Linearisable Third Order Ordinary Differential Equations and Generalised Sundman Transformations

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Abstract:
We calculate in detail the conditions which allow the most general third order ordinary differential equation to be linearised in $X'''(T) = 0$ under the transformation $X(T) = F(x,t),\ dT = G(x,t)dt$. Further generalisations are considered.

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1 Introduction

In the modelling of physical and other phenomena differential equations, be they ordinary or partial, scalar or a system, are a common outcome of the modelling process. The basic problem becomes the solution of these differential equations. One of the fundamental methods of solution relies upon the transformation of a given equation (or equations; hereinafter the singular will be taken to include the plural where appropriate) to another equation of standard form. The transformation may be to an equation of like order or of greater or lesser order. In the early days of the solution of differential equations at the beginning of the eighteenth century the methods for determining suitable transformations were developed very much on an *ad hoc* basis. With the development of symmetry methods, initiated by Lie towards the end of the nineteenth century and revived as well as developed ever since, the *ad hoc* methods were replaced by systematic approaches. About the same time classes of equations were established as equivalent to certain standard equations. In particular the possibility that a given equation could be linearised, *i.e.* transformed to a linear equation, was a most attractive proposition due to the special properties of linear differential equations. Already in his thesis of 1896 Tresse [22] showed that the most general second order ordinary differential equation which could be transformed to the simplest second order equation, *videlicet*

$$X''(T) = 0, \tag{1.1}$$

by means of the point transformation

$$X = F(x,t) \quad T = G(x,t) \tag{1.2}$$

has the form

$$\ddot{x} + \Lambda_3(x,t)\dot{x}^3 + \Lambda_2(x,t)\dot{x}^2 + \Lambda_1(x,t)\dot{x} + \Lambda_0(x,t) = 0, \tag{1.3}$$

where the overdot denotes differentiation with respect to the independent variable t whereas the primes refer to T derivatives. This notation is used throughout the paper. In terms of the transformation functions F and G the functions Λ_i are given by

$$\begin{aligned}
\Lambda_3(x,t) &= \left[F_{xx}G_x - F_xG_{xx}\right]/J \\
\Lambda_2(x,t) &= \left[F_{xx}G_t + 2F_{xt}G_x - 2F_xG_{xt} - F_tG_{xx}\right]/J
\end{aligned}$$

$$\begin{aligned}\Lambda_1(x,t) &= [F_{tt}G_x + 2F_{xt}G_t - 2F_tG_{xt} - F_xG_{tt}]/J \\ \Lambda_0(x,t) &= [F_{tt}G_t - F_tG_{tt}]/J, \end{aligned} \tag{1.4}$$

where $J(x,t) = F_xG_t - F_tG_x \neq 0$ is the Jacobian of the point transformation (1.2). As usual subscripts denote partial derivatives. Through the elimination of the transformation functions F and G it is found that the coefficient functions, $\Lambda_i,\ i = 0,\ldots,3$, must satisfy the conditions

$$\begin{aligned}&\Lambda_{1xx} - 2\Lambda_{2xt} + 3\Lambda_{3tt} + 6\Lambda_3\Lambda_{0x} + 3\Lambda_0\Lambda_{3x} - 3\Lambda_3\Lambda_{1t} - 3\Lambda_1\Lambda_{3t} \\ &\quad -\Lambda_2\Lambda_{1x} + 2\Lambda_2\Lambda_{2t} = 0 \end{aligned} \tag{1.5}$$

$$\begin{aligned}&\Lambda_{2tt} - 2\Lambda_{1xt} + 3\Lambda_{0xx} - 6\Lambda_0\Lambda_{3t} - 3\Lambda_3\Lambda_{0t} + 3\Lambda_0\Lambda_{2x} + 3\Lambda_2\Lambda_{0x} \\ &\quad +\Lambda_1\Lambda_{2t} - 2\Lambda_1\Lambda_{1x} = 0. \end{aligned} \tag{1.6}$$

The problem has attracted some interest since the work of Tresse. See [3, 20, 4, 5]. Note that the condition corresponding to (1.5) given in [4] omits the coefficient 2 in the final term.

In a practical context one identifies the coefficient functions from the given nonlinear equation and substitutes them in the compatibility conditions (1.5) and (1.6) to determine whether the equation is of the correct form. If this be the case, the transformation is determined from the solution of the system (1.4) and the solution to the nonlinear equation follows immediately. Since all second order linear ordinary differential equations are equivalent to (1.1) under a point transformation (*i.e.* of the type (1.2)), these formulæ answer the question of the class of second order equations linearisable under a point transformation.

The attraction of point transformations is that they preserve Lie point symmetries. Since a second order linear equation possesses eight Lie point symmetries, which is far in excess of the two required to reduce the equation to quadratures, there is no necessity to confine one's interest to point transformations if the matter of interest is the solution of the equation and not its point symmetries. One can then look towards some type of transformation which is more general than a point transformation. The advantage of point transformations is that they are fairly easy to work with. In looking towards a generalisation one wants to keep this property, if possible.

A convenient form of generalisation is the nonpoint transformation introduced by Duarte *et al* [4] which has the form

$$X(T) = F(t,x), \qquad dT = G(t,x)dt. \tag{1.7}$$

Without a knowledge of the functional form of $x(t)$ the transformation (1.7) is a particular type of nonlocal transformation. This transformation is a generalisation of a transformation proposed by Sundman [21]. Since the expression 'non-point transformation' has a meaning more general than that of (1.7), it may be better to refer to this type of transformation as a **generalised Sundman transformation**.

In their paper [4] Duarte, Moreira and Santos derived the most general conditions for which a second order ordinary differential equation may be transformed to the free particle equation $X'' = 0$ under the generalised Sundman transformation (1.7). Euler [5] studied the general anharmonic oscillator

$$\ddot{x} + f_1(t)\dot{x} + f_2(t)x + f_3(t)x^n = 0 \tag{1.8}$$

and derived conditions on the coefficient functions f_j for which (1.8) may be linearised under the generalised Sundman transformation (1.7). It follows [5] that (1.8) may be reduced to the linear equation

$$X'' + k = 0, \qquad k \in \Re\backslash\{0\} \tag{1.9}$$

by the transformation

$$X(T) = h(t)x^{n+1}, \qquad dT = \left(\frac{n+1}{k} f_3(t)h(t)\right)\ x^n\, dt, \tag{1.10}$$

where $n \in \mathcal{Q}\backslash\{-3,-2,-1,0,1\}$ and

$$h(t) = f_3^{(n+1)/(n+3)} \exp\left\{2\left(\frac{n+1}{n+3}\right)\int^t f_1(\rho)d\rho\right\}, \tag{1.11}$$

if and only if $f_1,\ f_2$ and f_3 satisfy the following condition:

$$\begin{aligned} f_2 = {} & \frac{1}{n+3}\frac{\ddot{f}_3}{f_3} - \frac{n+4}{(n+3)^2}\left(\frac{\dot{f}_3}{f_3}\right)^2 + \frac{n-1}{(n+3)^2}\left(\frac{\dot{f}_3}{f_3}\right) f_1 \\ & +2\frac{1}{n+3}\dot{f}_1 + 2\frac{n+1}{(n+3)^2} f_1^2. \end{aligned} \tag{1.12}$$

This leads to the invariant (time-dependent first integral) of (1.8) through the first integral of (1.9), which is

$$I(X, X') = X + \frac{1}{2k}\left(X'\right)^2.$$

Within the two classes of transformation given by (1.2) and (1.7) there are subclasses which have some particular interest if one is concerned about the type of transformation and preservation of types of symmetry. For example in the case that $G_x = 0$ both classes of transformations reduce to a transformation of Kummer-Liouville type [10, 16] which has the property of preserving symmetries of Cartan form, *i.e.* fibre-preserving transformations [9]. These transformations are of importance for Hamiltonian Mechanics and Quantum Mechanics.

In the studies of second order equations use was made of the fact that every linear second order equation is equivalent under a point transformation to (1.2). We recall [15] [p 405] that the number of Lie point symmetries of a second order equation can be 0, 1, 2, 3 or 8 and that all linear second order equations have eight Lie point symmetries with the Lie algebra $sl(2, R)$. Naturally any second order equation linearisable under a point transformation also has eight Lie point symmetries. In the case of third and higher order equations such economy of property does not persist. An n-th order linear differential equation can have $n + 1$, $n + 2$ or $n + 4$ Lie point symmetries [14]. Consequently, even under point transformations, there are three equivalence classes of linearisable n-th order equations. In addition there are the classes of equations corresponding to other numbers of Lie point symmetries or different algebras.

The manipulations required for the calculations of the classes of equations which are equivalent under either of these classes of transformation are nontrivial. Although, in principle, the same ideas – indeed more complicated types of transformation – can be applied to differential equations of all orders, the burden of calculation has in the past made the widespread use of these methods impracticable. These days with the ready availability of symbolic manipulation codes on personal computers of reasonable computational power the drudgery of these calculations has been removed or, at least, transferred to a third nonvocal party. The time has come to consider the possibilities of more complicated transformations of more complicated equations. In this paper we report the results for generalised Sundman transformations for third order ordinary differential equations. In our calculations we make use of the packages Crack written in the computer algebra system Reduce and Rif written in Maple. They are described in [23] and [19].

We concentrate on the basic third order equation

$$X'''(T) = 0, \tag{1.13}$$

but there is no reason why the ideas presented here cannot be extended to the other types of third order equation which constitute the set of equivalence classes of third order equations [17]. We give only some examples of more general linear third order equations (see Section 3). Note that (1.13) admits the two first integrals

$$I_1 = X'', \qquad I_2 = X''X - \frac{1}{2}(X')^2. \tag{1.14}$$

The generalised Sundman transformation (1.7) can then be used to express these first integrals to obtain the invariants of the nonlinear equation derived by the transformation.

In the case of scalar third order equations there can be 0, 1, 2, 3, 4, 5, 6 and 7 Lie point symmetries. When the maximum number of Lie point symmetries is found in an equation, this equation has in addition three irreducible contact symmetries. The ten symmetries possess the Lie algebra $sp(5)$ [1]. The equation (1.13) is a representative of this last class.

We recall that the most general linear differential equation of third order is

$$\dddot{x} + f_4(t)\ddot{x} + f_3(t)\dot{x} + f_2(t)x + f_1(t) = 0. \tag{1.15}$$

The condition for (1.15) to be reduced to

$$X'''(T) = 0 \tag{1.16}$$

by an invertible point transformation was found by Laguerre in 1898 [11] to be

$$\frac{1}{6}\frac{d^2 f_4}{dt^2} + \frac{1}{3}f_4\frac{df_4}{dt} - \frac{1}{2}\frac{df_3}{dt} + \frac{2}{27}f_4^3 - \frac{1}{3}f_4 f_3 + f_2 = 0. \tag{1.17}$$

We note in passing that it is possible to transform (1.15) to (1.16) without any conditions on the coefficient functions if contact transformations are used either directly or as an equivalent point transformation of the corresponding linear first-order system [18].

The article is organised as follows: In Section 2 we present the classes of equation equivalent to (1.13) under the generalised Sundman transformation

(1.7). In Section 3 we consider a special Sundman transformation and show how linearisable ordinary differential equations of second, third and fourth order and their invariants can be constructed. Section 4 is devoted to an extension of the generalised Sundman transformation, which is related to the so-called generalised hodograph transformation introduced recently for evolution equations [6]. We note that there has been work done recently on the problem of transitive fibre-preserving point symmetries of third order ordinary differential equations [7], as well as contact transformations and local reducibility of an ordinary differential equation to the form (1.13) [8]. The results presented here are complementary to the results presented in those two papers.

