

IWONA NARANIECKA, JAN SZYNAL and ANNA TATARCZAK

Linearly-invariant families and generalized Meixner–Pollaczek polynomials

ABSTRACT. The extremal functions $f_0(z)$ realizing the maxima of some functionals (e.g. $\max |a_3|$, and $\max \arg f'(z)$) within the so-called universal linearly invariant family U_α (in the sense of Pommerenke [10]) have such a form that $f_0'(z)$ looks similar to generating function for Meixner–Pollaczek (MP) polynomials [2], [8]. This fact gives motivation for the definition and study of the generalized Meixner–Pollaczek (GMP) polynomials $P_n^\lambda(x; \theta, \psi)$ of a real variable x as coefficients of

$$G^\lambda(x; \theta, \psi; z) = \frac{1}{(1 - ze^{i\theta})^{\lambda-ix}(1 - ze^{i\psi})^{\lambda+ix}} = \sum_{n=0}^{\infty} P_n^\lambda(x; \theta, \psi) z^n, \quad |z| < 1,$$

where the parameters λ, θ, ψ satisfy the conditions: $\lambda > 0$, $\theta \in (0, \pi)$, $\psi \in \mathbb{R}$. In the case $\psi = -\theta$ we have the well-known (MP) polynomials. The cases $\psi = \pi - \theta$ and $\psi = \pi + \theta$ leads to new sets of polynomials which we call quasi-Meixner–Pollaczek polynomials and strongly symmetric Meixner–Pollaczek polynomials. If $x = 0$, then we have an obvious generalization of the Gegenbauer polynomials.

The properties of (GMP) polynomials as well as of some families of holomorphic functions $|z| < 1$ defined by the Stieltjes-integral formula, where the function $zG^\lambda(x; \theta, \psi; z)$ is a kernel, will be discussed.

1. Linearly-invariant families of holomorphic functions

$$(1.1) \quad f(z) = z + a_2 z^2 + \dots, \quad z \in \mathbb{D}$$

2010 *Mathematics Subject Classification.* 30C45, 30C70, 42C05, 33C45.

Key words and phrases. Typically-real functions, univalent functions, local univalence, orthogonal polynomials, Meixner–Pollaczek polynomials.

in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ were introduced by Pommerenke in [10], and then were intensively studied by several authors (e.g. [14], [15] and [17]).

A family \mathfrak{M} of holomorphic functions of the form (1.1) is linearly-invariant if it satisfies two conditions:

- (a) $f'(z) \neq 0$ for any z in \mathbb{D} (local univalence),
- (b) for any linear fractional transformation

$$\phi(z) = e^{i\theta} \frac{z + a}{1 + \bar{a}z}, \quad a, z \in \mathbb{D}, \quad \theta \in \mathbb{R},$$

of \mathbb{D} onto itself, the function

$$\Lambda[f](z) = F(z) = \frac{f(\phi(z)) - f(\phi(0))}{f'(\phi(0))\phi'(0)} = z + \dots \in \mathfrak{M}.$$

The order of the linearly-invariant family \mathfrak{M} is defined as

$$\text{ord } \mathfrak{M} = \sup_{f \in \mathfrak{M}} |a_2(f)|.$$

Universal invariant family U_α is defined as

$$U_\alpha = \bigcup_{\text{ord } \mathfrak{M} \leq \alpha} \mathfrak{M}.$$

It is well known that $\alpha \geq 1$ and $U_1 \equiv S^c =$ the class of convex univalent functions in \mathbb{D} , and the familiar class S of all univalent functions is strictly included in U_2 . Moreover, for every $\alpha > 1$, the class U_α contains functions which are infinitely valent in \mathbb{D} [10], for example:

$$f_0(z) = \frac{1}{2i\gamma} \left[\left(\frac{1+z}{1-z} \right)^{i\gamma} - 1 \right],$$

$$f'_0(z) = \frac{1}{(1+z)^{1-i\gamma}(1-z)^{1+i\gamma}}, \quad \gamma = \sqrt{\alpha^2 - 1}.$$

Another example of such a function was presented in [15]:

$$(1.2) \quad f_0(z) = \frac{1}{(e^{it_2} - e^{it_1})i\sqrt{\alpha^2 - 1}} \left[\left(\frac{1 - ze^{it_1}}{1 - ze^{it_2}} \right)^{i\sqrt{\alpha^2 - 1}} - 1 \right], \quad t_1 \neq t_2 + 2k\pi,$$

for which

$$(1.3) \quad f'_0(z) = \frac{1}{(1 - ze^{it_1})^{1-i\sqrt{\alpha^2 - 1}}(1 - ze^{it_2})^{1+i\sqrt{\alpha^2 - 1}}}.$$

