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Linearised actions for $\mathcal{N}$-extended (higher-spin) superconformal gravity

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Abstract: The off-shell actions for $\mathcal{N}$-extended conformal supergravity theories in three dimensions were formulated in [1, 2] for $1 \leq \mathcal{N} \leq 6$ using a universal approach. Each action is generated by a closed super three-form which is constructed in terms of the constrained geometry of $\mathcal{N}$-extended conformal superspace. In this paper we initiate a program to recast these actions (and to formulate their higher-spin counterparts) in terms of unconstrained gauge prepotentials as integrals over the full superspace. We derive transverse projection operators in $\mathcal{N}$-extended Minkowski superspace and then use them to construct linearised rank-n super-Cotton tensors and off-shell $\mathcal{N}$-extended superconformal actions. We also propose off-shell gauge-invariant actions to describe massive higher-spin supermultiplets in $\mathcal{N}$-extended supersymmetry. Our analysis leads to general expressions for identically conserved higher-spin current multiplets in $\mathcal{N}$-extended supersymmetry.

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Contents

1 Introduction **1**

2 Conceptual setup and the main results **2**

3 $\mathcal{N} = 1$ supersymmetry **5**
- 3.1 Superprojectors 6
- 3.2 Linearised rank-n super-Cotton tensor 7

4 $\mathcal{N} = 2$ supersymmetry **7**
- 4.1 Superprojectors in the complex basis 8
- 4.2 Linearised rank-n super-Cotton tensor 10
- 4.3 Superprojectors and super-Cotton tensors in the real basis 10

5 $\mathcal{N} = 3$ supersymmetry **11**
- 5.1 Superprojectors 11
- 5.2 Linearised rank-n super-Cotton tensor 12
- 5.3 Superconformal gravitino multiplet 12

6 $\mathcal{N} = 4$ supersymmetry **13**
- 6.1 Superprojectors 13
- 6.2 Linearised rank-n super-Cotton tensor 14
- 6.3 Linearised $\mathcal{N} = 4$ conformal supergravity 14

7 $\mathcal{N} = 5$ supersymmetry **15**

8 $\mathcal{N} = 6$ supersymmetry **16**

9 Conclusion **17**

A Superconformal primary multiplets **20**

B Superhelicity **21**

1 Introduction

Superprojectors [3–6] are superspace projection operators which single out irreducible representations of supersymmetry. There are various applications of such operators in the literature, including the constructions of superfield equations of motion [7, 8] and gauge-invariant actions [9, 10]. In the case of $\mathcal{N} = 1$ anti-de Sitter supersymmetry in four spacetime dimensions, the superprojectors introduced in [5] define the two types of complex linear supermultiplets (transverse and longitudinal) that are used as the compensators in the massless supersymmetric higher-spin gauge theories proposed in [11] and generalised in [12].[1]

[1] The Ivanov-Sorin superprojectors [5] have a natural three-dimensional analogue in the framework of $(1, 1)$ anti-de Sitter supersymmetry [13].

In this paper we first derive transverse spin projection operators in three-dimensional $\mathcal{N}$-extended Minkowski superspace, $\mathbb{M}^{3|2\mathcal{N}}$, and then make use of these superprojectors to construct linearised off-shell actions for $\mathcal{N}$-extended (higher-spin) conformal supergravity in terms of unconstrained prepotentials. For the $1 \leq \mathcal{N} \leq 6$ cases, the complete nonlinear actions for $\mathcal{N}$-extended conformal supergravity were derived in [1, 2] using the off-shell formulation for $\mathcal{N}$-extended conformal supergravity developed in [14].[2] Since the complete nonlinear actions are known, it is natural to ask the following question: what is the point of constructing linearised conformal supergravity actions? The answer is that the supergravity actions proposed in [1] are realised using certain closed super three-forms which are constructed in terms of the constrained geometry of $\mathcal{N}$-extended conformal superspace [14]. However, it may be shown that the constraints can be solved in terms of unconstrained prepotentials. Modulo purely gauge degrees of freedom, the structure of unconstrained conformal gauge prepotentials are as follows: $H_{\alpha\beta\gamma}$ for $\mathcal{N}=1$ [20], $H_{\alpha\beta}$ for $\mathcal{N}=2$ [21, 22], H_α for $\mathcal{N}=3$ [14], and H for $\mathcal{N}=4$ [14]. The action for conformal supergravity, S_{CSG}, may be reformulated in terms of the gauge prepotential H (with indices suppressed), $S_{\text{CSG}} = S_{\text{CSG}}[H]$, and such a formulation is expected to be essential for doing quantum supergraph calculations (say, in off-shell $\mathcal{N}$-extended versions of topologically massive supergravity [23, 24]) and other applications. In order to determine the structure of $S_{\text{CSG}}[H]$, the starting point is to first construct a linearised conformal supergravity action, which is one of the aims of this paper.

In this paper we propose a universal approach to construct linearised actions for $\mathcal{N}$-extended superconformal gravity theories and their higher-spin extensions. Our conceptual setup will be described in the next section. Then it will be applied to theories with $1 \leq \mathcal{N} \leq 6$ in sections 3 to 8. Our main results and their implications and generalisations will be discussed in section 9. The main body of the paper is accompanied by two technical appendices.

2 Conceptual setup and the main results

The geometry of $\mathcal{N}$-extended conformal superspace [14] is formulated in terms of a single curvature superfield, which is the super-Cotton tensor $\mathcal{W}$ (with suppressed indices). This tensor is encoded in the action for conformal supergravity by the rule [2, 14]

$$\mathcal{W} \propto \frac{\delta S_{\text{CSG}}[H]}{\delta H}\,. \tag{2.1}$$

The functional structure of $\mathcal{W}$ depends on the choice of $\mathcal{N}$. The $\mathcal{N}=1$ super-Cotton tensor [25] is a primary symmetric rank-3 spinor superfield $\mathcal{W}_{\alpha\beta\gamma}$ of dimension $5/2$, which obeys the conformally invariant constraint [14]

$$\nabla^\alpha \mathcal{W}_{\alpha\beta\gamma} = 0\,. \tag{2.2}$$

[2] The $\mathcal{N}=1$ and $\mathcal{N}=2$ conformal supergravity theories were constructed for the first time by van Nieuwenhuizen [15] and Roček and van Nieuwenhuizen [16], respectively. The off-shell action for $\mathcal{N}=6$ conformal supergravity was independently derived by Nishimura and Tanii [17]. On-shell formulations for $\mathcal{N}$-extended conformal supergravity with $\mathcal{N}>2$ were given in [18, 19].

JHEP08(2019)077

In the $\mathcal{N}=2$ case, the super-Cotton tensor [21, 22] is a primary symmetric rank-2 spinor superfield $\mathcal{W}_{\alpha\beta}$ of dimension 2, which obeys the Bianchi identity [14]

$$\nabla^{\alpha I}\mathcal{W}_{\alpha\beta}=0\,. \tag{2.3}$$

In the $\mathcal{N}=3$ case, the super-Cotton tensor is a primary spinor superfield $\mathcal{W}_\alpha$ of dimension 3/2 constrained by [14]

$$\nabla^{\alpha I}\mathcal{W}_{\alpha}=0\,. \tag{2.4}$$

In the $\mathcal{N}=4$ case, the super-Cotton tensor is a primary scalar superfield $\mathcal{W}$ of dimension 1 constrained by [14]

$$\nabla^{\alpha I}\nabla^{J}_{\alpha}\mathcal{W}=\frac{1}{4}\delta^{IJ}\nabla^{\alpha K}\nabla^{K}_{\alpha}\mathcal{W}\,. \tag{2.5}$$

For $\mathcal{N}>4$, the super-Cotton tensor [26, 27] is a completely antisymmetric tensor $\mathcal{W}^{IJKL}$ of dimension 1 constrained by [14]

$$\nabla^{I}_{\alpha}\mathcal{W}^{JKLP}=\nabla^{[I}_{\alpha}\mathcal{W}^{JKLP]}-\frac{4}{\mathcal{N}-3}\nabla^{Q}_{\alpha}\mathcal{W}^{Q[JKL}\delta^{P]I}\,. \tag{2.6}$$

In the above relations, ∇^I_α denotes the spinor covariant derivative of $\mathcal{N}$-extended conformal superspace [14].

The above consideration implies that we need expressions for linearised super-Cotton tensors in terms of the gauge prepotentials, $W=W(H)$, in order to obtain linearised conformal supergravity actions for $1\leq\mathcal{N}\leq 4$. We will consider a more general problem and work out linearised rank-n super-Cotton tensors $W_{\alpha(n)}(H)$ as descendants of superconformal gauge prepotentials $H_{\alpha(n)}$ in the case of $\mathcal{N}$-extended supersymmetry.

In this paper we make use of the notation and conventions adopted in [28]. In particular, $\mathcal{N}$-extended Minkowski superspace $\mathbb{M}^{3|2\mathcal{N}}$ is parametrised by real coordinates $z^A=(x^a,\theta^\alpha_I)$, where the R-symmetry index of θ^α_I takes $\mathcal{N}$ values, $I=\underline{1},\underline{2},\dots,\underline{\mathcal{N}}$.[3] The spinor covariant derivatives D^I_α obey the anti-commutation relation

$$\{D^I_\alpha,D^J_\beta\}=2\mathrm{i}\,\delta^{IJ}\partial_{\alpha\beta}\,. \tag{2.7}$$

An important role in our analysis will be played by the operator

$$\Delta=-\frac{\mathrm{i}}{2\mathcal{N}}D^{\alpha I}D^I_\alpha \tag{2.8}$$

with the following properties

$$D^{\alpha I}\Psi_\alpha=0 \quad\Longrightarrow\quad \Delta\Psi_\alpha=\partial_\alpha{}^\beta\Psi_\beta\,, \tag{2.9a}$$
$$D^{\alpha I}\Psi_\alpha=0 \quad\Longrightarrow\quad D^{\alpha I}\Delta\Psi_\alpha=0\,. \tag{2.9b}$$

[3]Since the R-symmetry group is $\mathrm{SO}(\mathcal{N})$, and the corresponding indices are raised and lowered using the Kronecker delta, we do not distinguish between upper and lower $\mathrm{SO}(\mathcal{N})$ indices.

