Linearized Kerr and spinning massive bodies: An electrodynamics analogy

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We discuss the correspondence between spinning, charged spherical sources in electrodynamics and spinning, massive spherical sources in linearized general relativity and show that the form of the potentials and equations of motion are similar in the two cases in the slow motion limit. This similarity allows us to interpret the Kerr metric in analogy with a spinning sphere in electrodynamics and aids in understanding linearized general relativity, where the "forces" are effective and come from the intrinsic curvature of space-time. © *2007 American Association of Physics Teachers.*

I. INTRODUCTION

All classical field theories require two ingredients: field production and test particle response. That is, we need to construct the fields given a set of sources and know how particles interact with these fields. In electrodynamics (E&M) the two ingredients are represented by Maxwell's equations (sources generate fields) and the Lorentz force law (charged particles move under the influence of the fields). For the potentials V and $\mathbf{A}$ (in the Lorentz gauge), we have

$$\text{Maxwell:} \quad \begin{aligned} \nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} &= -\frac{\rho}{\epsilon_0}, \\ \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\mu_0 \mathbf{J}, \end{aligned} \tag{1}$$

$$\text{Lorentz:} \quad L = \tfrac{1}{2}mv^2 - q(V - \mathbf{v}\cdot\mathbf{A}). \tag{2}$$

We have expressed the Lagrangian in Eq. (2) for the Lorentz force in a form for comparison with general relativity. Here V and $\mathbf{A}$ are the usual potentials, $\mathbf{E}=-\nabla V-\partial\mathbf{A}/\partial t$ and $\mathbf{B}=\nabla\times\mathbf{A}$. If we vary the Lagrangian in Eq. (2), we recover the familiar expression $\mathbf{F}=q\mathbf{E}+q\mathbf{v}\times\mathbf{B}$. The fields in E&M are $\mathbf{E}$ and $\mathbf{B}$ (or V and $\mathbf{A}$), and any particle with charge q interacts with them. As usual, we make a distinction between field-producing distributions (ρ and $\mathbf{J}$) and test particles that are influenced by, but do not contribute to, the fields.

In general relativity we have the same logical structure—the field in this case is the metric $g_{\mu\nu}$ and, as with Maxwell's equations, it is related to sources through its derivatives. The notion of source is generalized, with the mass density and mass current density appearing together with the rest of the components of the stress-energy tensor, allowing both matter and light to generate the gravitational field $g_{\mu\nu}$. The analog of Maxwell's equations is Einstein's equation, and the Lorentz force Lagrangian has a partner in the four-dimensional Lagrangian appropriate to geodesic motion. Formally, the equations analogous to Eqs. (1) and (2) for general relativity are

$$\text{Einstein:} \quad G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{3}$$

$$\text{Geodesic:} \quad L = \tfrac{1}{2}m\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu, \tag{4}$$

where the Einstein tensor $G_{\mu\nu}$ is constructed from a specific combination of $g_{\mu\nu}$ and its first and second derivatives. Equation (3) relates the derivatives of the metric to sources, just as Maxwell's equation does for E&M, and Eq. (4) tells us how test particles move in the gravitational field $g_{\mu\nu}$. The geodesic Lagrangian involves the four-dimensional coordinates (time included), unlike the Lorentz force Lagrangian, which we have written in its nonrelativistic form. (We will work in the $v \ll c$ limit for the remainder of this paper.)

It is not obvious that general relativity should reduce to E&M in any limit. But the Newtonian form of gravity is comparable to Coulomb's law—both fall off as r^{-2} and both are generated by sources according to Poisson's equation (in the static limit). Because some modification of Newtonian gravity is required (due to the precession of the perihelion of Mercury, for example), it is reasonable to turn to the rest of E&M for guidance. If a single scalar potential generating a force of gravity similar to electrostatics is insufficient, perhaps introducing some sort of vector potential and using an analog of the magnetic force will work. In the "linearized" regime of general relativity, the analogy with E&M is demonstrable from the form of the field equations and is explained in several texts.[1–3]

The goal of this paper is to show explicitly how we can describe the gravitational "fields" (space-time structure) of spinning massive bodies in analogy with the electromagnetic fields of spinning charged bodies. In Sec. II we will consider a simple electrostatics problem and guess the correction to Newtonian gravity that is implied. In Secs. III and IV we take the unique, axisymmetric stationary vacuum space-time of general relativity, the Kerr metric, and show that it lends itself to a Newtonian potential plus a "gravito-magnetic" vector potential interpretation. In Sec. V we will return to the E&M case and introduce spin for our test particles. Using the correspondence developed in the preceding sections, we know that the analogous spin-orbit coupling should be observable in general relativity.