2 Generalised Sundman transformations for $X''' = 0$

We turn our attention to the equivalence class of third order nonlinear ordinary differential equations obtainable from (1.16), *videlicet*

$$X'''(T) = 0$$

by means of the generalised Sundman transformation (1.7), *videlicet*

$$X(T) = F(t,x), \qquad dT = G(t,x)dt,$$

where $F,\ G \in \mathcal{C}^3$ and are to be determined for the transformation of (1.16). Provided $F_xG^2 \neq 0$, the form of the representative equation of the equivalence class is

$$\dddot{x} + \Lambda_5(x,t)\ddot{x} + \Lambda_4(x,t)\dot{x}\ddot{x} + \Lambda_3(x,t)\dot{x}^3 + \Lambda_2(x,t)\dot{x}^2 + \Lambda_1(x,t)\dot{x} + \Lambda_0(x,t) = 0, \tag{2.1}$$

where the functions Λ_i are related to the transformation functions F and G by means of

$$\begin{aligned}
\Lambda_5(x,t) &= 3\frac{F_{xt}}{F_x} - 3\frac{G_t}{G} - \frac{F_t}{F_x}\frac{G_x}{G} \\
\Lambda_4(x,t) &= -4\frac{G_x}{G} + 3\frac{F_{xx}}{F_x}
\end{aligned}$$

$$
\begin{aligned}
\Lambda_3(x,t) &= \frac{F_{xxx}}{F_x} + 3\left(\frac{G_x}{G}\right)^2 - 3\frac{F_{xx}}{F_x}\frac{G_x}{G}\frac{G_{xx}}{G} \\
\Lambda_2(x,t) &= 3\frac{F_{xxt}}{F_x} + 3\frac{F_t}{F_x}\left(\frac{G_x}{G}\right)^2 - 6\frac{F_{xt}}{F_x}\frac{G_x}{G} - 3\frac{F_{xx}}{F_x}\frac{G_t}{G} \\
&\quad +6\frac{G_x}{G}\frac{G_t}{G} - \frac{F_t}{F_x}\frac{G_{xx}}{G} - 2\frac{G_{xt}}{G} \qquad (2.2) \\
\Lambda_1(x,t) &= 3\left(\frac{G_t}{G}\right)^2 + 3\frac{F_{xtt}}{F_x} - 3\frac{F_{tt}}{F_x}\frac{G_x}{G} - 6\frac{F_{xt}}{F_x}\frac{G_t}{G} + 6\frac{F_t}{F_x}\frac{G_x}{G}\frac{G_t}{G} \\
&\quad -2\frac{F_t}{F_x}\frac{G_{xt}}{G} - \frac{G_{tt}}{G} \\
\Lambda_0(x,t) &= -\frac{F_t}{F_x}\frac{G_{tt}}{G} - 3\frac{F_{tt}}{F_x}\frac{G_t}{G} + \frac{F_{ttt}}{F_x} + 3\frac{F_t}{F_x}\left(\frac{G_t}{G}\right)^2.
\end{aligned}
$$

In order to have the inverse transformation we must establish the compatibility conditions for (2.2). This is the major thrust of our code.

The above set of equations has the form

$$
\begin{aligned}
0 &= E_1 := 3F_{tx}G - 3F_xG_t - F_tG_x - F_xG\Lambda_5 && (2.3) \\
0 &= E_2 := 3F_{xx}G - 4F_xG_x - F_x\Lambda_4 && (2.4) \\
0 &= E_3 := F_{xxx}G^2 - 3F_{xx}G_xG - F_xG_{xx}G + 3F_xG_x^2 - F_xG^2\Lambda_3 && (2.5) \\
0 &= E_4 := 3F_{txx}G^2 - 6F_{tx}G_xG - F_tG_{xx}G + 3F_tG_x^2 - 3F_{xx}G_tG \\
&\qquad -2F_xG_{tx}G + 6F_xG_tG_x - F_xG^2\Lambda_2 && (2.6) \\
0 &= E_5 := -3F_{ttx}G^2 + 6F_{tx}G_tG + 3F_{tt}G_xG + 2F_tG_{tx}G - 6F_tG_tG_x \\
&\qquad +F_xG_{tt}G - 3F_xG_t^2 + F_xG^2\Lambda_1 && (2.7) \\
0 &= E_6 := F_{ttt}G^2 - 3F_{tt}G_tG - F_tG_{tt}G + 3F_tG_t^2 - F_xG^2\Lambda_0 && (2.8)
\end{aligned}
$$

In the following simplifications of this system the equation to be replaced in each step is multiplied with a nonvanishing factor so that the new system after each replacement is still necessary and sufficient.

$$
\begin{aligned}
E_3 &\rightarrow E_7 := E_{2x}G - G_xE_2 - 3E_3 \\
E_7 &\rightarrow E_8 := \left((5G_x - G\Lambda_4)E_2 - 3E_7\right)/F_x \\
E_4 &\rightarrow E_9 := E_{2t}G - G_tE_2 - E_4 \\
E_9 &\rightarrow E_{10} := -G_tE_2 + E_9 \\
E_{10} &\rightarrow E_{11} := (G\Lambda_4 - 2G_x)E_1 + 3E_{10}
\end{aligned}
$$

$$
\begin{aligned}
E_5 &\rightarrow E_{12} := GE_{1t} - G_t E_1 + E_5 \\
E_{12} &\rightarrow E_{13} := E_1(G\Lambda_5 - 3G_t) + 3E_{12} \\
E_{11} &\rightarrow E_{14} := F_t E_8 - E_{11}
\end{aligned}
$$

The new system consists of equations $E_1, E_2, E_6, E_8, E_{13}, E_{14}$ and is transformed to two new functions h and p which are related to F and G through the relations

$$
\begin{aligned}
F(x,t) &= p(x,t)h^{-1}(x,t) && (2.9) \\
G(x,t) &= h^{-3/2}(x,t). && (2.10)
\end{aligned}
$$

After making all equations free of a denominator and performing three simplifications

$$
\begin{aligned}
E_{13} &\rightarrow E_{15} := (E_{13} - 3h_t E_1)/h \\
E_{14} &\rightarrow E_{16} := (E_{14} - 3h_x E_1)/h \\
E_2 &\rightarrow E_{17} := (pE_8 - 3E_2)/h
\end{aligned}
$$

we introduce new functions $\Lambda_6, \Lambda_7, \Lambda_8$ through

$$
\begin{aligned}
\Lambda_6(x,t) &= -6\Lambda_{5t} + 6\Lambda_1 - 2\Lambda_5^2 && (2.11) \\
\Lambda_7(x,t) &= 6\Lambda_{4t} - 6\Lambda_2 + 2\Lambda_4\Lambda_5 && (2.12) \\
\Lambda_8(x,t) &= -6\Lambda_{4x} + 18\Lambda_3 - 2\Lambda_4^2. && (2.13)
\end{aligned}
$$

To compactify the display of the resulting system we use the notation

$$[A]_{p\leftrightarrow h} := A - A|_{p\leftrightarrow h}$$

where A is a differential expression in the functions p and h and $A|_{p\leftrightarrow h}$ is the expression after swapping p and h:

$$
\begin{aligned}
0 &= E_8 = 9h_{xx} - 3h_x\Lambda_4 + h\Lambda_8 && (2.14) \\
0 &= E_{17} = 9p_{xx} - 3p_x\Lambda_4 + p\Lambda_8 && (2.15) \\
0 &= E_6 = [2h_{ttt}p + 3h_{tt}p_t - 2h_x p\Lambda_0]_{p\leftrightarrow h} && (2.16) \\
0 &= E_1 = [6h_{tx}p - 3h_t p_x - 2h_x p\Lambda_5]_{p\leftrightarrow h} && (2.17) \\
0 &= E_{16} = [18h_{tx}p_x - h_t p\Lambda_8 + h_x p\Lambda_7]_{p\leftrightarrow h} && (2.18) \\
0 &= E_{15} = [9h_{tx}p_t - 18h_{tt}p_x + 3h_t p_x\Lambda_5 + h_x p\Lambda_6]_{p\leftrightarrow h}. && (2.19)
\end{aligned}
$$

Remarkably the system is symmetric under the exchange $p \leftrightarrow h$.

Remark: In view of the symmetry $p \leftrightarrow h$ and the relations (2.9) and (2.10) we obtain the transformation coefficients

$$\bar{F}(x,t) = F^{-1}(x,t)$$
$$\bar{G}(x,t) = F^{-3/2}(x,t)G(x,t)$$

for the generalised Sundman transformation (1.7). This does not lead to new linearisable third order ordinary differential equations, so we do not list here any conditions for this transformation.

We obtain a first integrability condition by differentiating equation (2.17) with respect to x and substituting h_{txx}, h_{xx} using equation (2.14), substituting p_{txx}, p_{xx} using equation (2.15), substituting $h_{tx}p_x$ using equation (2.18) and substituting $h_{tx}p$ using equations (2.17). The result is

$$0 = E_{18} := [(h_x(12\Lambda_{4t} - 12\Lambda_{5x} - \Lambda_7) - h_t\Lambda_8)p]_{p\leftrightarrow h}. \tag{2.20}$$

Before we give the most general solution we consider two special cases in the next two subsections.

2.1 The case $G_x = 0$,

i.e. the transformation

$$X(T) = F(x,t), \qquad dT = G(t)dt,$$

where $F(x,t) = p(x,t)h^{-1}(t)$ and $G(t) = h^{-3/2}(t)$. By investigating the case $G_x = 0$ (which is equivalent to $h_x = 0$) we cover the case $p_x = 0$ as well because of the $p \leftrightarrow h$ symmetry.

For $h_x = 0$ we have $h_t \neq 0$ and get $E_8 = 0 = \Lambda_8$ and further $E_{16} = 0 = \Lambda_7$. After substitution of p_{tx} from equation (2.17) into equation (2.19) the resulting system is

$$0 = -E_{18}/(12p_x) = \Lambda_{4t} - \Lambda_{5x} \tag{2.21}$$
$$0 = E_{19} := E_{17}/3 = 3p_{xx} - p_x\Lambda_4 \tag{2.22}$$
$$0 = E_1 = -6hp_{tx} - 3h_tp_x + 2hp_x\Lambda_5 \tag{2.23}$$
$$0 = E_{20} := (3h_tE_1 - 2hE_{15})/p_x = 36h_{tt}h - 9h_t^2 + 2h^2\Lambda_6 \tag{2.24}$$
$$0 = E_6 = 2h_{ttt}p + 3h_{tt}p_t - 3h_tp_{tt} - 2hp_{ttt} + 2hp_x\Lambda_0. \tag{2.25}$$

For equation (2.24) to have a solution for $h(t)$ the condition on Λ_6 is

$$\Lambda_{6x} = 0. \tag{2.26}$$

We need to derive one more integrability condition before being able to formulate a procedure to solve the above system. We reduce the condition

$$E_{6x} = 0 \tag{2.27}$$

by substituting $p_{tttx}, p_{ttx}, p_{tx}$ computed from equation (2.23), substituting p_{xx} from equation (2.22) and substituting h_{tt} from equation (2.24) to get

$$\begin{aligned} 0 = {} & (108h^2 E_{6x} - 126hh_{tt}E_1 + 54h_t^2 E_1 - 36hh_t E_{1t} + 18h_t p_x E_{20} \\ & -6hh_t E_1 \Lambda_5 + 36h^2 E_{1tt} + 12h^2 E_{1t}\Lambda_5 - 9hp_x E_{20t} + 24h^2 E_1 \Lambda_{5t} \\ & -6hp_x E_{20}\Lambda_5 - 72h^3 E_{19}\Lambda_0 + 4h^2 E_1 \Lambda_5^2)/(2h^3 p_x) \qquad (2.28) \\ = {} & 108\Lambda_{0x} - 36\Lambda_{5tt} - 36\Lambda_{5t}\Lambda_5 - 9\Lambda_{6t} + 36\Lambda_0\Lambda_4 \\ & -4\Lambda_5^3 - 6\Lambda_5\Lambda_6. \qquad (2.29) \end{aligned}$$

We summarize: The procedure for a given set of expressions $\Lambda_0, \Lambda_1, \ldots, \Lambda_5$ is as follows.

