

# Linearizable special cases of the QAP

ERANDA ÇELA\*    VLADIMIR G. DEINEKO†    GERHARD J. WOEGINGER‡

## Abstract

We consider special cases of the quadratic assignment problem (QAP) that are linearizable in the sense of Bookhold. We provide combinatorial characterizations of the linearizable instances of the weighted feedback arc set QAP, and of the linearizable instances of the traveling salesman QAP. As a by-product, this yields a new well-solvable special case of the weighted feedback arc set problem.

*Keywords:* combinatorial optimization; quadratic assignment problem; linear assignment problem; computational complexity.

## 1 Introduction

The *Quadratic Assignment Problem* (QAP) and the *Linear Assignment Problem* (LAP) are two important and well-studied problems in combinatorial optimization; we refer the reader to the books by Çela [6] and by Burkard, Dell’Amico & Martello [5] for comprehensive surveys on these problems. The QAP in Koopmans-Beckmann form [17] takes as input two  $n \times n$  square matrices  $A = (a_{ij})$  and  $B = (b_{ij})$  with real entries, and assigns to every permutation  $\pi \in S_n$  (where  $S_n$  denotes the set of permutations of  $\{1, 2, \dots, n\}$ ) the corresponding objective value

$$\text{QAP}(A, B, \pi) := \sum_{i=1}^n \sum_{j=1}^n a_{\pi(i)\pi(j)} b_{ij}. \quad (1)$$

The LAP takes as input a single  $n \times n$  matrix  $C = (c_{ij})$ , and assigns to every permutation  $\pi \in S_n$  the objective value

$$\text{LAP}(C, \pi) := \sum_{i=1}^n c_{i\pi(i)}. \quad (2)$$

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\*cela@opt.math.tu-graz.ac.at. Institut für Optimierung und Diskrete Mathematik, TU Graz, Steyrergasse 30, A-8010 Graz, Austria

†vladimir.deineko@wbs.ac.uk. Warwick Business School, The University of Warwick, Coventry CV4 7AL, United Kingdom

‡gwoegi@win.tue.nl. Department of Mathematics and Computer Science, TU Eindhoven, P.O. Box 513, 5600 MB Eindhoven, Netherlands

The usual goal in these optimization problems is to identify permutations  $\pi$  that minimize the objective values (1) and (2), respectively. The QAP is NP-hard and extremely difficult to solve, whereas the LAP is polynomially solvable and easy from the computational point of view [6, 5].

Bookhold [2] calls an instance of the QAP (that is, two  $n \times n$  matrices  $A$  and  $B$ ) *linearizable*, if there exists a corresponding instance of the LAP (that is, a single  $n \times n$  matrix  $C$ ) such that

$$\text{QAP}(A, B, \pi) = \text{LAP}(C, \pi) \quad \text{for all permutations } \pi \in S_n. \quad (3)$$

Of course linearizable instances of the QAP are polynomially solvable by simply solving the corresponding instance of the LAP. The Ph.D. Thesis [9] of Erdogan contains a wealth of information on linearizations of the QAP. Further relevant papers in the area are [10, 11] by Erdogan & Tansel.

In a tour de force, Kabadi & Punnen [16, 19] designed an  $O(n^2)$  polynomial time algorithm for recognizing linearizable instances of the QAP in Koopmans-Beckmann form. Furthermore, [19] derived a purely combinatorial characterization of all linearizable QAP instances with *symmetric* matrices  $A$  and  $B$ : such instances are linearizable if and only if one of the two matrices is a weak sum matrix (see Section 3 for a more precise statement of this result). Hence linearizable *symmetric* QAP instances are fully understood and carry a highly restrictive combinatorial structure. The structure of *asymmetric* linearizable QAP instances is much richer, and it seems to be very difficult to extend the algorithmic characterization of [19] to a clean combinatorial characterization. Asymmetric linearizable QAPs are the topic of the present paper.

**Results of this paper.** We perform a combinatorial study on Bookhold linearizations of two prominent and well-studied families of *asymmetric* QAP instances: the feedback arc set problem (FAS) and the traveling salesman problem (TSP). As our main results, we derive the following combinatorial characterizations for these problems.

