Invent. math. 176, 601–616 (2009)
DOI: 10.1007/s00222-008-0171-5

Inventiones mathematicae

Linearization of conservative toral homeomorphisms

T. Jäger

Collège de France, 3 rue d'Ulm, 75005 Paris, France
(e-mail: tobias.jager@college-de-france.fr)

Oblatum 14-VII-2008 & 19-XI-2008
Published online: 9 December 2008 –

Abstract. We give an equivalent condition for the existence of a semi-conjugacy to an irrational rotation for conservative homeomorphisms of the two-torus. This leads to an analogue of Poincaré's classification of circle homeomorphisms for conservative toral homeomorphisms with unique rotation vector and a certain bounded mean motion property. For minimal toral homeomorphisms, the result extends to arbitrary dimensions. Further, we provide a basic classification for the dynamics of toral homeomorphisms with all points non-wandering.

1 Introduction

One of the earliest, and still one of the most elegant, results in dynamical systems is Henri Poincaré's celebrated classification of the dynamics of circle homeomorphisms [1].

> *An orientation-preserving homeomorphism of the circle is semi-conjugate to an irrational rotation if and only if its rotation number is irrational, and if only if it has no periodic orbits.*

Ever since, the question of linearization has been one of the central themes of the subject – when can the dynamics of a given system be related to those of a linear model, as for example periodic or quasiperiodic motion on a torus? It seems natural to attempt to generalise Poincaré's result to higher dimensions. However, so far no results in this direction exist. Partly, this is explained by the fact that even on the two-torus, the situation which is best understood, obstructions to linearization other than the existence of periodic orbits appear. First of all, there does not have to be a uniquely defined rotation vector. Instead, it is only possible in general to define a rotation set, which is a compact convex subset of the plane [2] (see also (2.1) below for the definition). Further, even when this rotation set is reduced to a single, totally irrational rotation vector, a toral homeo- or diffeomorphism may have dynamics which are very different from quasiperiodic ones, for example it can exhibit weak mixing [3]. This is even true for toral flows. One way to bypass these problems is to use higher smoothness assumptions on the system, together with arithmetic conditions on the rotation vector, in order to guarantee the existence of a smooth conjugacy. This is the content of KAM-theory. However, in dimension greater than one, the price one has to pay for this is to restrict to perturbative results, meaning that the considered toral diffeomorphisms have to be close to the irrational rotation.

Here, we pursue a different direction. We show that whether or not a conservative[1] toral homeomorphism is (topologically) semi-conjugate to an irrational rotation is completely determined by the convergence properties of the rotation vector. The method is inspired by the one in [4], where an analogous result is given for skew products over irrational rotations. However, in order to overcome the lack of a fibred structure, a quite different implementation of the ideas is required. The fact that it is possible to carry these concepts over to the non-fibred setting also provides a new approach to study the dynamics of periodic point free toral homeomorphisms (see Theorem D below and also [5]), which are not yet very well understood in general (see [6–8] for some previous results and [9] for the statement of the related Franks–Misiurewicz conjecture).

Denote by $\mathrm{Homeo}_0(\mathbb{T}^d)$ the class of homeomorphisms of the d-dimensional torus which are homotopic to the identity. We say $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$ is an *irrational pseudo-rotation*, if there exists a totally irrational vector $\rho \in \mathbb{R}^d$ and a lift $F : \mathbb{R}^d \to \mathbb{R}^d$ of f, such that for all $z \in \mathbb{R}^d$ there holds

$$\lim_{n\to\infty} (F^n(z) - z)/n = \rho. \tag{1.1}$$

Similarly, when $K \subseteq \mathbb{T}$ is an invariant subset and (1.1) holds for all $z \in K$, then we say f is an irrational pseudo-rotation on K.

If f is semi-conjugate to the irrational rotation $R_\rho : z \mapsto z + \rho \bmod 1$, then it is further evident that there must be a certain rate of convergence in (1.1), namely an a priori error estimate of c/n, for some constant c independent of z. In order to reformulate this, let

$$D(n, z) := F^n(z) - z - n\rho. \tag{1.2}$$

We say an irrational pseudo-rotation f (on an invariant set $K \subseteq \mathbb{T}^d$) has *bounded mean motion*, with constant $c \geq 0$ (on K), if there holds $\|D(n, z)\| \leq c$ for all $n \in \mathbb{Z}$ and $z \in \mathbb{R}^d$ ($z \in K$).

Now, it is a natural question to ask whether these two obvious necessary conditions are already sufficient in order to guarantee the existence of a semi-conjugacy. This is not true in general, counter-examples are given in [5].

[1] By conservative, we mean that there exists an invariant probability measure of full topological support. Due to the Oxtoby–Ulam theorem, we can always assume that this measure is the Lebesgue measure on $\mathbb{T}^2$, but we will actually not make use of this fact.

However, these examples exhibit wandering open sets, such that one can still hope to obtain a positive result under additional recurrence assumptions on the system. A first, quite elementary observation is the following.

Proposition A. *Let* $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$*, and suppose* $K \subseteq \mathbb{T}^d$ *is a minimal set and* f *is an irrational pseudo-rotation with bounded mean motion on* K*. Then* $f_{|K}$ *is regularly semi-conjugate to the irrational rotation on* $\mathbb{T}^d$*.*[2]

In particular, when f has bounded mean motion on all of $\mathbb{T}^d$, then its restriction to any minimal subset is semi-conjugate to R_ρ. The analogue statement holds for toral flows.

The possibility of restricting to minimal subsets in Proposition A is particularly interesting in dimension two, since it can be combined with an old result by Misiurewicz and Ziemian [10] in order to obtain the following consequence.

Corollary B. *Suppose the rotation set of* $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ *has non-empty interior. Then for any totally irrational vector* ρ *in the interior of the rotation set, there exists a minimal subset* K_ρ*, such that* $f_{|K_\rho}$ *is regularly semi-conjugate to* R_ρ*.*

This can be seen as a natural analogue of the fact that rational rotation vectors in the interior of the rotation set are realised by periodic orbits [11].

In order to obtain an analogous result for conservative homeomorphisms of the two-torus, an important ingredient will be the concept of a *circloid*, which is a subset $C \subseteq \mathbb{T}^2$ which is (i) compact and connected, (ii) *essential* (not contained in any embedded topological disk), (iii) has a connected complement which contains an essential simple closed curve and (iv) does not contain any strictly smaller subset with properties (i)–(iii). The semi-conjugacy in the conservative case will be obtained by constructing a "lamination" on the torus consisting of pairwise disjoint circloids, on which f acts in the same way as the irrational rotation on the foliation into horizontal (or vertical) lines.

Apart from this technical purpose, circloids are also of an independent interest, since they may appear as invariant or periodic sets of a toral homeomorphism. This provides a natural generalisation of the concept of an invariant essential simple closed curve. Altogether, this leads to the following Poincaré-like classification of conservative pseudo-rotations with bounded mean motion.