II. UNIFORM ROTATING SPHERE IN E&M

We know that a spherically symmetric charge distribution generates the electric potential $V=Q/4\pi\epsilon_0 r$ for the total charge Q in SI units. If the charge distribution does not change in time, there is no magnetic field. This static configuration with charge replaced by mass gives the standard Newtonian scalar potential, and the E&M problem is identical (up to constant factors like G) to the gravitational problem, although the physical mechanisms are completely dif-
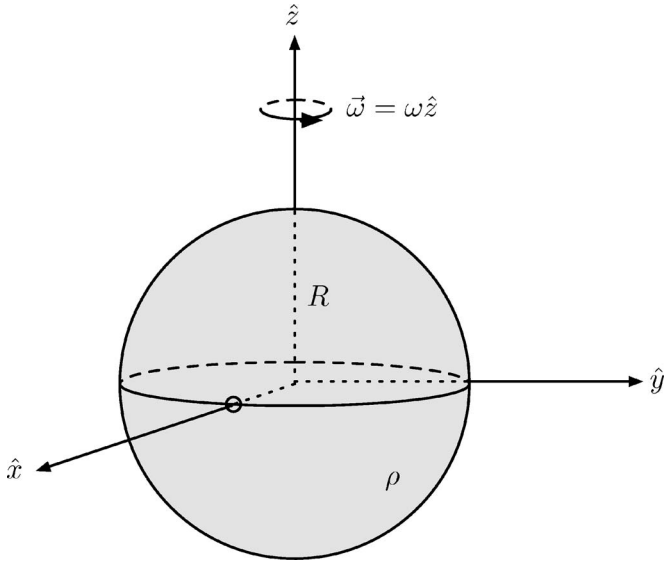
Fig. 1. A uniformly charged spinning sphere with radius R and angular frequency ω.

ferent. To extend the analogy beyond the Newtonian theory, we need a configuration in E&M that has a static magnetic field in addition to the static electric field.

For a uniformly charged sphere with radius R and charge density ρ, we can generate a magnetic field by spinning the sphere with constant angular velocity ω about the $\hat{z}$-axis that goes through its center (see Fig. 1). For this configuration, the potentials are, for $r > R$,[4]

$$V(r, \theta) = \frac{Q}{4\pi\epsilon_0 r}, \tag{5a}$$

$$\mathbf{A}(r, \theta) = \frac{1}{2}\frac{\mu_0 Q\ell}{4\pi}\frac{\sin\theta}{r^2}\hat{\phi}. \tag{5b}$$

We have written the potentials in terms of the total charge $Q = \frac{4}{3}\pi\rho R^3$, and the angular momentum per unit mass $\ell \equiv I\omega/M$ using the moment of inertia of a sphere $I = 2MR^2/5$. The scalar potential is the usual one for spherically symmetric distributions, and the magnetic vector potential is dipolar.

The motion of a test charge can be determined from the usual nonrelativistic Lagrangian:

$$L = \tfrac{1}{2}mv^2 - qV + q\mathbf{v}\cdot\mathbf{A}. \tag{6}$$

The Euler-Lagrange equations return the Lorentz force as the right-hand side of Newton's second law. In spherical coordinates L is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - \frac{qQ}{4\pi\epsilon_0 r}$$
$$+ \frac{\mu_0 qQ\ell}{8\pi}\frac{\sin^2\theta}{r}\dot{\phi}. \tag{7}$$

There are two immediate constants of the motion for this configuration: the z-component of the angular momentum and the total energy. There is no ϕ dependence in the Lagrangian, so the conjugate momentum $p_\phi \equiv \partial L/\partial\dot{\phi}$ is constant:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} - \frac{\partial L}{\partial\phi} = 0, \tag{8}$$

which implies that

$$J_z \equiv \frac{\partial L}{\partial\dot{\phi}} = \text{const}. \tag{9}$$

The other constant is the associated Hamiltonian:

$$H = \frac{\partial L}{\partial\dot{x}^j}\dot{x}^j - L \equiv E, \tag{10}$$

where the generalized coordinates are x^j, $j = 1, 2, 3$, with $x^1 = r$, $x^2 = \theta$, and $x^3 = \phi$.