Functions of the form (1.2) appear to be extremal for the long lasting problems:

$$\max_{f \in U_\alpha} |a_3| \quad \text{and} \quad \max_{f \in U_\alpha} |\arg f'(z)|,$$

recently solved by Starkov [14], [15], who proved that the extremal function for $\max |a_3|$ is of the form (1.2) with $t_1 = \theta$, $t_2 = -\theta$, where

$$e^{i\theta} = \sqrt{\frac{(3 - \alpha^2) + 3i\sqrt{\alpha^2 - 1}}{\alpha\sqrt{\alpha^2 + 3}}}, \quad f \in U_\alpha \quad ([14]).$$

However, the extremal function f_0 for $\max_{f \in U'_\alpha} |\arg f'(z)|$ is of the form (1.2) with

$$t_1 = \pi - \arctan \frac{1}{\alpha} - \arctan \frac{r}{\alpha},$$

$$t_2 = -\pi + \arcsin \frac{1}{\alpha} - \arcsin \frac{r}{\alpha}, \quad r = |z| < 1, \quad t_1 \neq -t_2 \quad ([15]).$$

We see that the extremal function for $\max_{f \in U_\alpha} |a_3|$ has a special form leading to (MP) polynomials, but the extremal function for $\max_{f \in U_\alpha} |\arg f'(z)|$ leads to (GMP) polynomials, defined below.

2. Comparing (1.3) with the generating function for Meixner–Pollaczek polynomials $P_n^\lambda(x; \theta)$ ([2]):

$$G^\lambda(x; \theta, -\theta; z) = \frac{1}{(1 - ze^{i\theta})^{\lambda-ix}(1 - ze^{-i\theta})^{\lambda+ix}} = \sum_{n=0}^{\infty} P_n^\lambda(x; \theta) z^n, \quad z \in \mathbb{D},$$

where $\lambda > 0, \theta \in (0, \pi), x \in \mathbb{R}$, we are motivated to introduce the generalized Meixner–Pollaczek (GMP) polynomials $P_n^\lambda(x; \theta, \psi)$ of variable $x \in \mathbb{R}$ and parameters $\lambda > 0, \theta \in (0, \pi), \psi \in \mathbb{R}$ via the generating function

$$(2.1) \quad G^\lambda(x; \theta, \psi; z) = \frac{1}{(1 - ze^{i\theta})^{\lambda-ix}(1 - ze^{i\psi})^{\lambda+ix}} = \sum_{n=0}^{\infty} P_n^\lambda(x; \theta, \psi) z^n,$$

$z \in \mathbb{D}$. Of course, we have $P_n^\lambda(x; \theta, -\theta) = P_n^\lambda(x; \theta)$. We will find the three-term recurrence relation, the explicit formula, the hypergeometric representation and the difference equation for (GMP) polynomials $P_n^\lambda(x; \theta, \psi)$.

Theorem 2.1. (i) *The polynomials $P_n^\lambda = P_n^\lambda(x; \theta, \psi)$ satisfy the three-term recurrence relation:*

$$(2.2) \quad P_{-1}^\lambda = 0,$$

$$P_0^\lambda = 1,$$

$$nP_n^\lambda = [(\lambda - ix)e^{i\theta} + (\lambda + ix)e^{i\psi} + (n - 1)(e^{i\theta} + e^{i\psi})]P_{n-1}^\lambda - [(2\lambda + n - 2)e^{i(\theta+\psi)}]P_{n-2}^\lambda, \quad n \geq 1.$$

(ii) *The polynomials $P_n^\lambda(x; \theta, \psi)$ are given by the formula:*

$$(2.3) \quad P_n^\lambda(x; \theta, \psi) = e^{in\theta} \sum_{j=0}^n \frac{(\lambda + ix)_j (\lambda - ix)_{n-j}}{j!(n-j)!} e^{ij(\psi-\theta)}, \quad n \in \mathbb{N} \cup \{0\}.$$

(iii) *The polynomials $P_n^\lambda(x; \theta, \psi)$ have the hypergeometric representation*

$$(2.4) \quad n!P_n^\lambda(x; \theta, \psi) = (2\lambda)_n e^{in\theta} F\left(-n, \lambda + ix, 2\lambda; 1 - \frac{e^{i\psi}}{e^{i\theta}}\right).$$

(iii) Let $y(x) = P_n^\lambda(x; \theta, \psi)$. The function $y(x)$ satisfies the following difference equation

$$(2.5) \quad e^{i\theta}(\lambda - ix)y(x+i) + [ix(e^{i\theta} + e^{i\psi}) - (n+\lambda)(e^{i\theta} - e^{i\psi})]y(x) - e^{i\psi}(\lambda + ix)y(x-i) = 0.$$

Proof. (i) We differentiate the formula (2.1) with respect to z and after multiplication by $(1 - ze^{i\theta})(1 - ze^{i\psi})$ we compare the coefficients at the power z^{n-1} .