Theorem C. *Suppose* $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ *is a conservative pseudo-rotation with rotation vector* $\rho \in \mathbb{R}^2$ *and bounded mean motion. Then the following hold.*

---

[2] See Sect. 2 for the definition of a *regular semi-conjugacy*. When $K = \mathbb{T}^d$, this just means that the semi-conjugacy is homotopic to the identity (and therefore, in particular, preserves the rotation vector and the bounded mean motion property).

(i) ρ *is totally irrational if and only if* f *is semi-conjugate to* R_ρ.
(ii) ρ *is neither totally irrational nor rational if and only if* f *has a periodic circloid.*
(iii) ρ *is rational if and only if* f *has a periodic point.*

Finally, the same concepts lead to the following trichotomy for the dynamics of non-wandering toral homeomorphisms. (We say a map f is *non-wandering* if there exists no non-empty open set U with $f^n(U) \cap U = \emptyset$ $\forall n \geq 1$.)

Theorem D. *Suppose* $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ *is non-wandering. Then one of the following holds.*

(i) f *is topologically transitive;*
(ii) f *has two disjoint periodic circloids;*
(iii) f *has a periodic point.*

We note that alternatives (i) and (ii) are mutually exclusive, but may both coexist with (iii). An equivalent way of expressing (ii) is to say that there exist two disjoint periodic embedded open annuli, both of which contain an essential simple closed curve.

The existence of a periodic circloid forces the rotation set to be contained in a line segment which contains no totally irrational rotation vectors (see Proposition 3.9 and Remark 3.10 below). Hence, we obtain the following corollary.

Corollary E. *Suppose* $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ *is a non-wandering irrational pseudo-rotation. Then* f *is topologically transitive.*

Acknowledgements. I would like to thank an anonymous referee for several thoughtful remarks. This work was supported by a research fellowship (Ja 1721/1-1) of the German Science Foundation (DFG).

2 The minimal case

The aim of this section is to prove a slightly more general version of Proposition A, which also takes into account the situation where the rotation set is not reduced to a single point, but contained in some lower-dimensional hyperplane. We define the rotation set of a toral homeomorphism $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$, with lift F, on a subset $K \subseteq \mathbb{T}^d$ as

(2.1)
$$\rho_K(F) := \left\{\rho \in \mathbb{R}^d \,\middle|\, \exists n_i \nearrow \infty,\ x_i \in K : \lim_{i\to\infty} \left(F^{n_i}(x_i) - x_i\right)/n_i = \rho\right\}.$$

When $K = \mathbb{T}^d$, this coincides with the standard definition (see [2]). Note that for a different lift F' of f, the rotation set $\rho_K(F')$ will be an integer translate of $\rho_K(F)$. However, this slight ambiguity will not cause any problems, and we will nevertheless call $\rho_K(F)$ *the* rotation set of f. Now, suppose $\rho_K(F)$ is contained in a $d-1$-dimensional hyperplane, that is $\rho_K(F) \subseteq \lambda v + \{v\}^{\perp}$ for some $v \in \mathbb{R}^d \setminus \{0\}$ and $\lambda \in \mathbb{R}$. In this case, we let

$$D_v(n, z) := \langle F^n(z) - z - n\rho, v\rangle, \tag{2.2}$$

where $\rho \in \rho_K(F)$ is arbitrary. We say f has *bounded mean motion parallel to v on K* if there exists a constant $c > 0$ such that

$$|D_v(n, z)| \leq c \quad \forall n \in \mathbb{Z},\ z \in K. \tag{2.3}$$

By $\|v\|$, we denote the Euclidean norm of a vector $v \in \mathbb{R}^d$, by π_i the projection to the i-th coordinate (on any product space). $\pi : \mathbb{R}^d \to \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ will denote the quotient map.

Recall that when φ and ψ are endomorphisms of topological spaces X and Y, respectively, then a continuous and onto map $h : X \to Y$ is called a semi-conjugacy from ϕ to ψ, if $h \circ \phi = \psi \circ h$. In general, the existence of a semi-conjugacy from $f_{|K}$ to an irrational rotation R_ρ does not have any implications for the rotation set. Therefore, we will use the notion of a *regular* semi-conjugacy, which we define as follows.

Suppose $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$ leaves $K \subseteq \mathbb{T}^d$ invariant and R_ρ is a rotation on the k-dimensional torus $\mathbb{T}^k$. If B is a $k \times d$ matrix with integer entries, then a semi-conjugacy $h : K \to \mathbb{T}^k$ from $f_{|K}$ to R_ρ is called *regular with respect to B* if it has a lift $H : \pi^{-1}(K) \to \mathbb{R}^k$ that semi-conjugates $F_{|\pi^{-1}(K)}$ to the translation $T_\rho : z \mapsto z + \rho$ and satisfies $\sup_{z\in\pi^{-1}(K)} \|H(z) - B(z)\| \leq \infty$. Note that in this case $\rho_K(F) \subseteq B^{-1}(\rho)$ and f has bounded mean motion orthogonal to $B^{-1}(\rho)$ (that is, parallel to all $v \in B^{-1}(\rho)^{\perp}$). Furthermore, if ρ is totally irrational, then B is surjective and hence $B^{-1}(\rho)$ is a $(d-k)$-dimensional hyperplane. When B is just the projection to the first k coordinates, we simply say that h is regular.

Proposition 2.1. *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$ and $K \subseteq \mathbb{T}^d$ be a minimal set of f. Suppose that there exists an integer vector $v \in \mathbb{Z}^d \setminus \{0\}$ with* $\gcd(v_1, \ldots, v_d) = 1$ *and a number $\rho_0 \in \mathbb{R} \setminus \mathbb{Q}$, such that*

$$\rho_K(F) \subseteq \frac{\rho_0}{\|v\|^2} \cdot v + \{v\}^{\perp}.$$

Further, assume that f has bounded mean motion parallel to v on K. Then $f_{|K}$ is regularly semi-conjugate to the one-dimensional irrational rotation $r_{\rho_0} : x \mapsto x + \rho_0 \mod 1$.

The statement can be obtained as a consequence of the Gottschalk–Hedlund theorem, but we prefer to give a short direct proof.