The equations of motion given by the Euler-Lagrange equations,

$$\left(\frac{d}{dt}\frac{\partial L}{\partial\dot{x}^j} - \frac{\partial L}{\partial x^j}\right) = 0, \tag{11}$$

are not easily solved. (Without a third constant of the motion, the solutions can be obtained only numerically.) We simplify the problem by restricting our attention to planar orbits. If we set $\theta = \pi/2$ and $\dot{\theta} = 0$, then $\ddot{\theta} = 0$, and a test particle that begins in the plane will remain there. Then we only have one free coordinate in the Hamiltonian, r. We use Eqs. (9) and (10) and write the equation for $\dot{r}$ as

$$\dot{r}^2 = \left(\frac{2E}{m} - \frac{J_z^2}{m^2 r^2} - \frac{qQ}{2\pi\epsilon_0 mr}\right) + \frac{\mu_0 qQ\ell}{4\pi m^2}\left(\frac{J_z}{r^3} - \frac{\mu_0 qQ\ell}{16\pi r^4}\right). \tag{12}$$

The first term in parentheses is the usual Coulomb expression; the second term gives the magnetic contribution. Notice that the magnetic terms involve r^{-3} and r^{-4} (and associated constants)—these terms will typically be smaller and can be viewed as a perturbation to the spherically symmetric Coulomb term, at least, for large r.

A. Numerical examples

From the Lagrangian we can rescale the radial coordinate by R: $r = R\tilde{r}$, so that $\tilde{r}$ is dimensionless. Then Eq. (7) becomes

$$L = R^2\left[\frac{1}{2}m(\dot{\tilde{r}}^2 + \tilde{r}^2\dot{\theta}^2 + \tilde{r}^2\sin^2\theta\dot{\phi}^2) - \frac{qQ}{4\pi\epsilon_0 R^3\tilde{r}}\right.$$
$$\left. + \frac{qQ}{4\pi\epsilon_0 R^3}\frac{\ell}{2c^2}\frac{\sin^2\theta}{\tilde{r}}\dot{\phi}\right]. \tag{13}$$

The overall factor of R^2 does not contribute to the equations of motion, so there are two parameters we need to choose: $qQ/(4\pi\epsilon_0 R^3)$ and $\ell/2c^2$. The charge term is shared by both the electric and magnetic potentials, and therefore cannot be used to vary the relative strength of the two. The other parameter $\ell/2c^2$ is adjustable and must be large for the magnetic effects to be significant. Hence, we are interested in large objects rotating with high frequency.

Bound orbits can be defined by Eq. (12). The turning points of elliptical motion occur when $\dot{r} = 0$, so we can solve for E and J_z (two unknowns) by picking two values of r for which $\dot{r}$ vanishes. These are constants of the motion, and J_z will give us the initial value for $v = \dot{\phi}r$. We return to the
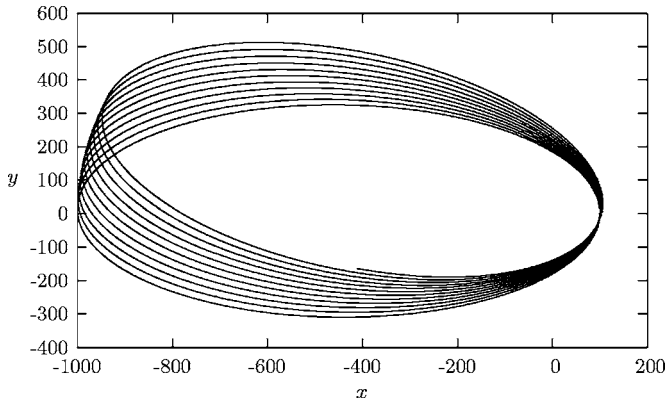
Fig. 2. Orbit with turning points $\tilde{r} = 100, 1000R$ for a nonspinning test body and a spinning charged central body.

