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LINEARIZATION TECHNIQUES FOR $\mathbb{L}^\infty$-CONTROL PROBLEMS AND DYNAMIC PROGRAMMING PRINCIPLES IN CLASSICAL AND $\mathbb{L}^\infty$-CONTROL PROBLEMS

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Abstract. The aim of the paper is to provide a linearization approach to the $\mathbb{L}^\infty$-control problems. We begin by proving a semigroup-type behaviour of the set of constraints appearing in the linearized formulation of (standard) control problems. As a byproduct we obtain a linear formulation of the dynamic programming principle. Then, we use the $\mathbb{L}^p$ approach and the associated linear formulations. This seems to be the most appropriate tool for treating $\mathbb{L}^\infty$ problems in continuous and lower semicontinuous setting.

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1. Introduction

Linear programming techniques proved to be very useful in dealing with deterministic and stochastic control problems. A wide literature is available on the subject both in the deterministic setting ([11,12,17,22]) and in the stochastic framework ([6–9,14]). Recently, the authors of [8,11,17] have provided linearized versions of the standard continuous infinite horizon discounted control problems. Their method is entirely based on Hamilton-Jacobi-Bellman equations and occupational measures. In [14], the authors give linearized primal and dual formulations for the stochastic Mayer or optimal stopping control problems. In finite horizon, the set of constraints turns out to be convex and compact with respect to the weak convergence of probability measures. These properties allow extending the control problem to a semicontinuous setting (see [14]). The method is used to characterize the (stochastic) value function as a viscosity solution of the associated HJB system in some generalized sense (*cf.* [16,18,19]).

For the different definition of discontinuous viscosity solutions see also Ishii solutions (exposed in [2]) based on semicontinuous envelopes of functions, semicontinuous solutions introduced by Frankowska in [10] and by Barron and Jensen in [5] for convex Hamiltonians, and envelope solutions [1] which are related to Subbotin minimax solution [20] (called bilateral solutions in [1]).

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In the first part of the paper, we investigate further properties of the set of constraints appearing in classical (deterministic) control problems. Identification of standard formulation to linearized problems leads to characterizing the set of constraints as the closed convex envelope of the family of occupational measures associated to the dynamic system. Using this property, we prove a semigroup behavior of the set of constraints. As application we provide a simple proof for the dynamic programming principle in classical control problems with bounded cost. In the semicontinuous and continuous framework, more precise assertions are proved.

In the second part, we aim at linearizing the value function of an $\mathbb{L}^\infty$-control problem with either continuous or lower semicontinuous cost. The study of this kind of problems is a very difficult task because, in general, it leads to strongly non-linear equations for which classical solutions may not and generally do not exist. Nevertheless, we succeed to deal with this problem by using an $\mathbb{L}^p$ approximating method. The first step consists in providing appropriate linearized primal and dual formulations for $\mathbb{L}^p$-problems. The dual formulations are somewhat different from those in the first part. They allow the passage to the limit as $p \to \infty$. Both Lipschitz-continuous and the lower semicontinuous $\mathbb{L}^p$-problems are considered. Identifying the limit of primal and dual problems gives an alternative characterization of the value function in $\mathbb{L}^\infty$-control problems. In the Lipschitz-continuous case, the standard $\mathbb{L}^\infty$-value function, its primal and dual formulations coincide. This is generally not the case for nonconvex dynamics and semicontinuous cost functions. We provide an Example in this direction. The dual problem is a linear formulation of the $\mathbb{L}^\infty$-control problem. The techniques in the proofs rely mainly on the theory of Hamilton-Jacob equations and occupational measures. We use the semigroup property of the set of constraints to prove a dynamic programming principle in bounded or continuous setting.

Our paper is organized as follows: in Section 2 we study the standard control problems with running and terminal cost. We begin by recalling the notion of occupational measures in finite-horizon in Section 2.1. The family of occupational measures is embedded in a more general set of probability measures which will be referred to as the set of constraints. This set is convex and compact with respect both to the weak convergence of probability measures and the Wasserstein distance of order 2. Its definition only depends on the coefficient function f. In Section 2.2, we recall some results on the linearized primal and dual formulation of standard control problems. Proof of these results can be found in [14], but, for reader's convenience, we provide a proof in Appendix A. As Corollary, we prove that the set of constraints is the closed convex envelope of the family of occupational measures. Using this characterization, a semigroup-type property of the set of constraints is given in Section 2.3, Proposition 2.9. We apply this result to prove dynamic programming principles (Thm. 2.11). In Section 3.1 we give linearized versions of the $\mathbb{L}^p$-control problems with Lipschitz continuous cost. Using a method based on inf convolutions, we extend the results to the lower semicontinuous setting in Section 3.2. Passing to the limit as $p \to \infty$, we give a primal and linear dual formulation for the $\mathbb{L}^\infty$-control problem (Sect. 3.3). In the continuous case, these formulations coincide with the usual value function (Thm. 3.5). A counter example is given for l.s.c. costs if no convexity assumption is made on the dynamics (Ex. 3.6). Finally, using the semigroup property of the set of constraints, we prove a dynamic programming principle for the $\mathbb{L}^\infty$-control problem (Sect. 3.4).

Throughout the paper, we will be dealing with the following control system

$$\begin{cases} dx_t^{t_0,x_0,u} = f\left(t, x_t^{t_0,x_0,u}, u_t\right) \mathrm{d}t, \ t_0 \le t \le T, \\ x_{t_0}^{t_0,x_0,u} = x_0 \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $T > 0$ is a finite time horizon, $t_0 \in [0, T]$ and the control u takes its values in a compact, metric space U. We recall that a control $(u_t)_{t_0 \le t \le T}$ is said to be admissible on $[t_0, T]$ if it is Lebesgue-measurable on $[t_0, T]$ and we let $\mathcal{U}$ denote the family of all admissible controls on $[0, T]$. We assume that the coefficient function $f : \mathbb{R} \times \mathbb{R}^N \times U \longrightarrow \mathbb{R}^N$ satisfies

$$\begin{cases} f \text{ is bounded and uniformly continuous on } \mathbb{R} \times \mathbb{R}^N \times U, \\ |f(s, x, u) - f(t, y, u)| \le c\left(|x - y| + |s - t|\right), \end{cases} \tag{1.2}$$

for all $(s, t, x, y, u) \in \mathbb{R}^2 \times \mathbb{R}^{2N} \times U$, for some positive real constant $c > 0$.

2. Linear formulations and the dynamic programing principle for classical control problems

2.1. Occupational measures

We begin by giving some properties of the linear formulations associated to control problems. Let us suppose that $T > 0$ is a fixed time horizon. We fix $t \geq 0$ and $x_0 \in \mathbb{R}^N$. To every $r > t$ and $u \in \mathcal{U}$, one can associate a couple of occupational measures $\gamma^{t,r,x_0,u} = \left(\gamma_1^{t,r,x_0,u}, \gamma_2^{t,r,x_0,u}\right) \in \mathcal{P}\left([t,r] \times \mathbb{R}^N \times U\right) \times \mathcal{P}\left(\mathbb{R}^N\right)$ defined by

$$\begin{cases} \gamma_1^{t,r,x_0,u}(A \times B \times C) = \frac{1}{r-t}\int_t^r 1_{A\times B\times C}\left(s, x_s^{t,x_0,u}, u_s\right) \mathrm{d}s, \\ \gamma_2^{t,r,x_0,u} = \delta_{x_r^{t,x_0,u}}, \end{cases}$$

for all Borel sets $A \subset [t,r]$, $B \subset \mathbb{R}^N$ and $C \subset U$. When $x \in \mathbb{R}^N$, δ_x stands for the Dirac measure. One can also define $\left(\gamma_1^{t,t,x_0,u}, \gamma_2^{t,t,x_0,u}\right) \in \mathcal{P}\left(\{t\} \times \mathbb{R}^N \times U\right) \times \mathcal{P}\left(\mathbb{R}^N\right)$ by setting

$$\gamma_1^{t,t,x_0,u} = \delta_{t,x_0,u_t}, \gamma_2^{t,t,x_0,u} = \delta_{x_0}.$$

For every $r \geq t$, the family of occupational measures

$$\Gamma(t,r,x_0) = \left\{ \left(\gamma_1^{t,r,x_0,u}, \gamma_2^{t,r,x_0,u}\right), \text{ for all } u \in \mathcal{U} \right\}, \tag{2.1}$$

can be embedded into a larger set

$$\Theta(t,r,x_0) = \left\{ \begin{array}{l} (\gamma_1,\gamma_2) \in \mathcal{P}\left([t,r]\times\mathbb{R}^N\times U\right)\times\mathcal{P}\left(\mathbb{R}^N\right), \ \forall \phi \in C_2^{1,1}\left(\mathbb{R}_+\times\mathbb{R}^N\right), \\ \int\limits_{[t,r]\times\mathbb{R}^N\times U\times\mathbb{R}^N} \left(\phi(t,x_0) + (r-t)\mathcal{L}^u\phi(s,y) - \phi(r,z)\right)\gamma_1(\mathrm{d}s\mathrm{d}y\mathrm{d}u)\,\gamma_2(\mathrm{d}z) = 0, \\ \int\limits_{\mathbb{R}^N} |y|^{2+\delta}\gamma_1([t,r],\mathrm{d}y,U) \leq c_{T,x_0}, \int\limits_{\mathbb{R}^N} |z|^{2+\delta}\gamma_2(\mathrm{d}z) \leq c_{T,x_0}, \end{array} \right\}$$

where

$$\mathcal{L}^u\phi(s,y) = \langle f(s,y,u), D\phi(s,y)\rangle + \partial_t\phi(s,y),$$

for all $\phi \in C^{1,1}\left(\mathbb{R}_+ \times \mathbb{R}^N\right)$ and all $s \geq 0, y \in \mathbb{R}^N$. Here, $\delta > 0$ is fixed and c_{T,x_0} is a positive constant depending on x_0 and, eventually, on T, δ. This constant can be chosen of the form $ce^{cT}\left(1 + |x_0|^{2+\delta}\right)$, where $c > 0$ depends only on the Lipschitz coefficient of f (but not on T, nor x_0). The set $C_2^{1,1}\left(\mathbb{R}_+ \times \mathbb{R}^N\right)$ stands for the set of continuously differentiable functions such that the function and its first order derivatives with respect to t and x have at most quadratic growth.