- An instance of the FAS is linearizable if and only if in the underlying arc weight matrix all the 3-cycles are balanced; this means that for every cycle on three vertices, the total weight of its clockwise traversal equals the total weight of its counter-clockwise traversal.
- An instance of the TSP is linearizable if and only if the underlying distance matrix is a weak sum matrix; this means that the (asymmetric) distances from city  $i$  to city  $j$  are given as the sum of two parameters, one of which only depends on  $i$  while the other one only depends on  $j$ .

For the TSP, our results indicate that linearizations will not lead to new well-solvable instances. In fact linearizations will not be able to add anything new to the TSP literature, as TSP instances on weak sum matrices have been fully analyzed a

long time ago. It is known that for weak sum matrices, all feasible solutions yield the same TSP objective value. Gabovich [13] further showed that weak sum matrices are the *only* matrices with that property.

For the FAS, our results indicate that linearizations are sometimes useful. There is one branch of research on the QAP that concentrates on the algorithmic behavior of strongly structured special cases; see for instance Burkard & al [3], Deineko & Woeginger [8], or Çela, Deineko & Woeginger [7] for typical results in this direction. Our results contribute a new well-solvable case to this research branch. Our proof method analyzes certain linear combinations of certain simple 0-1 matrices, and hence is similar in spirit to the approaches in [3, 8, 7].

**Organization of the paper.** Section 2 summarizes the relevant matrix classes and provides a characterization of balanced 3-cycle matrices. Section 3 states several observations and results on linearizable QAPs. Section 4 derives our results on the feedback arc set QAP, and Section 5 gives the results on the traveling salesman QAP. Section 6 completes the paper with a short conclusion.

## 2 The central matrix classes

In this section we summarize definitions and results around several matrix classes that will play a central role in our investigations. All matrices in this paper have real entries, and most of them are square matrices. An  $n \times n$  matrix  $A = (a_{ij})$  is a *sum matrix*, if there exist real numbers  $\alpha_1, \dots, \alpha_n$  and  $\beta_1, \dots, \beta_n$  such that

$$a_{ij} = \alpha_i + \beta_j \quad \text{for } 1 \leq i, j \leq n. \quad (4)$$

Matrix  $A$  is a *weak sum matrix*, if  $A$  can be turned into a sum matrix by appropriately changing the entries on its main diagonal. Matrix  $A$  is a *directed cut matrix*, if there exists a subset  $I \subseteq \{1, 2, \dots, n\}$  such that

$$a_{ij} = \begin{cases} 1 & \text{if } i \in I \text{ and } j \notin I \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

In graph theoretic terms, the entries in (5) encode the arcs of the directed cut from vertex set  $I$  to the complement of  $I$ . We will sometimes say that the directed cut matrix is *induced* by  $I$ .

Three indices  $i, j, k \in \{1, 2, \dots, n\}$  are said to form a *balanced 3-cycle* in an  $n \times n$  matrix  $A$ , if the corresponding entries satisfy

$$a_{ij} + a_{jk} + a_{ki} = a_{ji} + a_{kj} + a_{ik}. \quad (6)$$

This means that the total weight on the clockwise cycle  $i, j, k$  equals the total weight on the counter-clockwise cycle  $k, j, i$ . Matrix  $A$  is a *balanced 3-cycle matrix*, if every three indices  $i, j, k$  satisfy (6).

Note that (6) trivially holds whenever two of the indices  $i, j, k$  coincide. Note furthermore that the condition (6) is linear. Hence the class of balanced 3-cycle matrices is closed under addition and under multiplication by a scalar, and forms a subspace of the space of  $n \times n$  matrices. The following theorem derives a characterization of balanced 3-cycle matrices that is crucial for our arguments in Section 4.

