the fundamental theorem of projective geometry [1, p. 88, Theorem 2.26].

LEMMA 1 (Gerstenhaber [6]). *Suppose K has at least n elements and \mathfrak{N} is a space of nilpotent matrices. Then $\dim \mathfrak{N} \leq n(n-1)/2$. If $\dim \mathfrak{N} = n(n-1)/2$, then there exists a non-singular matrix S such that $\mathfrak{N} = S^{-1}\mathfrak{U}S$, where \mathfrak{U} is the algebra of strictly upper triangular matrices. Moreover, any matrix of nilindex n is contained in exactly one maximal nilpotent algebra.*

Tangent Spaces. Let $K[X] = K[X_{11}, \dots, X_{nn}]$ be the ring of polynomials in n^2 variables with coefficients in K . For $r = 1, 2, \dots, n$, let $E_r(X) \in K[X]$ be the r th elementary symmetric function of the matrix $X = (X_{ij})$, i.e. $E_r(X)$ is the sum of all principal $r \times r$ subdeterminants of X . We let J be the ideal in $K[X]$ generated by $E_1(X), \dots, E_n(X)$ and $\text{rad } J = \{F \in K[X]: F^k \in J \text{ for some positive integer } k\}$. Clearly we have $\mathfrak{N} = \{A \in M_n: F(A) = 0 \text{ for all } F \in J\}$. If $A \in \mathfrak{N}$ then

$$\tan(J, A) = \left\{ B \in M_n: \left. \frac{dF}{dt}(A + tB) \right|_{t=0} = 0 \text{ for all } F \in J \right\}$$

and

$$\tan(\text{rad } J, A) = \left\{ B \in M_n: \left. \frac{dF}{dt}(A + tB) \right|_{t=0} = 0 \text{ for all } F \in \text{rad } J \right\}.$$

Both of these are vector spaces and the second is the usual tangent space at the point A of the algebraic set \mathfrak{N} . Further, the second is a subspace of the first.

If A and B belong to \mathfrak{N} and are similar then their tangent spaces defined above are related by the appropriate similarity. Further note that $C \in \tan(J, A)$ if and only if $(d/dt)E_r(A + tC)|_{t=0} = 0$ for all $r = 1, 2, \dots, n$. If $A \in \mathfrak{N}$ is of nilindex n , then, by taking A into upper Jordan canonical form, one sees that the equations for $X \in \tan(J, A)$ are, up to a similarity,

$$0 = \sum_{i=0}^{n-j} X_{j+i, i+1}, \quad j = 1, 2, \dots, n.$$

Therefore $\dim \tan(J, A) = n^2 - n$. Since J is generated by n polynomials, if N is of nilindex n we have [7, p. 28, 37]

$$n^2 - n \leq \dim \mathcal{N} \leq \dim \tan(\text{rad } J, N) \leq \dim \tan(J, N) = n^2 - n.$$

So if N is of nilindex n then $\tan(\text{rad } J, N) = \tan(J, N)$.

LEMMA 2. *If $A, B \in \mathcal{N}$ are both of nilindex n then $AB = BA$ if and only if $\tan(\text{rad } J, A) = \tan(\text{rad } J, B)$.*

Proof. A is of nilindex n so its minimal and characteristic polynomials are equal. Therefore, if $AB = BA$, then B is a polynomial in A . By the above remarks, we may assume that

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

so

$$B = \begin{pmatrix} 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & 0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $a_i \in K$. Since B is of nilindex n , $a_1 \neq 0$. A direct computation shows that

$$\left. \frac{d}{dt} E_n(B + tX) \right|_{t=0} = a_1^{n-1} X_{n1}.$$

Hence the equation for B arising from E_n is $X_{n1} = 0$, which is the same as for A . One has that

$$\left. \frac{d}{dt} E_r(B + tX) \right|_{t=0} = a_1^{r-1} \sum_{i=0}^{n-r} X_{r+i+1} + \sum_{j=1}^{r-1} A_j \sum_{i=0}^{n-j} X_{j+i+1}$$

for suitable constants A_j depending on a_1, \dots, a_{n-1} . By induction, the equation for B arising from E_r is

$$a_1^{r-1} \sum_{i=0}^{n-r} X_{r+ii+1} = 0,$$

and since $a_1 \neq 0$ this is the same as for A . Since $\tan(\text{rad } J, A) = \tan(\text{rad } J, A)$ the results follows.