Remark 2.1. The set $\Theta(t,r,x_0)$ contains all occupational measures $\gamma^{t,r,x_0,u(\cdot)}$ issued from x_0 at time t. This follows from the following equality

$$\begin{aligned} -\phi(t,x_0) + \int\limits_{\mathbb{R}^N} \phi(r,z)\gamma_2^{t,r,x_0,u}(\mathrm{d}z) &= -\phi(t,x_0) + \phi\left(r, x_r^{t,x_0,u}\right) \\ &= \int_t^r \left(\partial_t\phi\left(s, x_s^{t,x_0,u}\right) + \left\langle f\left(s, x_s^{t,x_0,u}, u_s\right), D\phi\left(s, x_s^{t,x_0,u}\right)\right\rangle\right)\mathrm{d}s \\ &= \int\limits_{[t,r]\times\mathbb{R}^N\times U} (r-t)\mathcal{L}^u\phi(s,y)\,\gamma_1^{t,r,x_0,u}(\mathrm{d}s\mathrm{d}y\mathrm{d}u), \end{aligned}$$

for regular test functions $\phi \in C^{1,1}\left(\mathbb{R}_+ \times \mathbb{R}^N\right)$. Also, standard estimates and (1.2) show that

$$\sup_{s\in[t,T]} \left|x_s^{t,x_0,u}\right|^{2+\delta_0} \leq C\mathrm{e}^{CT}\left(T + |x_0|^{2+\delta}\right) \leq c\mathrm{e}^{cT}\left(1 + |x_0|^{2+\delta}\right),$$

for some constant $C > 0$ independent of T (and choosing $c = C + 1$). Therefore, the occupational measures satisfy the inequalities in the definition of $\Theta(t, r, x_0)$. In particular,

$$\begin{aligned}&\gamma_1^{t,r,x_0,u}\left([t,r], \left\{y \in \mathbb{R}^N : |y| > K\right\}, U\right) + \gamma_2^{t,r,x_0,u}\left(\left\{y \in \mathbb{R}^N : |y| > K\right\}\right)\\&\leq c\mathrm{e}^{cT}\left(1 + |x_0|^{2+\delta}\right)\frac{1}{K^{2+\delta_0}},\end{aligned}$$

for all $K > 0$. The set $\Theta(t, r, x_0)$ is also relatively compact with respect to the weak convergence of probability measures (due to Prohorov's Theorem). One notices that the equality constraint in the definition of $\Theta(t, r, x_0)$ can alternatively be written

$$\phi(t, x_0) + \int_{[t,r]\times\mathbb{R}^N\times U} (r - t)\mathcal{L}^u\phi(s, y)\gamma_1(\mathrm{d}s\mathrm{d}y\mathrm{d}u) - \int_{\mathbb{R}^N}\phi(r, z)\gamma_2(\mathrm{d}z) = 0.$$

It follows that $\Theta(t, r, x_0)$ is also convex.

We recall the following

Definition 2.2. Let $(\mathcal{X}, d)$ be a Polish metric space.

1. For any two probability measures $\gamma, \mu \in \mathcal{P}(\mathcal{X})$, the Wasserstein distance of order 2 between γ, μ is defined by the formula

$$W_2(\gamma, \mu) = \left(\inf_{\pi\in\Pi(\gamma,\mu)} \int_{\mathcal{X}\times\mathcal{X}} d(x, y)^2\pi(\mathrm{d}x\mathrm{d}y)\right)^{\frac{1}{2}},$$

where $\Pi(\gamma, \mu)$ is the family of probability measures π on $\mathcal{X} \times \mathcal{X}$ such that $\pi(\mathrm{d}x, \mathcal{X}) = \gamma(\mathrm{d}x)$ and $\pi(\mathcal{X}, \mathrm{d}y) = \mu(\mathrm{d}y)$.

2. The Wasserstein space of order 2 is defined as

$$\mathcal{P}_2(\mathcal{X}) = \left\{\gamma \in \mathcal{P}(\mathcal{X}) : \int_{\mathcal{X}} d(x_0, x)^2\gamma(\mathrm{d}x) < \infty\right\},$$

where $x_0 \in \mathcal{X}$ is arbitrary. It is known that W_2 defines a distance on $\mathcal{P}_2(\mathcal{X})$.

Remark 2.3. It follows that the set $\Theta(t, r, x_0) \subset \mathcal{P}_2\left([t, r] \times \mathbb{R}^N \times U\right) \times \mathcal{P}_2\left(\mathbb{R}^N\right)$. Moreover:

1. The family $\Theta(t, r, x_0)$ is weakly closed in $\mathcal{P}_2\left([t, r] \times \mathbb{R}^N \times U\right) \times \mathcal{P}_2\left(\mathbb{R}^N\right)$, hence, closed with respect to the distance W_2.

2. The set $\Theta(t, r, x_0)$ is compact with respect to the topology induced by W_2. This is a simple consequence of the fact that $\Theta(t, r, x_0)$ is relatively compact w.r.t. the weak convergence of probability measures, and that

$$\int_{|y|>R} |y|^2\gamma_1([t, r], \mathrm{d}y, U) \leq \frac{1}{R^\delta}c_{T,x_0} \text{ and } \int_{|z|>R} |z|^2\gamma_2(\mathrm{d}z) \leq \frac{1}{R^\delta}c_{T,x_0}.$$

3. One deduces that $\Theta(t, r, x_0)$ is also compact in $\mathcal{P}\left([t, r] \times \mathbb{R}^N \times U\right) \times \mathcal{P}\left(\mathbb{R}^N\right)$.(For more details on Wasserstein distance, the reader is referred to [21]).

2.2. Linearized formulation for classical control problems

We consider two functions $g:\mathbb{R}\times\mathbb{R}^N\times U\longrightarrow\mathbb{R}$ and $g':\mathbb{R}^N\longrightarrow\mathbb{R}$ assumed to satisfy

$$\left\{\begin{array}{l}\text{(i) the functions } g \text{ and } g' \text{ are bounded and uniformly continuous,}\\ \text{(ii) there exists a real constant } c>0 \text{ such that}\\ |g(s,x,u)-g(t,y,u)|+|g'(x)-g'(y)|\leq c\left(|x-y|+|t-s|\right),\end{array}\right. \tag{2.2}$$

for all $(s,t,x,y)\in\mathbb{R}^2\times\mathbb{R}^{2N}$ and all $u\in U$. To every $(t,x)\in[0,r]\times\mathbb{R}^N$ and $u\in\mathcal{U}$, one associates the criterion

$$J^r_{g,g'}(t,x,u)=\int_t^r g\left(s,x_s^{t,x,u},u_s\right)\mathrm{d}s+g'\left(x_r^{t,x,u}\right)$$

and the corresponding value function

$$V^r_{g,g'}(t,x)=\inf_{u\in\mathcal{U}}J^r_{g,g'}(t,x,u). \tag{2.3}$$

Under the assumptions (1.2) and (2.2), the value function V^r is the unique bounded, uniformly continuous viscosity solution of the HJ equation

$$\partial_t V^r_{g,g'}(t,x)+H\left(t,x,DV^r_{g,g'}(t,x)\right)=0, \tag{2.4}$$

for all $(t,x)\in(0,r)\times\mathbb{R}^N$, and $V^r_{g,g'}(r,\cdot)=g'(\cdot)$ on $\mathbb{R}^N$, where the Hamiltonian is given by

$$H(t,x,p)=\inf_{u\in U}\left\{\langle f(t,x,u),p\rangle+g(t,x,u)\right\}, \tag{2.5}$$

for all $(t,x,p)\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$. For proofs of the connection between $V^r_{g,g'}$ and (2.4), the reader is referred to [1] and the references therein.

We also consider the linearized problems

$$\Lambda^r_{g,g'}(t,x)=\inf_{\gamma=(\gamma_1,\gamma_2)\in\Theta(t,r,x)}\left((r-t)\int_{[t,r]\times\mathbb{R}^N\times U}g(s,y,u)\,\gamma_1(\mathrm{d}s\mathrm{d}y\mathrm{d}u)+\int_{\mathbb{R}^N}g'(z)\,\gamma_2(\mathrm{d}z)\right)$$

and its dual

$$\eta^r_{g,g'}(t,x)=\sup\left\{\begin{array}{l}\eta\in\mathbb{R}:\exists\phi\in C_2^{1,1}\left(\mathbb{R}_+\times\mathbb{R}^N\right)\text{ s.t. }\forall(s,y,v,z)\in[t,r]\times\mathbb{R}^N\times U\times\mathbb{R}^N,\\ \eta\leq(r-t)\left[\mathcal{L}^v\phi(s,y)+g(s,y,v)\right]+g'(z)-\phi(T,z)+\phi(t,x).\end{array}\right\}, \tag{2.6}$$

for all $(t,x)\in[0,r]\times\mathbb{R}^N$. The following result links the three quantities. Its proof follows the ideas in [8,13]. For reader's convenience, we give the proof in the appendix.

Proposition 2.4. *If (1.2) and (2.2) hold true, then, for every $(t,x)\in[0,r]\times\mathbb{R}^N$, one has*

$$V^r_{g,g'}(t,x)=\Lambda^r_{g,g'}(t,x)=\eta^r_{g,g'}(t,x).$$

As consequence, we have the following characterization of the set of constraints $\Theta(t,r,x_0)$:

Corollary 2.5. *The set $\Theta(t,r,x_0)$ is the closed convex hull of the family of occupational couples $\Gamma(t,r,x_0)$*

$$\Theta(t,r,x_0)=cl\ \left(co\left(\Gamma(t,r,x_0)\right)\right)=cl_{W_2}\left(co\left(\Gamma(t,r,x_0)\right)\right),$$

for all $r\geq t\geq 0$. The operator cl designates the closure with respect to the topology induced by the weak convergence of probability measures and cl_{W_2} designates the closure w.r.t. the metric W_2.

For further details, the reader is referred to [11].

Remark 2.6. In view of the Corollary 2.5, Proposition 2.4 extends to continuous functions g and g' with at most quadratic growth w.r.t. the state variable x. Similar arguments can be used for polynomial growth cost functions.