magnetic potential term in Eq. (13) and factor out the electrostatic constants (with $\sin\theta = 1$ for planar orbits):

$$\frac{qQ}{4\pi\epsilon_0 R^3 \tilde{r}} \frac{\ell}{2c^2} \dot\phi = \frac{1}{5} \frac{qQ}{4\pi\epsilon_0 R^3 \tilde{r}} \frac{\omega R}{c} \frac{\dot\phi R}{c}. \qquad (14)$$

The product of the angular velocity of the source $\omega R/c$ and of the orbit $\dot\phi R/c$ must be of order unity for the magnetic term to be comparable to $qQ/(4\pi\epsilon_0 R^3 \tilde{r})$. Neither the source nor the orbital angular velocity can be very close to unity in a relativistic setting because we have neglected relativistic effects. For both ωR and $\dot\phi R$ to be small compared to c, the magnetic term must be order $(v/c)^2$. Thus, the magnetic contribution to the forces will be perturbative, which is not surprising because the magnetic force is usually dominated by the electric force. The perturbative effects will accumulate over time, so that if we consider reasonable values of ωR and $\dot\phi R$ and look at the orbits for a sufficiently long time, we could see them. For visual clarity, it is easier to use larger values of the initial orbital velocity so that we need to plot fewer orbital cycles.

In Fig. 2 we plot $\tilde{r}$ for a spinning source with $R \approx 2$ km, $\omega \approx 33$ s^{-1}, $Q = 1$ C, and a test body with mass $m = 1$ kg and $q = -0.1$ C. These values lead to a velocity at the surface of the spinning sphere of $\omega R \sim 0.2c$, too large to take seriously from the point of view of special relativity, but, as we have mentioned, these parameters will allow us to see magnetic effects. If we choose the closest and furthest approach distances to be $r_p = 100R$ and $r_a = 1000R$ (perihelion and aphelion) with respect to the central body, then the initial linear velocity of the orbit is $\approx 25c$, well beyond the range of applicability of our nonrelativistic equations (both for the source and the test particle orbit). We take this large value only to show qualitative features of the orbit.

To generate Fig. 2 we used Mathematica's built-in numerical integrator (implicit Runge-Kutta) to solve the $\ddot{r}$ and $\ddot\phi$ equations simultaneously with the initial conditions $r(0) = 100R$, $\dot{r}(0) = 0$, and $\phi(0) = 0$. The initial value of $\dot\phi$ was obtained from the constant J_z, chosen to ensure turning points at $\tilde{r} = 100R$ and $1000R$. If we take the same parameters and initial data, but choose a nonzero $\dot\theta$ at $t = 0$, the motion will move out of the $\theta = \pi/2$ plane.

From Fig. 2 we see that the closest approach to the central body of the orbital ellipse precesses as is expected.

Bertrand's theorem[5] tells us that closed orbits only occur for r^{-1} and r^2 potentials. The physical basis for the precession in this case is the (small) magnetic term in the Lagrangian. If we had chosen the linear velocities in our numerical example to have realistic values (much less than c), we would need thousands of orbits to observe the precession of the closest or furthest approaches to the central body.

B. Correction to Newtonian gravity

General relativity is distinct from Newtonian gravity even at the static, spherically symmetric level. What is interesting is the new effect predicted by general relativity: that moving masses, like moving charges, will produce mass "currents" that behave (far away from the moving body) very much like charge currents in the sense that moving test masses respond to the mass currents just as moving charges respond to charge currents.

If we take the correspondence at face value, we have

$$\frac{qQ}{4\pi\epsilon_0 r} \rightarrow -\frac{GmM}{r} \qquad (15)$$

for the central charge Q, central mass M, and test charge (mass) q (m) and we have $qQ/(4\pi\epsilon_0) \rightarrow -GmM$. For the **A** term from Eq. (5a), we can form the "potential" appearing in the Lagrangian, $q\mathbf{v} \cdot \mathbf{A}$. Then using the replacement (15), we expect a spinning massive sphere to introduce the term:

$$-\frac{qQ}{4\pi\epsilon_0} \frac{\ell}{2c^2} \frac{\sin^2\theta}{r} \dot\phi \rightarrow \frac{GmM\ell}{2c^2} \frac{\sin^2\theta}{r} \dot\phi. \qquad (16)$$

A term that looks like the right-hand side of Eq. (16) in a Lagrangian appropriate to general relativity would be the hallmark of a gravito-magnetic field (the gravitational version of the magnetic field). Our task is to show that such a term arises in the appropriate setting.