**Theorem 2.1** *An  $n \times n$  matrix  $A$  is a balanced 3-cycle matrix, if and only if it can be written as the sum of a symmetric matrix and a linear combination of directed cut matrices.*

*Proof.* For the if part, first observe that any symmetric matrix  $A$  trivially satisfies (6). Next consider the case of a directed cut matrix  $A$  that is induced by  $I \subseteq \{1, 2, \dots, n\}$ , and let  $i, j, k$  be three indices. If all three of  $i, j, k$  are contained in  $I$  or if none of them is contained in  $I$ , then the values of the left hand side and right hand side in (6) both are 0. If exactly one or two of  $i, j, k$  are contained in  $I$ , then the values of the left hand side and right hand side in (6) both are 1. Hence any symmetric matrix and any directed cut matrix is a balanced 3-cycle matrix, and the linearity of (6) completes the first part of the proof.

For the only if part, we first subtract from  $A$  the symmetric matrix  $A'$  with entries  $a'_{ij} = a'_{ji} = \min\{a_{ij}, a_{ji}\}$ . After that subtraction, all entries in the resulting matrix  $A$  are non-negative and satisfy

$$a_{ij} a_{ji} = 0 \quad \text{for all } i, j \text{ with } 1 \leq i, j \leq n. \quad (7)$$

We fix two indices  $r$  and  $s$  such that the value  $a_{rs}$  is maximum among all the entries in matrix  $A$ . If  $a_{rs} = 0$ , then  $A$  is the all zero matrix and we are done. Otherwise  $a_{rs}$  is positive, and (7) implies  $a_{sr} = 0$ . We define set  $I$  to contain all indices  $i$  satisfying  $a_{ri} < \frac{1}{2}a_{rs}$ ; note that  $r \in I$  and  $s \notin I$ .

Now consider two arbitrary indices  $i \in I$  and  $j \notin I$ , which by definition fulfill  $a_{ri} < \frac{1}{2}a_{rs} \leq a_{rj}$ . By (7) we then have  $a_{jr} = 0$ . In case also  $a_{ij} = 0$  holds, (6) would yield

$$\frac{1}{2}a_{rs} \leq a_{rj} \leq a_{ji} + a_{rj} + a_{ir} = a_{ij} + a_{jr} + a_{ri} = a_{ri} < \frac{1}{2}a_{rs}. \quad (8)$$

This contradiction implies that

$$a_{ij} > 0 \quad \text{whenever } i \in I \text{ and } j \notin I. \quad (9)$$

Let  $A''$  be the directed cut matrix induced by  $I$ , and let  $p \in I$  and  $q \notin I$  be the indices with the smallest value  $a_{pq}$ ; then  $a_{pq} > 0$  by (9). The matrix  $A - a_{pq}A''$  has non-negative entries, satisfies (7), and has at least one more zero entry than matrix  $A$  (as it also has a zero entry at the crossing of row  $p$  and column  $q$ ).

We iterate this step and repeatedly subtract such matrices  $a_{pq}A''$  from  $A$  and thereby increase the number of zero entries. When we finally reach the all zero matrix, the subtracted matrices yield the desired representation of  $A$  as sum of a symmetric matrix and a linear combination of directed cut matrices.  $\square$

### 3 Linearizations of the QAP

In this section we collect some observations and results around linearizable QAPs. The following statement belongs to the QAP folklore and has been known (in slightly different formulations) for decades.

**Proposition 3.1** (*Folklore*) *If one of the matrices  $A$  and  $B$  is a weak sum matrix, then the QAP for  $A$  and  $B$  is linearizable.  $\square$*

If matrix  $A$  in some QAP instance is symmetric, then matrix  $B$  may also be made symmetric by replacing it by  $\frac{1}{2}(B + B^T)$ . Therefore the QAP literature only considers symmetric QAPs (where both matrices are symmetric) and asymmetric QAPs (where both matrices are asymmetric). The following result establishes the reverse of Proposition 3.1 for the case of symmetric matrices.

**Proposition 3.2** (*Punnen & Kabadi [19]*) *If the QAP for two symmetric matrices  $A$  and  $B$  is linearizable, then one of  $A$  and  $B$  is a weak sum matrix.  $\square$*

Propositions 3.1 and 3.2 provide a full combinatorial characterization of linearizable symmetric QAPs. In strong contrast to this, the structure of asymmetric linearizable QAPs is much richer, and in particular is not tied to weak sum matrices. For an illustration, consider the following three matrices:

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (10)$$

Note that matrices  $A$  and  $B$  are asymmetric and that neither of them is a weak sum matrix. Lemma 4.2 in Section 4 yields that the QAP for  $A$  and  $B$  is linearizable, and that matrix  $C$  is one possible linearization for it.