1. Compute $\Lambda_6, \Lambda_7, \Lambda_8$ from equations (2.11), (2.12), (2.13).
2. The following set of conditions for Λ_i is necessary and, as becomes clear below, also sufficient for a solution with $h_x = 0 = G_x$ to exist:

$$\Lambda_7 = 0, \quad \Lambda_8 = 0, \quad \Lambda_{4t} - \Lambda_{5x} = 0, \quad \Lambda_{6x} = 0,$$

$$108\Lambda_{0x} - 36\Lambda_{5tt} - 36\Lambda_{5t}\Lambda_5 - 9\Lambda_{6t} + 36\Lambda_0\Lambda_4 - 4\Lambda_5^3 - 6\Lambda_5\Lambda_6 = 0.$$

3. The function $h(t)$ is to be computed from the condition (2.24) which is an ordinary differential equation due to $\Lambda_{6x} = 0$ and which can be written as a linear equation for $h^{3/4}$:

$$24(h^{3/4})_{tt} + h^{3/4}\Lambda_6 = 0.$$

4. Compute a function $q(x,t)(=\log(p_x))$ from the two equations (2.22) and (2.23) as a line integral:

$$q_x = \frac{1}{3}\Lambda_4, \quad q_t = \frac{1}{3}\Lambda_5 - \frac{1}{2}\frac{h_t}{h}.$$

   The existence of q is guaranteed through condition (2.21).

5. Compute a function $r(x,t)$ from

$$r(x,t) = \int \exp\left[q(x,t)\right]\, dx.$$

6. The function $p(x,t)$ is computed from

$$p(x,t) = r(x,t) + s(t),$$

   where the function $s(t)$ is computed from equation (2.25) which after the substitution $p = r + s$ takes the form

$$2h_{ttt}(r+s) + 3h_{tt}(r+s)_t - 3h_t(r+s)_{tt} - 2h(r+s)_{ttt} + 2hr_x\Lambda_0 = 0.$$

   This is a linear third order equation for $s(t)$. It is an ordinary differential equation with a solution for $s(t)$ because the derivative of the left hand side of this equation with respect to x vanishes due to equations (2.28), (2.22), (2.23), (2.24) and (2.29).

7. With the last step we satisfied all conditions (2.21) - (2.25) and computed p and h which gives G and F through equations (2.10) and (2.9):

$$G(t) = h^{-3/2}, \quad F(x,t) = ph^{-1}.$$

2.2 The case $F_t = 0$,

i.e. the transformation

$$X(T) = F(x), \qquad dT = G(x,t)dt.$$

For all investigations below we can assume $h_x \neq 0, p_x \neq 0$. We consider $f_t = (p/h)_t = 0$ so that

$$h_t = h\frac{p_t}{p}. \tag{2.30}$$

A straightforward substitution of h_t from equation (2.30) into equation (2.17) gives

$$9p_t - 2p\Lambda_5 = 0. \tag{2.31}$$

Application of this condition on equation (2.30) yields

$$9h_t - 2h\Lambda_5 = 0 \tag{2.32}$$

and both together simplify equations (2.14) - (2.19) to

$$\Lambda_0 = 0 \tag{2.33}$$
$$4\Lambda_{5x} - \Lambda_7 = 0 \tag{2.34}$$
$$12\Lambda_{5t} + 2{\Lambda_5}^2 + 3\Lambda_6 = 0 \tag{2.35}$$
$$9p_{xx} - 3p_x\Lambda_4 + p\Lambda_8 = 0 \tag{2.36}$$
$$9h_{xx} - 3h_x\Lambda_4 + h\Lambda_8 = 0. \tag{2.37}$$

Substitution of h_t, p_t into the integrability condition (2.20) gives

$$12\Lambda_{4t} - 12\Lambda_{5x} - \Lambda_7 = 0$$

which when simplified with equation (2.34) yields

$$3\Lambda_{4t} - 4\Lambda_{5x} = 0. \tag{2.38}$$

The remaining integrability condition between equations (2.31) and (2.36) (and equally (2.32) and (2.37)) is computed by the differentiation of (2.36) with respect to t and the replacement of p_{txx}, p_{tx}, p_t with (2.31), p_{xx} with (2.36) and Λ_{4t} with (2.34). The result is

$$6\Lambda_{5xx} - 2\Lambda_{5x}\Lambda_4 + 3\Lambda_{8t} = 0. \tag{2.39}$$

We summarize: The procedure for a given set of expressions $\Lambda_0, \Lambda_1, \ldots, \Lambda_5$ is as follows.

1. Compute $\Lambda_6, \Lambda_7, \Lambda_8$ from equations (2.11), (2.12), (2.13).

2. The following set of conditions for Λ_i $(i = 0, \ldots, 5)$ is necessary and sufficient for a solution with $F_t = 0$, $h_x \neq 0$ to exist:

$$\Lambda_0 = 0, \quad 4\Lambda_{5x} - \Lambda_7 = 0, \quad 12\Lambda_{5t} + 2\Lambda_5^2 + 3\Lambda_6 = 0,$$
$$3\Lambda_{4t} - 4\Lambda_{5x} = 0, \quad 6\Lambda_{5xx} - 2\Lambda_{5x}\Lambda_4 + 3\Lambda_{8t} = 0.$$

3. A function $u(x,t)$ is to be computed as

$$u(x,t) = \exp\left[\frac{2}{9}\int \Lambda_5(x,t)\,dt\right].$$

4. The functions $p(t,x)$ and $h(t,x)$ are computed from

$$p(x,t) = v(x)u(x,t), \quad h(x,t) = w(x)u(x,t),$$

   where $v(x)$ and $w(x)$ are to be computed from the two linear second order conditions

$$9(vu)_{xx} - 3(vu)_x\Lambda_4 + (vu)\Lambda_8 = 0, \quad 9(wu)_{xx} - 3(wu)_x\Lambda_4 + (wu)\Lambda_8 = 0$$

   which when divided by u are purely x-dependent and therefore are ordinary differential equations for $v(x)$ and $w(x)$ as guaranteed by the integrability conditions above.

5. With the last step we satisfied all conditions (2.31) - (2.37) and computed p and h which give G and F through equations (2.10) and (2.9):

$$G(x,t) = h^{-3/2}, \quad F(x) = ph^{-1}.$$

2.3 The general conditions for $X''' = 0$

In order to invert the system (2.2) in general, that is to solve F and G from (2.2), we need to consider three different cases. The obvious conditions which apply in all cases are $p \neq 0,\ h \neq 0,\ F_x G \neq 0$. Recall further that $X''' = 0$ is transformed into (2.1), *videlicet*

$$\dddot{x} + \Lambda_5(x,t)\ddot{x} + \Lambda_4(x,t)\dot{x}\ddot{x} + \Lambda_3(x,t)\dot{x}^3 + \Lambda_2(x,t)\dot{x}^2 + \Lambda_1(x,t)\dot{x} + \Lambda_0(x,t) = 0,$$

and $\Lambda_6,\ \Lambda_7,\ \Lambda_8$ are defined in (2.11)-(2.13)), *videlicet*

$$\begin{aligned} \Lambda_6(x,t) &= -6\Lambda_{5t} + 6\Lambda_1 - 2\Lambda_5^2 \\ \Lambda_7(x,t) &= 6\Lambda_{4t} - 6\Lambda_2 + 2\Lambda_4\Lambda_5 \\ \Lambda_8(x,t) &= -6\Lambda_{4x} + 18\Lambda_3 - 2\Lambda_4^2. \end{aligned}$$

In providing the general conditions in this subsection we are not able to document each individual step like for the previous two subcases. Instead we give only the final result, *i.e.* we list the conditions on h and p and the conditions on the Λ_is which must be satisfied for a given third order ordinary differential equation in order to establish linearisation to $X'''(T) = 0$ by (1.7).

Below we use the notation

$$\{A(h) = 0\}_{h\leftrightarrow p} := \{A(h) = 0 \text{ and } A(h)|_{h\to p} = 0\},$$

where $A(h)$ denotes the differential expression in h. Thus $\{A\}_{h\leftrightarrow p}$ represents two differential expression; one in h and the same differential expression but with h replaced by p.

Case I. $\Lambda_8 \neq 0,\ -ph_x + p_x h \neq 0,\ 2\Lambda_8\Lambda_4 - 3\Lambda_{8x} \neq 0$:

The functions h and p for the transformation

$$X(T) = p(x,t)h^{-1}(x,t), \qquad dT = h^{-3/2}(x,t)dt$$

are to be solved from the following set of linear equations:

$$\left\{h_{xx} = \frac{1}{3}h_x\Lambda_4 - \frac{1}{9}h\Lambda_8\right\}_{h\leftrightarrow p}$$

$$\left\{h_t = \frac{1}{9\Lambda_8^2}\Big(108h_x\Lambda_{4t}\Lambda_8 - 108h_x\Lambda_{5x}\Lambda_8 - 9h_x\Lambda_7\Lambda_8 + 2h\Lambda_8^2\Lambda_5 - 2h\Lambda_8\Lambda_{7x} + 16h\Lambda_8\Lambda_{5x}\Lambda_4 - 4h\Lambda_8\Lambda_{8t} - 16h\Lambda_8\Lambda_{4t}\Lambda_4 - 2h\Lambda_7\Lambda_{8x} - 24h\Lambda_{5x}\Lambda_{8x} + 24h\Lambda_{4t}\Lambda_{8x} + 2h\Lambda_8\Lambda_7\Lambda_4\Big)\right\}_{h\leftrightarrow p}.$$