Proof. First, assume that $v = e^1 = (1, 0, \ldots, 0)$. Define $H : K \to \mathbb{R}$ by

$$H(z) = \pi_1(z) + \sup_{n\in\mathbb{Z}} D_{e^1}(n, z) = \sup_{n\in\mathbb{Z}} \big(\pi_1 \circ F^n(z) - n\rho_0\big). \tag{2.4}$$

Due to the bounded mean motion property H is well-defined, and it is easy to check that $H \circ F(z) = H(z) + \rho_0$. Furthermore $|H(z) - \pi_1(z)| \leq c$, where c is the bounded mean motion constant. It remains to show that H is continuous. In order to do so, note that the function $\varphi(z) = \sup_{n\in\mathbb{Z}} D_{e^1}(n, z)$ is lower semi-continuous, and $\psi(z) = \inf_{n\in\mathbb{Z}} D_{e^1}(n, z)$ is upper semi-continuous. Therefore $\varphi - \psi$ is lower semi-continuous, and a straightforward computation shows that it is furthermore invariant. Since $f_{|K}$ is minimal, this implies that $\varphi - \psi$ is equal to a constant on K, say c. It follows that $\varphi = c + \psi$ is also upper semi-continuous, hence continuous, and thus the same holds for $H(z) = \pi_1(z) + \varphi(z)$. Since H also satisfies $H(z+v) = H(z) + \pi_1(v)\ \forall v \in \mathbb{Z}^d$, its projection h to $\mathbb{T}^d$ yields the required regular semi-conjugacy. The surjectivity of h follows from the minimality of r_{ρ_0}.

In order to reduce the general case to the one treated above, let $\mathrm{Conv}^*(z_1, \ldots, z_n) := \mathrm{Conv}(z_1, \ldots, z_n) \setminus \{z_1, \ldots, z_n\}$, where Conv denotes the convex hull. Choose a basis $w^2, \ldots, w^d \in \mathbb{Z}^d$ of $\{v\}^\perp$ with the property that the $\mathrm{Conv}^*(w^2, \ldots, w^d)$ contains no integer vectors. Next, choose some vector w^1, such that $\mathrm{Conv}^*(w^1, \ldots, w^d)$ contains no integer vectors either. If we denote the matrix $(w^1, \ldots, w^d)$ by A, then the latter implies that the linear toral automorphism f_A induced by A is bijective, such that $\det A = 1$. Furthermore, $\tilde{F} = A^{-1} \circ F \circ A$ is the lift of a toral homeomorphism $\tilde{f}$. There holds

$$\rho_{f_A^{-1}(K)}(\tilde{F}) = A^{-1}(\rho_K(F)) \subseteq \frac{\rho_0}{\|v\|^2} \cdot A^{-1}(v) + \{e^1\}^\perp$$

and using $(A^{-1})^t e^1 \in (A(\{e^1\}^\perp))^\perp = \mathbb{R}v$ it is easy to check that $\tilde{f}$ has bounded deviations parallel to e^1. Thus, it only remains to show that $\langle A^{-1}(v), e^1\rangle = \|v\|^2$. Let $\tilde{v}$ be the vector representing the linear functional $x \mapsto \det(x, w^2, \ldots, w^d)$ on $\mathbb{R}^d$, that is $\det(x, w^2, \ldots, w^d) = \langle x, \tilde{v}\rangle$ $\forall x \in \mathbb{R}^d$. Then $\tilde{v} \perp w^i\ \forall i = 2, \ldots, d$, and hence $\tilde{v} \in \mathbb{R}v$. Furthermore, $x \mapsto \det(x, w^2, \ldots, w^d)$ maps integer vectors to integers, which implies $\tilde{v} \in \mathbb{Z}^d$. Finally, the existence of a vector $w^1 \in \mathbb{Z}^d$ with $\langle w^1, \tilde{v}\rangle = \det A = 1$ implies that the coordinates of $\tilde{v}$ are relatively prime, and hence $\tilde{v} = \pm v$. It follows that $|\det(v, w^2, \ldots, w^d)| = \langle v, v\rangle = \|v\|^2$, and since $\det A = 1$ we obtain

$$|\langle A^{-1}(v), e^1\rangle| = |\det(A^{-1}(v), e^2, \ldots, e^d)| = |\det(v, w^2, \ldots, w^d)| = \|v\|^2.$$

If the sign of $\langle A^{-1}v, e^1\rangle$ is negative, then we simply replace w_1 by $-w_1$.

Now, as we showed above, there exists a regular semi-conjugacy h from $\tilde{f}$ to r_{ρ_0}. Thus $h \circ A^{-1}$ yields the required semi-conjugacy from f to r_{ρ_0}, which is regular with respect to $B = \pi_1 \circ A^{-1}$. □

Remark 2.2. Even without the minimality assumption, the proof of Proposition A still yields the existence of a *'measurable semi-conjugacy'*, that is, a measurable map $h : K \to \mathbb{T}^1$ that satisfies $h \circ f_{|K} = r_{\rho_0} \circ h$. Since h must map any $f_{|K}$-invariant measure μ to the Lebesgue measure on $\mathbb{T}^1$, this is already sufficient to exclude certain exotic behaviour, like weak mixing (see [3] for examples of this type).

We obtain the following corollary, which in particular implies Proposition A.

Corollary 2.3. *Let F be a lift of $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$ and suppose there exist vectors $v^1, \dots, v^k$ with $\gcd(v^i_1, \dots, v^i_d) = 1\ \forall i = 1, \dots, k$ and a totally irrational vector $\rho \in R^k$, such that*

$$\rho_K(F) \subseteq \bigcap_{i=1}^{k} \left(\frac{\rho_i}{\|v^i\|^2} \cdot v^i + \{v^i\}^{\perp} \right).$$

Then f is regularly semi-conjugate to the k-dimensional irrational rotation R_ρ.

Proof. Let h_i be the semi-conjugacy between f and r_{ρ_i}, obtained from Proposition 2.1 with $v = v^i$. Then $h : \mathbb{T}^d \to \mathbb{T}^k$, $z \mapsto (h_1(z), \dots, h_k(z))$ yields the required semi-conjugacy between f and R_ρ. Again, the surjectivity of h follows from the minimality of R_ρ, and the regularity is inherited from that of $h_1, \dots, h_k$. □

The following result is contained in [10].

Theorem 2.4 (Theorem A in [10])**.** *Let F be a lift of $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ and suppose that $\rho(F)$ has non-empty interior. Then given any $\rho \in \mathrm{int}(\rho(F))$ there exists a minimal set M_ρ such that $\rho_{M_\rho}(F) = \{\rho\}$ and f has bounded mean motion on M_ρ.*

The bounded mean motion property is not explicity stated there, but contained in the proof (see formula (9) in [10]). Together with the preceeding statement, this yields Corollary B.

3 Invariant circloids

In the following, we collect a number of statements about circloids, both on the open annulus $\mathbb{A} = \mathbb{T}^1 \times \mathbb{R}$ and on $\mathbb{T}^2$. These results will be crucial for the proof of Theorem C in the next and of Theorem D at the end of this section. Before we start, we want to mention a well-known example, namely the so-called 'pseudo-circle' introduced by Bing [12], which shows that the structure of a circloid may be much more complicated than that of a simple closed curve. Later Handel [13] and Herman [14] showed that the pseudo-circle may appear as an invariant set of smooth surface diffeomorphisms. Nevertheless, we will see below that circloids have many 'nice' properties, which make them an interesting tool in the study of toral and annular homeomorphisms.