We close this section with a simple but useful observation.

**Lemma 3.3** *Let  $A_1$ ,  $A_2$  and  $B$  be  $n \times n$  matrices such that the QAP with matrices  $A_1$  and  $B$  as well as the QAP with matrices  $A_2$  and  $B$  are linearizable. Then for any real numbers  $\lambda_1$  and  $\lambda_2$ , also the QAP with matrices  $\lambda_1 A_1 + \lambda_2 A_2$  and  $B$  is linearizable.*

*Proof.* For  $k \in \{1, 2\}$  let  $C_k$  be a matrix such that  $\text{QAP}(A_k, B, \pi) = \text{LAP}(C_k, \pi)$  for all permutations  $\pi \in S_n$ . Then  $\text{QAP}(\lambda_1 A_1 + \lambda_2 A_2, B, \pi) = \text{LAP}(\lambda_1 C_1 + \lambda_2 C_2, \pi)$  for all  $\pi \in S_n$ .  $\square$

### 4 The feedback arc set QAP

A *feedback arc set* in a directed graph  $G = (V, E)$  is a subset  $E'$  of the arcs such that the subgraph  $(V, E - E')$  is a directed acyclic graph; in other words, the subset  $E'$

contains at least one arc from every directed cycle in  $G$ . The goal is to find a feedback arc set of minimum cardinality. We refer the reader to the survey article [12] by Festa, Pardalos & Resende for more information on this problem.

The problem of finding a feedback arc set of minimum cardinality can be modeled as a QAP of size  $n = |V|$ . Matrix  $A$  is the adjacency matrix of  $G$  (so that  $a_{ij} = 1$  whenever there is an arc from vertex  $i$  to vertex  $j$ , and  $a_{ij} = 0$  otherwise), and matrix  $B$  is the  $n \times n$  *feedback arc matrix*  $F_n = (f_{ij})$  whose entries are defined as follows:

$$f_{ij} = \begin{cases} 1 & \text{if } 1 \leq j < i \leq n \\ 0 & \text{if } 1 \leq i \leq j \leq n \end{cases} \quad (11)$$

In graph theoretic terms, matrix  $F_n$  is the adjacency matrix of the directed graph whose vertices are laid out on the integers  $1, 2, \dots, n$ , and whose arc set contains all possible backward arcs (that is, arcs going back from a vertex to another vertex with lower number). The permutation  $\pi$  in the QAP then specifies a topological ordering of the acyclic subgraph  $(V, E - E')$ . In the corresponding objective value (1), all the forward arcs (from vertices with low number to vertices with high number) are matched with a 0 entry in  $F_n$  and all the backward arcs (from vertices with high number to vertices with low number) are matched with a 1 entry in  $F_n$ . The backward arcs form a feedback arc set, and minimizing the cardinality of this set exactly corresponds to minimizing the objective value of the QAP.

The general *feedback arc set QAP* considers the arc-weighted version, where the goal is to find a feedback arc set of minimum weight. The first matrix  $A$  in the QAP has arbitrary real entries and encodes the arc weights, while the second matrix is the feedback arc matrix  $F_n$  as specified in (11). We will call this problem the *FAS-QAP for matrix  $A$* , or just *FAS-QAP* for short. The FAS-QAP is NP-hard, as it models the NP-hard feedback arc set problem in directed graphs [14]. In the following, we will concisely characterize all linearizable instances of the FAS-QAP.