The following conditions on Λ_i $(i = 1, \ldots, 8)$ are to be satisfied:

$$\Lambda_{4tt} = \frac{1}{12\Lambda_8^2}\Big(-24\Lambda_{8x}\Lambda_{5x}^2 - 24\Lambda_{8x}\Lambda_{4t}^2 - 2\Lambda_{8x}\Lambda_7\Lambda_{5x} + 48\Lambda_{8x}\Lambda_{4t}\Lambda_{5x} + 2\Lambda_{8x}\Lambda_7\Lambda_{4t} + 16\Lambda_8\Lambda_{4t}^2\Lambda_4 - 2\Lambda_8\Lambda_{4t}\Lambda_7\Lambda_4 + 16\Lambda_8\Lambda_{5x}^2\Lambda_4 - 2\Lambda_8\Lambda_{5x}\Lambda_{7x} - 4\Lambda_8^2\Lambda_5\Lambda_{4t} - \Lambda_8^2\Lambda_{6x} + 4\Lambda_8^2\Lambda_5\Lambda_{5x} + 2\Lambda_8\Lambda_{4t}\Lambda_{7x} + 2\Lambda_8\Lambda_{5x}\Lambda_7\Lambda_4 - 16\Lambda_8\Lambda_{8t}\Lambda_{5x} - 32\Lambda_8\Lambda_{5x}\Lambda_{4t}\Lambda_4 - \Lambda_8\Lambda_{8t}\Lambda_7 + 12\Lambda_8^2\Lambda_{5xt} + 16\Lambda_8\Lambda_{8t}\Lambda_{4t}\Big)$$

$$\Lambda_{5tt} = \frac{1}{72\Lambda_8^3}\Big(-144\Lambda_{8x}\Lambda_{5x}\Lambda_7\Lambda_{4t} + 38\Lambda_8\Lambda_7^2\Lambda_4\Lambda_{5x} - 96\Lambda_8\Lambda_7\Lambda_4\Lambda_{5x}^2$$
$$+6\Lambda_8\Lambda_{7x}\Lambda_{4t}\Lambda_7 - 96\Lambda_8\Lambda_{4t}^2\Lambda_4\Lambda_7 - 288\Lambda_8\Lambda_{7x}\Lambda_{4t}\Lambda_{5x} - 6\Lambda_8\Lambda_{7x}\Lambda_7\Lambda_{5x}$$
$$-84\Lambda_8^2\Lambda_5\Lambda_7\Lambda_{4t} - 3456\Lambda_8\Lambda_{4t}^2\Lambda_4\Lambda_{5x} + 60\Lambda_8\Lambda_7\Lambda_{5x}\Lambda_{8t} - 1440\Lambda_8\Lambda_{4t}\Lambda_{5x}\Lambda_{8t}$$
$$-38\Lambda_8\Lambda_7^2\Lambda_4\Lambda_{4t} + 84\Lambda_8^2\Lambda_7\Lambda_5\Lambda_{5x} - 60\Lambda_8\Lambda_7\Lambda_{4t}\Lambda_{8t} + 192\Lambda_8\Lambda_7\Lambda_4\Lambda_{4t}\Lambda_{5x}$$
$$+3456\Lambda_8\Lambda_{5x}^2\Lambda_4\Lambda_{4t} - 54\Lambda_{8x}\Lambda_7^2\Lambda_{5x} + 5184\Lambda_{8x}\Lambda_{5x}\Lambda_{4t}^2 + 72\Lambda_{8x}\Lambda_{5x}^2\Lambda_7$$
$$+72\Lambda_{8x}\Lambda_{4t}^2\Lambda_7 + 144\Lambda_8\Lambda_{4t}^2\Lambda_{7x} + 1152\Lambda_8\Lambda_{4t}^3\Lambda_4 + 12\Lambda_8^2\Lambda_7^2\Lambda_5 + 144\Lambda_8\Lambda_{7x}\Lambda_{5x}^2$$
$$-252\Lambda_8^2\Lambda_{4t}\Lambda_{7t} - 216\Lambda_{4t}\Lambda_8^2\Lambda_{6x} + 33\Lambda_7\Lambda_8^2\Lambda_{6x} - 1152\Lambda_8\Lambda_{5x}^3\Lambda_4 - 12\Lambda_8^3\Lambda_6\Lambda_5$$
$$+72\Lambda_0\Lambda_8^3\Lambda_4 - 72\Lambda_8^3\Lambda_5\Lambda_{5t} - 5184\Lambda_{8x}\Lambda_{4t}\Lambda_{5x}^2 - 5\Lambda_8\Lambda_7^2\Lambda_{8t} + 720\Lambda_8\Lambda_{5x}^2\Lambda_{8t}$$
$$+36\Lambda_8^2\Lambda_{7t}\Lambda_7 + 4\Lambda_8\Lambda_7^3\Lambda_4 + 54\Lambda_{8x}\Lambda_{4t}\Lambda_7^2 + 216\Lambda_{5x}\Lambda_8^2\Lambda_{6x} + 720\Lambda_8\Lambda_{4t}^2\Lambda_{8t}$$
$$-4\Lambda_8\Lambda_{7x}\Lambda_7^2 + 252\Lambda_8^2\Lambda_{7t}\Lambda_{5x} - 4\Lambda_{8x}\Lambda_7^3 - 1728\Lambda_{8x}\Lambda_{4t}^3 + 216\Lambda_8^3\Lambda_{0x}$$
$$-18\Lambda_8^3\Lambda_{6t} + 1728\Lambda_{8x}\Lambda_{5x}^3 - 8\Lambda_8^3\Lambda_5^3\Big)$$

$$\Lambda_{7tt} = \frac{1}{18\Lambda_8^3}\Big(2\Lambda_8^2\Lambda_7\Lambda_5\Lambda_{7x} + 252\Lambda_8^2\Lambda_{5x}\Lambda_7\Lambda_{4t} + 3\Lambda_7^2\Lambda_{8t}\Lambda_8\Lambda_4 - 3\Lambda_8\Lambda_{7x}\Lambda_7\Lambda_{8t}$$
$$-3\Lambda_8^2\Lambda_{6x}\Lambda_7\Lambda_4 - 36\Lambda_{8x}\Lambda_8\Lambda_{6x}\Lambda_{4t} + 48\Lambda_8^2\Lambda_{4t}\Lambda_4\Lambda_{7t} + 10\Lambda_8^2\Lambda_{8t}\Lambda_5\Lambda_7$$
$$-6\Lambda_8^2\Lambda_{7t}\Lambda_4\Lambda_7 + 36\Lambda_{8x}\Lambda_8\Lambda_{6x}\Lambda_{5x} + 36\Lambda_{8x}\Lambda_7\Lambda_{4t}\Lambda_{8t} + 6\Lambda_{8x}\Lambda_8\Lambda_{7t}\Lambda_7$$
$$+3\Lambda_{8x}\Lambda_8\Lambda_{6x}\Lambda_7 - 24\Lambda_8^2\Lambda_{6x}\Lambda_{5x}\Lambda_4 + 24\Lambda_8^2\Lambda_{6x}\Lambda_{4t}\Lambda_4 + 2\Lambda_{8x}\Lambda_8\Lambda_7^2\Lambda_5$$
$$+72\Lambda_{8x}\Lambda_8\Lambda_{7t}\Lambda_{5x} - 72\Lambda_{8x}\Lambda_8\Lambda_{4t}\Lambda_{7t} - 36\Lambda_{8x}\Lambda_7\Lambda_{5x}\Lambda_{8t} - 48\Lambda_8^2\Lambda_{7t}\Lambda_4\Lambda_{5x}$$
$$-2\Lambda_8^2\Lambda_7^2\Lambda_4\Lambda_5 + 24\Lambda_7\Lambda_{5x}\Lambda_{8t}\Lambda_8\Lambda_4 + 16\Lambda_8^2\Lambda_5\Lambda_7\Lambda_{4t}\Lambda_4 - 24\Lambda_{8x}\Lambda_8\Lambda_5\Lambda_7\Lambda_{4t}$$
$$-24\Lambda_7\Lambda_{4t}\Lambda_{8t}\Lambda_8\Lambda_4 + 24\Lambda_{8x}\Lambda_8\Lambda_7\Lambda_5\Lambda_{5x} - 16\Lambda_8^2\Lambda_7\Lambda_5\Lambda_{5x}\Lambda_4 - 18\Lambda_8^3\Lambda_{6xt}$$
$$-126\Lambda_8^2\Lambda_{5x}^2\Lambda_7 - 6\Lambda_8^3\Lambda_7\Lambda_{5t} + 21\Lambda_8^2\Lambda_{4t}\Lambda_7^2 - 126\Lambda_8^2\Lambda_{4t}^2\Lambda_7 - 648\Lambda_8^2\Lambda_{5x}\Lambda_{4t}^2$$
$$+648\Lambda_8^2\Lambda_{4t}\Lambda_{5x}^2 + 6\Lambda_8^3\Lambda_5^2\Lambda_{4t} + 6\Lambda_8^2\Lambda_{7t}\Lambda_{7x} + 30\Lambda_8^2\Lambda_{7t}\Lambda_{8t} - 6\Lambda_8^3\Lambda_{5x}\Lambda_5^2$$
$$-9\Lambda_8^3\Lambda_6\Lambda_{5x} + 9\Lambda_8^3\Lambda_6\Lambda_{4t} - 6\Lambda_8\Lambda_7\Lambda_{8t}^2 - 21\Lambda_8^2\Lambda_7^2\Lambda_{5x} - 3\Lambda_{8x}\Lambda_7^2\Lambda_{8t}$$
$$-36\Lambda_8^3\Lambda_{5x}\Lambda_{5t} - 4\Lambda_8^3\Lambda_7\Lambda_5^2 - 12\Lambda_8^3\Lambda_{6x}\Lambda_5 - 18\Lambda_8^3\Lambda_{7t}\Lambda_5 + 24\Lambda_8^2\Lambda_{6x}\Lambda_{8t}$$
$$+3\Lambda_8^2\Lambda_{6x}\Lambda_{7x} + 36\Lambda_8^3\Lambda_{5t}\Lambda_{4t} - 18\Lambda_0\Lambda_8^4 + 216\Lambda_8^2\Lambda_{4t}^3 - \Lambda_8^2\Lambda_7^3 - 216\Lambda_8^2\Lambda_{5x}^3\Big)$$

$$\Lambda_{8tt} = \frac{1}{36\Lambda_8^3}\Big(9\Lambda_8^2\Lambda_{7xx}\Lambda_7 - 108\Lambda_8^2\Lambda_{7xx}\Lambda_{4t} + 1296\Lambda_8\Lambda_{5x}^2\Lambda_{8xx} + 9\Lambda_8\Lambda_7^2\Lambda_{8xx}$$
$$-16\Lambda_8^2\Lambda_{8t}\Lambda_7\Lambda_4 + 12\Lambda_8\Lambda_{8t}\Lambda_{8x}\Lambda_7 + 108\Lambda_8^2\Lambda_{7xx}\Lambda_{5x} + 24\Lambda_8\Lambda_{4t}\Lambda_4\Lambda_7\Lambda_{8x}$$
$$-24\Lambda_8\Lambda_{5x}\Lambda_4\Lambda_7\Lambda_{8x} - 288\Lambda_8\Lambda_{5x}\Lambda_4\Lambda_{4t}\Lambda_{8x} - 192\Lambda_8^2\Lambda_{5x}\Lambda_4^2\Lambda_{4t} + 8\Lambda_8^3\Lambda_5\Lambda_7\Lambda_4$$