The definition of a circloid on the annulus is more or less the same as on the torus. However, for convenience we reformulate it, and introduce some more terminology. We say a subset $E \subseteq \mathbb{A}$ is an *annular continuum*, if it is compact and connected, and $\mathbb{A} \setminus E$ consists of exactly two connected components which are both unbounded. Note that each of the connected components will be unbounded in one direction (above or below), and bounded in the other. We say a subset $C \subseteq \mathbb{A}$ is a *circloid*, if it is an annular continuum and does not contain any strictly smaller annular continuum as a subset.

We call a set $E \subseteq \mathbb{A}$ *essential*, if its complement does not contain any connected component which is unbounded in both directions. (For compact sets, this coincides with the usual definition that E is not contained in any embedded topological disk). Now, suppose that $U \subseteq \mathbb{A}$ is bounded from below and its closure is essential. We will call such a set an *upper generating set* and define its *associated lower component* $\mathcal{L}(U)$ as the connected component of $\mathbb{A} \setminus \overline{U}$ which is unbounded from below. Similarly, we call a set $L \subseteq \mathbb{A}$ which is bounded from above and has essential closure a *lower generating set*, and define its *associated upper component* $\mathcal{U}(L)$ as the connected component of $\mathbb{A} \setminus \overline{L}$ which is unbounded from above. We call an open set $\mathcal{U}$ (respectively $\mathcal{L}$) an *upper* (*lower*) *hemisphere*, if $\mathcal{U} \cup \{+\infty\}$ is bounded from below ($\mathcal{L} \cup \{-\infty\}$ is bounded from above) and homeomorphic to the open unit disk in $\mathbb{C}$.[3] If U, respectively L, is connected, then $\mathcal{L}(U)$, respectively $\mathcal{U}(L)$, is a hemisphere in this latter sense. In order to see this, suppose Γ is a Jordan curve in $\mathcal{L}(U) \cup \{-\infty\}$. Let D be the Jordan domain in $\bar{\mathbb{A}} = \mathbb{A} \cup \{-\infty, +\infty\} \simeq \bar{\mathbb{C}}$ which is bounded by Γ and does not contain $+\infty$. Since $\overline{U}$ is connected and essential, $D \cap \overline{U} = \emptyset$. Hence D is contractible to a point in $\mathcal{L}(U) \cup \{-\infty\}$. This shows that $\mathcal{U}(L)$ is simply connected, and the assertion follows from Riemann's uniformisation theorem.

The following remark states a number of elementary properties of the above objects.

Remark 3.1. (a) If U is an upper generating set, then there exist disjoint essential simple closed curves $\Gamma_n \subseteq \mathcal{L}(U)$, such that $\bigcup_{n \in \mathbb{N}} \mathcal{L}(\Gamma_n) = \mathcal{L}(U)$. (For example, the curves Γ_n may be chosen as the images of the circles with radius $1 - 1/n$ under the homeomorphism from the unit disk to $\mathcal{L}(U) \cup \{-\infty\}$.) The analogous statement holds for lower generating sets.

(b) Any annular continuum E is the intersection of a countable nested sequence of annuli, bounded by essential simple closed curves. (Simply apply (a) to $U = L = E$.)

(c) Any upper (lower) hemisphere is an upper (lower) generating set. Hence, the expressions $\mathcal{U}\mathcal{L}(U)$, $\mathcal{L}\mathcal{U}(L)$, $\mathcal{L}\mathcal{U}\mathcal{L}(U)$ etc. make sense.

[3] In order to be absolutely correct, we should say *'punctured'* hemispheres, but we ignore this for the sake of brevity.

(d) If U and U' are upper generating sets, then $U' \subseteq U$ implies $\mathcal{L}(U) \subseteq \mathcal{L}(U')$. Similarly, if L and L' are lower generating sets and $L' \subseteq L$, then $\mathcal{U}(L) \subseteq \mathcal{U}(L')$.
(e) If U is an upper separating set, then $\mathcal{L}(U) \subseteq \mathcal{L}\mathcal{U}\mathcal{L}(U)$. (Note that $\mathcal{L}(U) \subseteq \mathcal{U}\mathcal{L}(U)^c$ by definition.) Similarly, if L is a lower separating set, then $\mathcal{U}(L) \subseteq \mathcal{U}\mathcal{L}\mathcal{U}(L)$.
(f) Suppose E is both an upper and a lower generating set, for example if E is an annular continuum. Then $\mathcal{L}(E) \subseteq \mathcal{L}\mathcal{U}(E)$ and $\mathcal{U}(E) \subseteq \mathcal{U}\mathcal{L}(E)$. (Note that $\mathcal{L}(E) \subseteq \mathcal{U}(E)^c$ and $\mathcal{U}(E) \subseteq \mathcal{L}(E)^c$.) Using (d), this further implies $\mathcal{U}\mathcal{L}\mathcal{U}(E) \subseteq \mathcal{U}\mathcal{L}(E)$ and $\mathcal{L}\mathcal{U}\mathcal{L}(E) \subseteq \mathcal{L}\mathcal{U}(E)$.

A general way to obtain circloids is the following.

Lemma 3.2. *Suppose U is an upper generating set. Then $\mathcal{C}^-(U) := \mathbb{A} \setminus (\mathcal{U}\mathcal{L}(U) \cup \mathcal{L}\mathcal{U}\mathcal{L}(U))$ is a circloid. Similarly, if L is a lower generating set, then $\mathcal{C}^+(L) := \mathbb{A} \setminus (\mathcal{L}\mathcal{U}(L) \cup \mathcal{U}\mathcal{L}\mathcal{U}(L))$ is a circloid.*

In particular, every annular continuum E contains a circloid. (Note that Remark 3.1(e) and (f) imply that $E = \mathbb{A} \setminus (\mathcal{U}(E) \cup \mathcal{L}(E))$ contains both $\mathcal{C}^+(E)$ and $\mathcal{C}^-(E)$.)

Proof of Lemma 3.2. First, note that since the operations $\mathcal{L}$ and $\mathcal{U}$ always produce hemispheres, $\mathcal{C}^-(U)$ and $\mathcal{C}^+(L)$ are annular continua.

Suppose E is an annular continuum which is contained in $\mathcal{C}^-(U)$. Then, by definition of $\mathcal{C}^-(U)$, there holds $\mathcal{U}\mathcal{L}(U) \subseteq \mathcal{U}(E)$ and $\mathcal{L}\mathcal{U}\mathcal{L}(U) \subseteq \mathcal{L}(E)$. Now $\mathcal{L}\mathcal{U}\mathcal{L}(U) \subseteq \mathcal{L}(E)$ implies, due to statement (e) in the preceding remark, $\mathcal{L}(U) \subseteq \mathcal{L}(E)$. Hence (d) yields $\mathcal{U}\mathcal{L}(E) \subseteq \mathcal{U}\mathcal{L}(U)$, and therefore $\mathcal{U}(E) \subseteq \mathcal{U}\mathcal{L}(U)$ by (f). Thus $\mathcal{U}(E) = \mathcal{U}\mathcal{L}(U)$.