JHEP08(2019)077

Another important property of Δ is

$$[\Delta\,, D^{\beta I} D^I_\alpha] = 0\,. \tag{2.10}$$

As a generalisation of the earlier $\mathcal{N}=1$ [29, 30] and $\mathcal{N}=2$ [31] results, in this paper we propose a superconformal gauge-invariant action of the form[4]

$$S^{(n|\mathcal{N})}[H_{\alpha(n)}] = \frac{\mathrm{i}^n}{2}\int \mathrm{d}^{3|2\mathcal{N}}z\, H^{\alpha(n)} W_{\alpha(n)}(H)\,, \qquad n>0\,, \tag{2.11}$$

where the dynamical superfield $H_{\alpha(n)} = H_{\alpha_1\dots\alpha_n} = H_{(\alpha_1\dots\alpha_n)}$ is a real symmetric rank-n spinor which is defined modulo gauge transformations

$$\delta_\zeta H_{\alpha(n)} = \mathrm{i}^n D^I_{(\alpha_1}\zeta^I_{\alpha_2\dots\alpha_n)}\,. \tag{2.12}$$

The field strength $W_{\alpha(n)}$ in (2.11) is a local descendant of $H_{\alpha(n)}$. It is a real completely symmetric rank-n spinor, which is required to obey several conditions:

1. $W_{\alpha(n)}$ is gauge invariant,
$$W_{\alpha(n)}(\delta_\zeta H) = 0\ ; \tag{2.13}$$

2. $W_{\alpha(n)}$ is transverse,
$$D^{\beta I} W_{\beta\alpha_1\dots\alpha_{n-1}} = 0\ ; \tag{2.14}$$

3. $W_{\alpha(n)}$ is a primary superconformal multiplet (all relevant technical details about the $\mathcal{N}$-extended superconformal group are collected in appendix A). The condition (2.14) uniquely fixes the dimension $d_{W_{\alpha(n)}}$ of $W_{\alpha(n)}$.

The dimension $d_{H_{\alpha(n)}}$ of $H_{\alpha(n)}$ is also fixed uniquely if we require $H_{\alpha(n)}$ and the gauge parameter $\zeta_{\alpha(n-1)}$ in (2.12) to be superconformal primary. The dimensions are:

$$d_{H_{\alpha(n)}} = 2-\mathcal{N}-\frac{n}{2}\,, \qquad d_{W_{\alpha(n)}} = 1+\frac{n}{2}\,. \tag{2.15}$$

In this paper we will demonstrate that the above conditions determine $W_{\alpha(n)}$, modulo an overall numerical factor, in the form

$$W_{\alpha(n)}(H) = \Delta^{n+\mathcal{N}-1}\Pi^\perp_{[n]} H_{\alpha(n)}\,, \tag{2.16}$$

where $\Pi^\perp_{[n]}$ is a transverse projector,

$$\Pi^\perp_{[n]}\Pi^\perp_{[n]} = \Pi^\perp_{[n]}\,. \tag{2.17}$$

By definition, the projection operator $\Pi^\perp_{[n]}$ acts on the space of real symmetric rank-n spinors $\Psi_{\alpha(n)} = \Psi_{\alpha_1\dots\alpha_n} = \Psi_{(\alpha_1\dots\alpha_n)}$ by the rule

$$\Pi^\perp_{[n]}\Psi_{\alpha(n)} := \Pi_{\alpha_1}{}^{\beta_1}\dots\Pi_{\alpha_n}{}^{\beta_n}\Psi_{\beta_1\dots\beta_n} \equiv \Psi^\perp_{\alpha_1\dots\alpha_n} = \Psi^\perp_{(\alpha_1\dots\alpha_n)}\,, \tag{2.18}$$

[4]The functional structure of (2.11) is reminiscent of the conformal higher-spin actions in four dimensions [32, 33].

JHEP08(2019)077

where the operator $\Pi_\alpha{}^\beta$ has the following universal properties

$$D^{\alpha I}\Pi_\alpha{}^\beta = 0\,, \tag{2.19a}$$

$$\Pi_\alpha{}^\beta D^I_\beta = 0\,, \tag{2.19b}$$

$$\Pi_\alpha{}^\beta \Pi_\beta{}^\gamma = \Pi_\alpha{}^\gamma\,, \tag{2.19c}$$

$$\big[\Pi_\alpha{}^\beta, \Pi_\gamma{}^\delta\big] = 0\,. \tag{2.19d}$$

The superfield $\Psi^\perp_{\alpha_1\dots\alpha_n}$ defined by (2.18) is completely symmetric, as a consequence of the identity (2.19d), and is transverse,

$$D^{I\beta}\Pi^\perp_{[n]}\Psi_{\beta\alpha_1\dots\alpha_{n-1}} = 0\,. \tag{2.20}$$

In general, a real symmetric rank-n spinor superfield $T_{\alpha(n)}$ is called transverse (or divergenceless) if it obeys the constraint (2.14). It holds that[5]

$$D^{I\beta}T_{\beta\alpha_1\dots\alpha_{n-1}} = 0 \quad\Longrightarrow\quad \Pi^\perp_{[n]}T_{\alpha(n)} = T_{\alpha(n)}\,. \tag{2.21}$$

In the case of $\mathcal{N}=2$ supersymmetry, one can define the so-called *complex transverse linear* superfields [31]. They obey a weaker constraint than (2.21), see section 4.

One of the main goals of this paper is the explicit construction of $\Pi_\alpha{}^\beta$ for different supersymmetry types, $1\leq\mathcal{N}\leq 6$. Our ansatz for $\Pi_\alpha{}^\beta$ is

$$\Pi_\alpha{}^\beta = D^{\beta I}D^I_\alpha\, F\big(\Delta,\Box\big)\,, \tag{2.22}$$

for some function $F\big(\Delta,\Box\big)$ to be determined. Due to (2.10) and the identity

$$D^{\beta I}D^I_\alpha = \mathrm{i}\mathcal{N}\big(\partial_\alpha{}^\beta + \delta_\alpha{}^\beta\Delta\big)\,, \tag{2.23}$$

the condition (2.19d) is satisfied.

This work is a natural continuation of the research described in the non-supersymmetric case in [34].

3 $\mathcal{N}=1$ supersymmetry

Let D_α be the spinor covariant derivative of $\mathcal{N}=1$ Minkowski superspace. Making use of (2.7) allows us to obtain a number of useful identities including the following:

$$D_\alpha D_\beta = \mathrm{i}\partial_{\alpha\beta} + \frac{1}{2}\varepsilon_{\alpha\beta}D^2\,, \tag{3.1a}$$

$$D^\alpha D_\beta D_\alpha = 0 \quad\Longrightarrow\quad [D_\alpha D_\beta, D_\gamma D_\delta] = 0\,, \tag{3.1b}$$

$$D^2 D_\alpha = -D_\alpha D^2 = 2\mathrm{i}\partial_{\alpha\beta}D^\beta\,, \tag{3.1c}$$

$$D^2D^2 = -4\Box\,, \tag{3.1d}$$

where we have denoted $D^2 = D^\alpha D_\alpha$ and $\Box = \partial^a\partial_a = -\frac{1}{2}\partial^{\alpha\beta}\partial_{\alpha\beta}$.

[5] For $n>1$ the spinor transverse condition (2.21) implies that $T_{\alpha(n)}$ is transverse in the usual sense, that is $\partial^{\beta\gamma}T_{\beta\gamma\alpha(n-2)} = 0$.

JHEP08(2019)077

3.1 Superprojectors

Let us consider the following operator

$$\Pi_\alpha{}^\beta = -\frac{D^2}{4\Box} D^\beta D_\alpha = -\frac{\mathrm{i}}{2}\frac{\Delta}{\Box} D^\beta D_\alpha \,, \tag{3.2}$$

which acts on the space of real spinor superfields, $\Psi_\alpha \to \Pi_\alpha{}^\beta \Psi_\beta$. It satisfies the projector property (2.19c), as a consequence of (3.1). The identities (3.1b) and (3.1c) imply that it also satisfies the conditions (2.19a) and (2.19b). If Ψ_α is transverse, $D^\alpha \Psi_\alpha = 0$, it holds that

$$D^\alpha \Psi_\alpha = 0 \quad \Longrightarrow \quad \Pi_\alpha{}^\beta \Psi_\beta = \Psi_\alpha \,. \tag{3.3}$$

We conclude that $\Pi_\alpha{}^\beta$ is the projection operator onto the space of transverse spinor superfields and thus $\Pi_\alpha{}^\beta$ can be called a transverse projector.

As a higher-rank generalisation of (3.2) we introduce a projection operator $\Pi^\perp_{[n]}$ which acts on the space of real symmetric rank-n spinors $\Psi_{\alpha(n)} = \Psi_{\alpha_1 \dots \alpha_n} = \Psi_{(\alpha_1 \dots \alpha_n)}$. It is defined by the rule (2.18). The superfield $\Psi^\perp_{\alpha_1 \dots \alpha_n}$ defined by (2.18) is completely symmetric as a consequence of the identity (3.1b). The same identity implies that

$$D^\beta \Pi^\perp_{[n]} \Psi_{\beta \alpha_1 \dots \alpha_{n-1}} = 0 \,, \tag{3.4}$$

and therefore $\Pi^\perp_{[n]} \Psi_{\alpha(n)}$ is transverse for every superfield $\Psi_{\alpha(n)}$. It is not difficult to see that $\Pi^\perp_{[n]}$ maps every transverse superfield to itself,

$$D^\beta \Psi_{\beta \alpha_1 \dots \alpha_{n-1}} = 0 \quad \Longrightarrow \quad \Pi^\perp_{[n]} \Psi_{\alpha(n)} = \Psi_{\alpha(n)} \,. \tag{3.5}$$

We conclude that $\Pi^\perp_{[n]}$ is the projector onto the space of transverse rank-n spinor superfields.