2.3. Semigroup property of the set of constraints. Application to the dynamic programing principle

We fix $x_0 \in \mathbb{R}^N$. Let us consider $t_1, t_2 \geq 0$ such that $t_1 + t_2 \leq T$, where $T > 0$ is a terminal time. If $\gamma \in \mathcal{P}\left([0, t_1] \times \mathbb{R}^N \times U\right) \times \mathcal{P}\left(\mathbb{R}^N\right)$, we define the set

$$\Theta\left(t_1, t_1 + t_2, \gamma\right) = \left\{ \begin{array}{l} \widetilde{\gamma} = (\widetilde{\gamma}_1, \widetilde{\gamma}_2) \in \mathcal{P}\left([t_1, t_1 + t_2] \times \mathbb{R}^N \times U\right) \times \mathcal{P}\left(\mathbb{R}^N\right) : \forall \phi \in C_2^{1,1}\left(\mathbb{R}_+ \times \mathbb{R}^N\right), \\ \int_{\mathbb{R}^N} \phi\left(t_1, x\right) \gamma_2\left(\mathrm{d}x\right) + \int_{[t_1, t_1+t_2] \times \mathbb{R}^N \times U} t_2 \mathcal{L}^u \phi\left(s, y\right) \widetilde{\gamma}_1\left(\mathrm{d}s\mathrm{d}y\mathrm{d}u\right) \\ \qquad - \int_{\mathbb{R}^N} \phi\left(t_1 + t_2, z\right) \widetilde{\gamma}_2\left(\mathrm{d}z\right) = 0, \\ \int_{\mathbb{R}^N} |y|^{2+\delta} \widetilde{\gamma}_1\left([t_1, t_1 + t_2], \mathrm{d}y, U\right) \leq c_{T,x_0}, \int_{\mathbb{R}^N} |z|^{2+\delta} \widetilde{\gamma}_2\left(\mathrm{d}z\right) \leq c_{T,x_0}. \end{array} \right\}$$

Proposition 2.7. *For every $t_1, t_2 \geq 0$ such that $t_1 + t_2 \leq T$ and every $\gamma = (\gamma_1, \gamma_2) \in \Theta\left(0, t_1, x_0\right)$, the set $\Theta\left(t_1, t_1 + t_2, \gamma\right)$ is nonempty, convex and compact (with respect to the weak convergence of probability measures and to W_2).*

Proof. We only need to prove that $\Theta\left(t_1, t_1 + t_2, \gamma\right)$ is nonempty. The fact that $\Theta\left(t_1, t_1 + t_2, \gamma\right)$ is convex and compact is obvious from the definition. Firstly, let us suppose that $\gamma = \gamma^{0,t_1,x_0,u}$ is an occupational couple associated to x_0 and $u \in \mathcal{U}$. We consider an arbitrary test function $\phi \in C_2^{1,1}\left(\mathbb{R}_+ \times \mathbb{R}^N\right)$. We define a couple of probability measures by setting

$$\widetilde{\gamma}_1\left(A, \mathrm{d}y\mathrm{d}u\right) = \frac{t_1 + t_2}{t_2} \gamma_1^{0,t_1+t_2,x_0,u}\left(A, \mathrm{d}y\mathrm{d}u\right), \ \widetilde{\gamma}_2 = \gamma_2^{0,t_1+t_2,x_0,u},$$

for all Borel sets $A \subset [t_1, t_1 + t_2]$. One has

$$\begin{aligned} &\int_{\mathbb{R}^N} \phi\left(t_1, x\right) \gamma_2\left(\mathrm{d}x\right) + \int_{[t_1, t_1+t_2] \times \mathbb{R}^N \times U} t_2 \mathcal{L}^u \phi\left(s, y\right) \widetilde{\gamma}_1\left(\mathrm{d}s\mathrm{d}y\mathrm{d}u\right) - \int_{\mathbb{R}^N} \phi\left(t_1 + t_2, z\right) \widetilde{\gamma}_2\left(\mathrm{d}z\right) \\ &\quad = \phi\left(t_1, x_{t_1}^{0,x_0,u}\right) + \int_{t_1}^{t_1+t_2} \mathcal{L}^{u_s} \phi\left(s, x_s^{0,x_0,u}\right) \mathrm{d}s - \phi\left(t_1 + t_2, x_{t_1+t_2}^{0,x_0,u}\right) = 0. \end{aligned} \tag{2.7}$$

Therefore, the assertion holds true for occupational couples.

To prove the result for general couples $\gamma = (\gamma_1, \gamma_2) \in \Theta\left(0, t_1, x_0\right)$, we will use Corollary 2.5. Whenever $\gamma = (\gamma_1, \gamma_2) \in \Theta\left(0, t_1, x_0\right)$, there exists a family of convex combinations

$$\left(\sum_{i=1}^{k_n} \alpha_n^i \gamma^{0,t_1,x_0,u_n^i}\right)_{n \geq 0}$$

converging to γ. One defines

$$\widetilde{\gamma}_{1,n} = \frac{t_1 + t_2}{t_2} \sum_{i=1}^{k_n} \alpha_n^i \gamma_1^{0,t_1+t_2,x_0,u_n^i}, \ \widetilde{\gamma}_{2,n} = \sum_{i=1}^{k_n} \alpha_n^i \gamma_2^{0,t_1+t_2,x_0,u_n^i},$$

for $n \geq 0$. There exists a subsequence (still denoted by $(\widetilde{\gamma}_{1,n}, \widetilde{\gamma}_{2,n})$) converging to some $\widetilde{\gamma} = (\widetilde{\gamma}_1, \widetilde{\gamma}_2) \in \mathcal{P}\left([t_1, t_1 + t_2] \times \mathbb{R}^N \times U\right) \times \mathcal{P}\left(\mathbb{R}^N\right)$. One notices that $\widetilde{\gamma} \in \Theta\left(t_1, t_1 + t_2, \gamma\right)$. □

Definition 2.8. Whenever $\gamma \in \Theta(0, t_1, x_0)$ and $\widetilde{\gamma} \in \Theta(t_1, t_1 + t_2, \gamma)$, we define

$$\widetilde{\gamma} \circ \gamma \in \mathcal{P}\left([0, t_1 + t_2] \times \mathbb{R}^N \times U\right) \times \mathcal{P}\left(\mathbb{R}^N\right) \text{ by setting:}$$
$$\begin{cases} (\widetilde{\gamma} \circ \gamma)_1 (A, dydu) = \frac{t_1}{t_1+t_2}\gamma_1 (A \cap [0, t_1], dydu) + \frac{t_2}{t_1+t_2}\widetilde{\gamma}_1 (A \cap [t_1, t_1 + t_2], dydu), \\ (\widetilde{\gamma} \circ \gamma)_2 = \widetilde{\gamma}_2, \end{cases}$$

for all Borel sets $A \subset [0, t_1 + t_2]$.

We introduce the following notation:

$$\Theta(t_1, t_1 + t_2, \cdot) \circ \Theta(0, t_1, x_0) = \{\widetilde{\gamma} \circ \gamma : \gamma \in \Theta(0, t_1, x_0), \widetilde{\gamma} \in \Theta(t_1, t_1 + t_2, \gamma)\}.$$

Proposition 2.9.

1. We have the following semigroup property

$$\Theta(t_1, t_1 + t_2, \cdot) \circ \Theta(0, t_1, x_0) = \Theta(0, t_1 + t_2, x_0),$$

for all $t_1, t_2 \geq 0$ *such that* $t_1 + t_2 \leq T$.

2. $\Theta(t_1, t_1 + t_2, \cdot) \circ \Gamma(0, t_1, x_0) \supset \Gamma(0, t_1 + t_2, x_0)$, *for all* $t_1, t_2 \geq 0$ *such that* $t_1 + t_2 \leq T$.

Proof. We only prove the first assertion. Whenever $\gamma \in \Theta(0, t_1, x_0)$ and $\widetilde{\gamma} \in \Theta(t_1, t_1 + t_2, \gamma)$,it is clear that $\widetilde{\gamma} \circ \gamma \in \Theta(0, t_1 + t_2, x_0)$. Indeed, if $\phi \in C_2^{1,1}\left(\mathbb{R}_+ \times \mathbb{R}^N\right)$, one gets

$$\phi(t, x_0) + \int_{[0,t_1] \times \mathbb{R}^N \times U} t_1 \mathcal{L}^u \phi(s, y)\, \gamma_1 (dsdydu) - \int_{\mathbb{R}^N} \phi(t_1, z)\, \gamma_2 (dz) = 0, \text{ and}$$
$$\int_{\mathbb{R}^N} \phi(t_1, z)\, \gamma_2 (dz) + \int_{[t_1,t_1+t_2] \times \mathbb{R}^N \times U} t_2 \mathcal{L}^u \phi(s, y)\, \widetilde{\gamma}_1 (dsdydu) - \int_{\mathbb{R}^N} \phi(t_1 + t_2, z)\, \widetilde{\gamma}_2 (dz) = 0.$$

The conclusion follows by summing the two equalities and recalling Definition 2.8.