$$
\begin{aligned}
&-144\Lambda_{5x}\Lambda_{8x}\Lambda_8\Lambda_{7x}+1728\Lambda_8^2\Lambda_{4t}\Lambda_{4x}\Lambda_{5x}+12\Lambda_8^2\Lambda_{7x}\Lambda_{4t}\Lambda_4+144\Lambda_8\Lambda_{4t}^2\Lambda_4\Lambda_{8x}\\
&-5\Lambda_7^2\Lambda_{8x}\Lambda_8\Lambda_4-12\Lambda_8^2\Lambda_{7x}\Lambda_{5x}\Lambda_4-5\Lambda_8^2\Lambda_{7x}\Lambda_4\Lambda_7+144\Lambda_8\Lambda_{5x}^2\Lambda_4\Lambda_{8x}\\
&+144\Lambda_{4t}\Lambda_{8x}\Lambda_8\Lambda_{7x}+180\Lambda_8^2\Lambda_{4t}\Lambda_{4x}\Lambda_7-12\Lambda_8^2\Lambda_5\Lambda_7\Lambda_{8x}-6\Lambda_7\Lambda_{8x}\Lambda_8\Lambda_{7x}\\
&+40\Lambda_8^2\Lambda_7\Lambda_4^2\Lambda_{5x}-180\Lambda_8^2\Lambda_7\Lambda_{4x}\Lambda_{5x}-40\Lambda_8^2\Lambda_7\Lambda_4^2\Lambda_{4t}-2592\Lambda_8\Lambda_{5x}\Lambda_{8xx}\Lambda_{4t}\\
&-216\Lambda_8\Lambda_7\Lambda_{8xx}\Lambda_{4t}+216\Lambda_8\Lambda_7\Lambda_{8xx}\Lambda_{5x}+1296\Lambda_8\Lambda_{4t}^2\Lambda_{8xx}+18\Lambda_8^3\Lambda_{6xx}\\
&+24\Lambda_8^3\Lambda_{7t}\Lambda_4-36\Lambda_8^2\Lambda_{7t}\Lambda_{8x}-864\Lambda_8^2\Lambda_{5x}^2\Lambda_{4x}+24\Lambda_8^2\Lambda_{8t}\Lambda_{7x}-216\Lambda_7\Lambda_{8x}^2\Lambda_{5x}\\
&+1656\Lambda_8^3\Lambda_{5x}\Lambda_{4t}+159\Lambda_8^3\Lambda_{4t}\Lambda_7+96\Lambda_8^2\Lambda_{5x}^2\Lambda_4^2+4\Lambda_8^2\Lambda_7^2\Lambda_4^2-144\Lambda_8^3\Lambda_{5x}\Lambda_7\\
&+3456\Lambda_{4t}\Lambda_{8x}^2\Lambda_{5x}+216\Lambda_7\Lambda_{8x}^2\Lambda_{4t}-12\Lambda_8^3\Lambda_{8t}\Lambda_5+18\Lambda_8^3\Lambda_{6x}\Lambda_4-36\Lambda_8^2\Lambda_{6x}\Lambda_{8x}\\
&+96\Lambda_8^2\Lambda_{4t}^2\Lambda_4^2-864\Lambda_8^2\Lambda_{4t}^2\Lambda_{4x}-9\Lambda_8^2\Lambda_7^2\Lambda_{4x}+6\Lambda_8^4\Lambda_6-1728\Lambda_{5x}^2\Lambda_{8x}^2-6\Lambda_7^2\Lambda_{8x}^2\\
&+4\Lambda_8^4\Lambda_5^2+24\Lambda_8^4\Lambda_{5t}-6\Lambda_8^3\Lambda_7^2-792\Lambda_8^3\Lambda_{5x}^2+48\Lambda_8^2\Lambda_{8t}^2-864\Lambda_8^3\Lambda_{4t}^2-1728\Lambda_{4t}^2\Lambda_{8x}^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_{4xt}=\frac{1}{18\Lambda_8}&\left(2\Lambda_8\Lambda_{7x}+2\Lambda_8\Lambda_{5x}\Lambda_4+18\Lambda_8\Lambda_{5xx}+\Lambda_8\Lambda_{8t}-2\Lambda_8\Lambda_{4t}\Lambda_4\right.\\
&\left.-\Lambda_7\Lambda_{8x}-12\Lambda_{5x}\Lambda_{8x}+12\Lambda_{4t}\Lambda_{8x}\right)
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_{7xt}=\frac{1}{36\Lambda_8^3}&\Big(4\Lambda_8^2\Lambda_{8t}\Lambda_7\Lambda_4-12\Lambda_8\Lambda_{8t}\Lambda_{8x}\Lambda_7-72\Lambda_8\Lambda_{4t}\Lambda_4\Lambda_7\Lambda_{8x}-4\Lambda_8^3\Lambda_5\Lambda_7\Lambda_4\\
&+72\Lambda_8\Lambda_{5x}\Lambda_4\Lambda_7\Lambda_{8x}+72\Lambda_{5x}\Lambda_{8x}\Lambda_8\Lambda_{7x}+48\Lambda_8^2\Lambda_{7x}\Lambda_{4t}\Lambda_4+8\Lambda_7^2\Lambda_{8x}\Lambda_8\Lambda_4\\
&-48\Lambda_8^2\Lambda_{7x}\Lambda_{5x}\Lambda_4-4\Lambda_8^2\Lambda_{7x}\Lambda_4\Lambda_7-72\Lambda_{4t}\Lambda_{8x}\Lambda_8\Lambda_{7x}+12\Lambda_8^2\Lambda_5\Lambda_7\Lambda_{8x}\\
&-16\Lambda_8^2\Lambda_7\Lambda_4^2\Lambda_{5x}+16\Lambda_8^2\Lambda_7\Lambda_4^2\Lambda_{4t}-36\Lambda_8^3\Lambda_{6xx}-12\Lambda_8^3\Lambda_{7t}\Lambda_4+36\Lambda_8^2\Lambda_{7t}\Lambda_{8x}\\
&+12\Lambda_8^2\Lambda_{8t}\Lambda_{7x}-72\Lambda_7\Lambda_{8x}^2\Lambda_{5x}-216\Lambda_8^3\Lambda_{5x}\Lambda_{4t}-66\Lambda_8^3\Lambda_{4t}\Lambda_7-2\Lambda_8^2\Lambda_7^2\Lambda_4^2\\
&+54\Lambda_8^3\Lambda_{5x}\Lambda_7+72\Lambda_7\Lambda_{8x}^2\Lambda_{4t}-12\Lambda_8^3\Lambda_{6x}\Lambda_4+36\Lambda_8^2\Lambda_{6x}\Lambda_{8x}-12\Lambda_8^3\Lambda_5\Lambda_{7x}\\
&+6\Lambda_8^2\Lambda_{7x}^2-3\Lambda_8^4\Lambda_6-6\Lambda_7^2\Lambda_{8x}^2-2\Lambda_8^4\Lambda_5^2-12\Lambda_8^4\Lambda_{5t}+3\Lambda_8^3\Lambda_7^2\\
&+72\Lambda_8^3\Lambda_{5x}^2+144\Lambda_8^3\Lambda_{4t}^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_{8xt}=-\frac{1}{18\Lambda_8^2}&\Big(-8\Lambda_8^2\Lambda_{4t}\Lambda_4^2-12\Lambda_7\Lambda_{8x}^2-144\Lambda_{5x}\Lambda_{8x}^2+144\Lambda_{4t}\Lambda_{8x}^2\\
&+9\Lambda_8^2\Lambda_{7xx}-36\Lambda_8^3\Lambda_{5x}-\Lambda_8^2\Lambda_{7x}\Lambda_4-9\Lambda_8^2\Lambda_7\Lambda_{4x}+12\Lambda_8\Lambda_{8x}\Lambda_{5x}\Lambda_4\\
&-12\Lambda_8\Lambda_{8x}\Lambda_{4t}\Lambda_4+5\Lambda_8\Lambda_{8x}\Lambda_7\Lambda_4+8\Lambda_8^2\Lambda_{5x}\Lambda_4^2+4\Lambda_8^2\Lambda_{8t}\Lambda_4+9\Lambda_8\Lambda_7\Lambda_{8xx}\\
&-24\Lambda_8\Lambda_{8t}\Lambda_{8x}-108\Lambda_8\Lambda_{4t}\Lambda_{8xx}+108\Lambda_8\Lambda_{5x}\Lambda_{8xx}+27\Lambda_8^3\Lambda_{4t}\\
&-12\Lambda_{8x}\Lambda_8\Lambda_{7x}+72\Lambda_8^2\Lambda_{4t}\Lambda_{4x}-72\Lambda_8^2\Lambda_{5x}\Lambda_{4x}\Big)\,.
\end{aligned}
$$

Case II. $\Lambda_8 \neq 0,\ -ph_x + hp_x \neq 0,\ 2\Lambda_8\Lambda_4 - 3\Lambda_{8x} = 0$:

The functions h and p for the transformation

$$X(T) = p(x,t)h^{-1}(x,t), \qquad dT = h^{-3/2}(x,t)dt$$

are to be solved from the following set of linear equations:

$$\left\{h_{xx} = \frac{1}{3}h_x\Lambda_4 - \frac{1}{9}h\Lambda_8\right\}_{h\leftrightarrow p}$$

$$\left\{h_t = \frac{1}{27\Lambda_8}\left(324h_x\Lambda_{4t} - 324h_x\Lambda_{5x} - 27h_x\Lambda_7 + 6h\Lambda_8\Lambda_5 \right.\right.$$
$$\left.\left. -6h\Lambda_{7x} - 12h\Lambda_{8t} + 2h\Lambda_7\Lambda_4\right)\right\}_{h\leftrightarrow p}.$$

The following conditions on Λ_i $(i = 1, \ldots, 8)$ are to be satisfied:

$$\Lambda_{6xxx} = -\frac{13}{27}\Lambda_7\Lambda_{8t} - \frac{4}{27}\Lambda_{5x}\Lambda_7\Lambda_4 - \frac{22}{9}\Lambda_{7x}\Lambda_{4t} + \frac{2}{3}\Lambda_{5x}^2\Lambda_4 + \frac{10}{9}\Lambda_{5x}\Lambda_{7x}$$
$$+2\Lambda_{4t}\Lambda_{5xx} - \frac{2}{27}\Lambda_7\Lambda_{7x} - 2\Lambda_{5x}\Lambda_{5xx} + \frac{1}{3}\Lambda_{4x}\Lambda_{6x} + \Lambda_4\Lambda_{6xx} - \frac{2}{3}\Lambda_7\Lambda_{5xx}$$
$$-\frac{4}{9}\Lambda_8\Lambda_{6x} - \frac{2}{9}\Lambda_4^2\Lambda_{6x} + \frac{82}{9}\Lambda_{8t}\Lambda_{4t} - \frac{79}{9}\Lambda_{8t}\Lambda_{5x} + \frac{22}{27}\Lambda_7\Lambda_4\Lambda_{4t}$$
$$+\frac{2}{81}\Lambda_7^2\Lambda_4 - \frac{2}{3}\Lambda_{4t}\Lambda_{5x}\Lambda_4$$

$$\Lambda_{4tt} = \frac{1}{36\Lambda_8}\left(-48\Lambda_{8t}\Lambda_{5x} + 12\Lambda_8\Lambda_5\Lambda_{5x} - 6\Lambda_{5x}\Lambda_{7x} + 6\Lambda_{7x}\Lambda_{4t} - 3\Lambda_8\Lambda_{6x}\right.$$
$$\left.+36\Lambda_8\Lambda_{5xt} - 2\Lambda_7\Lambda_4\Lambda_{4t} - 3\Lambda_7\Lambda_{8t} - 12\Lambda_8\Lambda_5\Lambda_{4t} + 48\Lambda_{8t}\Lambda_{4t} + 2\Lambda_{5x}\Lambda_7\Lambda_4\right)$$