**Lemma 4.1** *For any symmetric matrix  $A$ , the FAS-QAP for matrix  $A$  is linearizable.*

*Proof.* No matter whether vertex  $i$  comes before vertex  $j$  or after vertex  $j$  in the layout, the contribution of this vertex pair to the objective function exactly equals  $a_{ij}$ . Hence all permutations yield exactly the same objective value for this QAP instance, and the instance can be linearized trivially by a matrix  $C$  that yields the same constant LAP objective value for all permutations.  $\square$

**Lemma 4.2** *For any directed cut matrix  $A$ , the FAS-QAP for matrix  $A$  is linearizable.*

*Proof.* We assume without loss of generality that the  $n \times n$  directed cut matrix  $A$  is induced by  $I = \{1, \dots, k\}$ . For discussing the FAS-QAP, it is convenient to use the graph theoretic interpretation described at the beginning of this section. Consider a permutation  $\pi$  that assigns the  $k$  vertices of  $I$  to the  $k$  positions  $p_1 < p_2 < \dots < p_k$

in the layout. Then the vertex assigned to position  $p_i$  (with  $1 \leq i \leq k$ ) contributes  $p_i - i$  backward arcs to the objective value. Indeed, there are  $p_i - 1$  positions to the left of position  $p_i$ , of which  $i - 1$  are occupied by vertices in  $I$  while the remaining  $p_i - i$  positions are occupied by vertices not in  $I$ . There is a backward arc from the vertex at position  $p_i$  to each of these  $p_i - i$  vertices not in  $I$ . For the objective value in (1) this yields

$$\text{QAP}(A, F_n, \pi) = \sum_{i=1}^k (p_i - i). \quad (12)$$

For the linearization we use the  $n \times n$  matrix  $C$  whose first  $k$  rows are given by  $c_{ij} = j - i$  for  $i = 1, \dots, k$  and  $j = 1, \dots, n$ , and whose remaining  $n - k$  rows only contain zeroes; see (10) for an example. The objective value in (2) then becomes

$$\text{LAP}(C, \pi) = \sum_{i=1}^n c_{i\pi(i)} = \sum_{i=1}^k (\pi(i) - i). \quad (13)$$

Since the positions  $p_1 < p_2 < \dots < p_k$  are the values  $\pi(1), \dots, \pi(k)$  ordered by size, the objective values in (12) and (13) coincide.  $\square$

For an  $n \times n$  matrix  $A$  and a subset  $J \subseteq \{1, \dots, n\}$ , the principal submatrix  $A[J]$  results by removing from  $A$  all the rows and columns whose index is not in  $J$ .

**Lemma 4.3** *If the FAS-QAP for an  $n \times n$  matrix  $A$  is linearizable, then for any  $J \subseteq \{1, \dots, n\}$  the FAS-QAP for the principal submatrix  $A[J]$  is also linearizable.*

*Proof.* We assume without loss of generality that  $J = \{1, \dots, k\}$ . For a permutation  $\pi \in S_k$  we define its extension  $\pi^+ \in S_n$  by  $\pi^+(i) = \pi(i)$  for  $1 \leq i \leq k$  and  $\pi^+(i) = i$  for  $k + 1 \leq i \leq n$ . In other words, the graph layout corresponding to  $\pi^+$  starts with the vertices in  $J$  arranged according to  $\pi$ , followed by the vertices not in  $J$  arranged in strictly increasing order. Then the objective value of the FAS-QAP for  $\pi^+$  consists of three parts: the weight  $W_1^\pi$  of the backward arcs going from  $J$  into  $J$ , the weight  $W_2$  of the backward arcs going from the complement of  $J$  into the complement of  $J$ , and the weight  $W_3$  of the backward arcs going from the complement of  $J$  into  $J$ . We stress that the weights  $W_2$  and  $W_3$  only depend on  $J$  but do not depend on the choice of  $\pi$ . Hence we get for every permutation  $\pi \in S_k$  that

$$\text{QAP}(A, F_n, \pi^+) = \text{QAP}(A[J], F_k, \pi) + W_2 + W_3. \quad (14)$$

Let  $C$  be the  $n \times n$  matrix in the linearization of the FAS-QAP for  $A$ . Then

$$\text{LAP}(C, \pi^+) = \text{LAP}(C[J], \pi) + \sum_{i=k+1}^n c_{ii}. \quad (15)$$