(ii) The Cauchy product for the power series

$$(1 - ze^{i\theta})^{-(\lambda-ix)} = \sum_{n=0}^{\infty} \frac{(\lambda - ix)_n e^{in\theta}}{n!} z^n$$

and

$$(1 - ze^{i\psi})^{-(\lambda+ix)} = \sum_{n=0}^{\infty} \frac{(\lambda + ix)_n e^{in\psi}}{n!} z^n$$

gives (2.3).

(iii) We apply the formula from ([4], vol. 1, p. 82):

$$(1-s)^{a-c}(1-s+sz)^{-a} = \sum_{n=0}^{\infty} \frac{(c)_n}{n!} F(-n, a; c; z) s^n, \quad |s| < 1, |s(1-z)| < 1,$$

with $s = ze^{i\theta}$, $a = \lambda + ix$, $c = 2\lambda$, $z = 1 - e^{i(\psi-\theta)}$ and obtain

$$\begin{aligned} & (1 - ze^{i\theta})^{-(\lambda-ix)}(1 - ze^{i\psi})^{-(\lambda+ix)} \\ &= \sum_{n=0}^{\infty} \frac{z^n e^{in\theta} (2\lambda)_n}{n!} F(-n, \lambda + ix, 2\lambda; 1 - e^{i(\psi-\theta)}). \end{aligned}$$

Comparing the coefficients at the power z^n , we get (2.4).

(iii) Putting $(x+i)$ and $(x-i)$ instead of x into the generating function (2.1), we find that

$$\begin{aligned} y(x+i) &= \sum_{k=0}^{n-1} P_k^\lambda(x; \theta, \psi) [e^{i(n-k)\theta} - e^{i[(n-k-1)\theta+\psi]}] + P_n^\lambda \\ y(x-i) &= \sum_{k=0}^{n-1} P_k^\lambda(x; \theta, \psi) [e^{i(n-k)\psi} - e^{i[(n-k-1)\psi+\theta]}] + P_n^\lambda, \end{aligned}$$

which implies that

$$\begin{aligned}
 & e^{i\theta}(\lambda - ix)y(x + i) - e^{i\psi}(\lambda + ix)y(x - i) \\
 (2.6) \quad & = (e^{i\theta} - e^{i\psi}) \sum_{k=0}^{n-1} P_k^\lambda(x; \theta, \psi) [(\lambda - ix)e^{i(n-k)\theta} + (\lambda + ix)e^{i(n-k)\psi}] \\
 & \quad + [e^{i\theta}(\lambda - ix) - e^{i\psi}(\lambda + ix)]P_n^\lambda.
 \end{aligned}$$

Differentiation of the generating function (2.1) with respect to z and comparison of the coefficients at z^{n-1} yields:

$$nP_n^\lambda(x; \theta, \psi) = \sum_{k=0}^{n-1} P_k^\lambda(x; \theta, \psi) [(\lambda - ix)e^{i(n-k)\theta} + (\lambda + ix)e^{i(n-k)\psi}]$$

which together with (2.6) gives (2.5). \square

The first four polynomials P_n^λ are given by the formulas:

Corollary 1.

$$\begin{aligned}
 P_0^\lambda &= 1, \\
 P_1^\lambda &= ix(e^{i\psi} - e^{i\theta}) + \lambda(e^{i\theta} + e^{i\psi}), \\
 2P_2^\lambda &= -x^2(e^{i\psi} - e^{i\theta})^2 + ix(2\lambda + 1)(e^{2i\psi} - e^{2i\theta}) + \lambda[(1 + \lambda)e^{2i\psi} \\
 & \quad + 2\lambda e^{i(\psi+\theta)} + (1 + \lambda)e^{2i\theta}], \\
 6P_3^\lambda &= ix^3[3e^{i\theta}e^{i\psi}(e^{i\psi} - e^{i\theta}) - (e^{3i\psi} - e^{3i\theta})] \\
 & \quad + 3(1 + \lambda)x^2[e^{i\theta}e^{i\psi}(e^{i\psi} + e^{i\theta}) \\
 & \quad - (e^{3i\psi} + e^{3i\theta})] + ix[3\lambda^2e^{i\theta}e^{i\psi}(e^{i\psi} - e^{i\theta}) \\
 & \quad + (3\lambda^2 + 6\lambda + 2)(e^{3i\psi} - e^{3i\theta})] \\
 & \quad + \lambda(1 + \lambda)[3\lambda e^{i\theta}e^{i\psi}(e^{i\psi} + e^{i\theta}) + (\lambda + 2)(e^{3i\psi} + e^{3i\theta})], \\
 24P_4^\lambda &= x^4[(e^{i\psi} - e^{i\theta})^4 + 4e^{2i\psi}e^{2i\theta}] + 2ix^3(2\lambda + 3)(e^{2i\psi} - e^{2i\theta})(e^{i\psi} + e^{i\theta})^2 \\
 & \quad + x^2[-(6\lambda^2 + 18\lambda + 11)(e^{4i\psi} + e^{4i\theta}) + 4(3\lambda + 2)e^{i\psi}e^{i\theta}(e^{2i\psi} + e^{2i\theta}) \\
 & \quad + 6(2\lambda^2 + 2\lambda + 1)e^{2i\psi}e^{2i\theta}] + 2ix(e^{2i\psi} - e^{2i\theta})[(4\lambda^3 + 9\lambda^2 + 11\lambda + 3) \\
 & \quad \times (e^{2i\psi} + e^{2i\theta}) + 2\lambda(2\lambda + 3)e^{i\psi}e^{i\theta}] \\
 & \quad + \lambda(1 + \lambda)[(\lambda + 2)(\lambda + 3)(e^{4i\psi} + e^{4i\theta}) \\
 & \quad + 4\lambda(\lambda + 2)e^{i\psi}e^{i\theta}(e^{2i\psi} + e^{2i\theta}) + 6\lambda(\lambda + 1)e^{2i\psi}e^{2i\theta}].
 \end{aligned}$$