$$\Lambda_{5tt} = \frac{1}{216\Lambda_8^2}\left(288\Lambda_7\Lambda_4\Lambda_{4t}\Lambda_{5x} + 252\Lambda_8\Lambda_7\Lambda_{5x}\Lambda_5 + 99\Lambda_8\Lambda_{6x}\Lambda_7\right.$$
$$-648\Lambda_8\Lambda_{6x}\Lambda_{4t} - 252\Lambda_8\Lambda_5\Lambda_7\Lambda_{4t} + 6\Lambda_7^2\Lambda_4\Lambda_{5x} - 144\Lambda_7\Lambda_4\Lambda_{5x}^2 - 144\Lambda_{4t}^2\Lambda_4\Lambda_7$$
$$+648\Lambda_8\Lambda_{6x}\Lambda_{5x} - 36\Lambda_8^2\Lambda_6\Lambda_5 - 864\Lambda_{7x}\Lambda_{4t}\Lambda_{5x} - 18\Lambda_{7x}\Lambda_7\Lambda_{5x} + 36\Lambda_8\Lambda_7^2\Lambda_5$$
$$+216\Lambda_0\Lambda_8^2\Lambda_4 - 6\Lambda_7^2\Lambda_4\Lambda_{4t} - 216\Lambda_8^2\Lambda_5\Lambda_{5t} + 180\Lambda_{8t}\Lambda_7\Lambda_{5x} - 180\Lambda_{8t}\Lambda_7\Lambda_{4t}$$
$$-756\Lambda_8\Lambda_{7t}\Lambda_{4t} + 18\Lambda_{7x}\Lambda_{4t}\Lambda_7 + 108\Lambda_8\Lambda_7\Lambda_{7t} - 4320\Lambda_{8t}\Lambda_{4t}\Lambda_{5x} - 54\Lambda_8^2\Lambda_{6t}$$
$$+756\Lambda_8\Lambda_{7t}\Lambda_{5x} + 2160\Lambda_{8t}\Lambda_{4t}^2 + 648\Lambda_8^2\Lambda_{0x} - 12\Lambda_{7x}\Lambda_7^2 + 4\Lambda_7^3\Lambda_4 + 432\Lambda_{4t}^2\Lambda_{7x}$$

$$+432\Lambda_{7x}\Lambda_{5x}^2 - 24\Lambda_8^2\Lambda_5^3 + 2160\Lambda_{8t}\Lambda_{5x}^2 - 15\Lambda_{8t}\Lambda_7^2\Big)$$

$$\Lambda_{7tt} = -\frac{1}{54\Lambda_8^2}\Big(6\Lambda_8\Lambda_{7t}\Lambda_4\Lambda_7 + 2\Lambda_8\Lambda_7^2\Lambda_4\Lambda_5 - 30\Lambda_8\Lambda_5\Lambda_7\Lambda_{8t} + 54\Lambda_8^2\Lambda_{6xt}$$
$$-756\Lambda_8\Lambda_{4t}\Lambda_{5x}\Lambda_7 - 6\Lambda_8\Lambda_{7x}\Lambda_5\Lambda_7 + 3\Lambda_8\Lambda_{6x}\Lambda_7\Lambda_4 + 108\Lambda_8^2\Lambda_{5x}\Lambda_{5t}$$
$$+63\Lambda_8\Lambda_7^2\Lambda_{5x} + 18\Lambda_8^2\Lambda_7\Lambda_{5t} - 18\Lambda_8\Lambda_{7x}\Lambda_{7t} + 18\Lambda_8^2\Lambda_{5x}\Lambda_5^2 - 63\Lambda_8\Lambda_{4t}\Lambda_7^2$$
$$+378\Lambda_8\Lambda_{4t}^2\Lambda_7 + 1944\Lambda_8\Lambda_{5x}\Lambda_{4t}^2 + 9\Lambda_{7x}\Lambda_7\Lambda_{8t} - 3\Lambda_7^2\Lambda_{8t}\Lambda_4 + 54\Lambda_8^2\Lambda_{7t}\Lambda_5$$
$$+378\Lambda_8\Lambda_{5x}^2\Lambda_7 - 108\Lambda_8^2\Lambda_{5t}\Lambda_{4t} - 90\Lambda_8\Lambda_{7t}\Lambda_{8t} + 12\Lambda_8^2\Lambda_7\Lambda_5^2 - 1944\Lambda_8\Lambda_{4t}\Lambda_{5x}^2$$
$$-72\Lambda_8\Lambda_{6x}\Lambda_{8t} + 36\Lambda_8^2\Lambda_{6x}\Lambda_5 - 9\Lambda_8\Lambda_{6x}\Lambda_{7x} + 27\Lambda_{5x}\Lambda_8^2\Lambda_6 - 27\Lambda_{4t}\Lambda_8^2\Lambda_6$$
$$-18\Lambda_8^2\Lambda_5^2\Lambda_{4t} + 648\Lambda_8\Lambda_{5x}^3 + 54\Lambda_0\Lambda_8^3 + 3\Lambda_8\Lambda_7^3 - 648\Lambda_8\Lambda_{4t}^3 + 18\Lambda_7\Lambda_{8t}^2\Big)$$

$$\Lambda_{8tt} = \frac{1}{18\Lambda_8}\Big(24\Lambda_{8t}^2 - 6\Lambda_8\Lambda_{8t}\Lambda_5 + 60\Lambda_8\Lambda_{4t}\Lambda_7 - 54\Lambda_8\Lambda_7\Lambda_{5x} + 378\Lambda_8\Lambda_{4t}\Lambda_{5x}$$
$$-4\Lambda_{8t}\Lambda_7\Lambda_4 - 3\Lambda_8\Lambda_7^2 + 12\Lambda_8^2\Lambda_{5t} + 2\Lambda_8^2\Lambda_5^2 + 12\Lambda_{8t}\Lambda_{7x} + 3\Lambda_8^2\Lambda_6 - 198\Lambda_8\Lambda_{4t}^2$$
$$-180\Lambda_8\Lambda_{5x}^2 + 9\Lambda_8\Lambda_{6xx} - 3\Lambda_4\Lambda_8\Lambda_{6x}\Big)$$

$$\Lambda_{4xt} = \frac{1}{9}\Lambda_{7x} - \frac{1}{3}\Lambda_{5x}\Lambda_4 + \Lambda_{5xx} + \frac{1}{18}\Lambda_{8t} + \frac{1}{3}\Lambda_{4t}\Lambda_4 - \frac{1}{27}\Lambda_7\Lambda_4$$

$$\Lambda_{7xt} = -\frac{1}{108\Lambda_8}(12\Lambda_{7x}\Lambda_4\Lambda_7 + 36\Lambda_8\Lambda_5\Lambda_{7x} - 162\Lambda_8\Lambda_7\Lambda_{5x} + 648\Lambda_8\Lambda_{4t}\Lambda_{5x}$$
$$-12\Lambda_8\Lambda_5\Lambda_7\Lambda_4 - 2\Lambda_7^2\Lambda_4^2 + 9\Lambda_8^2\Lambda_6 + 6\Lambda_8^2\Lambda_5^2 - 18\Lambda_{7x}^2 + 36\Lambda_8^2\Lambda_{5t} - 9\Lambda_8\Lambda_7^2$$
$$+12\Lambda_{8t}\Lambda_7\Lambda_4 - 36\Lambda_4\Lambda_8\Lambda_{6x} + 108\Lambda_8\Lambda_{6xx} - 216\Lambda_8\Lambda_{5x}^2 - 36\Lambda_{8t}\Lambda_{7x}$$
$$-432\Lambda_8\Lambda_{4t}^2 - 36\Lambda_8\Lambda_{7t}\Lambda_4 + 198\Lambda_8\Lambda_{4t}\Lambda_7\Big)$$

$$\Lambda_{7xx} = -\frac{2}{9}\Lambda_7\Lambda_4^2 - \frac{13}{3}\Lambda_8\Lambda_{4t} + 4\Lambda_8\Lambda_{5x} + \Lambda_{7x}\Lambda_4 + \frac{1}{3}\Lambda_7\Lambda_{4x}.$$

Case III. $\Lambda_8 = 0,\ \Lambda_7 \neq 0,\ -ph_x + hp_x \neq 0, \quad p_x \neq 0$:

The functions h and p in the transformation

$$X(T) = p(x,t)h^{-1}(x,t), \qquad dT = h^{-3/2}(x,t)dt$$

are to be solved from the following set of linear equations:

$$\left\{h_{xx} = \frac{1}{3} h_x \Lambda_4\right\}_{h\leftrightarrow p}$$

$$\Big\{h_t = \frac{1}{9\Lambda_7^3}\Big(-12h\Lambda_{6x}\Lambda_7^2 - 12h\Lambda_{7t}\Lambda_7^2 - 2h\Lambda_5\Lambda_7^3 - 1296h_x\Lambda_{7t}\Lambda_{6x} - 432h_x\Lambda_{6x}^2$$
$$+72h_x\Lambda_7^2\Lambda_{5t} - 864h_x\Lambda_{7t}^2 - 144h_x\Lambda_7\Lambda_{6x}\Lambda_5 - 144h_x\Lambda_7\Lambda_5\Lambda_{7t} + 432h_x\Lambda_7\Lambda_{6xt}$$
$$+432h_x\Lambda_7\Lambda_{7tt} - 12h_x\Lambda_7^2\Lambda_5^2 - 18h_x\Lambda_7^2\Lambda_6\Big)\Big\}_{h\leftrightarrow p}.$$

The following conditions on Λ_i $(i = 1, \dots, 8)$ are to be satisfied:

$$\Lambda_{5ttt} = \frac{1}{\Lambda_7^4}\Big(-\frac{13}{9}\Lambda_7^3\Lambda_{5t}\Lambda_{7tt} + \frac{83}{270}\Lambda_7^3\Lambda_5^2\Lambda_{7tt} - \frac{10}{9}\Lambda_7^3\Lambda_{6xt}\Lambda_{5t} - \frac{38}{3}\Lambda_7^2\Lambda_{7tt}\Lambda_{6xt}$$
$$+\frac{136}{15}\Lambda_7\Lambda_{6xt}\Lambda_{6x}^2 + \frac{404}{15}\Lambda_7\Lambda_{7t}^2\Lambda_{7tt} + \frac{31}{36}\Lambda_7^3\Lambda_6\Lambda_{7tt} + \frac{34}{135}\Lambda_7^3\Lambda_{6xt}\Lambda_5^2$$
$$+\frac{7}{9}\Lambda_7^3\Lambda_{6xt}\Lambda_6 + \frac{166}{15}\Lambda_7\Lambda_{7tt}\Lambda_{6x}^2 + \frac{344}{15}\Lambda_7\Lambda_{7t}^2\Lambda_{6xt} - \frac{6}{5}\Lambda_7^4\Lambda_{5tt}\Lambda_5$$
$$+\frac{12}{5}\Lambda_7^3\Lambda_{5tt}\Lambda_{7t} + \frac{6}{5}\Lambda_7^3\Lambda_{6x}\Lambda_{5tt} - \frac{22}{3}\Lambda_7^2\Lambda_{7tt}^2 - \frac{1}{4}\Lambda_7^4\Lambda_{6tt} - \frac{16}{3}\Lambda_7^2\Lambda_{6xt}^2$$
$$+3\Lambda_7^4\Lambda_{0xt} - 64\Lambda_{7t}^3\Lambda_{6x} - \frac{1}{8}\Lambda_7^5\Lambda_0 - \frac{5}{216}\Lambda_7^4\Lambda_6^2 - \frac{908}{15}\Lambda_{7t}^2\Lambda_{6x}^2 - \frac{61}{2430}\Lambda_7^4\Lambda_5^4$$
$$-\frac{28}{27}\Lambda_7^4\Lambda_{5t}^2 - \frac{124}{5}\Lambda_{6x}^3\Lambda_{7t} + \frac{38}{45}\Lambda_7^2\Lambda_{5t}\Lambda_{6x}^2 - \frac{43}{810}\Lambda_7^4\Lambda_5^2\Lambda_6 + \frac{112}{45}\Lambda_7^2\Lambda_{7t}^2\Lambda_{5t}$$
$$+\frac{44}{15}\Lambda_7^2\Lambda_{7t}\Lambda_{6x}\Lambda_{5t} - \frac{6}{5}\Lambda_7^3\Lambda_0\Lambda_4\Lambda_{6x} - \frac{12}{5}\Lambda_7^3\Lambda_0\Lambda_4\Lambda_{7t} + \frac{194}{135}\Lambda_7^3\Lambda_{5t}\Lambda_{6x}\Lambda_5$$
$$-\frac{86}{9}\Lambda_7\Lambda_{6x}^2\Lambda_{7t}\Lambda_5 - \frac{67}{30}\Lambda_7^2\Lambda_{6x}\Lambda_{7t}\Lambda_6 - \frac{368}{15}\Lambda_{7t}^4 - \frac{56}{15}\Lambda_{6x}^4 + \Lambda_7^4\Lambda_{0t}\Lambda_4$$
$$-\frac{36}{5}\Lambda_7^3\Lambda_{0x}\Lambda_{7t} + \frac{3}{5}\Lambda_7^4\Lambda_{0x}\Lambda_5 - \frac{18}{5}\Lambda_7^3\Lambda_{0x}\Lambda_{6x} - \frac{8}{15}\Lambda_7^2\Lambda_5^2\Lambda_{6x}^2 + \Lambda_7^4\Lambda_0\Lambda_{5x}$$
$$+\frac{1}{5}\Lambda_7^4\Lambda_0\Lambda_4\Lambda_5 + \frac{97}{540}\Lambda_7^3\Lambda_6\Lambda_{7t}\Lambda_5 - \frac{7}{270}\Lambda_7^3\Lambda_5\Lambda_{6x}\Lambda_6 + \frac{353}{135}\Lambda_7^3\Lambda_5\Lambda_{5t}\Lambda_{7t}$$
$$-\frac{118}{9}\Lambda_7\Lambda_{7t}^2\Lambda_{6x}\Lambda_5 - \frac{35}{27}\Lambda_7^2\Lambda_{6x}\Lambda_{7t}\Lambda_5^2 - \frac{52}{9}\Lambda_7\Lambda_{7t}^3\Lambda_5 - \frac{73}{45}\Lambda_7^2\Lambda_{7t}^2\Lambda_6$$
$$+\frac{3}{5}\Lambda_7^3\Lambda_{6t}\Lambda_{7t} - \frac{413}{810}\Lambda_7^4\Lambda_{5t}\Lambda_5^2 - \frac{7}{108}\Lambda_7^4\Lambda_{5t}\Lambda_6 - \frac{13}{60}\Lambda_7^4\Lambda_{6t}\Lambda_5 - \frac{20}{9}\Lambda_7\Lambda_{6x}^3\Lambda_5$$
$$-\frac{32}{45}\Lambda_7^2\Lambda_{6x}^2\Lambda_6 + \frac{3}{10}\Lambda_7^3\Lambda_{6t}\Lambda_{6x} + \frac{169}{810}\Lambda_7^3\Lambda_5^3\Lambda_{7t} - \frac{112}{135}\Lambda_7^2\Lambda_{7t}^2\Lambda_5^2 + \frac{29}{405}\Lambda_7^3\Lambda_5^3\Lambda_{6x}$$