III. THE GEODESICS

The geodesic Lagrangian (4) is manifestly relativistically invariant. The vector $\dot{x}^\mu$ is now a four-vector, and the dots refer to derivatives with respect to the proper time. The Euler-Lagrange equations that come from this L reproduce straight lines in a curved space-time defined by the metric $g_{\mu\nu}$. The Lagrangian is familiar from force-free special relativity, where $g_{\mu\nu} = \eta_{\mu\nu}$ (the Minkowski metric). It is precisely the deviations from flat Minkowski space that lead to effective forces in general relativity.

To compare with Eq. (7), we must first switch to the coordinate-time parametrization of the three-dimensional spatial curve $\mathbf{x}(t)$. Then we can introduce the (linearized) Kerr metric as $g_{\mu\nu}$ and consider a special regime in which the form of the geodesic Lagrangian and the electromagnetic Lagrangian are the same. We know that far enough away from a central body, the gravitational effects should be described by Newtonian gravity. The idea is to move in to an intermediate distance where the gravito-magnetostatic effects can be seen in general relativity.

A. Reparametrizing in terms of coordinate time

In the geodesic Lagrangian (4), variation with respect to x^μ yields paths of minimal length as measured by the four-dimensional invariant $ds^2 = \dot{x}^\mu g_{\mu\nu} \dot{x}^\nu d\tau^2$, with $\dot{x}^\mu \equiv dx^\mu/d\tau$ the

derivative with respect to the proper time. In the language of special relativity τ can be interpreted as the time recorded on a particle's own watch (which is also its meaning in general relativity.[6]) To connect the four-vector $\dot{x}^\mu \equiv dx^\mu/d\tau$ to our everyday velocity [the one found in Eq. (2), for example], the spatial dx^j/dt, we perform a change of variables in the action $S = \int L\, d\tau$. Let $x'^\mu = dx^\mu/dt$ and consider τ as a function of t (an inversion of $t(\tau)$). The action becomes

$$S = \frac{1}{2}m \int \dot{x}^\mu g_{\mu\nu} \dot{x}^\nu \, d\tau = \frac{1}{2}m \int x'^\mu g_{\mu\nu} x'^\nu \frac{dt}{d\tau} \, dt. \qquad (17)$$

The integrand suggests that the Lagrangian appropriate to general relativity expressed in terms of the coordinate time t is

$$L = \frac{1}{2}m x'^\mu g_{\mu\nu} x'^\nu \frac{dt}{d\tau}$$

$$= \frac{1}{2}m(g_{00}c^2 + 2g_{0j}x'^j c + x'^j g_{jk} x'^k) \frac{dt}{d\tau}. \qquad (18)$$

In the above, the Roman indices represent the spatial (1, 2, 3) components of the four-vector x^μ, so g_{jk} is the spatial portion of the metric. We have simplified the sums by noting that

$$\rho^2 \equiv r^2 + \left(\frac{a}{c}\right)^2 \cos^2 \theta, \qquad (22)$$

$$\Delta \equiv \left(\frac{a}{c}\right)^2 - 2\frac{MGr}{c^2} + r^2. \qquad (23)$$

For this coordinate choice $x^\mu \sim (ct, r, \theta, \phi)$; the coordinates are labeled in the familiar spherical notation. The metric has two parameters a and M. Note that if we set $a = M = 0$, the resulting metric represents flat Minkowski space-time in spherical coordinates. We can also recover flat space-time by letting $r \to \infty$. In this sense we can solve (if possible) the geodesic equations of motion and make plots of the particle motion because we are on a viewing platform very far away. Close to the central body, the structure of the space-time is complicated, and it is unclear what geometrical interpretation to attach to r. We can overcome this difficulty by writing our geodesics in terms of physical observables such as the orbital period.