We now turn to studying the $\mathcal{N} = 1$ projection operator $\Pi^\parallel_{[n]} := \mathbb{1}_{[n]} - \Pi^\perp_{[n]}$. Given an arbitrary symmetric real rank-n spinor $\Psi_{\alpha(n)}$, we obtain

$$\big(\mathbb{1}_{[n]} - \Pi^\perp_{[n]}\big) \Psi_{\alpha(n)} = \mathrm{i}^n D_{(\alpha_1} \lambda_{\alpha_2 \dots \alpha_n)} \,, \tag{3.6a}$$

where we have denoted

$$\begin{aligned} \lambda_{\alpha_1 \dots \alpha_{n-1}} := -(-\mathrm{i})^n \sum_{j=1}^{n} \frac{1}{(4\Box)^j} \binom{n}{j} (D^2)^j D^{\beta_{n-1}} D_{(\alpha_{n-1}} \dots D^{\beta_{n-j+1}} D_{\alpha_{n-j+1}} \\ \times D^{\beta_n} \Psi_{\alpha_1 \dots \alpha_{n-j})\beta_{n-j+1} \dots \beta_n} \,. \end{aligned} \tag{3.6b}$$

In order to prove (3.6), it is useful to rewrite $\Pi_\alpha{}^\beta$ in the form

$$\Pi_\alpha{}^\beta = \delta_\alpha{}^\beta - \frac{D^2}{4\Box} D_\alpha D^\beta \,. \tag{3.7}$$

Any symmetric rank-n spinor of the form $\Phi_{\alpha(n)} = D_{(\alpha_1} \Upsilon_{\alpha_2 \dots \alpha_n)}$ is said to be longitudinal. The projector $\Pi^\parallel_{[n]}$ maps every longitudinal superfield to itself, $\big(\mathbb{1}_{[n]} - \Pi^\perp_{[n]}\big)\Phi_{\alpha(n)} = \Phi_{\alpha(n)}$. Thus $\Pi^\parallel_{[n]}$ is the projector onto the space of longitudinal rank-n spinor superfields.

JHEP08(2019)077

3.2 Linearised rank-n super-Cotton tensor

Given a positive integer n, the rank-n super-Cotton tensor [29] (see also [30, 35]) is

$$W_{\alpha(n)}(H) = \Big(-\frac{\mathrm{i}}{2}\Big)^n D^{\beta_1} D_{\alpha_1} \dots D^{\beta_n} D_{\alpha_n} H_{\beta_1 \dots \beta_n}\,. \tag{3.8}$$

Its fundamental properties are the following: (i) it is invariant under the gauge transformations (2.12); and (ii) it obeys the conservation identity (2.14).

The choice $n = 1$ in (3.8) corresponds to the gauge-invariant field strength of an Abelian vector multiplet [36]. The case $n = 2$ corresponds to the super-Cottino tensor [29] which is the gauge-invariant field strength of a superconformal gravitino multiplet.[6] Choosing $n = 3$ in (3.8) gives the linearised version of the $\mathcal{N} = 1$ super-Cotton tensor [25]. Finally, for $n > 3$ the component fields of $W_{\alpha(n)}$ contain linearised bosonic [39] and fermionic [29] higher-spin Cotton tensors.

The super-Cotton tensor (3.8) can be expressed in terms of the transverse projection operator $\Pi^{\perp}_{[n]}$ in the form

$$W_{\alpha(n)} = \Delta^n \Pi^{\perp}_{[n]} H_{\alpha(n)}\,, \tag{3.9}$$

which is a special case of (2.16). In order to demonstrate that (3.9) is equivalent to (3.8), it suffices to note that, in accordance with (3.1d), Δ is a square root of the d'Alembertian,

$$\Delta^2 = \Box\,. \tag{3.10}$$

This property allows us to obtain alternative expressions for $W_{\alpha(n)}$, depending on whether an explicit value of n is even or odd. These expressions are:

$$W_{\alpha(2s)} = \Box^s \Pi^{\perp}_{[2s]} H_{\alpha(2s)}\,, \qquad s = 1, 2, \dots \tag{3.11a}$$

$$W_{\alpha(2s+1)} = \Box^s \Delta \Pi^{\perp}_{[2s+1]} H_{\alpha(2s+1)}\,, \qquad s = 0, 1, \dots \tag{3.11b}$$

4 $\mathcal{N} = 2$ supersymmetry

In the case of $\mathcal{N} = 2$ supersymmetry, it is often useful to work with a complex basis for the spinor covariant derivatives. Such a basis is introduced [28] by replacing the real covariant derivatives $D^I_\alpha = (D^1_\alpha, D^2_\alpha)$ with the complex operators D_α and $\bar{D}_\alpha$ defined by:

$$D_\alpha = \frac{1}{\sqrt{2}}(D^1_\alpha - \mathrm{i}D^2_\alpha)\,, \qquad \bar{D}_\alpha = -\frac{1}{\sqrt{2}}(D^1_\alpha + \mathrm{i}D^2_\alpha)\,. \tag{4.1}$$

As follows from (2.7), the complex spinor covariant derivatives satisfy the anti-commutation relations

$$\{D_\alpha, D_\beta\} = 0\,, \qquad \{\bar{D}_\alpha, \bar{D}_\beta\} = 0\,, \qquad \{D_\alpha, \bar{D}_\beta\} = -2\mathrm{i}\partial_{\alpha\beta}\,. \tag{4.2}$$

In terms of the new covariant derivatives, one naturally defines important off-shell supermultiplets including (i) a chiral superfield Φ constrained by $\bar{D}_\alpha \Phi = 0$; (ii) a complex linear superfield Γ constrained by $\bar{D}^2 \Gamma = 0$; and (iii) a real linear superfield $L = \bar{L}$ constrained by $\bar{D}^2 L = 0$.

[6] Among the component fields of $W_{\alpha\beta}$ is the so-called Cottino tensor $C_{\alpha\beta\gamma} = C_{(\alpha\beta\gamma)}$, which is the gauge-invariant field strength of a conformal gravitino [23, 37, 38].

JHEP08(2019)077

4.1 Superprojectors in the complex basis

We introduce the operator

$$\Pi_\alpha{}^\beta = \frac{\mathrm{i}}{4\Box}\Delta\big(\bar{D}^\beta D_\alpha + D^\beta \bar{D}_\alpha\big)\,, \tag{4.3a}$$

where Δ denotes the $\mathcal{N}=2$ version of (2.8) written in the complex basis,

$$\Delta = \frac{\mathrm{i}}{2}D^\alpha \bar{D}_\alpha = \frac{\mathrm{i}}{2}\bar{D}^\alpha D_\alpha\,. \tag{4.3b}$$

Making use of the anti-commutation relations (4.2), $\Pi_\alpha{}^\beta$ can be rewritten in the form

$$\Pi_\alpha{}^\beta = \frac{\Delta}{2\Box}\big(\partial_\alpha{}^\beta + \delta_\alpha{}^\beta \Delta\big)\,, \tag{4.4}$$

which implies the validity of (2.19d). One may also check that $\Pi_\alpha{}^\beta$ satisfies the other three conditions in (2.19), keeping in mind that D^I_α stands for D_α and $\bar{D}_\alpha$. For this it suffices to make use of identities of the type $\bar{D}^\beta\Delta = \frac{\mathrm{i}}{4}\bar{D}^2 D^\beta$. Thus $\Pi_\alpha{}^\beta$ is the projection operator onto the space of transverse spinor superfields.

Given the projector $\Pi_\alpha{}^\beta$, we define the transverse projection operator $\Pi^\perp_{[n]}$ by the rule (2.18). It acts on the space of real symmetric rank-n spinors, $\Psi_{\alpha(n)}$, and projects it onto the subspace of transverse superfields such that

$$\bar{D}^\beta \Pi^\perp_{[n]}\Psi_{\alpha_1\dots\alpha_{n-1}\beta} = D^\beta \Pi^\perp_{[n]}\Psi_{\alpha_1\dots\alpha_{n-1}\beta} = 0\,. \tag{4.5}$$

At this point it is worth pausing in order to make a few comments. According to the terminology of [31], every real symmetric rank-n spinor $T_{\alpha(n)} = \bar{T}_{\alpha(n)}$ constrained by

$$\bar{D}^\beta T_{\alpha_1\dots\alpha_{n-1}\beta} = 0 \quad \Longleftrightarrow \quad D^\beta T_{\alpha_1\dots\alpha_{n-1}\beta} = 0 \tag{4.6}$$

is said to be *real transverse linear*. It is linear since the above constraint implies

$$\bar{D}^2 T_{\alpha(n)} = 0 \quad \Longleftrightarrow \quad D^2 T_{\alpha(n)} = 0\,, \tag{4.7}$$

as a consequence of (4.2). There also exist *complex transverse linear* superfields $\Gamma_{\alpha(n)}$ which obey the only constraint

$$\bar{D}^\beta \Gamma_{\alpha_1\dots\alpha_{n-1}\beta} = 0 \quad \Longrightarrow \quad \bar{D}^2\Gamma_{\alpha(n)} = 0\,. \tag{4.8}$$

The general solution to this constraint proves to be

$$\Gamma_{\alpha(n)} = \bar{D}^\beta \xi_{\beta\alpha_1\dots\alpha_n}\,, \tag{4.9}$$

with the prepotential $\xi_{\alpha(n+1)}$ being complex unconstrained. A general solution to (4.6) is more complicated, and will be discussed below.