$$+\frac{118}{45}\Lambda_7^2\Lambda_{7t}\Lambda_{6xt}\Lambda_5+\frac{148}{45}\Lambda_7^2\Lambda_{7t}\Lambda_5\Lambda_{7tt}+\frac{148}{5}\Lambda_7\Lambda_{7t}\Lambda_{6x}\Lambda_{6xt}$$
$$+\frac{178}{5}\Lambda_7\Lambda_{7t}\Lambda_{7tt}\Lambda_{6x}+\frac{124}{45}\Lambda_7^2\Lambda_{6xt}\Lambda_{6x}\Lambda_5+\frac{154}{45}\Lambda_7^2\Lambda_{7tt}\Lambda_{6x}\Lambda_5\Big)$$

$$\Lambda_{7ttt}=-\frac{1}{\Lambda_7^2}\Big(-\frac{14}{5}\Lambda_7\Lambda_{6x}\Lambda_{6xt}-\frac{1}{5}\Lambda_7^2\Lambda_{6xt}\Lambda_5-\frac{33}{5}\Lambda_{7x}\Lambda_{7t}\Lambda_{7tt}-\frac{19}{5}\Lambda_7\Lambda_{7tt}\Lambda_{6x}$$
$$-\frac{28}{5}\Lambda_7\Lambda_{7t}\Lambda_{6xt}-\frac{1}{5}\Lambda_7^2\Lambda_5\Lambda_{7tt}+\frac{4}{15}\Lambda_7^3\Lambda_{5tt}+\Lambda_7^2\Lambda_{6xtt}+6\Lambda_{7t}\Lambda_{6x}^2-\frac{3}{10}\Lambda_7^3\Lambda_{0x}$$
$$-\frac{1}{60}\Lambda_7^3\Lambda_{6t}+\frac{62}{5}\Lambda_{7t}^2\Lambda_{6x}+\frac{4}{5}\Lambda_{6x}^3+\frac{36}{5}\Lambda_{7t}^3+\frac{1}{90}\Lambda_7^3\Lambda_6\Lambda_5+\frac{1}{15}\Lambda_7^3\Lambda_{5t}\Lambda_5$$
$$-\frac{1}{10}\Lambda_7^3\Lambda_0\Lambda_4-\frac{1}{45}\Lambda_7^2\Lambda_5^2\Lambda_{6x}+\frac{3}{5}\Lambda_7\Lambda_{7t}^2\Lambda_5+\frac{2}{15}\Lambda_7\Lambda_5\Lambda_{6x}^2+\frac{11}{15}\Lambda_7\Lambda_{7t}\Lambda_{6x}\Lambda_5$$
$$+\frac{1}{135}\Lambda_7^3\Lambda_5^3+\frac{1}{30}\Lambda_7^2\Lambda_{6x}\Lambda_6-\frac{7}{15}\Lambda_7^2\Lambda_{6x}\Lambda_{5t}+\frac{1}{15}\Lambda_7^2\Lambda_6\Lambda_{7t}-\frac{3}{5}\Lambda_7^2\Lambda_{7t}\Lambda_{5t}\Big)$$

$$\Lambda_{5xtt}=\frac{1}{\Lambda_7}\Big(\frac{4}{3}\Lambda_{7t}\Lambda_{6x}+\frac{1}{216}\Lambda_7^2\Lambda_6-\frac{5}{324}\Lambda_7^2\Lambda_5^2-\frac{17}{108}\Lambda_7^2\Lambda_{5t}-\frac{7}{9}\Lambda_7\Lambda_{6xt}$$
$$-\frac{19}{36}\Lambda_7\Lambda_{7tt}+\frac{8}{9}\Lambda_{7t}^2+\frac{4}{9}\Lambda_{6x}^2-\frac{2}{27}\Lambda_7\Lambda_{6x}\Lambda_5+\frac{7}{108}\Lambda_7\Lambda_5\Lambda_{7t}+\Lambda_7\Lambda_0\Lambda_{4x}$$
$$-\Lambda_7\Lambda_{5t}\Lambda_{5x}+3\Lambda_7\Lambda_{0xx}-\frac{1}{6}\Lambda_7\Lambda_6\Lambda_{5x}-\Lambda_7\Lambda_5\Lambda_{5xt}-\frac{1}{3}\Lambda_7\Lambda_5^2\Lambda_{5x}+\Lambda_7\Lambda_4\Lambda_{0x}\Big)$$

$$\Lambda_{6xx}=-\frac{1}{2}\Lambda_7\Lambda_{5x}-\frac{5}{72}\Lambda_7^2+\frac{1}{3}\Lambda_4\Lambda_{6x}$$

$$\Lambda_{4t}=\Lambda_{5x}+\frac{1}{12}\Lambda_7$$

$$\Lambda_{7x}=\frac{1}{3}\Lambda_7\Lambda_4.$$

Remark: From our general calculations two more cases were derived which are not listed in this subsection, namely the case $\Lambda_7=0$, $\Lambda_8=0$, $\Lambda_{6x}=0$, $-ph_x+hp_x\neq 0$, $p_x\neq 0$ *and the case* $\Lambda_7=0$, $\Lambda_8=0$, $\Lambda_{6x}=0$, $-ph_x+hp_x\neq 0$, $p_x=0$. *Both these cases provide identical conditions on the* Λ_i*s already given in subsection 2.1., i.e. for* $G_x=0$. *This means that for these ordinary differential equations (specified by these* Λ_i*s) we already have a simpler linearisation procedure with* $p_x=0$. *Recall that our aim is not to provide all linearisation procedures for a given ordinary differential equation but rather* *to find all linearisable equations allowed by (1.7).*

We end this subsection with an example of a third order ordinary differential equation which we construct to be linearisable by the generalised Sundman transformation. We do not claim any physical relevance for this equation.

Example 2.1: The equation

$$\dddot{x} + \frac{3xe^x - e^x}{x(1+te^x)}\ddot{x} + \frac{3xte^x - 4te^x - 4}{x(1+te^x)}\dot{x}\ddot{x} + \frac{x^2te^x - 3xte^x + 3te^x + 3}{x^2(1+te^x)}\dot{x}^3 + \frac{3x^2e^x - 6xe^x + 3e^x}{x^2(1+te^x)}\dot{x}^2 = 0 \tag{2.40}$$

may be reduced to $X''' = 0$ by the transformation

$$X(T) = te^x + x, \qquad dT = xdt,$$

that is

$$h(x,t) = x^{-2/3}, \qquad p(x,t) = tx^{-2/3}e^x + x^{1/3}.$$

By (1.14) and the transformation given here two time dependent first integrals follow for (2.40). This example corresponds to Case I above.

2.4 On the computations

Some remarks about the computations with computer algebra are in order. The computational part of this section consists in determining all solutions $F(x,t)$ and $G(x,t)$ of the system (2.2), *i.e.* for each class of solutions a set of conditions for $\Lambda_0, \ldots, \Lambda_5$ and a list of instructions of how to compute F and G. Both are obtained by investigating integrability conditions, like $\partial_x F_{txx} = \partial_t F_{xxx}$ with F_{txx} and F_{xxx} being replaced by corresponding expressions from two equations. An algorithm for completing this task in a finite and systematic way is the well-known (Pseudo-)Differential Gröbner Basis algorithm. To apply it one has to specify a total ordering of all partial derivatives of all functions by firstly specifying a lexicographical ordering between all functions and also between all variables. In addition one can require that the total derivative of a function should have a higher priority than lexicographical ordering which was the case in these computations.

One of the programs that has this algorithm implemented is the package RIF by Allan Wittkopf and Greg Reid which was used for the general cases I - III. Its strength is that it is very efficient, especially for larger nonlinear problems. It can be downloaded from the World Wide Web site with URL `http://www.cecm.sfu.ca/~wittkopf/rif.html`.

For the two special cases $G_x = 0$ and $F_t = 0$ the package CRACK of TW was used. With its possibility to record the 'history' of any derived equation, *i.e.* how it results from other equations, and with its optional fully interactive mode of operation it provided a convenient way for simplifying system (2.2) to system (2.14) - (2.19). This made it easily possible to prove all conclusions for these two cases as one could explicitly describe how each condition is generated. For the harder more general cases computed by RIF tracing the history of equations would be too costly, so one has to trust the computation, although the remaining equations for F and G and the conditions for the Λ_i have been tested with several third order linearisable ordinary differential equations available to the authors. The program CRACK can be downloaded from `http://lie.math.brocku.ca/twolf/home/crack.html`.

3 A special Sundman transformation

In this section we consider a Sundman transformation of a special form and construct several examples of linearisable ordinary differential equations of second, third and fourth order. Some well-known equations are shown to be included in this construction. Note that we allow here linearisation to a more general linear form than $X'''(T) = 0$, whereby the transformation functions F and G in the generalised Sundman transformation (1.7) are special.