Equations (14) and (15) show that the FAS-QAP for  $A[J]$  is linearizable. The corresponding linearization matrix is  $C[J]$  plus another linearization matrix that yields a constant LAP objective value of  $\sum_{i=k+1}^n c_{ii} - (W_2 + W_3)$ .  $\square$

|                     | QAP( $A[J], F_3, \pi$ )    | LAP( $C[J], \pi$ )         |
|---------------------|----------------------------|----------------------------|
| $\pi_1 = (i, j, k)$ | $a_{ji} + a_{kj} + a_{ki}$ | $c_{ii} + c_{jj} + c_{kk}$ |
| $\pi_2 = (i, k, j)$ | $a_{ji} + a_{jk} + a_{ki}$ | $c_{ii} + c_{jk} + c_{kj}$ |
| $\pi_3 = (j, k, i)$ | $a_{ij} + a_{kj} + a_{ik}$ | $c_{ik} + c_{ji} + c_{kj}$ |
| $\pi_4 = (j, i, k)$ | $a_{ij} + a_{kj} + a_{ki}$ | $c_{ij} + c_{ji} + c_{kk}$ |
| $\pi_5 = (k, i, j)$ | $a_{ji} + a_{jk} + a_{ik}$ | $c_{ij} + c_{jk} + c_{ki}$ |
| $\pi_6 = (k, j, i)$ | $a_{ij} + a_{jk} + a_{ik}$ | $c_{ik} + c_{jj} + c_{ki}$ |

Table 1: The objective values of the six permutations in the proof of Theorem 4.4.

**Theorem 4.4** *The FAS-QAP for matrix  $A$  is linearizable, if and only if  $A$  is a balanced 3-cycle matrix.*

*Proof.* For the if part, we first use Theorem 2.1 to decompose  $A$  into the sum of a symmetric matrix and a linear combination of directed cut matrices. Lemmas 4.1 and 4.2 imply that the FAS-QAP is linearizable for each of the summands, and then Lemma 3.3 shows that the FAS-QAP is linearizable for matrix  $A$  itself.

For the only if part, consider a matrix  $A$  for which the FAS-QAP is linearizable. Lemma 4.3 yields that the FAS-QAP for every principal  $3 \times 3$  submatrix  $A[J]$  defined by some  $J = \{i, j, k\}$  with  $i < j < k$  is linearizable. We denote the corresponding linearization by  $C[J]$ , and for convenience we index the rows and columns of  $C[J]$  also by  $i < j < k$ . Table 1 lists the objective values of the QAP and the LAP for the six permutations  $\pi_1 = (i, j, k)$ ,  $\pi_2 = (i, k, j)$ ,  $\pi_3 = (j, k, i)$ ,  $\pi_4 = (j, i, k)$ ,  $\pi_5 = (k, i, j)$ , and  $\pi_6 = (k, j, i)$ . Note that the sum of the LAP objective values for the three permutations  $\pi_1, \pi_3, \pi_5$  equals the sum of LAP objective values for the three permutations  $\pi_2, \pi_4, \pi_6$  (as both sums coincide with the sum of all the entries in matrix  $C[J]$ ). Consequently the two corresponding sums of QAP objective values are equal to each other as well, which yields

$$\begin{aligned}
& (a_{ji} + a_{kj} + a_{ki}) + (a_{ij} + a_{kj} + a_{ik}) + (a_{ji} + a_{jk} + a_{ik}) = \\
& = (a_{ji} + a_{jk} + a_{ki}) + (a_{ij} + a_{kj} + a_{ki}) + (a_{ij} + a_{jk} + a_{ik})
\end{aligned} \tag{16}$$

Some algebraic simplifications turn (16) into (6). As the choice of  $i, j, k$  was arbitrary, matrix  $A$  indeed is a balanced 3-cycle matrix.  $\square$