On the other hand, suppose $\tan(\text{rad } J, A) = \tan(\text{rad } J, B)$. We may assume A is as above. Let E_{ij} be the matrix with 1 in the (i, j) position and zeros elsewhere. Then

$$E_{ji} \in \tan(\text{rad } J, A), \quad i > j,$$

and

$$E_{ji} - E_{j+1i+1} \in \tan(\text{rad } J, A), \quad i \leq j.$$

Writing $B = (b_{ij})$, we have

$$\left. \frac{d}{dt} E_2(B + tE_{ji}) \right|_{t=0} = b_{ij}, \quad \text{if } i > j,$$

$$\left. \frac{d}{dt} E_2(B + t(E_{ji} - E_{j+1i+1})) \right|_{t=0} = \pm (b_{ij} - b_{i+1j+1}) \quad \text{if } i \leq j.$$

Therefore $b_{ij} = 0$ if $i > j$ and $b_{ij} = b_{i+1,j+1}$ if $i \leq j$, and B is a polynomial in A .

LEMMA 3. *If $L: \mathfrak{sl}_n \rightarrow \mathfrak{sl}_n$ is a nonsingular linear transformation with the property that $L(\mathfrak{U}) = \mathfrak{U}$, and $A \in \mathfrak{U}$, then $L(\tan(\text{rad } J, A)) = \tan(\text{rad } J, L(A))$.*

Proof. The map $\tilde{L}: K[X] \rightarrow K[X]$ defined by $\tilde{L}(f)(A) = f(L(A))$ is a K -algebra homomorphism. Since L is nonsingular and $L(\mathfrak{U}) = \mathfrak{U}$ and $\text{rad } J = \{f \in K[X]: f(N) = 0, \text{ for all } N \in \mathfrak{U}\}$, we have $\tilde{L}(\text{rad } J) = \text{rad } J$. Thus

$$\begin{aligned} \tan(\text{rad } J, L(A)) &= \left\{ B \in M_n: \left. \frac{df}{dt}(L(A) + tB) \right|_{t=0} \text{ for all } f \in \text{rad } J \right\} \\ &= \left\{ L(C) \in M_n: \left. \frac{df}{dt}(L(A) + tL(C)) \right|_{t=0} \text{ for all } f \in \text{rad } J \right\} \\ &= \left\{ L(C) \in M_n: \left. \frac{d\tilde{L}(f)}{dt}(A + tC) \right|_{t=0} = 0 \text{ for all } f \in \text{rad } J \right\} \\ &= L(\tan(\text{rad } J, A)). \end{aligned}$$

Proof of theorem. First we observe that $L(\mathfrak{N}) = \mathfrak{N}$. This follows from Lemma 1 of Dixon [5] and the fact that L is nonsingular.

We now show that L preserves nilindex n . If $A \in \mathfrak{N}$ and $\text{rank } A \leq n - 2$, then A kills two linearly independent vectors v, w . Let $\mathfrak{N}_1, \mathfrak{N}_2$ be maximal nilpotent algebras containing A and killing v, w respectively. Every maximal nilpotent algebra kills exactly one line, so $\mathfrak{N}_1 \neq \mathfrak{N}_2$. By Lemma 1, L maps maximal nilpotent algebras to maximal nilpotent algebras and again by lemma 1, L preserves the matrices of nilindex n .

Now we show that if $A \in \mathfrak{N}$ has rank one, then so does $L(A)$. Let U be the unit auxiliary matrix $E_{12} + \dots + E_{n-1,n}$.