We consider the following special form of the Sundman transformation:

$$\begin{aligned} X(T(t)) &= x(t)^p \\ dT(t) &= x(t)^n dt, \end{aligned} \tag{3.1}$$

where $p, n \in \mathcal{Q}\backslash\{0\}$. The first three prolongations are

$$X' = px^{p-n-1}\dot{x}$$

$$X'' = p(p-n-1)x^{p-2n-2}\dot{x}^2 + px^{p-2n-1}\ddot{x}$$
$$X''' = p(p-n-1)(p-2n-2)x^{p-3n-3}\dot{x}^3 + p(3p-4n-3)x^{p-3n-2}\dot{x}\ddot{x}$$
$$+px^{p-3n-1}\dddot{x}\,.$$

We consider several examples.

Example 3.1: Let

$$X'' = I_0, \tag{3.2}$$

where I_0 is an arbitrary constant. By the Sundman transformation (3.1) we obtain in terms of the coordinates (x,t) the following equation:

$$p(p-n-1)x^{p-2n-2}\dot{x}^2 + px^{p-2n-1}\ddot{x} = I_0. \tag{3.3}$$

Equation (3.3) can thus be linearised by (3.1). The first integral for (3.2) is

$$I_1(X,X') = X - \frac{1}{2I_0}(X')^2$$

so that a first integral for (3.3) takes the form

$$I_1(x,\dot{x}) = x^p - \frac{p^2}{2I_0}x^{2p-2n-2}\dot{x}^2.$$

As a special case we consider $p = n+1$. Then (3.3) simplifies to

$$\ddot{x} = \frac{I_0}{n+1}x^n. \tag{3.4}$$

Solving (3.4) for I_0 and considering this I_0 this as a first integral of the third order equation which results upon differentiation I_0 with respect to t, we obtain

$$\dddot{x} - nx^{-1}\dot{x}\ddot{x} = 0. \tag{3.5}$$

Thus (3.5) can be linearised by

$$X = x^{n+1}, \qquad dT = x^n dt \tag{3.6}$$

to the equation

$$X'''(T) = 0.$$

Note that (3.5) is a special case of the generalised Sundman transformation (1.7) with $F_t = 0$, derived in general in subsection 2.2. Equation (3.5) admits two first integrals, namely I_0 (solved from (3.4)) as well as

$$I_1(X, X', X'') = XX'' - \frac{1}{2}(X')^2$$

expressed in the coordinates (x, t) by (3.6).

Example 3.2: Let

$$X'''(T) = I_0, \tag{3.7}$$

where I_0 is an arbitrary constant. By the Sundman transformation (3.1) we obtain in terms of the coordinates (x, t) the following equation:

$$\begin{aligned} p(p-n-1)(p-2n-2)x^{p-3n-3}\dot{x}^3 + p(3p-4n-3)x^{p-3n-2}\dot{x}\ddot{x} \\ +px^{p-3n-1}\dddot{x} = I_0 \end{aligned} \tag{3.8}$$

Equation (3.8) can thus be linearised by (3.1). Two first integrals for (3.7) are

$$\begin{aligned} I_1(X', X'') &= X' - \frac{1}{2I_0}(X'')^2 \\ I_2(X, X', X'') &= X + \frac{1}{3}I_0^{-2}(X'')^3 - I_0^{-1}X'X'' \end{aligned}$$

so that two first integrals for (3.8) are obtained by expressing the above first integrals in (x, t) variables using the corresponding Sundman transformation (3.1).

As a special case we let $p = -1$ and $n = -3/2$. Equation (3.8) then simplifies to

$$\dddot{x} = -I_0 x^{-5/2} \tag{3.9}$$

and it follows that (3.9) can be linearised by

$$X = x^{-1}, \qquad dT = x^{-3/2}dt \tag{3.10}$$

to the equation

$$X'''(T) = I_0.$$

Solving (3.9) for I_0 and considering I_0 as a first integral of the fourth order equation which results upon differentiation I_0 to t, we obtain

$$2xx^{(4)} + 5\dot{x}\,\dddot{x} = 0. \tag{3.11}$$

Equation (3.11) plays an important role in the symmetry classification of the anharmonic oscillator [12]. By the above construction (3.11) can be linearised by (3.10) to the equation

$$X^{(4)} = 0.$$

Equation (3.11) admits three first integrals, namely I_0 (solved from (3.9)), and

$$I_1(X', X'', X''') = X'X''' - \frac{1}{2}(X'')^2$$
$$I_2(X, X', X'', X''') = X\,(X''')^2 + \frac{1}{3}(X'')^3 - X'X''X'''$$

which are to be expressed in the coordinates (x, t) by (3.10).

Example 3.3: We consider

$$\dddot{x} - x^{-1}\dot{x}\ddot{x} - 4ax^2\dot{x} = 0, \tag{3.12}$$

where a is an arbitrary constant. Equation (3.12) is of Rikitake type [13]. By the Sundman transformation (3.1) with $p = 2$ and $n = 1$, *i.e.*

$$X = x^2, \qquad dT = xdt, \tag{3.13}$$

it follows that (3.12) linearises to

$$X''' - 4aX' = 0. \tag{3.14}$$

More examples of linearisable equations using the special Sundman transformation (3.1) are given in [2].

4 An extended Sundman transformation

In this final section we propose a further extension of the generalised Sundman transformation (1.7).

Consider the following exact one-form:

$$dT(t, x(t)) = G_1(t, x)dt + G_2(t, x, \dot{x}, \ddot{x}, \ldots)dx,$$

where G_1 is a smooth function of x and t, whereas G_2 is a smooth function of x, t and a finite number of derivatives of x with respect to t. Here d is the exterior derivative and by the Lemma of Poincaré it is known that

$$d(dT) \equiv 0.$$

To find the relations between G_1 and G_2 we calculate

$$\begin{aligned} d(dT(x,t)) &= dG_1 \wedge dt + dG_2 \wedge dx \\ &= \frac{\partial G_1}{\partial x} dx \wedge dt + \frac{\partial G_2}{\partial t} dt \wedge dx + \frac{\partial G_2}{\partial \dot{x}} d\dot{x} \wedge dx + \frac{\partial G_2}{\partial \ddot{x}} d\ddot{x} \wedge dx + \cdots \\ &= \frac{\partial G_1}{\partial x} dx \wedge dt + \frac{\partial G_2}{\partial t} dt \wedge dx + \frac{\partial G_2}{\partial \dot{x}} \ddot{x} dt \wedge dx + \frac{\partial G_2}{\partial \ddot{x}} \dddot{x}\, dt \wedge dx + \cdots \\ &\equiv 0. \end{aligned} \tag{4.1}$$

It follows that

$$\frac{\partial G_1}{\partial x} = \frac{\partial G_2}{\partial t} + \ddot{x} \frac{\partial G_2}{\partial \dot{x}} + \dddot{x}\, \frac{\partial G_2}{\partial \ddot{x}} + \cdots.$$

Also

$$\frac{\partial T}{\partial t} = G_1(t, x) \tag{4.2}$$

$$\frac{\partial T}{\partial x} = G_2(t, x, \dot{x}, \ddot{x}, \ldots). \tag{4.3}$$

We now propose the transformation

$$\begin{aligned} X(T(t,x)) &= F(x,t) \\ dT(t,x) &= G_1(t,x)dt + G_2(t, x, \dot{x}, \ddot{x}, \ldots)dx, \end{aligned} \tag{4.4}$$

with

$$\frac{\partial G_1}{\partial x} = \frac{\partial G_2}{\partial t} + \ddot{x}\frac{\partial G_2}{\partial \dot{x}} + \dddot{x}\,\frac{\partial G_2}{\partial \ddot{x}} + \cdots \tag{4.5}$$

and name this the **extended Sundman transformation**. Note that with $G_2 = 0$, (4.4) takes the form of the generalised Sundman transformation (1.7).

Next we derive the first two prolongations of (4.4):

$$X'(G_1 + \dot{x}G_2) = F_t + \dot{x}F_x \tag{4.6}$$

$$\begin{aligned} &X''(G_1 + \dot{x}G_2)^2 \\ &\quad + X'\left[G_{1t} + \dot{x}G_{1x} + \dot{x}\left(G_{2t} + \dot{x}G_{2x} + \ddot{x}G_{2\dot{x}} + \dddot{x}\,G_{2\ddot{x}} + \cdots\right) + \ddot{x}G_2\right] \\ &\quad = F_{tt} + 2\dot{x}F_{xt} + \dot{x}^2F_{xx} + \ddot{x}F_x. \end{aligned} \tag{4.7}$$

Substituting G_{2t} from (4.5) in (4.7) we obtain

$$\begin{aligned} &X''(G_1 + \dot{x}G_2)^2 + X'\left[G_{1t} + 2\dot{x}G_{1x} + \dot{x}^2G_{2x} + \ddot{x}G_2\right] \\ &\quad = F_{tt} + 2\dot{x}F_{xt} + \dot{x}^2F_{xx} + \ddot{x}F_x. \end{aligned} \tag{4.8}$$

We give two examples.

Example 5.1: Consider the first order ordinary differential equation

$$X' = X$$

and let

$$F(t) = t, \qquad G_1(x,t) = xt$$

so that $x(t) = t^{-2}$. By (4.6) we obtain

$$G_2(t,x,\dot{x}) = \left(\frac{\dot{F}}{F} - G_1\right)\frac{1}{\dot{x}} = \left(\frac{1}{t} - xt\right)\frac{1}{\dot{x}}. \tag{4.9}$$

Insert G_2 given by (4.9) and $G_1 = xt$ in (4.5) to obtain the second order ordinary differential equation

$$\left(-\frac{1}{t^2} - x\right)\frac{1}{\dot{x}} - \frac{\ddot{x}}{\dot{x}^2}\left(\frac{1}{t} - xt\right) = t$$

which may be written in the form

$$\ddot{x}\left(\frac{1}{t}-xt\right)+\dot{x}^2t+\dot{x}\left(\frac{1}{t^2}+x\right)=0.$$

Obviously $x = t^{-2}$ satisfies this second order ordinary differential equation. The transformation is

$$x = (X')^{-1}\, e^{-T}, \qquad t = e^T.$$

Example 5.2: Consider the second order ordinary differential equation

$$X''(T) = 0$$

and let

$$F(t) = t, \qquad G_1(x,t) = xt$$

so that $x(t) = t^{-1}$. By (4.8) we obtain

$$x + 2\dot{x}t + \ddot{x}G_2 + \ddot{x}^2 G_{2x} = 0, \tag{4.10}$$

where $G_2 = G_2(t, x, \dot{x}, \ddot{x})$. We now solve G_2 from (4.10) by integration. For example, a solution is

$$G_2(t,x,\dot{x},\ddot{x}) = -\frac{x}{\ddot{x}} + \left(\frac{\dot{x}}{\ddot{x}}\right)^2 - 2\frac{\dot{x}t}{\ddot{x}}. \tag{4.11}$$

Insertion of G_2, given by (4.11), and $G_1 = xt$ into (4.5) gives the third order ordinary differential equation

$$x\ddot{x}\,\dddot{x} + 2t\dot{x}\ddot{x}\,\dddot{x} - 2\dot{x}^2\,\dddot{x} - 3t\ddot{x}^3 = 0$$

for which $x = t^{-1}$ is a solution. The transformation is

$$x = (X')^{-1}\, T^{-1}, \qquad t = T.$$

We note that the linearisations which follow from the extended Sundman transformation (4.4) given in the examples above are 'downwards', *i.e.* to a lower order ordinary differential equation.

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