In the case of $\mathcal{N}=1$ supersymmetry, Δ is an invertible operator, eq. (3.10). This is no longer true for $\mathcal{N}=2$, in particular due to the relation

$$\Box = \Delta^2 + \frac{1}{16}\big\{\bar{D}^2, D^2\big\}\,, \tag{4.10}$$

JHEP08(2019)077

which is the three-dimensional analogue of a famous result in four dimensions [3]. This relation can be rewritten in the form

$$\begin{aligned}
&\mathbb{1} = \mathcal{P}_{(\ell)} + \mathcal{P}_{(+)} + \mathcal{P}_{(-)} \,, \\
\mathcal{P}_{(\ell)} = \frac{1}{\Box}\Delta^2 \,, \qquad &\mathcal{P}_{(+)} = \frac{1}{16\Box}\bar{D}^2 D^2 \,, \qquad \mathcal{P}_{(-)} = \frac{1}{16\Box} D^2 \bar{D}^2 \,,
\end{aligned} \tag{4.11}$$

where $\mathcal{P}_i = \big(\mathcal{P}_{(\ell)}, \mathcal{P}_{(+)}, \mathcal{P}_{(-)}\big)$ are orthogonal projectors, $\mathcal{P}_i \mathcal{P}_j = \delta_{ij} \mathcal{P}_i$. The operator $\mathcal{P}_{(\ell)}$ is the projection operator onto the space of real linear superfields,

$$\bar{D}^2 \mathcal{P}_{(\ell)} V = D^2 \mathcal{P}_{(\ell)} V = 0 \,, \tag{4.12}$$

for every real scalar V. The operator $\mathcal{P}_{(+)}$ is the projection operator onto the space of chiral superfields,

$$\bar{D}_\alpha \mathcal{P}_{(+)} U = 0 \,. \tag{4.13}$$

Finally, the operator $\mathcal{P}_{(-)}$ is the projection operator onto the space of antichiral superfields.

We now turn to studying the $\mathcal{N}=2$ projection operator $\Pi^{\parallel}_{[n]} := \mathbb{1}_{[n]} - \Pi^{\perp}_{[n]}$. For an arbitrary real symmetric rank-n superfield $H_{\alpha(n)}$, we obtain

$$\big(\mathbb{1} - \Pi^{\perp}\big) H_{\alpha(n)} = \bar{D}_{(\alpha_n} \Lambda_{\alpha_1 \dots \alpha_{n-1})} - (-1)^n D_{(\alpha_n} \bar{\Lambda}_{\alpha_1 \dots \alpha_{n-1})} \,, \tag{4.14a}$$

where we have denoted

$$\begin{aligned}
\Lambda_{\alpha_1 \dots \alpha_{n-1}} = &-\sum_{j=1}^{n} \left(\frac{\mathrm{i}}{4\Box}\right)^j \binom{n}{j} D^{\beta_n} A_{(\alpha_{n-1}}{}^{\beta_{n-1}} \dots A_{\alpha_{n-j+1}}{}^{\beta_{n-j+1}} \Delta^j H_{\alpha_1 \dots \alpha_{n-j})\beta_{n-j+1} \dots \beta_n} \\
&+ \frac{1}{8\Box} D^\beta \bar{D}^2 H_{\beta \alpha_1 \dots \alpha_{n-1}} \,.
\end{aligned} \tag{4.14b}$$

Here the operator $A_\alpha{}^\beta$ is defined by

$$A_\alpha{}^\beta := D_\alpha \bar{D}^\beta + \bar{D}_\alpha D^\beta \tag{4.15}$$

and satisfies the property

$$[\Delta, A_\alpha{}^\beta] = 0 \,, \tag{4.16}$$

which is crucial for our analysis. In order to prove the relation (4.14), it is useful to rewrite $\Pi_\alpha{}^\beta$ in the form

$$\Pi_\alpha{}^\beta = \delta_\alpha{}^\beta - \delta_\alpha{}^\beta \big(\mathcal{P}_{(+)} + \mathcal{P}_{(-)}\big) + \frac{\mathrm{i}}{4\Box} \Delta A_\alpha{}^\beta \,, \tag{4.17}$$

where $\mathcal{P}_{(+)}$ and $\mathcal{P}_{(-)}$ are the chiral and antichiral projection operators (4.11). It is natural to call the operator $\Pi^{\parallel}_{[n]} := \mathbb{1}_{[n]} - \Pi^{\perp}_{[n]}$ a longitudinal superprojector.

The relation (4.14) naturally leads to the gauge transformation law

$$\delta H_{\alpha(n)} = g_{\alpha(n)} + \bar{g}_{\alpha(n)} \,, \qquad g_{\alpha(n)} = \bar{D}_{(\alpha_1} L_{\alpha_2 \dots \alpha_n)} \,, \tag{4.18}$$

with the gauge parameter $L_{\alpha(n-1)}$ being complex unconstrained. This transformation law was postulated in [31] to describe the gauge freedom of a superconformal gauge prepotential $H_{\alpha(n)}$. The parameter $g_{\alpha(n)}$ in (4.18) is an example of a *complex longitudinal linear* superfield [31]. In general, such a superfield $G_{\alpha(n)}$ is constrained by

$$\bar{D}_{(\alpha_1}G_{\alpha_2\dots\alpha_{n+1})} = 0 \quad \Longrightarrow \quad \bar{D}^2 G_{\alpha(n)} = 0\,. \tag{4.19}$$

This constraint can be compared with (4.8). For $n = 0$ this constraint is equivalent to the chirality condition.

4.2 Linearised rank-n super-Cotton tensor

In this subsection we derive a new representation for the linearised $\mathcal{N} = 2$ rank-n super-Cotton tensor, with $n > 1$. This real tensor superfield is a descendant of the superconformal gauge superfield $H_{\alpha(n)}$, which was constructed in [31] in the form

$$\begin{aligned} W_{\alpha(n)}(H) := \frac{1}{2^{n-2}}\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor}\Bigg\{ &\binom{n}{2j}\Delta\Box^j \partial_{(\alpha_1}{}^{\beta_1}\dots\partial_{\alpha_{n-2j}}{}^{\beta_{n-2j}} H_{\alpha_{n-2j+1}\dots\alpha_n)\beta_1\dots\beta_{n-2j}} \\ &+\binom{n}{2j+1}\Delta^2\Box^j\partial_{(\alpha_1}{}^{\beta_1}\dots\partial_{\alpha_{n-2j-1}}{}^{\beta_{n-2j-1}} H_{\alpha_{n-2j}\dots\alpha_n)\beta_1\dots\beta_{n-2j-1}}\Bigg\}\,. \end{aligned} \tag{4.20}$$

The fundamental properties of $W_{\alpha(n)}(H)$ are: (i) it is invariant under the gauge transformations (4.18); and (ii) it is transverse,

$$D^\beta W_{\beta\alpha_1\dots\alpha_{n-1}} = \bar{D}^\beta W_{\beta\alpha_1\dots\alpha_{n-1}} = 0\,. \tag{4.21}$$

The case $n = 1$ corresponds to the super-Cottino tensor [29] which is the gauge-invariant field strength of a superconformal gravitino multiplet. The choice $n = 2$ gives the linearised version of the $\mathcal{N} = 2$ super-Cotton tensor [21, 22].

Making use of the representation (4.4), one may show that $W_{\alpha(n)}(H)$ can be constructed using the transverse superprojectors $\Pi^\perp_{[n]}$ in the form

$$W_{\alpha(n)}(H) = \Delta^{n+1}\Pi^\perp_{[n]}H_{\alpha(n)}\,, \tag{4.22}$$

which is a special case of (2.16). In deriving (4.22), we have made use of the special properties of the operator Δ:

$$\Delta^{2k} = \Box^{k-1}\Delta^2\,, \qquad \Delta^{2k+1} = \Box^k\Delta\,, \qquad k = 1, 2, \dots \tag{4.23}$$

4.3 Superprojectors and super-Cotton tensors in the real basis

In the real basis for the spinor covariant derivatives, our transverse superprojector (4.3) takes the form

$$\Pi_\alpha{}^\beta = -\frac{\mathrm{i}\Delta}{4\Box}D^{\beta I}D^I_\alpha\,. \tag{4.24}$$

It satisfies all the properties (2.19).

We now derive another representation for the rank-n super-Cotton tensor (4.20) using the transverse superprojector $\Pi^{\perp}_{[n]}$ formulated in the real basis. Direct calculations give

$$W_{\alpha(n)}(H) = \Delta^{n+1}\,\Pi^{\perp}_{[n]} H_{\alpha(n)}\,, \tag{4.25}$$

which is a special case of (2.16). Making use of the definition (4.24) and the property (2.10), the expression (4.25) turns into

$$W_{\alpha(n)}(H) = \Big(-\frac{\mathrm{i}}{4\Box}\Big)^n \Delta^{2n+1} D^{\beta_1 I_1} D^{I_1}_{\alpha_1} \dots D^{\beta_n I_n} D^{I_n}_{\alpha_n} H_{\beta_1\dots\beta_n}\,. \tag{4.26}$$

It can be shown that the above is equivalent to

$$W_{\alpha(n)}(H) = \Big(-\frac{\mathrm{i}}{4}\Big)^n \Delta D^{\beta_1 I_1} D^{I_1}_{\alpha_1} \dots D^{\beta_n I_n} D^{I_n}_{\alpha_n} H_{\beta_1\dots\beta_n}\,, \tag{4.27}$$

where we have made use of the property (4.23). The representation (4.27) for the rank-n super-Cotton tensor is clearly much simpler than the expression (4.20) originally given in [31].

5 $\mathcal{N} = 3$ supersymmetry

In order to construct superprojectors in the case of $\mathcal{N} \geq 3$ supersymmetry, we will only make use of the real basis for the spinor covariant derivatives, which satisfy the anti-commutation relation (2.7).

5.1 Superprojectors

Our $\mathcal{N} = 3$ superprojector $\Pi_\alpha{}^\beta$ is given by the operator

$$\Pi_\alpha{}^\beta = -\frac{\mathrm{i}\Delta}{48\Box^2} D^{\beta I} D^I_\alpha \big(9\Delta^2 - \Box\big)\,, \tag{5.1}$$

which acts on the space of real spinor superfields. It is possible to show that the projector satisfies the properties (2.19). These properties can be proved using (2.10) and (2.23), in conjunction with the following identity:

$$\Delta^4 = \frac{1}{9}\Big\{10\Box\Delta^2 - \Box^2\Big\}\,. \tag{5.2}$$

It should be pointed out that in order to prove (2.19a), we recall that the operator Δ preserves the transversality condition (2.9b). Thus, it suffices to show that the following relation holds

$$D^{\alpha J}\Big\{D^{\beta I} D^I_\alpha \big(9\Delta^2 - \Box\big)\Big\} = 0\,. \tag{5.3}$$

JHEP08(2019)077

5.2 Linearised rank-n super-Cotton tensor

A linearised version $W_\alpha(H)$ of the $\mathcal{N}=3$ super-Cotton tensor [14] has never been computed. Our goal in this subsection is to construct $W_\alpha(H)$ and its higher spin extension $W_{\alpha(n)}(H)$ using the transverse superprojector $\Pi^{\perp}_{[n]}$.