The four special cases of $P_n^\lambda(x; \theta, \psi)$ corresponding to the choice:

$$(a) \quad \psi = -\theta, \quad (b) \quad \psi = \pi - \theta, \quad (c) \quad \psi = \pi + \theta, \quad (d) \quad \psi = \theta$$

lead to some interesting families of polynomials. Namely, we define:

$$(a) \quad G^\lambda(x; \theta, -\theta; z) = \frac{1}{(1 - ze^{i\theta})^{\lambda - ix} (1 - ze^{-i\theta})^{\lambda + ix}} = \sum_{n=0}^{\infty} P_n^\lambda(x; \theta) z^n, \quad z \in \mathbb{D},$$

and of course $P_n^\lambda(x; \theta)$ are the well-known (MP) polynomials of variable $x \in \mathbb{R}$ with parameters $\lambda > 0$, $\theta \in (0, \pi)$;

$$(b) \quad G^\lambda(x; \theta, \pi - \theta; z) = \frac{1}{(1 - ze^{i\theta})^{\lambda - ix} (1 + ze^{-i\theta})^{\lambda + ix}} = \sum_{n=0}^{\infty} Q_n^\lambda(x; \theta) z^n, \quad z \in \mathbb{D},$$

where $Q_n^\lambda(x; \theta)$ we call quasi-Meixner–Pollaczek (QMP) polynomials;

$$(c) \quad G^\lambda(x; \theta, \pi + \theta; z) = \frac{1}{(1 - ze^{i\theta})^{\lambda - ix} (1 + ze^{i\theta})^{\lambda + ix}} = \sum_{n=0}^{\infty} S_n^\lambda(x; \theta) z^n, \quad z \in \mathbb{D},$$

where $S_n^\lambda(x; \theta)$ we call strongly symmetric Meixner–Pollaczek (SSMP) polynomials.

Observe that the special cases: $i^{-n} Q_n^\lambda(x; 0)$ and $S_n^\lambda(x; \frac{\pi}{2})$ represent symmetric (MP) polynomials studied in [1], [6], [8] and [9].

$$(d) \quad G^\lambda(x; \theta, \theta; z) = \frac{1}{(1 - ze^{i\theta})^{2\lambda}} = \sum_{n=0}^{\infty} H_n^\lambda(\theta) z^n, \quad z \in \mathbb{D},$$

where $H_n^\lambda(\theta) = \frac{(2\lambda)_n}{n!} e^{in\theta}$.

From Theorem 2.1 we have as the corollaries the following formulas for the polynomials

$$P_n^\lambda(x; \theta) = P_n^\lambda(x; \theta, -\theta), \quad Q_n^\lambda(x; \theta) = P_n^\lambda(x; \theta, \pi - \theta), \quad S_n^\lambda(x; \theta) = P_n^\lambda(x; \theta, \pi + \theta).$$

Corollary 2. (i) *The (MP) polynomials $P_n^\lambda(x; \theta)$ satisfy the three-term recurrence relation:*

$$\begin{aligned} P_{-1}^\lambda(x; \theta) &= 0, \\ P_0^\lambda(x; \theta) &= 1, \\ nP_n^\lambda(x; \theta) &= 2[x \sin \theta + (n - 1 + \lambda) \cos \theta] P_{n-1}^\lambda(x; \theta) \\ &\quad - (2\lambda + n - 2) P_{n-2}^\lambda(x; \theta), \quad n \geq 1. \end{aligned}$$

(ii) *The polynomials $P_n^\lambda(x; \theta)$ are given by the formula:*

$$P_n^\lambda(x; \theta) = e^{in\theta} \sum_{j=0}^n \frac{(\lambda + ix)_j (\lambda - ix)_{n-j}}{j!(n-j)!} e^{-2ij\theta}, \quad n \in \mathbb{N} \cup \{0\}.$$