Let us suppose that $\overline{\gamma} = (\overline{\gamma}_1, \overline{\gamma}_2) \in \Theta(0, t_1 + t_2, x_0)$ is an occupational measure associated to $(x_0, u) \in \mathbb{R}^N \times \mathcal{U}$. We define

$$\begin{cases} \gamma_1 (A, dydu) = \gamma_1^{0,t_1,x_0,u} (A, dydu) = \frac{t_1+t_2}{t_1}\overline{\gamma}_1 (A, dydu), \\ \gamma_2 = \gamma_2^{0,t_1,x_0,u}, \\ \widetilde{\gamma}_1 (A', dydu) \overset{def}{=} \frac{t_1+t_2}{t_2}\gamma_1^{0,t_1+t_2,x_0,u} (A', dydu) = \frac{t_1+t_2}{t_2}\overline{\gamma}_1 (A', dydu), \\ \widetilde{\gamma}_2 = \gamma_2^{0,t_1+t_2,x_0,u} = \gamma_2, \end{cases}$$

for all Borel sets $A \subset [0, t_1]$, $A' \subset [t_1, t_1 + t_2]$. It is clear that

$$\gamma = (\gamma_1, \gamma_2) \in \Theta(\ 0, t_1, x_0), \widetilde{\gamma} = (\widetilde{\gamma}_1, \widetilde{\gamma}_2) \in \Theta(t_1, t_1 + t_2, \gamma) \text{ and } \overline{\gamma} = \widetilde{\gamma} \circ \gamma.$$

Whenever $(\overline{\gamma}_1, \overline{\gamma}_2) \in \Theta(0,\ t_1 + t_2, x_0)$, there exists a sequence of convex combination of occupational measures $\left(\sum_{i=1}^{m_n} \alpha_n^i \gamma^{0,t_1+t_2,x_0,u_n^i}\right)_n$ converging to $\overline{\gamma}$ in the weak topology. We define

$$\gamma_1^n = \frac{t_1 + t_2}{t_1} \sum_{i=1}^{m_n} \alpha_n^i \gamma_1^{0,t_1,x_0,u_n^i}, \gamma_2^n = \sum_{i=1}^{m_n} \alpha_n^i \gamma_2^{0,t_1,x_0,u_n^i},$$
$$\widetilde{\gamma}_1^n (A', dydu) = \frac{t_1 + t_2}{t_2} \sum_{i=1}^{m_n} \alpha_n^i \gamma^{0,t_1+t_2,x_0,u_n^i} (A', dydu), \widetilde{\gamma}_2^n = \sum_{i=1}^{m_n} \alpha_n^i \gamma_2^{0,t_1+t_2,x_0,u_n^i}.$$

It follows that $\gamma^n = (\gamma_1^n, \gamma_2^n) \in \Theta(0, t_1, x_0)$ and $\widetilde{\gamma}^n = (\widetilde{\gamma}_1^n, \widetilde{\gamma}_2^n) \in \Theta(t_1, t_1 + t_2, \gamma^n)$ for all $n \geq 0$. The conclusion follows using compactness arguments and passing to the limit as $n \to \infty$. □

We recall the linear formulation of the value function $\Lambda_{g,g'}^{r}$

$$\Lambda_{g,g'}^{r}(t_0,x_0)=\inf_{(\gamma_1,\gamma_2)\in\Theta(t_0,r,x_0)}\left((r-t_0)\int_{[t,r]\times\mathbb{R}^N\times U}g(s,y,u)\,\gamma_1(\mathrm{d}s\mathrm{d}y\mathrm{d}u)+\int_{N}g'(z)\,\gamma_2(\mathrm{d}z)\right), \tag{2.8}$$

for all $0\le t_0\le r\le T$. If the functions g and g' are l.s.c. (respectively u.s.c.), the function $\Lambda_{g,g'}^{r}$ inherits the semicontinuity. Also, in the u.s.c. case, the infimum can be taken on $\Gamma(t_0,r,x_0)$. To prove these results, one can use an inf (resp. sup) convolution to approximate the cost functional (see [14]). If $\gamma=(\gamma_1,\gamma_2)\in\Theta(t_0,t_0,x)$, then $\gamma_2=\delta_x$ and it seems natural to impose $\Lambda_{g,g'}^{r}(t_0,\gamma)=\Lambda_{g,g'}^{r}(t_0,x_0)$. If $t_0<t\le r$ and $\gamma\in\Theta(t_0,t,x_0)$ we make the following natural notation

$$\Lambda_{g,g'}^{r}(t,\gamma)=\inf_{\widetilde{\gamma}\in\Theta(t,r,\gamma)}\left((r-t)\int_{[t,r]\times\mathbb{R}^N\times U}g(s,y,u)\,\widetilde{\gamma}_1(\mathrm{d}s\mathrm{d}y\mathrm{d}u)+\int_{\mathbb{R}^N}g'(z)\,\widetilde{\gamma}_2(\mathrm{d}z)\right). \tag{2.9}$$

Remark 2.10. If $\gamma=\gamma^{t_0,t,x_0,u}$ is an occupational couple associated to x_0 and $u\in\mathcal{U}$, then, $\Theta(t,T,\gamma)=\Theta\left(t,T,x_t^{t_0,x_0,u}\right)$ and, by definition,

$$\int_{\mathbb{R}^N}\Lambda_{g,g'}^{T}(t,x)\,\gamma_2(\mathrm{d}x)=\Lambda_{g,g'}^{T}\left(t,x_t^{t_0,x_0,u}\right)=\Lambda_{g,g'}^{T}(t,\gamma). \tag{2.10}$$

This result gives a simple proof for the dynamic programming principle for the value function (2.3).

Theorem 2.11 (dynamic programming principles)**.**

1. Let us suppose that the functions g and g' are bounded. Then, the following equality holds true

$$\Lambda_{g,g'}^{T}(t_0,x_0)=\inf_{\gamma\in\Theta(t_0,t,x_0)}\left((t-t_0)\int_{[t_0,t]\times\mathbb{R}^N\times U}g(s,y,u)\,\gamma_1(\mathrm{d}s\mathrm{d}y\mathrm{d}u)+\Lambda_{g,g'}^{T}(t,\gamma)\right), \tag{2.11}$$

for all $x_0\in\mathbb{R}^N$ and all $t\in(t_0,T)$. Also

$$\Lambda_{g,g'}^{T}(t_0,x_0)\le\inf_{\gamma\in\Gamma(t_0,t,x_0)}\left((t-t_0)\int_{[t_0,t]\times\mathbb{R}^N\times U}g(s,y,u)\,\gamma_1(\mathrm{d}s\mathrm{d}y\mathrm{d}u)+\int_{\mathbb{R}^N}\Lambda_{g,g'}^{T}(t,x)\,\gamma_2(\mathrm{d}x)\right),$$

for all $x_0\in\mathbb{R}^N$ and all $t\in(t_0,T)$.

2. If the functions g and g' are bounded and l.s.c., one also has

$$\Lambda_{g,g'}^{T}(t_0,x_0)\le\inf_{\gamma\in co\Gamma(t_0,t,x_0)}\left((t-t_0)\int_{[t_0,t]\times\mathbb{R}^N\times U}g(s,y,u)\,\gamma_1(\mathrm{d}s\mathrm{d}y\mathrm{d}u)+\int_{\mathbb{R}^N}\Lambda_{g,g'}^{T}(t,x)\,\gamma_2(\mathrm{d}x)\right),$$

for all $x_0\in\mathbb{R}^N$ and all $t\in(t_0,T)$.

3. If g and g' are bounded and upper semicontinuous, one also has

$$\Lambda_{g,g'}^{T}(t_0,x_0)=\inf_{\gamma\in\Gamma(t_0,t,x_0)}\left((t-t_0)\int_{[t_0,t]\times\mathbb{R}^N\times U}g(s,y,u)\,\gamma_1(\mathrm{d}s\mathrm{d}y\mathrm{d}u)+\int_{\mathbb{R}^N}\Lambda_{g,g'}^{T}(t,x)\,\gamma_2(\mathrm{d}x)\right),$$

for all $x_0\in\mathbb{R}^N$ and all $t\in(t_0,T)$.

4. If g and g' are bounded and continuous,

$$\Lambda_{g,g'}^{T}(t_0,x_0)=\inf_{\gamma\in\Theta(0,t,x_0)}\left((t-t_0)\int\limits_{[t_0,t]\times\mathbb{R}^N\times U}g(s,y,u)\gamma_1(\mathrm{d}s\mathrm{d}y\mathrm{d}u)+\int\limits_{\mathbb{R}^N}\Lambda_{g,g'}^{T}(t,x)\gamma_2(\mathrm{d}x)\right),$$

for all $x_0\in\mathbb{R}^N$ and all $t\in(t_0,T)$.

Proof. Let us fix $x_0\in\mathbb{R}^N$ and $t\in(0,T]$. To simplify notation, we assume that $t_0=0$.

1. Proposition 2.9 and Definition 2.8 yield

$$\begin{aligned}\Lambda_{g,g'}^{T}(0,x_0)&=\inf_{\substack{\gamma\in\Theta(0,t,x_0)\\ \widetilde{\gamma}\in\Theta(t,T,\gamma)}}\left(T\int\limits_{[0,T]\times\mathbb{R}^N\times U}g(s,y,u)(\widetilde{\gamma}\circ\gamma)_1(\mathrm{d}s\mathrm{d}y\mathrm{d}u)+\int\limits_{\mathbb{R}^N}g'(z)(\widetilde{\gamma}\circ\gamma)_2(\mathrm{d}z)\right)\\&=\inf_{\gamma\in\Theta(0,t,x_0)}\left(t\int\limits_{[0,t]\times\mathbb{R}^N\times U}g(s,y,u)\gamma_1(\mathrm{d}s\mathrm{d}y\mathrm{d}u)+\Lambda_{g,g'}^{T}(t,\gamma)\right).\end{aligned}$$

The second inequality follows from the previous Remark, by recalling that $\Gamma(0,t,x_0)\subset\Theta(0,t,x_0)$.

2. We introduce the functional $\chi:\mathcal{P}\left([t,T]\times\mathbb{R}^N\times U\right)\times\mathcal{P}\left(\mathbb{R}^N\right)\longrightarrow\mathbb{R}$ given by

$$\chi(\widetilde{\gamma})=(T-t)\int\limits_{[t,T]\times\mathbb{R}^N\times U}g(s,y,u)\widetilde{\gamma}_1(\mathrm{d}s\mathrm{d}y\mathrm{d}u)+\int\limits_{\mathbb{R}^N}g'(z)\widetilde{\gamma}_2(\mathrm{d}z),$$

for all $\widetilde{\gamma}\in\mathcal{P}\left([t,T]\times\mathbb{R}^N\times U\right)\times\mathcal{P}\left(\mathbb{R}^N\right)$. Then

$$\Lambda_{g,g'}^{T}(t,\gamma)=\inf_{\widetilde{\gamma}\in\Theta(t,T,\gamma)}\chi(\widetilde{\gamma}),$$

for all $\gamma\in\Theta(0,t,x_0)$. The functional χ is convex and l.s.c. (consequence of the l.s.c. of g and g'). Whenever $\gamma\in\Theta(0,t,x_0)$, using the compactness of $\Theta(t,T,\gamma)$, one gets the existence of some $\widetilde{\gamma}\in\Theta(t,T,\gamma)$ such that

$$\Lambda_{g,g'}^{T}(t,\gamma)=\chi(\widetilde{\gamma}).$$

If $\gamma^1,\gamma^2\in\Theta(0,t,x_0)$ and $\alpha\in[0,1]$, then there exist $\widetilde{\gamma}^i\in\Theta\left(t,T,\gamma^i\right)$ such that $\Lambda_{g,g'}^{T}\left(t,\gamma^i\right)=\chi\left(\widetilde{\gamma}^i\right)$. Moreover, $\alpha\widetilde{\gamma}^1+(1-\alpha)\widetilde{\gamma}^2\in\Theta\left(t,T,\alpha\gamma^1+(1-\alpha)\gamma^2\right)$. This implies that the functional $\gamma\mapsto\Lambda_{g,g'}^{T}(t,\gamma)$ is convex. As consequence, using (2.10), we have