In accordance with (2.15), the dimensions of the $\mathcal{N}=3$ gauge prepotential $H_{\alpha(n)}$ and its corresponding field strength $W_{\alpha(n)}(H)$ are $(-1-\frac{n}{2})$ and $(1+\frac{n}{2})$, respectively. The dimensional analysis (2.15) and the conditions (2.13) and (2.14) imply that the field strength $W_{\alpha(n)}(H)$ is fixed, modulo an overall constant, in the form

$$W_{\alpha(n)}(H) = \Delta^{n+2}\,\Pi^{\perp}_{[n]} H_{\alpha(n)}\,, \tag{5.4}$$

which is a special case of (2.16). When $W_{\alpha(n)}$ is represented in the form (5.4), both conditions (2.13) and (2.14) are made manifest as a consequence of (2.9b), (2.19) and (2.20).

Making use of (5.1) and the property (2.10), the expression (5.4) turns into

$$W_{\alpha(n)}(H) = \Big(-\frac{\mathrm{i}}{48\Box^2}\Big)^n \Delta^{2(n+1)}\big(9\Delta^2-\Box\big)^n D^{\beta_1 I_1} D^{I_1}_{\alpha_1}\dots D^{\beta_n I_n} D^{I_n}_{\alpha_n} H_{\beta_1\dots\beta_n}\,. \tag{5.5}$$

The expression (5.5) contains $\Box^{2n}$ in the denominator. However, it is possible to simplify it by making use of the following identities which can be derived from (5.2)

$$(9\Delta^2-\Box)^n = (8\Box)^{n-1}(9\Delta^2-\Box)\,, \tag{5.6a}$$

$$\Delta^{2n}(9\Delta^2-\Box) = \Box^n(9\Delta^2-\Box)\,, \qquad n=1,2,\dots \tag{5.6b}$$

Then, it follows from (5.6a) and (5.6b) that (5.5) is equivalent to

$$W_{\alpha(n)}(H) = \frac{1}{8}\Big(-\frac{\mathrm{i}}{6}\Big)^n \big(9\Delta^2-\Box\big) D^{\beta_1 I_1} D^{I_1}_{\alpha_1}\dots D^{\beta_n I_n} D^{I_n}_{\alpha_n} H_{\beta_1\dots\beta_n}\,. \tag{5.7}$$

In the $n=1$ case, the field strength W_α corresponds to the linearised version of the $\mathcal{N}=3$ super-Cotton tensor. Thus, the field strength (5.4) (or equivalently (5.7)) can be referred to as the rank-n super-Cotton tensor.

5.3 Superconformal gravitino multiplet

Let us point out that (5.6a) implies that the following operator

$$\mathcal{P} = \frac{1}{8\Box}(9\Delta^2-\Box) \tag{5.8}$$

is a projector, $\mathcal{P}^2=\mathcal{P}$. This projector is relevant in the context of a superconformal gravitino multiplet.

In accordance with the analysis of $\mathcal{N}=3$ supermultiplets of conserved currents [40], the superconformal gravitino multiplet should be described by a real scalar gauge prepotential H of dimension -1, which is defined modulo under gauge transformations

$$\delta_\zeta H = \mathrm{i}D^{\alpha I}D^J_\alpha \zeta^{IJ}\,, \qquad \zeta^{IJ}=\zeta^{JI}\,, \quad \zeta^{II}=0\,. \tag{5.9}$$

JHEP08(2019)077

Associated with H is a primary descendant $W(H)$ of dimension $+1$, which has the following properties: (i) it is gauge invariant,

$$W(\delta_\zeta H) = 0 \;; \tag{5.10}$$

and (ii) it obeys the constraint

$$(D^{\alpha I} D^J_\alpha - 2\mathrm{i}\delta^{IJ}\Delta) W = 0\,. \tag{5.11}$$

We normalise this super-Cottino as

$$W = \frac{1}{8}(9\Delta^2 - \Box) H\,. \tag{5.12}$$

Let us also note that acting with $D^{\beta J}$ on the constraint (5.11) leads to

$$D^{\alpha I}\Big\{D^{\beta J} D^J_\alpha\big(9\Delta^2 - \Box\big)\Big\} = 0\,, \tag{5.13}$$

which is the transverse condition (5.3). In deriving (5.13), we have made use of (2.7), (2.23) and the following identity $[\Delta, D_\beta{}^I] = \frac{2}{3}\partial_{\beta\gamma} D^{\gamma I}$.

A linearised gauge-invariant action for the $\mathcal{N} = 3$ superconformal gravitino multiplet is fixed up to an overall constant. We propose the following action:

$$S[H] = \frac{1}{2}\int \mathrm{d}^{3|6}z\, H W(H) \tag{5.14}$$

to describe the dynamics of the superconformal gravitino multiplet.

6 $\mathcal{N} = 4$ supersymmetry

In this section we introduce $\mathcal{N} = 4$ superprojectors and construct the superconformal field strength $W_{\alpha(n)}(H)$ following approaches similar to those developed in section 5. An expression for the $\mathcal{N} = 4$ super-Cotton tensor will also be presented.

6.1 Superprojectors

The $\mathcal{N} = 4$ transverse projector is given by

$$\Pi_\alpha{}^\beta = -\frac{\mathrm{i}\Delta}{24\Box^2} D^{\beta I} D^I_\alpha\big(4\Delta^2 - \Box\big)\,. \tag{6.1}$$

The properties (2.19) can be proved using (2.10) and (2.23), in conjunction with the following identity:

$$\Delta^5 = \frac{1}{4}\Big\{5\Box\Delta^3 - \Box^2\Delta\Big\}\,. \tag{6.2}$$

Unlike in the $\mathcal{N} = 3$ case (5.3), we are now required to use the full expression of the projector $\Pi_\alpha{}^\beta$ in order to prove (2.19a).

JHEP08(2019)077

6.2 Linearised rank-n super-Cotton tensor

In accordance with (2.15), the dimensions of the $\mathcal{N}=4$ gauge prepotential $H_{\alpha(n)}$ and its corresponding field strength $W_{\alpha(n)}(H)$ are $(-2-\frac{n}{2})$ and $(1+\frac{n}{2})$, respectively. The dimensional analysis (2.15) and the conditions (2.13) and (2.14) imply that the field strength $W_{\alpha(n)}(H)$ is fixed, modulo an overall constant, in the form

$$W_{\alpha(n)}(H) = \Delta^{n+3}\,\Pi^{\perp}_{[n]}H_{\alpha(n)}\,, \tag{6.3}$$

which is a special case of (2.16).

Making use of (6.1) and the property (2.10), the expression (6.3) turns into

$$W_{\alpha(n)}(H) = \Big(-\frac{\mathrm{i}}{24\Box^2}\Big)^n\Delta^{2(n+1)}\Delta\big(4\Delta^2-\Box\big)^n D^{\beta_1 I_1}D^{I_1}_{\alpha_1}\dots D^{\beta_n I_n}D^{I_n}_{\alpha_n}H_{\beta_1\dots\beta_n}\,. \tag{6.4}$$

The expression (6.4) contains $\Box^{2n}$ in the denominator. However, it is possible to simplify it further using the following observations. First, equation (6.2) leads to the following relation

$$\Delta(4\Delta^2-\Box)^n = (3\Box)^{n-1}\Delta(4\Delta^2-\Box)\,,\qquad n=1,2,\dots \tag{6.5}$$

Next, equation (2.9a) implies that for every transverse superfield Ψ_α, we have

$$\Delta^2\Psi_\alpha = \Box\Psi_\alpha\,. \tag{6.6}$$

As a result, one may show that it is possible to cancel $\Box^{2n}$ in the denominator of (6.4) and thus arrive at

$$W_{\alpha(n)}(H) = \frac{1}{3}\Big(-\frac{\mathrm{i}}{8}\Big)^n\Delta\big(4\Delta^2-\Box\big)D^{\beta_1 I_1}D^{I_1}_{\alpha_1}\dots D^{\beta_n I_n}D^{I_n}_{\alpha_n}H_{\beta_1\dots\beta_n}\,. \tag{6.7}$$

The superconformal field strength (6.3), or equivalently (6.7), can be called the rank-n super-Cotton tensor.

As a direct consequence of (6.2) and (6.5), we deduce that the following operator

$$\mathcal{P} = \frac{\Delta^2}{3\Box^2}(4\Delta^2-\Box) \tag{6.8}$$

is a projector, $\mathcal{P}^2=\mathcal{P}$.

6.3 Linearised $\mathcal{N}=4$ conformal supergravity

In the $\mathcal{N}=4$ case, the linearised super-Cotton tensor proves to be a real scalar superfield, which obeys the following condition

$$(D^{\alpha I}D^J_\alpha - 2\mathrm{i}\delta^{IJ}\Delta)W = 0\,. \tag{6.9}$$

In accordance with (2.15), both W and H are primary superfields of dimension 1 and -2, respectively. It is worth pointing out that if we act with $D^{\beta J}$ on both sides of equation (6.9) and making use of (2.7) along with $[\Delta, D_\beta{}^I] = \frac{1}{2}\partial_{\beta\gamma}D^{\gamma I}$, the resulting equation is

$$D^{\alpha I}D^{\beta J}D^J_\alpha\Delta\big(4\Delta^2-\Box\big) = 0\,. \tag{6.10}$$

This is exactly the transversality condition of the projector $\Pi_\alpha{}^\beta$ (2.19a). The super-Cotton tensor is given by

$$W(H) = \frac{\Delta}{3}(4\Delta^2 - \Box)H\,, \tag{6.11}$$

which is the solution to (6.9).