(iii) *The polynomials $P_n^\lambda(x; \theta)$ have the hypergeometric representation*

$$P_n^\lambda(x; \theta) = e^{in\theta} \frac{(2\lambda)_n}{n!} F(-n, \lambda + ix, 2\lambda; 1 - e^{-2i\theta}).$$

(iii) *The polynomials $y(x) = P_n^\lambda(x; \theta)$ satisfy the following difference equation*

$$e^{i\theta}(\lambda - ix)y(x + i) + 2i[x \cos \theta - (n + \lambda) \sin \theta]y(x) - e^{-i\theta}(\lambda + ix)y(x - i) = 0.$$

Corollary 3. (i) *The (QMP) polynomials $Q_n^\lambda = Q_n^\lambda(x; \theta)$ satisfy the three-term recurrence relation:*

$$\begin{aligned} Q_{-1}^\lambda &= 0, \\ Q_0^\lambda &= 1, \\ nQ_n^\lambda &= 2i[(\lambda + n - 1) \sin \theta - x \cos \theta]Q_{n-1}^\lambda + (2\lambda + n - 2)Q_{n-2}^\lambda, \quad n \geq 1. \end{aligned}$$

(ii) *The polynomials $Q_n^\lambda = Q_n^\lambda(x; \theta)$ are given by the formula:*

$$Q_n^\lambda(x; \theta) = e^{in\theta} \sum_{j=0}^n (-1)^j \frac{(\lambda + ix)_j (\lambda - ix)_{n-j}}{j!(n-j)!} e^{-2ij\theta}, \quad n \in \mathbb{N} \cup \{0\}.$$

(iii) *The polynomials $Q_n^\lambda = Q_n^\lambda(x; \theta)$ have the hypergeometric representation*

$$Q_n^\lambda(x; \theta) = e^{in\theta} \frac{(2\lambda)_n}{n!} F(-n, \lambda + ix, 2\lambda; 1 + e^{-2i\theta}).$$

(iii) *The polynomials $y(x) = Q_n^\lambda(x; \theta)$ satisfy the following difference equation*

$$e^{i\theta}(\lambda - ix)y(x + i) - 2[x \sin \theta + (n + \lambda) \cos \theta]y(x) + e^{-i\theta}(\lambda + ix)y(x - i) = 0.$$

Corollary 4. (i) *The (SSMP) polynomials $S_n^\lambda = S_n^\lambda(x; \theta)$ satisfy the three-term recurrence relation:*

$$\begin{aligned} S_{-1}^\lambda &= 0, \\ S_0^\lambda &= 1, \\ nS_n^\lambda &= -2ixe^{i\theta}S_{n-1}^\lambda + (2\lambda + n - 2)e^{2i\theta}S_{n-2}^\lambda, \quad n \geq 1. \end{aligned}$$

(ii) *The polynomials $S_n^\lambda = S_n^\lambda(x; \theta)$ are given by the formula:*

$$S_n^\lambda(x; \theta) = e^{in\theta} \sum_{j=0}^n (-1)^j \frac{(\lambda + ix)_j (\lambda - ix)_{n-j}}{j!(n-j)!}, \quad n \in \mathbb{N} \cup \{0\}.$$

(iii) *The polynomials $S_n^\lambda = S_n^\lambda(x; \theta)$ have the hypergeometric representation*

$$S_n^\lambda(x; \theta) = e^{in\theta} \frac{(2\lambda)_n}{n!} F(-n, \lambda + ix, 2\lambda; 2).$$

(iii) *The polynomials $y(x) = S_n^\lambda(x; \theta)$ satisfy the following difference equation*

$$(\lambda - ix)y(x + i) - 2(n + \lambda)y(x) + (\lambda + ix)y(x - i) = 0.$$

Theorem 2.2. *The polynomials $Q_n^\lambda(x; \theta)$ are orthogonal on $(-\infty, +\infty)$ with the weight*

$$w_\theta^\lambda(x) = \frac{1}{2\pi} e^{2\theta x} |\Gamma(\lambda + ix)|^2 \quad \text{if } \lambda > 0 \quad \text{and } \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

and

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{2\theta x} |\Gamma(\lambda + ix)|^2 Q_n^\lambda(x; \theta) \overline{Q_m^\lambda(x; \theta)} dx = \delta_{mn} \frac{\Gamma(n + 2\lambda)}{(2 \cos \theta)^{2\lambda} n!}.$$

In the proof we use the following lemmas.