$$\Lambda_{g,g'}^{T}(t,\gamma)\leq\int\limits_{\mathbb{R}^N}\Lambda_{g,g'}^{T}(t,x)\gamma_2(\mathrm{d}x), \tag{2.12}$$

for all $\gamma\in co(\Gamma(0,t,x_0))$. Using the equality (2.11) and the inequality (2.12), one gets

$$\begin{aligned}\Lambda_{g,g'}^{T}(0,x_0)&\leq\inf_{\gamma\in co(\Gamma(0,t,x_0))}\left(t\int\limits_{[0,t]\times\mathbb{R}^N\times U}g(s,y,u)\gamma_1(\mathrm{d}s\mathrm{d}y\mathrm{d}u)+\Lambda_{g,g'}^{T}(t,\gamma)\right)\\&\leq\inf_{\gamma\in co(\Gamma(0,t,x_0))}\left(t\int\limits_{[0,t]\times\mathbb{R}^N\times U}g(s,y,u)\gamma_1(\mathrm{d}s\mathrm{d}y\mathrm{d}u)+\int\limits_{\mathbb{R}^N}\Lambda_{g,g'}^{T}(t,x)\gamma_2(\mathrm{d}x)\right).\end{aligned}$$

3. Whenever g and g' are upper semicontinuous, one easily proves that $\Lambda^T_{g,g'}$ is u.s.c. Moreover, using Proposition 2.9 2. and the previous remark, we have

$$\begin{aligned}\Lambda^T_{g,g'}(0,x_0) &= \inf_{\overline{\gamma}\in\Gamma(0,T,x_0)}\left(T\int_{[0,T]\times\mathbb{R}^N\times U} g(s,y,u)\,\overline{\gamma}_1(\mathrm{d}s\mathrm{d}y\mathrm{d}u) + \int_{\mathbb{R}^N} g'(z)\,\overline{\gamma}_2(\mathrm{d}z)\right)\\ &\geq \inf_{\substack{\gamma\in\Gamma(0,t,x_0)\\ \widetilde{\gamma}\in\Theta(t,T,\gamma)}}\left(t\int_{[0,t]\times\mathbb{R}^N\times U} g(s,y,u)\,\gamma_1(\mathrm{d}s\mathrm{d}y\mathrm{d}u) + \chi(\widetilde{\gamma})\right)\\ &\geq \inf_{\gamma\in\Gamma(0,t,x_0)}\left(t\int_{[0,t]\times\mathbb{R}^N\times U} g(s,y,u)\,\gamma_1(\mathrm{d}s\mathrm{d}y\mathrm{d}u) + \int_{\mathbb{R}^N}\Lambda^T_{g,g'}(t,x)\,\gamma_2(\mathrm{d}x)\right).\end{aligned}$$

4. In the continuous case, the proof follows by combining 2. and 3. and recalling that $\Lambda^T_{g,g'}$ is continuous and $\Theta(0,t,x_0) = cl(co(\Gamma(0,t,x_0)))$. The proof of our theorem is now complete. □

3. The $\mathbb{L}^\infty$-control problem

We let $h : \mathbb{R}\times\mathbb{R}^N\times U \longrightarrow \mathbb{R}$ be a bounded function. We are interested in characterizing the following value function:

$$V_h^\infty(t_0,x_0) = \inf_{u\in\mathcal{U}} \operatorname*{ess\,sup}_{t\in[t_0,T]} \left|h\left(t, x_t^{t_0,x_0,u}, u_t\right)\right|,$$

for every $t_0\in[0,T]$ and every $x_0\in\mathbb{R}^N$. Firstly, let us suppose that

$$\begin{cases} h \text{ is uniformly continuous on } \mathbb{R}\times\mathbb{R}^N\times U \text{ and}\\ |h(s,x,u) - h(t,y,u)| \leq c(|x-y| + |s-t|),\end{cases} \tag{3.1}$$

for all $(s,t,x,y)\in\mathbb{R}^2\times\mathbb{R}^{2N}$, and all $u\in U$. Then, the value function can be approximated by considering the following sequence of value functions with $\mathbb{L}^p$-cost

$$V_h^p(t_0,x_0) = \inf_{u\in\mathcal{U}}\left(\int_{t_0}^T \left|h\left(t, x_t^{t_0,x_0,u}, u(t)\right)\right|^p \mathrm{d}t\right)^{\frac{1}{p}},$$

for every $t_0\in[0,T]$, every $x_0\in\mathbb{R}^N$ and for $p\geq 1$. Under the Assumption (3.1), V_h^p is known to satisfy, in the viscosity sense, the associated Hamilton Jacobi partial differential equation

$$\begin{cases} \partial_t V(t,x) + \min_{u\in U}\left(\langle\partial_x V(t,x), f(t,x,u)\rangle + \frac{1}{p}\left(\frac{|h(t,x,u)|}{V(t,x)}\right)^p V(t,x)\right) = 0,\\ \text{if } t\in[0,T),\ x\in\mathbb{R}^N,\\ V(T,x) = 0,\ x\in\mathbb{R}^N.\end{cases} \tag{3.2}$$

For further comments and proofs on the previous assertions, the reader is referred to [4] (Prop. 4.1).

3.1. **Linear formulation of continuous $\mathbb{L}^p$-control problems**

We begin by giving linear formulations for the $\mathbb{L}^p$ control problem. For the primal problem, the formulation is similar to the previous Section. However, for the purpose of our paper, the dual formulation will be slightly different.

We assume that

$$h^0 \geq \sup_{(t,x,u)\in\mathbb{R}\times\mathbb{R}^N\times U} |h(t,x,u)| \geq \inf_{(t,x,u)\in\mathbb{R}\times\mathbb{R}^N\times U} |h(t,x,u)| \geq 2h_0 > 0.$$

It is easy to verify the following equality

$$\begin{aligned} V_h^p(t_0,x_0) &= \inf_{u\in\mathcal{U}} \left(\int_{t_0}^T \left| h\left(t, x_t^{t_0,x_0,u}, u_t\right) \right|^p \mathrm{d}t \right)^{\frac{1}{p}} \\ &= \inf_{u\in\mathcal{U}} \left((T-t_0) \int\limits_{[t_0,T]\times\mathbb{R}^N\times U} |h(s,y,u)|^p \, \gamma_1^{t_0,T,x_0,u}(\mathrm{d}s\mathrm{d}y\mathrm{d}u) \right)^{\frac{1}{p}}, \end{aligned}$$

for all $(t_0, x_0) \in [0,T] \times \mathbb{R}^N$.

We also consider the linearized problem

$$\Lambda_h^p(t_0,x_0) = \inf_{\gamma\in\Theta(t_0,T,x_0)} \left((T-t_0) \int\limits_{[t_0,T]\times\mathbb{R}^N\times U} |h(s,y,u)|^p \, \gamma_1(\mathrm{d}s\mathrm{d}y\mathrm{d}u) \right)^{\frac{1}{p}}, \tag{3.3}$$

together with its dual formulation

$$\eta_h^p(t_0,x_0) = \sup \left\{ \begin{array}{l} \phi(t_0,x_0) - \sup_{z\in\mathbb{R}^N} \phi(T,z) : \phi \in C_b^{1,1}\left(\mathbb{R}_+\times\mathbb{R}^N;\mathbb{R}_+\right) \text{ s.t.} \\ \phi < \left(1+\frac{T}{p}\right)^{-1}(1+T)(2T\vee 1)\,h^0, \ \inf_{(s,y)\in[0,T]\times\mathbb{R}^N} \phi(s,y) > 0, \text{ and} \\ \forall (s,y,v) \in [t_0,T]\times\mathbb{R}^N\times U, \\ 0 \leq \mathcal{L}^v\phi(s,y) + \frac{1}{p}\left(\frac{|h(s,y,v)|}{\phi(s,y)}\right)^p \phi(s,y), \end{array} \right\}, \tag{3.4}$$

for all $(t_0, x_0) \in [0,T] \times \mathbb{R}^N$.

The main result of this section is the following:

Proposition 3.1. *We have that*

$$V_h^p(t_0,x_0) = \Lambda_h^p(t_0,x_0) = \eta_h^p(t_0,x_0),$$

for all $(t_0, x_0) \in [0,T] \times \mathbb{R}^N$.

Proof. We have

$$V_h^p(t_0,x_0) \geq \Lambda_h^p(t_0,x_0),$$

because $\Gamma(t_0,T,x_0) \subset \Theta(t_0,T,x_0)$.