We define the linearised action for $\mathcal{N}=4$ conformal supergravity to be

$$S[H] = \frac{1}{2}\int \mathrm{d}^{3|8}z\, HW(H)\,. \tag{6.12}$$

It is invariant under the gauge transformations

$$\delta_\zeta H = \mathrm{i}D^{\alpha I}D^J_\alpha \zeta^{IJ}\,, \qquad \zeta^{IJ} = \zeta^{JI}\,, \quad \zeta^{II} = 0\,. \tag{6.13}$$

7 $\mathcal{N}=5$ supersymmetry

The $\mathcal{N}=5$ transverse projection operator is given by

$$\Pi_\alpha{}^\beta = -\frac{\mathrm{i}\Delta}{3840\Box^3}D^{\beta I}D^I_\alpha\big(625\Delta^4 - 250\Box\Delta^2 + 9\Box^2\big)\,. \tag{7.1}$$

The projector properties (2.19) can be proved using eqs. (2.10) and (2.23), in conjunction with the following identity:

$$\Delta^6 = \frac{1}{625}\Box\Big\{875\Delta^4 - 259\Box\Delta^2 + 9\Box^2\Big\}\,. \tag{7.2}$$

As in the $\mathcal{N}=3$ case, to prove the transversality condition (2.19a), it suffices to show that the following relation holds

$$D^{\alpha J}\Big\{D^{\beta I}D^I_\alpha\big(625\Delta^4 - 250\Box\Delta^2 + 9\Box^2\big)\Big\} = 0\,. \tag{7.3}$$

In accordance with (2.16), the field strength $W_{\alpha(n)}(H)$ takes the following form

$$W_{\alpha(n)}(H) = \Delta^{n+4}\,\Pi^\perp_{[n]}H_{\alpha(n)}\,. \tag{7.4}$$

Making use of (7.1) and the property (2.10), the expression (7.4) turns into

$$\begin{aligned} W_{\alpha(n)}(H) = \Big(-\frac{\mathrm{i}}{3840\Box^3}\Big)^n \Delta^4\Delta^{2n}\big(625\Delta^4 - 250\Box\Delta^2 + 9\Box^2\big)^n \\ \times D^{\beta_1 I}D^I_{\alpha_1}\dots D^{\beta_n J}D^J_{\alpha_n}H_{\beta_1\dots\beta_n}\,. \end{aligned} \tag{7.5}$$

It is possible to simplify (7.5) further. First, equation (7.2) leads to the following relation

$$\Delta^{2n}\big(625\Delta^4 - 250\Box\Delta^2 + 9\Box^2\big)^n = 384^{n-1}\Box^{3(n-1)}\Delta^2\big(625\Delta^4 - 250\Box\Delta^2 + 9\Box^2\big) \tag{7.6}$$

for $n \geq 1$. As a result, using (7.6) and (6.6) one may show that it is possible to cancel $\Box^{3n}$ in the denominator of (7.5) to obtain

$$\begin{aligned} W_{\alpha(n)}(H) = \frac{1}{384}\Big(-\frac{\mathrm{i}}{10}\Big)^n\big(625\Delta^4 - 250\Box\Delta^2 + 9\Box^2\big) \\ \times D^{\beta_1 I_1}D^{I_1}_{\alpha_1}\dots D^{\beta_n I_n}D^{I_n}_{\alpha_n}H_{\beta_1\dots\beta_n}\,. \end{aligned} \tag{7.7}$$

JHEP08(2019)077

As a final observation, from equations (7.2) and (7.6) we deduce that the following operator

$$\mathcal{P} = \frac{1}{384\Box^2}\big(625\Delta^4 - 250\Box\Delta^2 + 9\Box^2\big) \tag{7.8}$$

is a projector, $\mathcal{P}^2 = \mathcal{P}$.

8 $\mathcal{N} = 6$ supersymmetry

The $\mathcal{N} = 6$ transverse projection operator $\Pi_\alpha{}^\beta$ is given by

$$\Pi_\alpha{}^\beta = -\frac{\mathrm{i}\Delta}{480\Box^3} D^{\beta I} D^I_\alpha \big(81\Delta^4 - 45\Box\Delta^2 + 4\Box^2\big)\,. \tag{8.1}$$

The superprojector properties (2.19) can be proved using (2.10) and (2.23), in conjunction with the following identity:

$$\Delta^7 = \frac{1}{81}\Box\Big\{126\Delta^5 - 49\Box\Delta^3 + 4\Box^2\Delta\Big\}\,. \tag{8.2}$$

In accordance with (2.16), the field strength $W_{\alpha(n)}(H)$ takes the following form

$$W_{\alpha(n)}(H) = \Delta^{n+5}\,\Pi^\perp_{[n]} H_{\alpha(n)}\,. \tag{8.3}$$

Making use of (8.1) and the property (2.10), the expression (8.3) turns into

$$\begin{aligned} W_{\alpha(n)}(H) = \Big(-\frac{\mathrm{i}}{480\Box^3}\Big)^n \Delta^{2n+5}\big(81\Delta^4 - 45\Box\Delta^2 + 4\Box^2\big)^n \\ \times D^{\beta_1 I_1} D^{I_1}_{\alpha_1} \dots D^{\beta_n I_n} D^{I_n}_{\alpha_n} H_{\beta_1 \dots \beta_n}\,. \end{aligned} \tag{8.4}$$

It is possible to simplify (8.4) further. First, equation (8.2) leads to the following relation

$$\Delta^2\big(81\Delta^4 - 45\Box\Delta^2 + 4\Box^2\big)^n = 40^{n-1}\Box^{2(n-1)}\Delta^2\big(81\Delta^4 - 45\Box\Delta^2 + 4\Box^2\big)\,. \tag{8.5}$$

As a result, using (8.5) and (6.6) one may show that it is possible to cancel $\Box^{3n}$ in the denominator of (8.4) and thus arrive at

$$W_{\alpha(n)}(H) = \frac{1}{40}\Big(-\frac{\mathrm{i}}{12}\Big)^n \Delta\big(81\Delta^4 - 45\Box\Delta^2 + 4\Box^2\big) D^{\beta_1 I_1} D^{I_1}_{\alpha_1} \dots D^{\beta_n I_n} D^{I_n}_{\alpha_n} H_{\beta_1 \dots \beta_n}\,. \tag{8.6}$$

Finally, let us point out that (8.5) allows us to show that the following operator

$$\mathcal{P} = \frac{\Delta^2}{40\Box^3}\big(81\Delta^4 - 45\Box\Delta^2 + 4\Box^2\big) \tag{8.7}$$

is a projector, $\mathcal{P}^2 = \mathcal{P}$.

JHEP08(2019)077

9 Conclusion

In this paper we have presented a universal approach to construct linearised gauge-invariant actions for higher-spin $\mathcal{N}$-extended superconformal gravity in terms of the unconstrained prepotentials $H_{\alpha(n)}$, $n > 0$. These superconformal actions have the form (2.11). Our method was based on the use of the transverse superprojectors $\Pi^{\perp}_{[n]}$, which exist for arbitrary $\mathcal{N}$ and have been explicitly constructed in this paper for $1 \leq \mathcal{N} \leq 6$. We have demonstrated that the rank-n super-Cotton tensor $W_{\alpha(n)}$ is given in terms of the prepotential $H_{\alpha(n)}$ by the universal expression (2.16), and $W_{\alpha(n)}$ has been computed explicitly for $1 \leq \mathcal{N} \leq 6$. In particular, a new expression (4.27) for the rank-n super-Cotton tensor in the case of $\mathcal{N} = 2$ supersymmetry has been obtained. This expression is much simpler than the one originally given in [31].

The rank-n super-Cotton tensors $W_{\alpha(n)}$ for $3 \leq \mathcal{N} \leq 6$ were derived in this paper for the first time. The corresponding results are given by eqs. (5.7), (6.7), (7.7) and (8.6), respectively. The linearised super-Cotton tensor of $\mathcal{N} = 4$ conformal supergravity requires special attention, since it is a scalar superfield W. It has also been computed in this paper for the first time and is given by eq. (6.11). Making use of W allowed us to construct the linearised action for $\mathcal{N} = 4$ conformal supergravity, which is given by eq. (6.12). In the $\mathcal{N} = 3$ case, we also constructed the gauge-invariant action (5.14) which describes the dynamics of the superconformal gravitino multiplet.

In the case of conformal supergravity with $\mathcal{N} \geq 5$, the super-Cotton tensor has a different tensorial form than that of $W_{\alpha(n)}$, see eq. (2.6). It cannot be directly obtained from our results and will be studied elsewhere.

A natural direction for future work is to extend the superconformal actions under study beyond the linearised level. To start with, one can consider the action (2.11) for values of $n = 3, 2, 1$ corresponding to the linearised $\mathcal{N}$-extended conformal supergravity with $\mathcal{N} = 1, 2, 3$, respectively, and develop the Noether procedure to construct the higher-order terms in H. From the previous work [1, 2] we know that $\mathcal{N}$-extended conformal supergravity does exist for $1 \leq \mathcal{N} \leq 6$ as a nonlinear theory, however it is formulated in terms of *constrained* supergeometry. This means that the Noether procedure should lead to the full nonlinear theory, now formulated in terms of an *unconstrained* prepotential. A more ambitious generalisation is to attempt to apply the same technique to the higher-spin theories corresponding to greater values of n. Hence, our approach might be a useful laboratory to study vertices for higher spin superfields.

Our analysis in $\mathcal{N}$-extended Minkowski superspace can be generalised to arbitrary conformally flat backgrounds by applying the approach advocated in [41].