Similarly, $\mathcal{U}\mathcal{L}(U) \subseteq \mathcal{U}(E)$ implies $\mathcal{L}\mathcal{U}(E) \subseteq \mathcal{L}\mathcal{U}\mathcal{L}(U)$ by (d) and thus $\mathcal{L}(E) \subseteq \mathcal{L}\mathcal{U}\mathcal{L}(U)$ by (f). Hence $\mathcal{L}(E) = \mathcal{L}\mathcal{U}\mathcal{L}(U)$. Together, we obtain

$$E = \mathbb{A} \setminus (\mathcal{U}(E) \cup \mathcal{L}(E)) = \mathbb{A} \setminus (\mathcal{U}\mathcal{L}(U) \cup \mathcal{L}\mathcal{U}\mathcal{L}(U)) = \mathcal{C}^-(U).$$

Of course, the same argument applies to $\mathcal{C}^+(L)$. □

This leads to a nice equivalent characterisation of circloids. We call an upper hemisphere U or a lower hemisphere L *reflexive*, if $\mathcal{U}\mathcal{L}(U) = U$ or $\mathcal{L}\mathcal{U}(L) = L$, respectively. We call (U, L) a *reflexive pair* of hemispheres, if $\mathcal{U}(L) = U$ and $\mathcal{L}(U) = L$.

Corollary 3.3. *An annular continuum C is a circloid if and only if $(\mathcal{U}(C), \mathcal{L}(C))$ is a reflexive pair of hemispheres.*

Lemma 3.4. *Suppose A is an annular continuum with empty interior. Then*

$$\mathcal{C}^-(A) = \mathcal{C}^+(A) = \partial\mathcal{U}(A) \cap \partial\mathcal{L}(A),$$

and this is the only circloid contained in A.

Proof. Let $C := \partial\mathcal{U}(A) \cap \partial\mathcal{L}(A)$. Since $\mathcal{U}(A)$ and $\mathcal{L}(A)$ are open and disjoint, we have

(3.1)
$$C = \overline{\mathcal{U}(A)} \cap \overline{\mathcal{L}(A)} = \left(\overline{\mathcal{U}(A)}^c \cup \overline{\mathcal{L}(A)}^c\right)^c = \mathbb{A} \setminus (\mathcal{L}\mathcal{U}(A) \cup \mathcal{U}\mathcal{L}(A)).$$

(The last inequality follows from the fact that $\mathrm{int}(A) = \emptyset$.) We first show that C is an annular continuum. Since the sets $\mathcal{L}\mathcal{U}(A)$ and $\mathcal{U}\mathcal{L}(A)$ are hemispheres, it suffices to prove that their union $V = C^c$ is not connected. Suppose for a contradiction that it is, and fix two points $z_1 \in \mathcal{L}(A) \subseteq \mathcal{L}\mathcal{U}(A)$ and $z_2 \in \mathcal{U}(A) \subseteq \mathcal{U}\mathcal{L}(A)$. Then, since V is open and connected, we can find an arc $\gamma : [0, 1] \to V$ that joins z_1 and z_2. However, the sets $\{t \in [0, 1] \mid \gamma(t) \notin \overline{\mathcal{U}(A)}\}$ and $\{t \in [0, 1] \mid \gamma(t) \notin \overline{\mathcal{L}(A)}\}$ are both open strict subsets of $[0, 1]$ and their union covers the interval, but they are disjoint (since $\overline{\mathcal{U}(A)} \cup \overline{\mathcal{L}(A)} = \mathbb{A}$). This contradicts the connectedness of $[0, 1]$. We conclude that V cannot be connected, and hence C is an annular continuum.

Now $\mathcal{L}(C) = \mathcal{L}\mathcal{U}(A)$ and $\mathcal{U}\mathcal{L}\mathcal{U}(A) \subseteq \mathcal{U}\mathcal{L}(A) = \mathcal{U}(C)$ by (3.1) and Remark 3.1(f). Hence $C \subseteq \mathcal{C}^+(A)$, and Lemma 3.2 therefore yields $C = \mathcal{C}^+(A)$. The same argument shows $C = \mathcal{C}^-(A)$. In particular, C is a circloid.

Finally, suppose C' is another circloid contained in A. Then $\mathcal{L}(A) \subseteq \mathcal{L}(C')$, and thus $\mathcal{L}(A) \cap \mathcal{U}(C') = \emptyset$. Therefore

$$\mathcal{U}(C') \subseteq \mathcal{U}\mathcal{L}(A) = \mathcal{U}(C).$$

In the same way, we obtain $\mathcal{L}(C') \subseteq \mathcal{L}(C)$, and hence $C' \subseteq C$. Since C is a circloid, we have $C' = C$. □

Next, we turn to study circloids which are invariant sets of non-wandering annular homeomorphisms. Let $\mathrm{Homeo}_0(\mathbb{A})$ denote the set of homeomorphisms of $\mathbb{A}$ which are homotopic to the identity. Given $f \in \mathrm{Homeo}_0(\mathbb{A})$, an open subset $U \subseteq \mathbb{A}$ is called *f-wandering*, if $f^n(U) \cap U = \emptyset\ \forall n \geq 1$. We call $f \in \mathrm{Homeo}_0(\mathbb{A})$ *non-wandering*, if it does not admit any non-empty wandering open set, and let $\mathrm{Homeo}_0^{\mathrm{nw}}(\mathbb{A}) := \{f \in \mathrm{Homeo}_0(\mathbb{A}) \mid f \text{ is non-wandering}\}$. Similarly, we let $\mathrm{Homeo}_0^{\mathrm{nw}}(\mathbb{T}^2) := \{f \in \mathrm{Homeo}_0(\mathbb{T}^2) \mid f \text{ is non-wandering}\}$. Finally, we call $f \in \mathrm{Homeo}_0(\mathbb{A})$ an *irrational pseudo-rotation*, if there exists an irrational number ρ, such that for all $z \in \mathbb{A}$ there holds

(3.2)
$$\lim_{n\to\infty} \pi_1(F^n(z) - z)/n = \rho.$$

Let $p : \mathbb{R}^2 \to \mathbb{A}$ be the canonical projection. The following lemma will turn out to be useful several times.