One difficulty in general relativity is its interpretation. Because the theory is coordinate invariant, no two frames need use the same coordinates. It is also not clear what the physi-

$dx^0/dt = c\,dt/dt = c$ in these units. Finally, from the definition of proper time in this setting, $\left(\frac{d\tau}{dt}\right)^2 = -\frac{1}{c^2} x'^\mu g_{\mu\nu} x'^\nu$, we have

$$L = -\frac{1}{2}mc^2 \left[-g_{00} - 2g_{0j}\frac{x'^j}{c} - \frac{x'^j x'^k}{c^2}g_{jk} \right]^{1/2}. \qquad (18a)$$

B. Kerr metric

The Kerr metric in general relativity is the unique, axisymmetric, stationary space-time that satisfies Einstein's equation in vacuum.[7] Vacuum, in this case, means that we are away from sources, not that no source exists. Thus, we are describing a situation outside of a compact central body. In this source-free region, the stress tensor on the right-hand side of Eq. (3) is zero so that the entire set of field equations (3) can be written as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = 0, \qquad (19)$$

which implies that the Ricci tensor $R_{\mu\nu}$ satisfies

$$R_{\mu\nu} = 0 \qquad (20)$$

We next give the form of the Kerr metric; our interpretation will come from the E&M toy problem.

The Kerr metric in Boyer-Lindquist coordinates[8] is

$$g_{\mu\nu} \doteq \begin{pmatrix} -\left(1 - \dfrac{2MGr}{c^2\rho^2}\right) & 0 & 0 & -\dfrac{2aMGr\sin^2\theta}{c^3\rho^2} \\[2ex] 0 & \dfrac{\rho^2}{\Delta} & 0 & 0 \\[2ex] 0 & 0 & \rho^2 & 0 \\[2ex] -\dfrac{2aMGr\sin^2\theta}{c^3\rho^2} & 0 & 0 & \left[\left(r^2 + \left(\dfrac{a}{c}\right)^2\right) + \dfrac{2a^2MGr\sin^2\theta}{c^4\rho^2}\right]\sin^2\theta \end{pmatrix}, \qquad (21)$$

cal significance is, if any, of the parameters a and M in Eq. (21). (There are other ways in the context of general relativity to understand these parameters that are discussed in most texts.[9]) One way to understand a space-time is to examine its geodesic structure—what do test bodies do? If we view the motion from infinity where space-time is flat, we can make the usual identification $x = r \sin\theta \cos\phi$, etc., and consider the motion as if it occurred in a truly Euclidean space.

We have provided the Kerr metric without showing that $R_{\mu\nu} = 0$, but this equality can be verified using a computer algebra package. From our point of view, Eq. (21) is just a metric to be used with Eq. (18). From this use we can gain insight into its interpretation. We leave the details to the texts.[10] The time independence of the linearized Kerr metric appears naturally in the current context, but can also be understood from the point of view of dynamical degrees of freedom.[11]

IV. THE KERR GEODESIC LAGRANGIAN

At this point we could take the metric (21) and form the Lagrangian (18), but because we will look at a particular

regime, it is slightly easier to simplify the metric before forming the Lagrangian. The Kerr metric is valid outside the central body, but we can approximate the geometry far from the source by linearizing the metric directly. The parameters a and M will be involved in the linearization, and although we do not yet have a physical interpretation of these parameters, we can define two natural lengths, a/c and MG/c^2. We take both to be small compared to our location (our distance from the central body r) so that $a/c \ll r$ and $MG/c^2 \ll r$. Then the Kerr metric can be linearized in a/c and MG/c^2:[12]

$$
g_{\mu\nu} \approx \underbrace{\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}}_{=\eta_{\mu\nu}} + \begin{pmatrix} \dfrac{2MG}{c^2 r} & 0 & 0 & -\dfrac{2aMG \sin^2 \theta}{c^3 r} \\ 0 & \dfrac{2MG}{c^2 r} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\dfrac{2aMG \sin^2 \theta}{c^3 r} & 0 & 0 & 0 \end{pmatrix}.
$$
(24)

This form of $g_{\mu\nu}$ no longer satisfies the vacuum Einstein equation exactly, that is, $R_{\mu\nu} \neq 0$; rather

$$
R_{\mu\nu} = 0 + O\left(\frac{a}{c}\right)^2 + O\left(\frac{MG}{c^2 r}\right)^2,
$$
(25)

but it is still accurate to first order in each length and their product. The linearization separates the metric into a flat portion and an additive perturbation, allowing us to give the coordinates their flat interpretation and view the extra terms in the metric as an effective potential. This point of view is fine for describing the qualitative features of the motion of test bodies, but will not allow us to determine the precise numerical factors in general relativity. The situation is akin to the perihelion precession of Mercury in the usual Schwarzschild setting—the radial coordinate there (in which the correct value for the precession appears) is not really the flat one.