Lemma 1 ([4], vol. I p. 12). *If $\alpha > 0$ and $p > 0$, then*

$$\int_0^{+\infty} u^{\alpha-1} e^{-pu} e^{-iqu} du = \Gamma(\alpha) (p^2 + q^2)^{-\frac{\alpha}{2}} e^{-i\alpha \arctan(\frac{q}{p})}.$$

Lemma 2 ([11]). *Let $F(s)$ and $G(s)$ be Mellin transforms of $f(x)$ and $g(x)$, i.e.*

$$F(s) = \int_0^{+\infty} f(x) x^{s-1} dx, \quad G(s) = \int_0^{+\infty} g(x) x^{s-1} dx.$$

Then the following formula (Parseval's identity) holds:

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) G(1-s) ds = \int_0^{+\infty} f(x) g(x) dx.$$

Corollary 5. *If $f(x) = x^{2(\lambda+j)} e^{-x^2}$ and $g(x) = x^{2(\lambda+k)-1} e^{-x^2}$, then*

$$F(s) = \Gamma\left(\lambda + j + \frac{s}{2}\right), \quad G(s) = \Gamma\left(\lambda + k + \frac{s-1}{2}\right).$$

Lemma 3. *For any $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\lambda > 0$, $j, k = 1, 2, \dots$ we have*

$$I = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (\lambda + ix)_j (\lambda - ix)_k |\Gamma(\lambda + ix)|^2 e^{2\theta x} dx = \frac{e^{i(j-k)\theta} \Gamma(2\lambda + k + j)}{(2 \cos \theta)^{2\lambda+k+j}}.$$

Proof. Putting $x = \frac{t}{2}$ and next $it = s$ we have:

$$\begin{aligned} I &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\lambda + i\frac{t}{2}\right)_j \left(\lambda - i\frac{t}{2}\right)_k \left|\Gamma\left(\lambda + i\frac{t}{2}\right)\right|^2 e^{\theta t} dt \\ &= \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} \left(\lambda + \frac{s}{2}\right)_j \left(\lambda - \frac{s}{2}\right)_k \left|\Gamma\left(\lambda + \frac{s}{2}\right)\right|^2 e^{-i\theta s} ds, \end{aligned}$$

where we use the well-known formula for Pochhammer symbol: $(a)_j = \frac{\Gamma(a+j)}{\Gamma(a)}$, $j = 1, 2, \dots$ \square

Lemma 4. *For arbitrary polynomial $Q_n^\lambda(x; \theta)$, $\lambda > 0$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$; $k, n = 1, 2, \dots$ we have*

$$J = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{2\theta x} (\lambda - ix)_k |\Gamma(\lambda + ix)|^2 Q_n^\lambda(x; \theta) dx = \frac{e^{i\theta} \Gamma(2k + \lambda) e^{-i\theta k}}{(2 \cos \theta)^{2\lambda+k}} (-k)_n.$$

Proof. Using hypergeometric representation for $Q_n^\lambda(x; \theta)$ we can write

$$\begin{aligned} Q_n^\lambda(x; \theta) &= \frac{e^{in\theta}(2\lambda)_n}{n!} F(-n, \lambda + ix; 2\lambda; 1 + e^{-2i\theta}) \\ &= \frac{e^{in\theta}(2\lambda)_n}{n!} \sum_{j=0}^n \frac{(-n)_j(\lambda + ix)_j}{(2\lambda)_j j!} (1 + e^{-2i\theta})^j. \end{aligned}$$

Therefore

$$\begin{aligned} J &= \frac{e^{in\theta}(2\lambda)_n}{n!} \sum_{j=0}^n \frac{(-n)_j(1 + e^{-2i\theta})^j}{(2\lambda)_j j!} \cdot I \quad (\text{by Lemma 3}) \\ &= \frac{e^{in\theta}(2\lambda)_n}{n!} e^{-ik\theta} \Gamma(2\lambda + k) \frac{1}{(4 \cos^2 \theta)^{\frac{2\lambda+k}{2}}} \\ &\quad \times \sum_{j=0}^n \frac{(-n)_j(2\lambda + k)_j}{(2\lambda)_j j!} \left(\frac{1 + e^{-2i\theta}}{(4 \cos^2 \theta)^{\frac{1}{2}} \cdot e^{-i\theta}} \right)^j \\ &= \frac{e^{in\theta}(2\lambda)_n \Gamma(2\lambda + k) e^{-ik\theta}}{n! (2 \cos \theta)^{2\lambda+k}} \cdot F(-n; 2\lambda + k; 2\lambda; 1). \end{aligned}$$

Using the well-known formula:

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

we obtain

$$(2.7) \quad J = \frac{e^{i(n-k)\theta} \Gamma(2\lambda + k)}{n! (2 \cos \theta)^{2\lambda+k}} \cdot (-k)_n. \quad \square$$