We notice that the value function V_h^p can be approximated by a sequence of regular subsolutions of (3.2) denoted by $(\phi_n)_n \subset C_b^{1,1}\left(\mathbb{R}_+ \times \mathbb{R}^N; \mathbb{R}_+\right)$ (see for instance [3]). By construction, ϕ_n can be chosen such that

$$\begin{cases} 0 < \inf_{(s,y)\in[0,T]\times\mathbb{R}^N} \phi_n(s,y) \le \sup_{(s,y)\in[0,T+n^{-1}]\times\mathbb{R}^N} \phi_n(s,y) < (2T\vee 1)\, h^0, \text{ and} \\ \left|\phi_n(s,y) - V_h^p(s,y)\right| \le cn^{-\frac{1}{p}}, \text{ for all } (s,y)\in[0,T]\times\mathbb{R}^N, \\ \left|\phi_n(s,y)\right| \le cn^{-\frac{1}{p}}, \text{ for all } (s,y)\in\left[T, T+n^{-1}\right]\times\mathbb{R}^N, \end{cases}$$

for all $n \geq 1$. The constant c can be chosen independent of n. Moreover, every ϕ_n satisfies the following inequality

$$\mathcal{L}^v\phi_n(s,y) + \frac{|h(s,y,v)|^p}{p\phi_n^p(s,y)}\phi_n(s,y) \ge 0$$

for all $(s,y,v)\in[t_0,T]\times\mathbb{R}^N\times U$. One notices that

$$-\sup_{z\in\mathbb{R}^N}\phi_n(T,z) + \phi_n(t_0,x_0) \ge \phi_n(t_0,x_0) - cn^{-\frac{1}{p}}.$$

We obtain that

$$\phi_n(t_0,x_0) - cn^{-\frac{1}{p}} \le \eta^p(t_0,x_0).$$

Consequently,

$$V_h^p(t_0,x_0) = \lim_n \phi_n(t_0,x_0) \le \eta^p(t_0,x_0).$$

We consider an arbitrary $\gamma\in\Theta(t_0,T,x_0)$ and $\phi\in C_b^{1,1}\left(\mathbb{R}_+\times\mathbb{R}^N\right)$ such that

$$\forall (s,y,v)\in[t_0,T]\times\mathbb{R}^N\times U, 0 \le p\phi^{p-1}(s,y)\,\mathcal{L}^v\phi(s,y) + |h(s,y,v)|^p. \tag{3.5}$$

If $\phi(t_0,x_0) - \sup_{z\in\mathbb{R}^N}\phi(T,z) < 0$, then, one has

$$\phi(t_0,x_0) - \sup_{z\in\mathbb{R}^N}\phi(T,z) < \Lambda_h^p(t_0,x_0).$$

Otherwise,

$$\begin{aligned} \left(\phi(t_0,x_0) - \sup_{z\in\mathbb{R}^N}\phi(T,z)\right)^p &\le -\sup_{z\in\mathbb{R}^N}\phi^p(T,z) + \phi^p(t_0,x_0) \\ &\le \int_{[t_0,T]\times\mathbb{R}^N\times U\times\mathbb{R}^N} \Big((T-t_0)\left[p\phi^{p-1}(s,y)\,\mathcal{L}^v\phi(s,y) + |h(s,y,v)|^p\right] \\ &\quad - \phi^p(T,z) + \phi^p(t_0,x_0)\Big)\,\gamma_1(\mathrm{d}s\mathrm{d}y\mathrm{d}v)\,\gamma_2(\mathrm{d}z) \\ &\le (T-t_0)\int_{[t_0,T]\times\mathbb{R}^N\times U} |h(s,y,v)|^p\,\gamma_1(\mathrm{d}s\mathrm{d}y\mathrm{d}v). \end{aligned}$$

Consequently,

$$\eta_h^p(t_0,x_0) \le \Lambda_h^p(t_0,x_0).$$

The proof of our Proposition is now complete. □

3.2. Linear formulation of lower semicontinuous $\mathbb{L}^p$-control problems

Due to the first assertion of Proposition 3.1, it appears natural to extend the $\mathbb{L}^p$-control problem to a general bounded cost h by considering the linear formulation (3.3) for all $(t_0, x_0) \in [0,T] \times \mathbb{R}^N$ and its dual formulation (3.4). Throughout the subsection, we assume that the function $h : \mathbb{R} \times \mathbb{R}^N \times U \longrightarrow \mathbb{R}$ satisfies

$$\begin{cases} h^0 \geq \sup_{(t,x,u)\in\mathbb{R}\times\mathbb{R}^N\times U} |h(t,x,u)| \geq \inf_{(t,x,u)\in\mathbb{R}\times\mathbb{R}^N\times U} |h(t,x,u)| \geq 2h_0 > 0, \\ |h(t,\cdot,u)| \text{ is lower semicontinuous, for all } (t,u) \in \mathbb{R}\times U, \\ \sup_{x\in\mathbb{R}^N} |h(s,x,u) - h(t,x,v)| \leq c|t-s| + \omega(|u-v|), \end{cases} \tag{3.6}$$

for some real constant $c > 0$ and some continuity modulus ω. The main result of the Subsection is the following:

Proposition 3.2. *If* (1.2) *and* (3.6) *hold true, then*

$$\Lambda_h^p(t_0, x_0) = \eta_h^p(t_0, x_0),$$

for all $(t_0, x_0) \in [0,T] \times \mathbb{R}^N$.

Proof. It is obvious that, for every $t \in \mathbb{R}$, $x \in \mathbb{R}^N$ and $u \in U$,

$$\left(h_n(t,x,u) = \inf_{y\in\mathbb{R}^N} (|h(t,y,u)| + n|y-x|) \right)_{n\in\mathbb{N}^*}$$

is a nondecreasing real sequence and $\lim_{n\to\infty} h_n(t,x,u) = |h(t,x,u)|$. Moreover, h_n are bounded, uniformly continuous on $\mathbb{R}\times\mathbb{R}^N\times U$ and Lipschitz-continuous w.r.t. the time and the space variable x, uniformly w.r.t. the control variable u. Proposition 3.1 yields

$$V_{h_n}^p(t,x) = \inf_{\gamma\in\Theta(t,T,x)} \left((T-t) \int_{[t,T]\times\mathbb{R}^N\times U} (h_n(s,y,u))^p \gamma_1(\mathrm{d}s\mathrm{d}y\mathrm{d}u) \right)^{\frac{1}{p}} = \eta_{h_n}^p(t,x),$$

for all $(t,x) \in [0,T) \times \mathbb{R}^N$ and all $n \geq 1$.

One easily notices that

$$V_{h_n}^p(t,x) \leq \eta_h^p(t,x), \tag{3.7}$$

for all $(t,x) \in [0,T) \times \mathbb{R}^N$. Also, arguments similar to those in the proof of Proposition 3.1 yield

$$\eta_h^p(t,x) \leq \Lambda_h^p(t,x),$$

for all $(t,x) \in [0,T) \times \mathbb{R}^N$. Combining the previous inequalities, one has

$$\lim_n V_{h_n}^p(t,x) \leq \eta_h^p(t,x) \leq \Lambda_h^p(t,x), \tag{3.8}$$

for all $(t,x) \in [0,T) \times \mathbb{R}^N$.

Let us fix $(t,x) \in [0,T) \times \mathbb{R}^N$. Due to the compactness of $\Theta(t,T,x)$, for every $n \geq 1$, there exists $\gamma^n \in \Theta(t,T,x)$ such that

$$V_{h_n}^p(t,x) = \left((T-t) \int_{[t,T]\times\mathbb{R}^N\times U} (h_n(s,y,u))^p \gamma_1^n(\mathrm{d}s\mathrm{d}y\mathrm{d}u) \right)^{\frac{1}{p}}.$$

For every $m \geq n$,

$$V_{h_m}^p(t,x) \geq \left((T-t) \int\limits_{[t,T]\times\mathbb{R}^N\times U} (h_n(s,y,u))^p \gamma_1^m(\mathrm{d}s\mathrm{d}y\mathrm{d}u) \right)^{\frac{1}{p}}.$$

Again, due to the compactness of $\Theta(t,T,x)$, there exists a subsequence (still denoted by $(\gamma^n)_n$) converging to some $\gamma \in \Theta(t,T,x)$. Then,

$$\begin{aligned}\lim_m V_{h_m}^p(t,x) &\geq \lim_{m\to\infty} \left((T-t) \int\limits_{[t,T]\times\mathbb{R}^N\times U} (h_n(s,y,u))^p \gamma_1^m(\mathrm{d}s\mathrm{d}y\mathrm{d}u) \right)^{\frac{1}{p}} \\ &= \left((T-t) \int\limits_{[t,T]\times\mathbb{R}^N\times U} (h_n(s,y,u))^p \gamma_1(\mathrm{d}s\mathrm{d}y\mathrm{d}u) \right)^{\frac{1}{p}}.\end{aligned}$$

Passing to the limit as $n \to \infty$ and using a dominated convergence argument, one obtains

$$\lim_m V_{h_m}^p(t,x) \geq \left((T-t) \int\limits_{[t,T]\times\mathbb{R}^N\times U} (h(s,y,u))^p \gamma_1(\mathrm{d}s\mathrm{d}y\mathrm{d}u) \right)^{\frac{1}{p}} \geq \Lambda_h^p(t,x). \tag{3.9}$$

The assertion of our proposition follows from (3.8) and (3.9). □

3.3. $\mathbb{L}^\infty$-control problems

In the previous Subsections, we have linearized the $\mathbb{L}^p$-control problem. Both the primal and dual formulation will allow a passage to the limit as the parameter $p \to \infty$. The limit dual problem will be similar to those in the case of $\mathbb{L}^p$-control problems and provides a linear formulation for the $\mathbb{L}^\infty$-control problem. Motivated by the previous results, we introduce, for bounded functions h, the $\mathbb{L}^\infty$-value function Λ_h^∞ given by

$$\Lambda_h^\infty(t_0,x_0) = \inf_{\gamma\in\Theta(t_0,T,x_0)} \|h\|_{\mathbb{L}^\infty([t_0,T]\times\mathbb{R}^N\times U,\gamma_1)}, \tag{3.10}$$

for all $(t_0,x_0) \in [0,T]\times\mathbb{R}^N$. Here,

$$\|h\|_{\mathbb{L}^\infty([t_0,T]\times\mathbb{R}^N\times U,\gamma_1)} = \inf\{r \geq 0 : \gamma_1(\{(s,y,u) : |h(s,y,u)| > r\}) = 0\}$$

is the usual *ess* sup norm with respect to γ_1.