We can use the approximation (24) in the Lagrangian (18a),

$$
L = -\frac{1}{2} mc^2 \left(1 - \frac{2MG}{c^2 r} + \frac{4aMG \sin^2 \theta}{c^4 r} \dot\phi - \frac{\dot{x}^j g_{jk} \dot{x}^k}{c^2} \right)^{1/2}, \quad (26)
$$

where dots refer to coordinate time t derivatives. Note that

$$
\dot{x}^j g_{jk} \dot{x}^k = \dot{r}^2 + r^2 \dot\theta^2 + r^2 \sin^2 \theta \dot\phi^2 + \frac{2MG}{c^2 r} \dot{r}^2.
$$
(27)

If we assume that the motion of the test body is small compared to the speed of light, then we have a cubic perturbation in $2MG\dot{r}/(c^4 r)$, so we can approximate Eq. (27) as

$$
\dot{x}^j g_{jk} \dot{x}^k \approx v^2 = \dot{r}^2 + r^2 \dot\theta^2 + r^2 \sin^2 \theta \dot\phi^2,
$$
(28)

which is the usual velocity squared in spherical coordinates. Now the Lagrangian becomes

$$
L = -\frac{1}{2} mc^2 \left(1 - \frac{2MG}{c^2 r} + \frac{4aMG \sin^2 \theta}{c^4 r} \dot\phi - \frac{v^2}{c^2} \right)^{1/2}, \quad (29a)
$$

$$
\approx \frac{1}{2} \left(-mc^2 + \frac{mMG}{r} - \frac{2amMG \sin^2 \theta}{c^2 r} \dot\phi + + \frac{1}{2} mv^2 \right),
$$
(29b)

for small test particle velocity (in addition to our assumptions for a/c and MG/c^2). Again we consider the slow motion limit, just as with the spinning spherical ball in E&M (which is explicitly nonrelativistic). The process of expanding L in Eq. (29b) further is the well-known post-Newtonian expansion procedure (a description can be found in, for example, Ref. 13).

If we expand the Lagrangian in Eq. (29b), we have

$$
L = \underbrace{\frac{1}{2} m(\dot{r}^2 + r^2 \dot\theta^2 + r^2 \sin^2 \theta \dot\phi^2)}_{\text{kinetic}} + \underbrace{\frac{mMG}{r} - 2\frac{amMG \sin^2 \theta}{rc^2} \dot\phi}_{\text{potential}}.
$$
(30)

We can compare Eq. (30) directly with Eq. (7). The parameter M in the metric corresponds to the mass of a central body, appearing in the Newtonian form, and the parameter $a \sim \ell$ is the angular momentum per unit mass as expected from Eq. (16). Our toy problem in E&M has given us an interpretation—we associate $a \sim \ell$ with a spinning massive source distribution, just as in electrostatics. It is the spin of the distribution that contributes this gravito-magnetostatic term to the Lagrangian. The only issue is a numerical factor for the current term that differs from the electrostatic case (different by a factor of 4) and is associated with the fact that $g_{\mu\nu}$ is a spin 2 field, but the physical interpretation is unchanged.

We conclude that the linearized Kerr metric represents the space-time generated by a rotating, massive, spherically symmetric body. The two parameters that appear in the derivation of the full metric take on physical significance in the linearized, slow-moving test body regime we have studied.

V. A SPINNING TEST MASS

We return to the electromagnetic problem and give the test body some additional structure. Suppose the test charge is a charged, spinning sphere. Because it is a test mass, we are not concerned with the electric or magnetic fields it generates, but the spin will introduce charge motion in addition to the orbital motion. Spinning test particles are usually introduced in E&M in the context of Larmor precession to provide a classical model for quantum spin. In the usual examples, there is a uniform magnetic field, a circular orbit for the center of mass, and the familiar precession comes directly from the angular equations of motion.