First note that the only members of \mathfrak{U} which commute with U and E_{12} are multiples of E_{1n} . Thus the centre of any maximal nilpotent algebra is one-dimensional and is generated by a rank one matrix.

Let $A \in \mathfrak{N}$ have rank one. Then for some nonsingular S , $S^{-1}AS = E_{1n}$. Let $\mathfrak{N} = S\mathfrak{U}S^{-1}$. Then A generates the centre of \mathfrak{N} . Let $V \in \mathfrak{N}$ have nilindex n . Then V and $A + V$ have nilindex n and commute. It follows from Lemmas 2 and 3 that $L(A + V)$ commutes with $L(V)$. Hence $L(A)$ commutes with $L(V)$. Since the nilindex n matrices in \mathfrak{N} generate \mathfrak{N} , $L(A)$ is in the centre of the maximal nilpotent algebra $L(\mathfrak{N})$. Hence $L(A)$ has rank one.

We next define two bijections on the lines (through the origin) of K^n and use the fundamental theorem of projective geometry. For each line $\langle v \rangle \in K^n$, define two $n - 1$ dimensional subspaces of \mathfrak{N} by

$$M(v) = \{X \in \mathfrak{N} \mid \text{Im } X = \langle v \rangle\},$$

$$M'(v) = \{X^t \mid X \in M(v)\}.$$

We will show that $L(M(v)) = M(w)$ or $M'(w)$ and $L(M'(v)) = M(w')$ or $M'(w')$ for some $w, w' \in K^n$. The bijections will be $\varphi(v) = w$ and $\theta(v) = w'$.

We note a few facts about $M(v)$. Any nonzero member of $M(v)$ has rank one. If $v, w \in K^n$, and are nonzero, then $M(v)$ and $M(w)$ are conjugate, and if $w = Av$, A nonsingular, then $M(w) = AM(v)A^{-1}$. In tensor notation, $M(v) = v \otimes v^\perp$ and $M'(v) = v^\perp \otimes v$. (Here, \perp means orthogonal complement with respect to the dot product.) It is easily verified that $M(u) \cap M(v) = M(u) = M(v)$ if u and v are linearly dependent and 0 otherwise, and that $M(u) \cap M'(v) = \langle u \otimes v \rangle$ if $u \cdot v = 0$ and is 0 otherwise. Finally, observe that any $n - 1$ dimensional subspace of \mathfrak{N} with all of its nonzero matrices having rank one must be an $M(v)$ or an $M'(v)$. It follows that for $v \in K^n$, there is a $w \in K^n$ such that $L(M(v)) = M(w)$ or $M'(w)$.

Suppose we have $v, w \in K^n$ with $L(M(v)) = M(v')$ and $L(M(w)) = M'(w')$. Since $n \geq 3$, pick u orthogonal to v and w . Then $M(v) \cap M'(u)$

and $M(w) \cap M'(u)$ are one dimensional. If $L(M'(u)) = M(u')$ then $M(u') \cap M(v') = L(M'(u) \cap M(v))$ has dimension 1; which is impossible, as $M(u') \cap M(v')$ has dimension 0 or $n - 1 \geq 2$. If $L(M'(u)) = M'(u')$, we reach a similar contradiction. A similar argument holds when we examine the images of $M'(v)$ and $M'(w)$. Thus, by replacing L with the map $X \rightarrow L(X)^t$ if necessary, we may assume that for any nonzero $v \in K^n$, $L(M(v)) = M(w)$ and $L(M'(v)) = M'(u)$ for appropriate $u, w \in K^n$.

Thus we define two maps φ, θ induced by L on the lines of K^n . We have $L(M(v)) = M(\varphi(v))$ and $L(M'(v)) = M'(\theta(v))$ for $v \in K^n$.

Since $L(\mathcal{U}) = \mathcal{U}$, L^{-1} also preserves nilpotence and hence φ and θ are bijections on the lines of K^n .