Proof of Theorem 2.2. Let $m \leq n$. Observe that $(-k)_n = 0$ if $k < n$. Therefore by (2.7)

$$J = \frac{\Gamma(2\lambda + n)(-n)_n}{n! (2 \cos \theta)^{2\lambda+n}}, \quad \text{if } k = n$$

and

$$J = 0, \quad \text{if } k < n.$$

Using hypergeometric representation for $Q_n^\lambda(x; \theta)$ we can write

$$Q_n^\lambda(x; \theta) = \frac{e^{in\theta}(2\lambda)_n}{n!} \sum_{j=0}^n \frac{(-n)_j(1 + e^{-2i\theta})^j}{(2\lambda)_j j!} (\lambda + ix)_j = \sum_{j=0}^n A_j (\lambda + ix)_j.$$

Therefore

$$\int_{-\infty}^{+\infty} Q_n^\lambda(x; \theta) \overline{Q_m^\lambda(x; \theta)} w_\theta^\lambda(x) dx = \delta_{nm} \overline{A_n} \frac{\Gamma(2\lambda + n)(-n)_n}{n! (2 \cos \theta)^{2\lambda+n}},$$

where

$$\overline{A}_n = \frac{(-n)_n(1 + e^{2i\theta})^n}{(2\lambda)_n n!} \cdot \frac{e^{-in\theta(2\lambda)_n}}{n!},$$

which ends the proof after some obvious simplifications. \square

Remark 1. In the case $x = 0$ we can obtain “more pleasant” sets of “polynomials”:

$$\begin{aligned} Q_n^\lambda(0; \theta) &= Q_n^\lambda(\theta), \\ S_n^\lambda(0, \theta) &= S_n^\lambda(\theta), \quad \theta \in [0, \pi], \quad \lambda > 0. \end{aligned}$$

for which one can prove the following.

Theorem 2.3. *The function $y = y(\theta) = Q_n^\lambda(\theta) = Q_n^\lambda(0; \theta)$, $\lambda > 0$ satisfies the following second order differential equation:*

$$\cos \theta (Q_n^\lambda)'' - 2\lambda \sin \theta (Q_n^\lambda)' + n(n + 2\lambda) \cos \theta Q_n^\lambda = 0.$$

In particular, if $\lambda = 1$ we have:

$$\cos \theta (Q_n^1)'' - 2 \sin \theta (Q_n^1)' + n(n + 2) \cos \theta Q_n^1 = 0.$$

Theorem 2.4. *The sets of functions $Q_{2k}^\lambda(\theta)$ and $Q_{2k-1}^\lambda(\theta)$ form (separately) the orthogonal systems with the weight function $w^\lambda(\theta) = \cos^{2\lambda} \theta$, $\theta \in [0, \pi]$, $\lambda > 0$.*

3. The generating function for (MP) polynomials allows us to define the generalization of the well-known class \mathbb{T} of holomorphic function (1.1) which are typically-real in \mathbb{D} ($\text{Im}f(z) \cdot \text{Im}z \geq 0$, $z \in \mathbb{D}$) and have the following integral representation

$$f(z) = \int_0^\pi \frac{z}{(1 - ze^{i\theta})(1 - ze^{-i\theta})} d\mu(\theta),$$

where μ is a probability measure on $[0, \pi]$ (e.g. [3], [5], [12], [13]).

Namely, we are going to study the extremal problems within the class $\mathbb{T}(\lambda, \tau)$, $\lambda > 0, \tau \in \mathbb{R}$ of holomorphic functions f of the form (1.1) given by the following integral representation

$$f(z) = \int_0^\pi \frac{z}{(1 - ze^{i\theta})^{\lambda-i\tau}(1 - ze^{-i\theta})^{\lambda+i\tau}} d\mu(\theta),$$

where μ is a probability measure on $[0, \pi]$.

We have in particular $\mathbb{T}(\lambda, 0) = \mathbb{T}(\lambda)$ (e.g. [16], [7]) and $\mathbb{T}(1, 0) = \mathbb{T}(\lambda, \tau)$. In parallel way we are going to study the extremal problems within the classes $\mathbf{T}(\lambda, \tau)$ and $\mathcal{T}(\lambda, \tau)$, $\lambda > 0, \tau \in \mathbb{R}$ of holomorphic functions of the form (1.1) which have the integral representation

$$f(z) = \int_0^\pi \frac{z}{(1 - ze^{i\theta})^{\lambda-i\tau}(1 + ze^{-i\theta})^{\lambda+i\tau}} d\mu(\theta),$$

and

$$f(z) = \int_0^\pi \frac{z}{(1 - ze^{i\theta})^{\lambda-i\tau}(1 + ze^{i\theta})^{\lambda+i\tau}} d\mu(\theta),$$

where μ is a probability measure on $[0, \pi]$.

The classes $\mathbb{T}(\lambda, \tau)$, $\mathbf{T}(\lambda, \tau)$ and $\mathcal{T}(\lambda, \tau)$ differ pretty much, for instance all coefficients a_k of $f \in \mathbb{T}(\lambda, \tau)$ are real, however the odd coefficients of $f \in \mathbb{T}(\lambda, \tau)$ are real and even coefficients of $f \in \mathbf{T}(\lambda, \tau)$ are purely imaginary.