Because most astrophysically relevant bodies carry very little charge, from the E&M point of view, the problem of macroscopic spin-orbit and spin-spin coupling is just a toy problem, although in the microscopic setting, the model leads to fine and hyperfine splitting. But in the context of gravity for a relatively dense, fast spinning macroscopic source (supermassive black holes, see for example Ref. 14), the orbital motion of spinning test particles is more relevant.

Spin in general relativity is tricky. In the strong-field regime, we are stuck with the full geodesic equations of motion, now including sources (forces) that involve a spin tensor (which is difficult to interpret) and the Riemann tensor (the full Papapetrou equations of motion[15]). If we retreat slightly, we can easily give spin to a test charge and see what happens in E&M; the qualitative results will be similar via the electromagnetic analogy. By interpreting Eq. (30) as a point Lagrangian and building up macroscopic bodies in the usual way, we will obtain the same final form as applying this procedure to the E&M Lagrangian.

The Lagrangian in E&M for a spinning test body comes from the addition of the rotational kinetic energy term and the dipole energy in a magnetic field $-\mathbf{m} \cdot \mathbf{B}$. These terms lead to the familiar torque $\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}$ and force $\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$ in the equations of motion. The full Lagrangian is

$$L = \tfrac{1}{2}Mv_{cm}^2 + \tfrac{1}{2}I\Omega^2 - qV + q\mathbf{v}_{cm} \cdot \mathbf{A} + \mathbf{m} \cdot (\nabla \times \mathbf{A}). \quad (31)$$

It is easy to see that the form of the linearized Kerr metric with the nonrelativistic Lagrangian will also pick up the dipole-field interaction term, and the physical trajectories are similar to the ones that come from Eq. (31). In the general relativity setting, the E&M Larmor precession is present and is known as Lense-Thirring precession; the Gravity Probe B experiment was sent up in 2004 to measure it.[16]

VI. CONCLUSIONS

We have explored the almost identical form of spinning charged particles in E&M and spinning massive bodies in general relativity via the linearized Kerr metric. The advantage is that electricity and magnetism take place in a static arena (for our purposes, the familiar flat Euclidean space) where interpretation and analysis are relatively simple. We took the Kerr metric, linearized it, and found the Lagrangian governing the motion of slow-moving test bodies. In this setting the two parameters (a, M) from the metric can be understood in terms of a spinning massive ball. The important shift in focus is that we have taken a geometric theory—metric determines geometry which governs motion via geodesics—and reinterpreted it as an effective force on a flat (Minkowski) background.

There are interesting differences between the geodesics of exact Kerr space-time[10] and the linearized form. In the linearized setting (as with the E&M toy problem), we have only two immediate constants of the motion, the total energy E and J_z, the z-component of the angular momentum. The Kerr space-time has an additional constant of the motion associated with its (somewhat surprising) geometric structure—Carter's constant.[17] This additional constant of the motion is associated with the separability of Hamilton's principal function for the geodesic Lagrangian in Kerr space-time and is lost in the linearization. We hope to further understand the physical significance of Carter's constant[18] by examining the separability of both Kerr and its linear form.

We also considered the problem of test particle spin in the E&M context, and the results there carried over to the linear general relativity setting as well. For full general relativity, we lose the flat space interpretation of spin, the equations of motion are no longer geodesics, and a force term is added to the right-hand side of Eq. (4).[15,19] It is now possible to numerically solve these "Papapetrou" equations. We hope that the simpler form of the Lagrangian in Eq. (31) that we have discussed is helpful to readers who are looking for an introduction to spin in general relativity. Spinning spheres are the most astrophysically relevant space-time structure (most massive objects are spinning) and test body approximation (most massive objects are spinning).

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Adjustable Angle Mirrors. An object placed between two plane mirrors set at a 90° angle to each other will show three images, one in each quadrant formed by the mirror and their intersections. When the angle is 60°, there are five images, while a 45° produces seven images. The analysis is easily done using the notion of *virtual mirrors*; see Thomas B. Greenslade, Jr., "Multiple Images in Plane Mirrors," Phys. Teach. **20**, 29–33 and cover (1982). If you don't have this device use 12 inch mirror tiles and set it up on a turntable so that an entire class can see. It is at the University of Utah (Photograph and Notes by Thomas B. Greenslade, Jr., Kenyon College).