Proposition 3.3. *Let us fix $(t_0,x_0) \in [0,T)\times\mathbb{R}^N$ and suppose that (1.2) and (3.6) hold true. Then*

$$\lim_{p\to\infty} \Lambda_h^p(t_0,x_0) = \lim_{p\to\infty} \uparrow \ (T-t_0)^{-\frac{1}{p}} \Lambda_h^p(t_0,x_0) = \Lambda_h^\infty(t_0,x_0).$$

Proof. It is obvious that $\Lambda_h^\infty(t_0,x_0) \geq (T-t_0)^{-\frac{1}{p}} \Lambda_h^p(t_0,x_0)$, for all $p \geq 1$. We recall that $\Theta(t_0,T,x_0)$ is compact w.r.t. the weak convergence of probability measures and h is lower semicontinuous. Then, for every $p \geq 1$, there exists some $\gamma^p \in \Theta(t_0,T,x_0)$ such that

$$(T-t_0)^{-\frac{1}{p}} \Lambda_h^p(t_0,x_0) = \left(\int\limits_{[t_0,T]\times\mathbb{R}^N\times U} |h(s,y,u)|^p \gamma_1^p(\mathrm{d}s\mathrm{d}y\mathrm{d}u) \right)^{\frac{1}{p}}.$$

For every $q \geq p \geq 1$,

$$(T-t_0)^{-\frac{1}{q}} \Lambda_h^q(t_0,x_0) \geq \left(\int\limits_{[t_0,T]\times\mathbb{R}^N\times U} |h(s,y,u)|^p \gamma_1^q(\mathrm{d}s\mathrm{d}y\mathrm{d}u) \right)^{\frac{1}{p}}.$$

The compactness of $\Theta(t_0,T,x_0)$ yields the existence of some $\gamma \in \Theta(t_0,T,x_0)$ such that γ^p (or, at least some subsequence of $(\gamma^p)_p$) converges weakly to γ as $p \to \infty$. Then, using the l.s.c. of h, one gets

$$\liminf_{q\to\infty} (T-t_0)^{-\frac{1}{q}} \Lambda_h^q(t_0,x_0) \geq \left(\int\limits_{[t_0,T]\times\mathbb{R}^N\times U} |h(s,y,u)|^p \gamma_1(\mathrm{d}s\mathrm{d}y\mathrm{d}u) \right)^{\frac{1}{p}}.$$

The conclusion follows by letting $p \to \infty$. Notice that γ is optimal for the linearized $\mathbb{L}^\infty$-control problem. □

We also introduce the dual problem

$$\eta_h^\infty(t_0,x_0) = \sup \left\{ \begin{array}{l} \phi(t_0,x_0) - \sup_{z\in\mathbb{R}^N} \phi(T,z) : \phi \in C_b^{1,1}\left(\mathbb{R}_+\times\mathbb{R}^N;\mathbb{R}_+\right) \text{ s.t.} \\ \phi < (1+T)(2T\vee 1) h^0, \quad \inf\limits_{(s,y)\in[0,T]\times\mathbb{R}^N} \phi(s,y) > 0, \\ 0 \leq \mathcal{L}^v \phi(s,y), \ \forall (s,y,v) \in [t_0,T]\times\mathbb{R}^N\times U \text{ s.t. } |h(s,y,v)| \leq \phi(s,y), \end{array} \right\} \tag{3.11}$$

for $(t_0,x_0) \in [0,T)\times\mathbb{R}^N$.

Before stating the main result, we recall the following proposition due to [3]:

Proposition 3.4. *We assume that* (1.2) *and* (3.1) *hold true. Then, the following assertions hold true:*

(*i*) $\lim_{p\to\infty} (T-t)^{-\frac{1}{p}} V_h^p(t,x) = V_h^\infty(t,x)$ *for all* $t<T$ *and all* $x\in\mathbb{R}^N$,

(*ii*) $(T-t)^{-\frac{1}{p}} V_h^p(t,x) \leq (T-t)^{-\frac{1}{p'}} V_h^{p'}(t,x)$ *for all* $t<T$ *and all* $x\in\mathbb{R}^N$, *whenever* $p<p'$,

(*iii*) $\lim_{t\to T} (T-t)^{-\frac{1}{p}} V_h^p(t,x) = \min\limits_{u\in U} |h(T,x,u)|$.

The main result of the subsection is

Theorem 3.5.

1. (L.s.c. case) We assume that (1.2) *and* (3.6) *hold true. Then,*

$$\Lambda_h^\infty(t_0,x_0) = \eta_h^\infty(t_0,x_0),$$

for all $(t_0,x_0)\in[0,T)\times\mathbb{R}^N$.

2. (Continuous case) Moreover, if (3.1) *holds true, then*

$$V_h^\infty(t_0,x_0) = \Lambda_h^\infty(t_0,x_0) = \eta_h^\infty(t_0,x_0),$$

for all $(t_0,x_0)\in[0,T)\times\mathbb{R}^N$.

Proof. Let us fix $(t_0,x_0)\in[0,T)\times\mathbb{R}^N$. Firstly, we prove that $\Lambda_h^\infty(t_0,x_0) \leq \eta_h^\infty(t_0,x_0)$. Let us fix $p>1$. We let $\phi \in C_b^{1,1}\left(\mathbb{R}_+\times\mathbb{R}^N;\mathbb{R}_+\right)$ s.t.

$$\phi < \left(1+\frac{T}{p}\right)^{-1} (1+T)(2T\vee 1) h^0, \quad \inf_{(s,y)\in[0,T]\times\mathbb{R}^N} \phi(s,y) > 0,$$

and

$$[c]l\forall (s,y,v) \in [t_0,T] \times \mathbb{R}^N \times U,$$
$$0 \leq \mathcal{L}^v \phi(s,y) + \frac{1}{p}\left(\frac{|h(s,y,v)|}{\phi(s,y)}\right)^p \phi(s,y).$$

We define $\psi(s,y) = \phi(s,y) + \frac{s}{p}\|\phi\|$. Then, $\psi < (1+T)(2T \vee 1) h^0$. Moreover,

$$0 \leq \mathcal{L}^v \phi(s,y) + \frac{1}{p}\left(\frac{|h(s,y,v)|}{\phi(s,y)}\right)^p \phi(s,y) \leq \mathcal{L}^v \psi(s,y) + \frac{\phi(s,y) - \|\phi\|}{p} \leq \mathcal{L}^v \psi(s,y),$$

for every $(s,y,v) \in [t_0,T] \times \mathbb{R}^N \times U$ such that $|h(s,y,v)| \leq \phi(s,y)$. It follows that

$$\phi(t_0,x_0) - \sup_{z\in\mathbb{R}^N} \phi(T,z) - \frac{T(1+T)(2T\vee 1)h^0}{p} \leq \psi(t_0,x_0) - \sup_{z\in\mathbb{R}^N}\psi(T,z) \leq \eta_h^\infty(t_0,x_0).$$

The conclusion follows from Propositions 3.2 and 3.3.

For the converse, let us consider $\phi \in C_b^{1,1}(\mathbb{R}_+ \times \mathbb{R}^N; \mathbb{R}_+)$ s.t. $\phi < (1+T)(2T\vee 1)h^0$, $\inf_{(s,y)\in[0,T]\times\mathbb{R}^N} \phi(s,y) \geq 2\phi_0 > 0$, and

$$\forall (s,y,v) \in [t_0,T] \times \mathbb{R}^N \times U, \text{ whenever } |h(s,y,v)| \leq \phi(s,y), \ 0 \leq \mathcal{L}^v\phi(s,y).$$

Since $\phi < (1+T)(2T\vee 1)h^0$, for every p great enough, $\phi < \left(1+\frac{T}{p}\right)^{-1}(1+T)(2T\vee 1)h^0$. Also, for every p great enough,

$$\frac{\|\phi\|}{\sqrt{p}+1} = \frac{\sup_{(s,y)\in[t_0,T]\times\mathbb{R}^N} \phi(s,y)}{\sqrt{p}+1} < \phi_0.$$

The function ψ given by

$$\psi(s,y) = \phi(s,y) - \frac{\|\phi\|}{\sqrt{p}+1}$$

is such that $\psi < \left(1+\frac{T}{p}\right)^{-1}(1+T)(2T\vee 1)h^0$ and $\inf_{(s,y)\in[t_0,T]\times\mathbb{R}^N}\psi(s,y) > \phi_0 > 0$. Due to the fact that $\phi \in C_b^{1,1}(\mathbb{R}_+\times\mathbb{R}^N;\mathbb{R}_+)$, there exists some real constant c such that

$$\mathcal{L}^v\phi(s,y) > c, \text{ for all } (s,y,v) \in [t_0,T]\times\mathbb{R}^N\times U.$$

One notices

$$\begin{aligned}
&\mathcal{L}^v\psi(s,y) + \frac{1}{p}\left(\frac{|h(s,y,v)|}{\psi(s,y)}\right)^p \psi(s,y)\\
&\geq \mathcal{L}^v\phi(s,y)\,1_{|h(s,y,v)|\leq\phi(s,y)} + \left(c + \frac{1}{p}\left(1+\frac{1}{\sqrt{p}}\right)^p \phi_0\right)1_{|h(s,y,v)|>\phi(s,y)}\\
&\geq 0,
\end{aligned}$$

for all $(s,y,v) \in [0,T]\times\mathbb{R}^N\times U$ and all p great enough. One gets

$$\phi(t_0,x_0) - \sup_{z\in\mathbb{R}^N}\phi(T,z) = \psi(t_0,x_0) - \sup_{z\in\mathbb{R}^N}\psi(T,z) \leq \lim_{p\to\infty}\Lambda_h^p(t_0,x_0) = \Lambda_h^\infty(t_0,x_0),$$

and the proof of our assertion follows by recalling the definition of $\eta_h^\infty(t_0,x_0)$.

2. In the continuous case (Assumption 3.1),

$$\Lambda_h^\infty(t_0,x_0)=\lim_{p\to\infty}(T-t)^{-\frac{1}{p}}\Lambda_h^p(t_0,x_0)=\lim_{p\to\infty}(T-t)^{-\frac{1}{p}}V_h^p(t_0,x_0)=V_h^\infty(t_0,x_0).\qquad\square$$

The equality between V_h^∞ and Λ_h^∞ is generally not true if the cost functional is not continuous and no convexity assumption is made on the dynamics.