Now we show that φ and θ preserve coplanarity of lines in K^n and thus satisfy the hypothesis of the fundamental theorem of projective geometry. Let $\langle u_1 \rangle, \langle u_2 \rangle, \langle u_3 \rangle$ be three distinct coplanar lines in K^n . Then

$$\begin{aligned} 2n - 1 &= \dim(M(u_1) + M(u_2) + M(u_3)) \\ &= \dim L(M(u_1) + M(u_2) + M(u_3)) \\ &= \dim(M(\varphi(u_1)) + M(\varphi(u_2)) + M(\varphi(u_3))). \end{aligned}$$

If $\varphi(u_1), \varphi(u_2), \varphi(u_3)$ are linearly independent then

$$\dim(M(\varphi(u_1)) + M(\varphi(u_2)) + M(\varphi(u_3))) = 3n - 3$$

and this is impossible since $n \geq 3$. Thus $\varphi(u_1), \varphi(u_2), \varphi(u_3)$ are coplanar and φ satisfies the hypothesis of the fundamental theorem of projective geometry. So does θ . Thus there exist semilinear maps S and T on K^n such that $\varphi(u) = \langle Su \rangle$ and $\theta(u) = \langle Tu \rangle$, for all nonzero u in K^n .

There are linear maps P and Q on K^n and automorphisms σ and τ on K such that $Sv = P(\sigma v)$ and $Tv = Q(\tau v)$. (The automorphisms act componentwise.) Then

$$L(M(v)) = M(P\sigma v) = PM(\sigma v)P^{-1}$$

and

$$L(M'(v)) = M'(Q\tau v) = Q^t M'(\tau v) Q^t.$$

Suppose $u \cdot v = 0$. Then $\dim(M(u) \cap M'(v)) = 1$ and so

$$\dim(M(P\sigma u) \cap M'(Q\tau v)) = 1$$

and thus $(P\sigma u) \cdot (Q\tau v) = 0$, i.e.,

$$u \cdot \sigma^{-1}(P'Q\tau v) = 0.$$

Let R be the semilinear map defined by

$$Rv = \sigma^{-1}(P'Q\tau v).$$

Then $u \cdot v = 0$ implies $u \cdot Rv = 0$. Thus $R = dI$ is a scalar map, $\sigma = \tau$ and $P'Q = dI$.

Replace the map L with the map $X \rightarrow P^{-1}L(X)P$. Then $L(M(v)) = M(\sigma v)$ and $L(M'(v)) = M'(\sigma v)$, for all nonzero v in K^n . Thus if $u \cdot v = 0$ then $L(u \otimes v) = c(u \otimes v)\sigma(u \otimes v)$, where c is a scalar valued function. If $v \in \langle u_1, u_2 \rangle^\perp$, then by comparing $L((u_1 + u_2) \otimes v)$ with $L(u_1 \otimes v) + L(u_2 \otimes v)$ we get $c(u_1 \otimes v) = c(u_2 \otimes v)$. Similarly if $u \in \langle v_1, v_2 \rangle$, then $c(u \otimes v_1) = c(u \otimes v_2)$.

Now we show that c is a constant function. Suppose that $u_1 \cdot v_1 = 0$ and $u_2 \cdot v_2 = 0$. Pick $v_3 \in \langle u_1, u_2 \rangle^\perp$. Then $c(u_1 \otimes v_1) = c(u_1 \otimes v_3) = c(u_2 \otimes v_3) = c(u_2 \otimes v_2)$. Thus c is a constant function say $c(u \otimes v) = k$. Then $L(u \otimes v) = k\sigma(u \otimes v)$, for all u, v with $u \cdot v = 0$.

Since L is linear, σ is the identity automorphism. The rank one nilpotent matrices span sl_n and so the theorem is proved.

REMARK. When $n = 2$, the same result is obtained by a simple computation.

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