Lemma 3.5. *Suppose $f \in \mathrm{Homeo}_0^{\mathrm{nw}}(\mathbb{A})$ or $f \in \mathrm{Homeo}_0^{\mathrm{nw}}(\mathbb{T}^2)$ has no periodic points. Then any open f-invariant set contains an essential simple closed curve.*

Proof. We give the proof for the case of the annulus, the modifications needed on the torus are minor. Suppose that $f \in \mathrm{Homeo}_0^{\mathrm{nw}}(\mathbb{A})$ has no periodic points and $V \subseteq \mathbb{A}$ is an open f-invariant set. Fix a small open ball $B \subseteq V$. Since B is non-wandering, there exists some $k \geq 1$ with $f^k(B) \cap B \neq \emptyset$. Choose a lift $G : \mathbb{R}^2 \to \mathbb{R}^2$ of f^k and a connected component $\hat{B}$ of $p^{-1}(B)$, such that $G(\hat{B}) \cap \hat{B} \neq \emptyset$. Since G has no periodic points, a sufficiently small ball $D \subseteq \hat{B}$ will satisfy $G(D) \cap D = \emptyset$. It follows from a result by Franks [15, Proposition 1.3], that $G^n(D) \cap D = \emptyset \ \forall n \in \mathbb{Z}$. Thus, as $p(D)$ is non-wandering for f^k, the G-orbit of D has to intersect one of its integer translates. (Note that for any $k \geq 1$, f is non-wandering if and only if f^k is non-wandering.) The same then certainly holds for $\hat{B}$. Since $\bigcup_{n \in \mathbb{Z}} G^n(\hat{B}) \subseteq p^{-1}(V)$ is connected, this shows that V contains an essential closed curve, which can be chosen simple. $\square$

Since essential simple closed curves are circloids themselves, we obtain the following corollary.

Corollary 3.6. *Suppose $f \in \mathrm{Homeo}_0^{\mathrm{nw}}(\mathbb{A})$ or $f \in \mathrm{Homeo}_0^{\mathrm{nw}}(\mathbb{T}^2)$ has no periodic points and C is an invariant circloid. Then C has empty interior.*

Now we can prove an important property of invariant circloids.

Proposition 3.7. *Suppose $f \in \mathrm{Homeo}_0^{\mathrm{nw}}(\mathbb{A})$ has no periodic points and C_1 and C_2 are f-invariant circloids. Then either $C_1 = C_2$, or $C_1 \cap C_2 = \emptyset$.*

Again, a similar statement holds on the torus, but we will not make use thereof.

Proof. First, suppose that $\mathcal{U}(C_1) \cap \mathcal{L}(C_2) = \mathcal{L}(C_1) \cap \mathcal{U}(C_2) = \emptyset$. Then $\mathcal{U}(C_1) \subseteq \overline{\mathcal{L}(C_2)}^c$ and therefore $\mathcal{U}(C_1) \subseteq \mathcal{U}\mathcal{L}(C_2) = \mathcal{U}(C_2)$ (the equality comes from Corollary 3.3). In the same way, we see that $\mathcal{U}(C_2) \subseteq \mathcal{U}(C_1)$ and thus $\mathcal{U}(C_1) = \mathcal{U}(C_2)$. The same argument yields $\mathcal{L}(C_1) = \mathcal{L}(C_2)$, such that $C_1 = C_2$.

Otherwise, one of the two intersections is nonempty, we may assume without loss of generality that $A = \mathcal{U}(C_1) \cap \mathcal{L}(C_2) \neq \emptyset$. Since A is open and invariant, Lemma 3.5 implies that it contains an essential simple closed curve Γ. It is now easy to see that Γ separates C_1 and C_2, that is $C_1 \subseteq \mathcal{L}(\Gamma)$ and $C_2 \subseteq \mathcal{U}(\Gamma)$, which implies the disjointness of the two sets. $\square$

In order to apply these results to toral maps, we need the following basic lemma, whose simple proof is omitted.

Lemma 3.8. *Let $f \in \mathrm{Homeo}_0^{\mathrm{nw}}(\mathbb{T}^2)$ and suppose $\rho(F) \subseteq \mathbb{R} \times \{0\}$ and f has bounded mean motion parallel to $e^2 = (0, 1)$. Let $\tilde{F} : \mathbb{A} \to \mathbb{A}$ be the (uniquely defined) lift of f, such that* $\sup_{n \in \mathbb{Z}, z \in \mathbb{A}} |\pi_2 \circ \tilde{F}(z)| < \infty$. *Then $\tilde{F} \in \mathrm{Homeo}_0^{\mathrm{nw}}(\mathbb{A})$.*

We call $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ *rationally bounded*, if there exists an integer vector v and some $\lambda \in \mathbb{Q}$, such that $\rho(F) \subseteq \lambda v + \{v\}^\perp$ and f has bounded mean motion parallel to v.

Proposition 3.9. *Suppose $f \in \mathrm{Homeo}_0^{\mathrm{nw}}(\mathbb{T}^2)$ has no periodic points. Then f is rationally bounded if and only if it has a periodic circloid.*

Proof. Suppose f is rationally bounded. Using a linear change of coordinates (as in the proof of Proposition 2.1), we may assume without loss of generality that $v = e^2$. Suppose $\lambda = p/q$ with $p, q \in \mathbb{Z}$. Let $\tilde{G} : \mathbb{A} \to \mathbb{A}$ be the non-wandering lift of f^q provided by Lemma 3.8. Then $A := \bigcup_{n \in \mathbb{Z}} \tilde{G}^n(\mathbb{T}^1 \times \{0\})$ is invariant, bounded and essential, and thus $C = \mathcal{C}^+(A)$ is an $\tilde{F}$-invariant circloid. Furthermore, Proposition 3.7 yields $C \cap (C + (0,1)) = \emptyset$. This implies that there is a simple closed curve Γ contained in the region between C and $C + (0,1)$, whose projection $p(\Gamma)$ will consequently be contained in $p(C)^c$. Thus $p(C)$ is the required f^q-invariant circloid.

Conversely, suppose that there exists a q-periodic circloid C. Then $\pi^{-1}(C) \subseteq \mathbb{R}$ consists of a countable number of connected components, separated by the lifts of the essential simple closed curve Γ contained in the complement of C. A suitable lift G of f^q will leave these connected components invariant, and it is easy to see that this implies $\rho(G) \subseteq \mathbb{R}v$, where $v \in \mathbb{Z}^2 \setminus \{0\}$ is the homotopy vector of Γ. □

Remark 3.10. Note that in the above proof, the non-existence of periodic points and wandering open sets is only used to ensure that the invariant circloid in $\mathbb{A}$ projects down to a circloid in $\mathbb{T}^2$, via Proposition 3.7. However, this can equally be ensured by projecting down only to a sufficiently large finite cover of $\mathbb{T}^2$. Hence, even if these assumptions are omitted, we obtain that $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ is rationally bounded if and only if there exists a lift $\tilde{f}$ of f to a finite cover of $\mathbb{T}^2$, such that $\tilde{f}$ has a periodic circloid.

Theorem D now follows quite easily from the above results.