Example 3.6. We consider the case of $\mathbb{R}^2$, the control set is $U=\{-1,1\}$ and $f:\mathbb{R}^3\times U\longrightarrow\mathbb{R}^2$ is given by

$$f(t,x,y,u)=\left(u,x^2\wedge 1\right),$$

for all $t,x,y\in\mathbb{R}$ and all $u\in U$. We consider the control system

$$\begin{cases} dx_t^{t_0,x_0,y_0,u}=u_t\mathrm{d}t,\\ dy_t^{t_0,x_0,y_0,u}=\left(x_t^{t_0,x_0,y_0,u(\cdot)}\right)^2\wedge 1\mathrm{d}t,\end{cases}$$

for every $t_0\in[0,1]$, all $(x_0,y_0)\in\mathbb{R}^2$ and all U-valued measurable functions $u(\cdot)$. We also consider the lower semicontinuous cost function $h:\mathbb{R}\times\mathbb{R}^2\times U\longrightarrow\mathbb{R}$ defined by

$$h(t,x,y,u)=\begin{cases}1, \text{ if } (x,y)\neq(0,0),\\ \frac{1}{2}, \text{ if } (x,y)=(0,0),\end{cases}$$

for all $t\in\mathbb{R}$ and all $u\in U$. One notices that, $(0,0)\notin\left\{x_t^{0,0,0,,u}:t\in(0,1]\right\}$. Thus,

$$V_h^\infty(0,0,0)=1.$$

On the other hand, for $n\geq 1$, we introduce the control

$$u^n=\sum_{k=0}^{n-1}1_{\left[\frac{2k}{2n},\frac{(2k+1)}{2n}\right)}-1_{\left[\frac{(2k+1)}{2n},\frac{(2k+2)}{2n}\right)}+1_{\{1\}}.$$

Then $\mathrm{Supp}\left(\gamma_1^{0,0,0,u^n}\right)\subset[0,t_0]\times\left[0,\frac{1}{n}\right]\times\left[0,\frac{1}{n^2}\right]\times U$. We recall that $\Theta(0,1,(0,0))$ is compact and get the existence of some subsequence $\left(\gamma^{0,0,0,u^n}\right)_n$ converging to some $\gamma\in\Theta(0,1,(0,0))$. It follows that $\mathrm{Supp}(\gamma_1)\subset[0,t_0]\times\{(0,0)\}\times U$ and

$$\Lambda_h^\infty(0,0,0)=\frac{1}{2}<V_h^\infty(0,0,0).$$

3.4. Dynamic programming principle for $\mathbb{L}^\infty$-control problems

Using the primal formulation for the $\mathbb{L}^\infty$-control problem and the semigroup property of the set of constraints, we give a simple proof for the dynamic programming principle (DPP).

If $t_0<t<T$, $x_0\in\mathbb{R}^N$ and $\gamma\in\Theta(t_0,t,x_0)$ we make the following natural notation

$$\Lambda_h^\infty(t,\gamma)=\inf_{\widetilde{\gamma}\in\Theta(t,T,\gamma)}\|h\|_{\mathbb{L}^\infty([t,T]\times\mathbb{R}^N\times U,\widetilde{\gamma}_1)}.\tag{3.12}$$

Remark 3.7. If $\gamma=\gamma^{t_0,t,x_0,u}$ is an occupational couple associated to x_0 and $u\in\mathcal{U}$, then, $\Theta(t,T,\gamma)=\Theta\left(t,T,x_t^{t_0,x_0,u}\right)$ and, by definition,

$$\int_{\mathbb{R}^N}\Lambda_h^\infty(t,x)\gamma_2(\mathrm{d}x)=\Lambda_h^\infty\left(t,x_t^{t_0,x_0,u}\right)=\Lambda_h^\infty(t,\gamma).$$

Theorem 3.8.

1. Let us assume that (1.2) holds true and h is a bounded function. For every $t_0 < t < T$, and every $x_0 \in \mathbb{R}^N$,

$$\Lambda_h^{\infty}(t_0, x_0) = \inf_{\gamma \in \Theta(t_0,t,x_0)} \max \left(\|h\|_{\mathbb{L}^{\infty}([t_0,t]\times\mathbb{R}^N\times U,\gamma_1)}, \Lambda_h^{\infty}(t,\gamma) \right).$$

Moreover,

$$\Lambda_h^{\infty}(t_0, x_0) \le \inf_{\gamma \in \Gamma(t_0,t,x_0)} \max \left(\|h\|_{\mathbb{L}^{\infty}([t_0,t]\times\mathbb{R}^N\times U,\gamma_1)}, \int_{\mathbb{R}^N} \Lambda_h^{\infty}(t,x)\, \gamma_2(\mathrm{d}x) \right)$$

2. Moreover, if (1.2) and (3.1) hold true, then,

$$\Lambda_h^{\infty}(t_0, x_0) = \inf_{\gamma \in \Gamma(t_0,t,x_0)} \max \left(\|h\|_{\mathbb{L}^{\infty}([t_0,t]\times\mathbb{R}^N\times U,\gamma_1)}, \int_{\mathbb{R}^N} \Lambda_h^{\infty}(t,x)\, \gamma_2(\mathrm{d}x) \right),$$

for every $t_0 < t < T$, and every $x_0 \in \mathbb{R}^N$.

Proof. We assume, without loss of generality, that $t_0 = 0$.

1. We recall that whenever $\gamma \in \Theta(0,t,x_0), \widetilde{\gamma} \in \Theta(t,T,\gamma)$, one has

$$(\widetilde{\gamma}\circ\gamma)_1(A, \mathrm{d}y\mathrm{d}u) = \frac{t}{T}\gamma_1(A \cap [0,t], \mathrm{d}y\mathrm{d}u) + \frac{T-t}{T}\widetilde{\gamma}_1(A\cap[t,T], \mathrm{d}y\mathrm{d}u),$$

for all Borel sets $A \subset [0,T]$. Using Proposition 2.9 and the previous Remark, one gets

$$\begin{aligned}\Lambda_h^{\infty}(0,x_0) &= \inf_{\substack{\gamma\in\Theta(0,t,x_0)\\ \widetilde{\gamma}\in\Theta(t,T,\gamma)}} (\widetilde{\gamma}\circ\gamma)_1\text{-}ess\sup\ |h| \\ &= \inf_{\gamma\in\Theta(0,t,x_0)} \max\left(\gamma_1\text{-}ess\sup\ |h|, \Lambda_h^{\infty}(t,\gamma)\right).\end{aligned}$$

The inequality follows from the previous remark and the fact that $\Gamma(0,t,x_0) \subset \Theta(0,t,x_0)$.

2. In the continuous case, Theorem 3.5 and Proposition 2.9 2. yield

$$\begin{aligned}\Lambda_h^{\infty}(0,x_0) = V_h^{\infty}(0,x_0) &= \inf_{\overline{\gamma}\in\Gamma(0,T,x_0)} \|h\|_{\mathbb{L}^{\infty}([0,T]\times\mathbb{R}^N\times U,\overline{\gamma}_1)} \\ &\ge \inf_{\substack{\gamma\in\Gamma(0,T,x_0)\\ \widetilde{\gamma}\in\Theta(t,T,\gamma)}} \max\left(\gamma_1\text{-}ess\sup\ |h|, \Lambda_h^{\infty}(t,\gamma)\right) \\ &= \inf_{\gamma\in\Gamma(0,T,x_0)} \left(\|h\|_{\mathbb{L}^{\infty}([0,t]\times\mathbb{R}^N\times U,\gamma_1)}, \Lambda_h^{\infty}(t,\gamma)\right).\end{aligned}$$

Using the previous remark and the inequality in the first part, one has

$$\Lambda_h^{\infty}(0, x_0) = \inf_{\gamma \in \Gamma(0,t,x_0)} \max \left(\|h\|_{\mathbb{L}^{\infty}([0,t]\times\mathbb{R}^N\times U,\gamma_1)}, \int_{\mathbb{R}^N} \Lambda_h^{\infty}(t,x)\, \gamma_2(\mathrm{d}x) \right). \qquad \square$$

Appendix A.

We prove the Proposition 2.4. We will make use of the following result due to Krylov (*cf.* [15], Thm. 2.1):

Proposition A.1. *There exists a constant $C > 0$ such that, for every $\varepsilon \in (0,1]$, there exists a function $V^\varepsilon \in C_b^{1,1}\left([0, r+\varepsilon] \times \mathbb{R}^N\right)$ (classical) subsolution of (2.4) defined on $[0, r+\varepsilon] \times \mathbb{R}^N$ satisfying*

$$(i)\, |V^\varepsilon(t,\cdot) - g'(\cdot)| \leq C\varepsilon, \text{ for } t \in [r, r+\varepsilon], \text{ and}$$
$$(ii)\, \left|V^\varepsilon(\cdot) - V^r_{g,g'}(\cdot)\right| \leq C\varepsilon, \text{ on } [0,r] \times \mathbb{R}^N.$$

Proof of Proposition 2.4. We have seen that $\gamma^{t,r,x,u} \in \Theta(t, r, x)$, for all $(t,x) \in [0,r] \times \mathbb{R}^N$ and all $u \in \mathcal{U}$. It follows that

$$V^r_{g,g'}(t,x) \geq \Lambda^r_{g,g'}(t,x). \tag{A.1}$$

For any $\gamma \in \Theta(t,r,x)$, whenever $(\eta, \phi) \in \mathbb{R} \times C_2^{1,1}\left(\mathbb{R}_+ \times \mathbb{R}^N\right)$ satisfies

$$\eta \leq (r-t)\left(\mathcal{L}^v \phi(s,y) + g(s,y,v)\right) + g'(z) - \phi(r,z) + \phi(t,x),$$

for all $(s,y,v,z) \in [t,r] \times \mathbb{R}^N \times U \times \mathbb{R}^N$, we have

$$\eta \leq (r-t) \int\limits_{[t,r]\times\mathbb{R}^N\times U} g(s,y,u)\,\gamma_1(\mathrm{d}s\mathrm{d}y\mathrm{d}u) + \int\limits_{\mathbb{R}^N} g'(z)\,\gamma_2(\mathrm{d}z).$$

Hence,

$$\eta^r_{g,g'}(t,x) \leq \Lambda^r_{g,g'}(t,x) < \infty. \tag{A.2}$$

To complete the proof, one only needs to show that

$$V^r_{g,g'}(t,x) \leq \eta^r_{g,g'}(t,x). \tag{A.3}$$

To this purpose, we apply the previous proposition and get, for every $\varepsilon > 0$, the existence of some regular V^ε such that

$$\partial_t V^\varepsilon(t,x) + H(t,x,DV^\varepsilon(t,x)) \geq 0,$$

for all $(t,x) \in [0,r] \times \mathbb{R}^N$. Thus, choosing C as in Proposition A.1, for every $(t,s,x,y,z) \in [0,r]^2 \times \mathbb{R}^{3N}$ and every $v \in U$, one has

$$V^\varepsilon(t,x) - C\varepsilon \leq (T-t)\left(\mathcal{L}^v V^\varepsilon(s,y) + g(s,y,v)\right) + g'(z) - V^\varepsilon(r,z) + V^\varepsilon(t,x).$$

Hence,

$$V^\varepsilon(t,x) - C\varepsilon \leq \eta^r_{g,g'}(t,x).$$

The inequality (A.3) follows by passing to the limit as $\varepsilon \to 0$ and recalling that Proposition A.1 (ii) holds true. The proof of our theorem is now complete. □

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