Making use of the superprojectors $\Pi^{\perp}_{[n]}$ naturally leads to a supersymmetric extension of the Fierz-Pauli equations [43]. More specifically, by applying the projection operator $\Pi^{\perp}_{[n]}$ to a superfield $\Psi_{\alpha(n)}$ that obeys the Klein-Gordon equation, $(\Box - m^2)\Psi_{\alpha(n)} = 0$, yields the transverse superfield $\Psi^{\perp}_{\alpha(n)} = \Pi^{\perp}_{[n]}\Psi_{\alpha(n)}$, which obeys the $\mathcal{N}$-extended super Fierz-Pauli equations

$$(\Box - m^2)\Psi^{\perp}_{\alpha(n)} = 0\,, \qquad D^{\beta I}\Psi^{\perp}_{\beta\alpha_1\ldots\alpha_{n-1}} = 0\,. \tag{9.1}$$

For $n > 1$ the latter implies the ordinary conservation condition

$$\partial^{\beta\gamma}\Psi^{\perp}_{\beta\gamma\alpha_1\dots\alpha_{n-2}} = 0\,. \tag{9.2}$$

The general solution of (9.1) describes two superhelicity states, see appendix B for the definition of the $\mathcal{N}$-extended superhelicity operator. If we are interested in an on-shell massive supermultiplet of a definite superhelicity, we must deal with the following superhelicity projection operators

$$\mathbb{P}^{\pm}_{[n]} := \frac{1}{2}\left(\mathbb{1} \pm \frac{\Delta}{\sqrt{\Box}}\right)\Pi^{\perp}_{[n]} \tag{9.3}$$

with the property

$$\Delta\mathbb{P}^{\pm}_{[n]} = \pm\sqrt{\Box}\mathbb{P}^{\pm}_{[n]}\,. \tag{9.4}$$

The operators $\mathbb{P}^{+}_{[n]}$ and $\mathbb{P}^{-}_{[n]}$ are orthogonal projectors:

$$\mathbb{P}^{+}_{[n]}\mathbb{P}^{-}_{[n]} = 0\,, \qquad \mathbb{P}^{+}_{[n]}\mathbb{P}^{+}_{[n]} = \mathbb{P}^{+}_{[n]}\,, \qquad \mathbb{P}^{-}_{[n]}\mathbb{P}^{-}_{[n]} = \mathbb{P}^{-}_{[n]}\,. \tag{9.5}$$

Applying the projector $\mathbb{P}^{+}_{[n]}$ or $\mathbb{P}^{-}_{[n]}$ to a superfield $\Psi_{\alpha(n)}$ that obeys the Klein-Gordon equation, $(\Box - m^2)\Psi_{\alpha(n)} = 0$, we will end up with an on-shell supermultiplet of a definite superhelicity. We conclude this paper by giving a general definition of such supermultiplets.

For $n > 0$ a massive on-shell $\mathcal{N}$-extended superfield is defined by the conditions

$$D^{\beta I}T_{\beta\alpha_1\dots\alpha_{n-1}} = 0\,, \tag{9.6a}$$

$$\Delta T_{\alpha_1\dots\alpha_n} = m\sigma T_{\alpha_1\dots\alpha_n}\,, \qquad \sigma = \pm 1\,, \tag{9.6b}$$

of which the former implies $\partial^{\beta\gamma}T_{\beta\gamma\alpha_1\dots\alpha_{n-2}} = 0$ for $n > 1$. This definition generalises those given earlier in the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ cases [30, 31, 42].

In conclusion, we propose an off-shell gauge-invariant model in which the equations of motion are equivalent to (9.6). It is a deformation of the superconformal action (2.11) given by

$$S^{(n|\mathcal{N})}_{\text{massive}}[H_{\alpha(n)}] \propto \frac{\mathrm{i}^n}{2}\int \mathrm{d}^{3|2\mathcal{N}}z\, H^{\alpha(n)}\Big(\Delta - m\sigma\Big)W_{\alpha(n)}(H)\,, \qquad n > 0\,. \tag{9.7}$$

Its invariance under the gauge transformation (2.12) follows from (2.9b). This model is a generalisation of the following massive gauge-invariant higher-spin theories: (i) the non-supersymmetric models in Minkowski space [34, 44, 45] and anti-de Sitter space AdS_3 [35]; and (ii) the supersymmetric models in AdS_3 with (1,0) [35] and (2,0) [46] anti-de Sitter supersymmetry.[7]

[7]There exist two alternative gauge-invariant formulations, off-shell and on-shell, for massive higher-spin supermultiplets. The off-shell formulations have been developed for the cases $\mathcal{N} = 1$ and $\mathcal{N} = 2$ and are given in terms of the topologically massive higher-spin actions proposed in [13, 30, 31, 35]. The on-shell formulations for massive higher-spin $\mathcal{N} = 1$ supermultiplets in $\mathbb{R}^{2,1}$ and AdS_3 were developed in [47, 48] by combining the massive bosonic and fermionic higher-spin actions described in [49, 50]. The formulations given in [47–50] are based on the gauge-invariant approaches to the dynamics of massive higher-spin fields, which were advocated by Zinoviev [51] and Metsaev [52].

In $\mathcal{N}$-extended supersymmetry, the conformal supercurrent has the same multiplet structure as the super-Cotton tensor [2]. The Bianchi identities (2.2)–(2.5) naturally lead to two types of supermultiplets of conserved currents in $\mathbb{M}^{3|2\mathcal{N}}$. One of them corresponds to the family of real symmetric rank-n spinor superfields $J_{\alpha(n)}$, $n = 1, 2, \ldots$, subject to the conservation condition [1, 2, 53]

$$D^{\beta I} J_{\beta\alpha_1\ldots\alpha_{n-1}} = 0\,, \qquad n > 0\,. \tag{9.8}$$

The second type is described by a real scalar superfield subject to the constraint [1, 2, 55]

$$\left(D^{\alpha I} D^J_\alpha - \frac{1}{\mathcal{N}}\delta^{IJ} D^{\alpha K} D^K_\alpha\right) J = 0\,, \qquad \mathcal{N} > 1\,. \tag{9.9}$$

The two types of conserved current supermultiplets are related to each other via the procedure of superspace $\mathcal{N} \to (\mathcal{N}-1)$ reduction described in [55]. Let us split the Grassmann coordinates θ^α_I of $\mathcal{N}$-extended Minkowski superspace $\mathbb{M}^{3|2\mathcal{N}}$ into two subsets: (i) the coordinates $\theta^\alpha_{\hat{I}}$, with $\hat{I} = \underline{1}, \ldots, \underline{\mathcal{N}-1}$, corresponding to $(\mathcal{N}-1)$-extended Minkowski superspace $\mathbb{M}^{3|2(\mathcal{N}-1)}$; and (ii) two additional coordinates $\theta^\alpha_{\underline{\mathcal{N}}}$. The corresponding splitting of the spinor derivatives D^I_α is $D^{\hat{I}}_\alpha$ and $D^{\underline{\mathcal{N}}}_\alpha$. Given a superfield V on $\mathbb{M}^{3|2\mathcal{N}}$, its projection to $\mathbb{M}^{3|2(\mathcal{N}-1)}$ is defined by $V| := V|_{\theta_{\underline{\mathcal{N}}}=0}$. The current supermultiplet $J_{\alpha(n)}$ produces two independent real superfields in $\mathbb{M}^{3|2(\mathcal{N}-1)}$, which are defined by the rule

$$\mathfrak{J}_{\alpha(n)} := J_{\alpha(n)}|\,, \qquad \mathfrak{J}_{\alpha_1\ldots\alpha_{n+1}} := \mathrm{i}^{n+1} D^{\underline{\mathcal{N}}}_{\alpha_1} J_{\alpha_2\ldots\alpha_{n+1}}| = \mathfrak{J}_{(\alpha_1\ldots\alpha_{n+1})} \tag{9.10}$$

and obey the conservation equations

$$D^{\beta\hat{I}} \mathfrak{J}_{\beta\alpha_1\ldots\alpha_{n-1}} = 0\,, \qquad D^{\beta\hat{I}} \mathfrak{J}_{\beta\alpha_1\ldots\alpha_n} = 0\,. \tag{9.11}$$

The scalar current supermultiplet J produces two independent real superfields in $\mathbb{M}^{3|2(\mathcal{N}-1)}$, which are defined by the rule

$$\mathfrak{J} := J|\,, \qquad \mathfrak{J}_\alpha := \mathrm{i} D^{\underline{\mathcal{N}}}_\alpha J|\,, \tag{9.12}$$

which obey the conservation equations

$$D^{\beta\hat{I}} \mathfrak{J}_\beta = 0\,, \qquad \left(D^{\alpha\hat{I}} D^{\hat{J}}_\alpha - \frac{1}{\mathcal{N}-1}\delta^{\hat{I}\hat{J}} D^{\alpha\hat{K}} D^{\hat{K}}_\alpha\right)\mathfrak{J} = 0\,. \tag{9.13}$$

This consideration shows that a current supermultiplet $J_{\alpha(n)}$ in $\mathbb{M}^{3|2\mathcal{N}}$ can be generated via Grassmann dimensional reduction from a scalar current supermultiplet in $\mathbb{M}^{3|2(\mathcal{N}+n)}$.

The results of this paper provide general expressions for identically conserved (higher-spin) current supermultiplets. Identically conserved supercurrents $J_{\alpha(n)}$ are given by eq. (2.16) in which $W_{\alpha(n)}$ has to be identified with $J_{\alpha(n)}$ and $H_{\alpha(n)}$ should be viewed as a local operator constructed in terms of the dynamical superfields and their derivatives. Expressions for the identically conserved supercurrent J follow from the relations (5.12) and (6.11) in the cases of $\mathcal{N} = 3$ and $\mathcal{N} = 4$ supersymmetry, respectively. The identically conserved $\mathcal{N} = 2$ scalar current supermultiplet is given by

$$J \propto \Delta H\,, \qquad \mathcal{N} = 2\,, \tag{9.14}$$

which is a real linear superfield. It is natural to use the name "$\mathcal{N}$-extended linear multiplet" for the real linear superfield J constrained by eq. (9.9). Finally, in the cases $\mathcal{N}=5$ and $\mathcal{N}=6$ scalar current supermultiplet has the form

$$J \propto \big(625\Delta^4 - 250\Box\Delta^2 + 9\Box^2\big)H\,, \qquad \mathcal{N}=5, \tag{9.15}$$

$$J \propto \Delta\big(81\Delta^4 - 45\Box\Delta^2 + 4\Box^2\big)H\,, \qquad \mathcal{N}=6\,. \tag{9.16}$$

It is an instructive exercise to check that these expressions satisfy the constraint (9.9). One may check that the currents (9.14), (9.15) and (9.16) are invariant under gauge transformations of the form

$$\delta_\zeta H = \mathrm{i}D^{\alpha I}D^J_\alpha \zeta^{IJ}\,, \qquad \zeta^{IJ}=\zeta^{JI}\,, \quad \zeta^{II}=0\,. \tag{9.17}$$