Proof of Theorem D. Suppose that $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ has no wandering open sets. Further, assume that f has no periodic points and is not topologically transitive. Then there exist two open sets U_1, U_2 with disjoint orbit, that is $\tilde{U}_1 \cap \tilde{U}_2 = \emptyset$, where $\tilde{U}_i = \bigcup_{n \in \mathbb{Z}} f^n(U_i)$. By Lemma 3.5, both $\tilde{U}_1$ and $\tilde{U}_2$ contain an essential simple closed curve, which we denote by Γ_1 and Γ_2, respectively. By means of a linear change of coordinates, we may assume that the homotopy type of these curves is $(1,0)$ (note that since Γ_1 and Γ_2 are disjoint, they have the same homotopy vector). Hence, they lift to essential simple closed curves in $\mathbb{A}$. Furthermore, any connected component $\hat{U}_1$ of $\pi^{-1}(\tilde{U}_1)$ will be contained between two successive lifts of Γ_2, and consequently be bounded. A suitable lift G of a suitable iterate of f will leave $\hat{U}_1$ invariant. Hence, using Lemma 3.2 we obtain the existence of two G-invariant circloids $\mathcal{C}^+(\hat{U}_1)$ and $\mathcal{C}^-(\hat{U}_1)$. These project to invariant or periodic circloids of f. They cannot project down to the same circloid, because they are both contained in the region between two successive lifts of Γ_2. □

4 The conservative case: Proof of Theorem C

Suppose that $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ is a conservative pseudo-rotation with bounded mean motion. Then the existence of a periodic orbit forces the unique rotation vector to be rational. Conversely, if the unique rotation vector is rational then the existence of a periodic orbit follows from a result of Franks [16, Theorem 3.5]. This yields the equivalence in (iii). (In fact, this holds for pseudo-rotations in general, even without the conservativity and bounded mean motion hypotheses.) The equivalence in (ii) follows from Proposition 3.9 above. Further, if f is semi-conjugate to a totally irrational rotation on $\mathbb{T}^2$, then the rotation vector evidently has to be totally irrational. Hence, it remains to prove the existence of a semi-conjugacy in (i).

Let $\tau : \mathbb{A} \to \mathbb{T}^2$ denote the canonical projection and let $T : \mathbb{A} \to \mathbb{A}$, $(x, y) \mapsto (x, y + 1)$. When A is an annular continuum and B is an arbitrary subset of $\mathbb{A}$, we will use the notation

$$\begin{aligned} A \preccurlyeq B \quad &:\Leftrightarrow \quad B \cap \mathcal{L}(A) = \emptyset; \\ A \prec B \quad &:\Leftrightarrow \quad B \subseteq \mathcal{U}(A). \end{aligned}$$

The reverse inequalities are defined analogously. If both A and B are annular continua and $A \preccurlyeq B$, then we let

$$\begin{aligned} (A, B) &:= \mathcal{U}(A) \cap \mathcal{L}(B); \\ [A, B] &:= \mathbb{A} \setminus (\mathcal{L}(A) \cup \mathcal{U}(B)). \end{aligned}$$

(Thus (A, B) is the open region strictly between A and B and $[A, B] = (A, B) \cup A \cup B$. One may think of these sets as open and closed ‘intervals’ with ‘endpoints’ A and B.) Now, suppose $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ is an irrational pseudo-rotation with rotation vector ρ and bounded mean motion with constant c. Let $\hat{f}$ be the lift of f to $\mathbb{A}$ with average vertical displacement ρ_2, such that $|\pi_2 \circ \hat{f}^n(z) - \pi_2(z) - n\rho_2| \leq c \; \forall n \in \mathbb{Z}, z \in \mathbb{A}$. We define

$$A_r := \bigcup_{n\in\mathbb{Z}} \hat{f}^n\big(\mathbb{T}^1 \times \{r - n\rho_2\}\big) \tag{4.1}$$

and

$$C_r := \mathcal{C}^+(A_r). \tag{4.2}$$

Note that due to the bounded mean motion property,

$$A_r \subseteq \mathbb{T}^1 \times [r - c, r + c]. \tag{4.3}$$

Since A_r is also essential, it is a lower generating set, and hence the definition of C_r makes sense. Further, Lemma 3.2 implies that the sets C_r are all circloids. The following properties hold and are easy to verify.

$$C_{r+1} = T(C_r) \tag{4.4}$$

$$\hat{f}(C_r) = C_{r+\rho_2} \tag{4.5}$$

$$C_r \preccurlyeq C_s \quad \text{if } r < s. \tag{4.6}$$

We claim that the circloids C_r are also disjoint, such that

$$C_r \prec C_s \quad \text{if } r < s. \tag{4.7}$$

This is in fact the crucial point in the proof, and also the part which strongly relies on the existence of the f-invariant measure μ of full topological support. In fact, the argument can be seen as a metric version of the one used in the proof of Lemma 3.5. Once we have established this assertion, the required semi-conjugacy can be constructed quite easily.

Disjointness of the circloids C_r. Note that by going over to a finite cover of $\mathbb{T}^2$ and rescaling, we may assume $c < 1/4$. This implies that $C_r \prec C_{r+1}\ \forall r \in \mathbb{R}$, such that the C_r project down to circloids on $\mathbb{T}^2$. Let $r < s$, and suppose first that $A = [C_r, C_s]$ has empty interior. In this case Lemma 3.4 shows that A contains only one circloid, and thus $C_r = C_s$. It follows that $C_{r'} = C_{s'}\ \forall r', s' \in [r, s]$. Choosing $r', s' \in [r, s]$ with $s' = r' + n\rho_2 \bmod 1$ we obtain $F^n(C_{r'-k}) = C_{r'}$ for some $k \in \mathbb{Z}$. This implies that f has an invariant or periodic circloid, and is therefore rationally bounded by Proposition 3.9, contradicting the irrationality of ρ.

Thus, we may assume that A has non-empty interior. We claim that $\mathrm{int}(A)$ contains an essential simple closed curve, which certainly implies the disjointness of C_r and C_s. In order to prove our claim, let $t = (r+s)/2$ and note that, without loss of generality, we may assume $\mathrm{int}(A') \neq \emptyset$, where $A' = [C_r, C_t]$ (otherwise, we work with $[C_t, C_s]$; one of the two sets always has non-empty interior by Baire's theorem.) Fix some open ball $V \subseteq \mathrm{int}(A')$ of diameter $\mathrm{diam}(V) \leq 1/8$ and let $V_0 = \tau(V)$. Choose some integer

$$M_1 \geq \max\left\{\frac{2\mu(\mathrm{int}(\tau(A)))}{\mu(V_0)}, 16(c+1)\right\}.$$

Further, choose some integer m, such that $(\rho_1', \rho_2') = m\rho \bmod 1$ satisfies $\rho_2' \in (0, \frac{t-r}{2M_1^3})$ and $\rho_1' \in (S\rho_2', 2S\rho_2')$, where

$$S = \frac{4M_1(c+1)}{t-r}.$$

The fact that such an m exists follows simply from the minimality of the irrational rotation R_ρ.