In special case $\tau = 0$, $\lambda = 1$, i.e. $\mathbf{T} = \mathbf{T}(1, 0)$, we are able to find explicitly the radius of local univalence and the radius of univalence of \mathbf{T} which differ from the corresponding values in the class $\mathbb{T} = \mathbb{T}(1, 0)$.

The classes $\mathbb{T}(0, \tau)$, $\mathbf{T}(0, \tau)$ and $\mathcal{T}(0, \tau)$ appear to be of special interest when $\lambda \rightarrow 0^+$.

The same remarks concern also the sets of polynomials

$$S^0(x, \theta) = \lim_{\lambda \rightarrow 0^+} S^\lambda(x, \theta) \quad \text{and} \quad Q^0(x, \theta) = \lim_{\lambda \rightarrow 0^+} Q^\lambda(x, \theta),$$

which generalize the special symmetric Pollaczek polynomials [1].

Remark 2. Due to definition (2.1) of the polynomials $P_n^\lambda(\tau; \theta, \psi)$, $\tau \in \mathbb{R}$, $\theta \in (0, \pi)$, $\psi \in \mathbb{R}$ we can as well consider the extremal problems for more general class of the holomorphic function f of the form (1.1) which have the integral representation

$$f(z) = \int \int_{\Delta} z G^\lambda(\tau; \theta, \psi; z) d\mu(\theta, \psi),$$

where μ is a probability measure on $\Delta = (0, \pi) \times \mathbb{R}$.

REFERENCES

- [1] Araaya, T. K., *The symmetric Meixner–Pollaczek polynomials*, Uppsala Dissertations in Mathematics, Department of Mathematics, Uppsala University, 2003.
- [2] Chihara, T. S., *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.
- [3] Duren, P. L., *Univalent Functions*, Springer, New York, 1983.
- [4] Erdélyi, A., Magnus, W., Oberhettinger, F., Tricomi, F. G., *Higher Transcendental Functions*, vol. I, McGraw-Hill Book Company, New York, 1953.
- [5] Golusin, G., *Geometric Theory of Functions of a Complex Variable*, Translations of Mathematical Monographs, no. 26, Amer. Math. Soc., Providence, R.I., 1969.
- [6] Ismail, M., *On sieved ultraspherical polynomials I: Symmetric Pollaczek analogues*, SIAM J. Math. Anal. **16** (1985), 1093–1113.
- [7] Kiepiela, K., Naraniecka, I., Szynal, J., *The Gegenbauer polynomials and typically real functions*, J. Comp. Appl. Math **153** (2003), 273–282.
- [8] Koekoek, R., Swarttouw, R. F., *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, Report 98-17, Delft University of Technology, 1998.
- [9] Koornwinder, T. H., *Meixner–Pollaczek polynomials and the Heisenberg algebra*, J. Math. Phys. **30** (4) (1989), 767–769.
- [10] Pommerenke, Ch., *Linear-invariant Familien analytischer Funktionen*, Mat. Ann. **155** (1964), 108–154.

- [11] Poularikas, A. D., *The Mellin Transform, The Handbook of Formulas and Tables for Signal Processing*, CRC Press LLC, Boca Raton, 1999.
- [12] Robertson, M. S., *On the coefficients of typically-real functions*, Bull. Amer. Math. Soc. **41** (1935), 565–572.
- [13] Rogosinski, W. W., *Über positive harmonische Entwicklungen und typisch-reelle Potenzreihen*, Math. Z. **35** (1932), 93–121.
- [14] Starkov, V. V., *The estimates of coefficients in locally-univalent family U'_α* , Vestnik Lenin. Gosud. Univ. **13** (1984), 48–54 (Russian).
- [15] Starkov, V. V., *Linear-invariant families of functions*, Dissertation, Ekatiirenburg, 1989, 1–287 (Russian).
- [16] Szynal, J., *An extension of typically-real functions*, Ann. Univ. Mariae Curie-Skłodowska, Sect. A **48** (1994), 193–201.
- [17] Szynal, J., Waniurski, J., *Some problems for linearly invariant families*, Ann. Univ. Mariae Curie-Skłodowska, Sect. A **30** (1976), 91–102.

Iwona Naraniecka
Department of Mathematics
Faculty of Economics
Maria Curie-Skłodowska University
20-031 Lublin
Poland
e-mail: inaraniecka@gmail.com

Jan Szynal
University of Economics and Innovation in Lublin
20-209 Lublin
Poland
e-mail: jan.szynal3@gmail.com

Anna Tatarczak
Department of Mathematics
Faculty of Economics
Maria Curie-Skłodowska University
20-031 Lublin
Poland
e-mail: antatarczak@gmail.com

Received September 28, 2012