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A Superconformal primary multiplets

The $\mathcal{N}$-extended superconformal symmetry in three dimensions was studied in detail by Park [54]. In this appendix our presentation follows [28, 55]. In $\mathcal{N}$-extended Minkowski superspace $\mathbb{M}^{3|2\mathcal{N}}$, superconformal transformations, $z^A \to z^A + \delta z^A = z^A + \xi^A(z)$, are generated by superconformal Killing vectors. By definition, a superconformal Killing vector

$$\xi = \xi^a(z)\,\partial_a + \xi^\alpha_I(z)\,D^I_\alpha \tag{A.1}$$

is a real vector field obeying the condition $[\xi, D^I_\alpha] \propto D^J_\beta$. This condition implies

$$[\xi, D^I_\alpha] = -(D^I_\alpha \xi^\beta_J)D^J_\beta = \frac{1}{2}\omega_\alpha{}^\beta(z)D^I_\beta + \Lambda^{IJ}(z)D^J_\alpha - \frac{1}{2}\sigma(z)D^I_\alpha\,, \tag{A.2}$$

where we have defined

$$\omega_{\alpha\beta} := -\frac{2}{\mathcal{N}}D^J_{(\alpha}\xi^J_{\beta)} = -\frac{1}{2}\partial^\gamma{}_{(\alpha}\xi_{\beta)\gamma}\,, \tag{A.3a}$$

$$\Lambda^{IJ} := -2D^{[I}_\alpha \xi^{J]\alpha}\,, \tag{A.3b}$$

$$\sigma := \frac{1}{\mathcal{N}}D^I_\alpha \xi^{\alpha I} = \frac{1}{3}\partial_a \xi^a\,. \tag{A.3c}$$

Here the parameters $\omega_{\alpha\beta}=\omega_{\beta\alpha}$, $\Lambda^{IJ}=-\Lambda^{JI}$ and σ correspond to z-dependent Lorentz, $\mathrm{SO}(\mathcal{N})$ and scale transformations. These transformation parameters are related to each other as follows:

$$D^I_\alpha \omega_{\beta\gamma} = 2\varepsilon_{\alpha(\beta}D^I_{\gamma)}\sigma\,, \tag{A.4a}$$

$$D^I_\alpha \Lambda^{JK} = -2\delta^{I[J}D^{K]}_\alpha \sigma\,. \tag{A.4b}$$

JHEP08(2019)077

Let $\Phi^{\mathcal{I}}_{\mathcal{A}}(z)$ be a superfield that transforms in a representation T of the Lorentz group with respect to its index $\mathcal{A}$ and in a representation D of the R-symmetry group $\mathrm{SO}(\mathcal{N})$ with respect to the index $\mathcal{I}$. Such a superfield is called primary of dimension d if its superconformal transformation law is

$$\delta\Phi^{\mathcal{I}}_{\mathcal{A}} = -\xi\Phi^{\mathcal{I}}_{\mathcal{A}} - d\,\sigma\Phi^{\mathcal{I}}_{\mathcal{A}} + \frac{1}{2}\omega^{\alpha\beta}(M_{\alpha\beta})_{\mathcal{A}}{}^{\mathcal{B}}\Phi^{\mathcal{I}}_{\mathcal{B}} + \frac{1}{2}\Lambda^{IJ}(R^{IJ})^{\mathcal{I}}{}_{\mathcal{J}}\Phi^{\mathcal{J}}_{\mathcal{A}}\,. \tag{A.5}$$

The Lorentz generator $M_{\alpha\beta} = M_{\beta\alpha}$ is defined to act on a spinor Φ_γ by the rule

$$M_{\alpha\beta}\Phi_\gamma = \varepsilon_{\gamma(\alpha}\Phi_{\beta)}\,. \tag{A.6}$$

The $\mathrm{SO}(\mathcal{N})$ generator R^{IJ} acts on an $\mathrm{SO}(\mathcal{N})$-vector V^K as

$$R^{IJ}V^K = 2\delta^{K[I}V^{J]}\,. \tag{A.7}$$

Making use of this transformation law allows one to determine the dimensions in (2.15).

B Superhelicity

In this appendix we demonstrate how the operator Δ defined by (2.8) emerges in the framework of superhelicity.

Let P_a, $J_{ab} = -J_{ba}$, Q_α be the generators of the $\mathcal{N}$-extended super-Poincaré group in three dimensions. Important for our discussion are the following graded commutation relations:

$$\big[P_{\alpha\beta}, P_{\gamma\delta}\big] = 0\,, \tag{B.1a}$$
$$\big\{Q^I_\alpha, Q^J_\beta\big\} = 2\delta^{IJ}P_{\alpha\beta}\,, \tag{B.1b}$$
$$\big[J_{\alpha\beta}, Q^K_\gamma\big] = \mathrm{i}\varepsilon_{\gamma(\alpha}Q^K_{\beta)}\,, \tag{B.1c}$$
$$\big[J_{\alpha\beta}, P_{\gamma\delta}\big] = \mathrm{i}\varepsilon_{\gamma(\alpha}P_{\beta)\delta} + \mathrm{i}\varepsilon_{\delta(\alpha}P_{\beta)\gamma}\,. \tag{B.1d}$$

The supersymmetric extension of the Pauli-Lubanski scalar $\mathbb{W} = \frac{1}{2}\varepsilon^{abc}P_aJ_{bc} = -\frac{1}{2}P^{\alpha\beta}J_{\alpha\beta}$ is the superhelicity operator [56]

$$\mathbb{Z} = \mathbb{W} - \frac{\mathrm{i}}{8}Q^{\alpha I}Q^I_\alpha\,, \tag{B.2}$$

which commutes with the supercharges,

$$\big[\mathbb{Z}, Q^I_\alpha\big] = 0\,. \tag{B.3}$$

It is worth pointing out that the structure of (B.2) is analogous to the superhelicity operator in four-dimensional $\mathcal{N} = 1$ supersymmetry [57]. Given an irreducible unitary representation of the super-Poincaré group, the quantum numbers of mass m and superhelicity κ are defined by

$$P^aP_a = -m^2\mathbb{1}\,, \qquad \mathbb{Z} = m\kappa\mathbb{1}\,. \tag{B.4}$$

For superfield representations of the $\mathcal{N}$-extended super-Poincaré group, the infinitesimal super-Poincaré transformation of a tensor superfield T (with suppressed indices) is given by

$$\delta T = \mathrm{i}\left(-b^a P_a + \frac{1}{2}\omega^{ab} J_{ab} + \mathrm{i}\epsilon^{\alpha I} Q_\alpha^I\right) T = \mathrm{i}\left(\frac{1}{2} b^{\alpha\beta} P_{\alpha\beta} + \frac{1}{2}\omega^{\alpha\beta} J_{\alpha\beta} + \mathrm{i}\epsilon^{\alpha I} Q_\alpha^I\right) T\,, \qquad \text{(B.5)}$$

where the generators of spacetime translations ($P_{\alpha\beta}$), Lorentz transformations ($J_{\alpha\beta}$) and supersymmetry transformations (Q_α^I) are

$$P_{\alpha\beta} = -\mathrm{i}\partial_{\alpha\beta}\,, \qquad \partial_{\alpha\beta} = (\gamma^a)_{\alpha\beta}\partial_a\,, \qquad \text{(B.6a)}$$

$$J_{\alpha\beta} = -\mathrm{i}x^\gamma{}_{(\alpha}\partial_{\beta)\gamma} + \mathrm{i}\theta^I_{(\alpha}\partial^I_{\beta)} - \mathrm{i}M_{\alpha\beta}\,, \qquad \text{(B.6b)}$$

$$Q_\alpha^I = \partial_\alpha^I - \mathrm{i}\theta^{\beta I}\partial_{\alpha\beta}\,, \qquad \partial_\alpha^I = \frac{\partial}{\partial\theta^{\alpha I}}\,. \qquad \text{(B.6c)}$$

These expressions allow us to represent the superhelicity operator (B.2) in a manifestly supersymmetric form

$$\mathbb{Z} = \frac{1}{2}\partial^{\alpha\beta} M_{\alpha\beta} + \frac{\mathcal{N}}{4}\Delta\,, \qquad [\mathbb{Z}, D_\alpha^I] = 0\,, \qquad \text{(B.7)}$$

with the operator Δ defined by (2.8). For completeness, we recall the explicit form of the spinor covariant derivative

$$D_\alpha^I = \partial_\alpha^I + \mathrm{i}\theta^{\beta I}\partial_{\alpha\beta}\,. \qquad \text{(B.8)}$$

Consider a massive on-shell superfield of the type (9.6). Its superhelicity is equal to

$$\kappa = \frac{1}{2}\left(n + \frac{1}{2}\mathcal{N}\right)\sigma\,. \qquad \text{(B.9)}$$

The independent component fields of $T_{\alpha(n)}$ may be chosen as

$$\Phi^{I_1\dots I_k}_{\alpha_1\dots\alpha_{n+k}}(x) := \mathrm{i}^{nk+\frac{1}{2}k(k+1)} D^{[I_1}_{\alpha_1}\dots D^{I_k]}_{\alpha_k} T_{\alpha_{k+1}\dots\alpha_{n+k}}\Big|_{\theta=0}\,, \qquad 0\le k\le\mathcal{N}\,. \qquad \text{(B.10)}$$

Each of these fields is completely symmetric in its spinor indices, $\Phi^{I_1\dots I_k}_{\alpha_1\dots\alpha_{n+k}} = \Phi^{I_1\dots I_k}_{(\alpha_1\dots\alpha_{n+k})}$, and proves to be transverse,

$$\partial^{\beta\gamma}\Phi^{I_1\dots I_k}_{\alpha_1\dots\alpha_{n+k-2}\beta\gamma} = 0\,, \qquad n+k>1\,. \qquad \text{(B.11)}$$

Its helicity is equal to $\frac{1}{2}\Big(n+k\Big)\sigma$.

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