Let $G_0 : \mathbb{A} \to \mathbb{A}$ be the lift of f^m with rotation vector (ρ_1', ρ_2'), and note that for all $i \leq \frac{t-r}{\rho_2'}$ there holds $G_0^i(A') = [C_{r+i\rho_2'}, C_{t+i\rho_2'}] \subseteq A$. Consequently $f^{im}(\tau(A')) \subseteq \tau(A)$, and thus $f^{im}(V_0) \subseteq \mathrm{int}(\tau(A))$. Since f^m preserves μ, it follows that there exists some $k \leq M_1$, such that $f^{km}(V_0) \cap V_0 \neq \emptyset$. If $(\rho_1'', \rho_2'') = k(\rho_1', \rho_2')$, then $\rho_2'' \in (0, \frac{t-r}{2M_1^2})$ and $\rho_1'' \in (S\rho_2'', 2S\rho_2'')$.

Fix a connected component $\hat{V}_0 \subseteq \mathbb{R}^2$ of $\pi^{-1}(V_0)$ and let $G_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be the lift of f^{km} with $G_1(\hat{V}_0) \cap \hat{V}_0 \neq \emptyset$. Then the bounded mean motion property with constant $c \leq 1/4$, which G_1 inherits from f, implies that $\rho(G_1) = (\rho_1''', \rho_2''') \in \mathbb{R}^2$ satisfies $\rho_2''' = \rho_2'' \in (0, \frac{t-r}{2M_1^2})$ and $\rho_1''' \in (S\rho_2''', 2S\rho_2''')$.[4] Now, choose $n \in \mathbb{N}$ with $n \in [\frac{t-r}{4kM_1\rho_2'}, \frac{t-r}{2kM_1\rho_2'}]$. Then $n\rho_2''' \in [\frac{t-r}{4M_1}, \frac{t-r}{2M_1}]$ and $|\rho_1'''| \geq Sn\rho_2''' \geq c+1$. The bounded mean motion of G_1^n therefore implies $G_1^{jn}(\hat{V}_0) \cap \hat{V}_0 = \emptyset\ \forall j \in \mathbb{N}$. However, by the same argument as before there must be some $l \leq M_1$, such that $f^{lnkm}(V_0) \cap V_0 \neq \emptyset$. This implies that $G_1^{ln}(\hat{V}_0)$ has to intersect some integer translate of $\hat{V}_0$. Since the set $W := \bigcup_{j=1}^{ln} G_1^j(\hat{V}_0)$ is open and connected and $\pi(W) \subseteq \tau(\mathrm{int}(A))$, this proves our claim.

Construction of the semi-conjugacy. We now define

$$H_2(z) := \sup\{r \in \mathbb{R} \mid z \succ C_r\}. \tag{4.8}$$

Using (4.4) and (4.5), it can easily be checked that

$$H_2 \circ T(z) = H(z) + 1; \tag{4.9}$$

$$H_2 \circ \hat{f}(z) = H(z) + \rho_2. \tag{4.10}$$

In order to see that H_2 is continuous, suppose $(a, b) \subseteq \mathbb{R}$ is an open interval and $z \in H_2^{-1}(a, b)$. Let $c = H_2(z)$, and choose $s \in (a, c)$ and $t \in (c, b)$. Then z is contained in the open set (C_s, C_t). ($z \succ C_s$ is obvious, and $z \succcurlyeq C_t$ would imply $z \succ C_{t'}$ for all $t' < t$ by (4.7), hence $H_2(z) \geq t$.) However, $H_2(C_s, C_t) \subseteq [s, t] \subseteq (a, b)$, such that $H_2^{-1}(a, b)$ contains the open neighbourhood (C_s, C_t) of z. Since z was arbitrary, this proves that $H_2^{-1}(a, b)$ is open, and as a, b were arbitrary we obtain the continuity of H_2.

Due to (4.9) and (4.10), H_2 projects to a semi-conjugacy h_2 between f and the irrational rotation $r_{\rho_2} : x \mapsto x + \rho_2 \bmod 1$. In the same way, we can construct a semi-conjugacy h_1 between f and the irrational rotation r_{ρ_1}, and $h = (h_1, h_2)$ then yields the required semi-conjugacy between f and R_ρ on $\mathbb{T}^2$. □

[4] It is here were we use $\mathrm{diam}(V) \leq 1/8$ and $M_1 \geq 16(c+1)$. Otherwise (ρ_1''', ρ_2''') could also be a different lift of (ρ_1'', ρ_2''). (Note that ρ_1'' is only defined mod 1.)

References

1. Poincaré, H.: Mémoire sur les courbes définies par une équation différentielle. J. Math. Pures Appl., IV Sér. **1**, 167–244 (1885)
2. Misiurewicz, M., Ziemian, K.: Rotation sets for maps of tori. J. Lond. Math. Soc. **40**, 490–506 (1989)
3. Fayad, B.: Weak mixing for reparameterized linear flows on the torus. Ergodic Theory Dyn. Syst. **22**(1), 187–201 (2002)
4. Jäger, T., Stark, J.: Towards a classification for quasiperiodically forced circle homeomorphisms. J. Lond. Math. Soc. **73**(3), 727–744 (2006)
5. Jäger, T.: The concept of bounded mean motion for toral homeomorphisms. Preprint, arXiv:0802.4187 (2008)
6. Kwapisz, J.: A priori degeneracy of one-dimensional rotation sets for periodic point free torus maps. Trans. Am. Math. Soc. **354**(7), 2865–2895 (2002)
7. Kwapisz, J.: Combinatorics of torus diffeomorphisms. Ergodic Theory Dyn. Syst. **23**, 559–586 (2003)
8. Béguin, F., Crovisier, S., Le Roux, F.: Pseudo-rotations of the closed annulus: variation on a theorem of J. Kwapisz. Nonlinearity **17**(4), 1427–1453 (2004)
9. Franks, J., Misiurewicz, M.: Rotation sets of toral flows. Proc. Am. Math. Soc. **109**(1), 243–249 (1990)
10. Misiurewicz, M., Ziemian, K.: Rotation sets and ergodic measures for torus homeomorphisms. Fundam. Math. **137**(1), 45–52 (1991)
11. Franks, J.: Realizing rotation vectors for torus homeomorphisms. Trans. Am. Math. Soc. **311**(1), 107–115 (1989)
12. Bing, R.H.: A homogeneous indecomposable plane continuum. Duke Math. J. **15**, 729–742 (1948)
13. Handel, M.: A pathological area preserving C^∞ diffeomorphism of the plane. Proc. Am. Math. Soc. **86**(1), 163–168 (1982)
14. Herman, M.: Construction of some curious diffeomorphisms of the Rieman sphere. J. Lond. Math. Soc. **34**, 375–384 (1986)
15. Franks, J.: Generalizations of the Poincaré–Birkhoff theorem. Ann. Math. (2) **128**(1), 139–151 (1988)
16. Franks, J.: Recurrence and fixed points of surface homeomorphisms. Ergodic Theory Dyn. Syst. **8***, 99–107 (1988)