## 5 The traveling salesman QAP

An instance of the traveling salesman problem (TSP) consists of  $n$  cities together with an  $n \times n$  distance matrix  $A = (a_{ij})$ . The goal is to find a *cyclic* permutation  $\pi \in S_n$



that minimizes the linear assignment function  $\text{LAP}(A, \pi)$  in (2). We refer the reader to the book [18] for a wealth of information on the TSP, and to Burkard & al [4] for a survey on its well-solvable special cases. The TSP can easily be formulated as a special case of the QAP, by choosing the first matrix  $A$  as the underlying distance matrix and by choosing the second matrix as the  $n \times n$  adjacency matrix  $H_n = (h_{ij})$  of a directed Hamiltonian cycle whose entries are defined as follows:

$$h_{ij} = \begin{cases} 1 & \text{if } j = i + 1, \text{ or if } i = n \text{ and } j = 1 \\ 0 & \text{otherwise} \end{cases} \quad (17)$$

We will call this problem the *TSP-QAP for matrix  $A$* , or just *TSP-QAP* for short. We stress that in the QAP formulation, all permutations  $\pi \in S_n$  (and not just the cyclic ones) constitute feasible solutions.

In Theorem 5.2, we will concisely characterize all linearizable instances of the TSP-QAP. The proof of this theorem is based on the following result.

**Proposition 5.1** (*Gabovich [13], and independently Berenguer [1]*) *The following two statements are equivalent:*

- (i) *For the distance matrix  $A$ , all permutations  $\pi$  yield the same TSP objective value.*
- (ii) *Matrix  $A$  is a weak sum matrix.  $\square$*

Gilmore, Lawler & Shmoys [15] present a very simple and concise proof of Proposition 5.1 by means of linear algebra.

**Theorem 5.2** *The TSP-QAP for matrix  $A$  is linearizable, if and only if  $A$  is a weak sum matrix.*

*Proof.* For the if part, consider a weak sum matrix  $A$ . By Proposition 5.1 all permutations yield the same QAP objective value, and it can be linearized trivially by a matrix  $C$  that yields the same constant LAP objective value for all permutations.

For the only if part, consider an  $n \times n$  matrix  $A$  for which the TSP-QAP is linearizable and let  $C$  be the corresponding linearization. For a permutation  $\pi \in S_n$ , its cyclic shift is the permutation  $\pi^{[1]}$  defined by  $\pi^{[1]}(i) = \pi(i + 1)$  for  $1 \leq i \leq n - 1$  and  $\pi^{[1]}(n) = \pi(1)$ . For  $0 \leq k \leq n - 1$ , the  $k$ th cyclic shift of  $\pi$  results by cyclically shifting it  $k$  times; note that  $\pi^{[0]} = \pi$ . Now let us consider the total objective value of all  $n$  cyclic shifts  $\pi^{[0]}, \dots, \pi^{[n-1]}$  of permutation  $\pi$  for QAP and LAP. In the QAP, every cyclic shift  $\pi^{[k]}$  has the same objective value. All cyclic shifts correspond to the same tour through the cities, and they only differ in the choice of their starting point. This yields

$$\sum_{k=0}^{n-1} \text{QAP}(A, H_n, \pi^{[k]}) = n \cdot \text{QAP}(A, H_n, \pi). \quad (18)$$

In the LAP, the  $n$  shifts cover every element of matrix  $C$  exactly once. This yields

$$\sum_{k=0}^{n-1} \text{LAP}(C, \pi^{[k]}) = \sum_{i=1}^n \sum_{j=1}^n c_{ij}. \quad (19)$$

Since the values in (18) and (19) coincide, this implies that all tours in the traveling salesman have the same length  $(\sum_{i=1}^n \sum_{j=1}^n c_{ij})/n$ . Then Proposition 5.1 yields that  $A$  indeed is a weak sum matrix.  $\square$

## 6 Conclusion

We have given combinatorial characterizations of the linearizable instances for two classes of asymmetric QAPs: the weighted feedback arc set QAP, and the traveling salesman QAP. Similarly as in the symmetric case, all these linearizable asymmetric instances carry a very strong and very restrictive combinatorial structure.

Our results (together with the known results on the symmetric case) might indicate that linearizable instances of the QAP are rare events and will essentially never show up in real world situations. It would be interesting to support these intuitions by means of a probabilistic analysis in some reasonable stochastic model.

Another line for future research is to identify further linearizable families for the asymmetric case. A more ambitious goal would be to get a complete combinatorial characterization of all linearizable asymmetric QAP instances.

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