LINES AND OSCULATING LINES OF HYPERSURFACES

J.M. LANDSBERG AND C. ROBLES

Abstract. This is a detailed study of the infinitesimal variation of the varieties of lines and osculating lines through a point of a low degree hypersurface in projective space. The motion is governed by a system of partial differential equations which we describe explicitly.

1. Introduction

1.1. **Motivation and context.** Let $X^n \subset \mathbb{CP}^{n+1}$ be a hypersurface with $deg(X) \leq n$. Given a point $x \in X$, let $\mathcal{C}_x \subset \mathbb{P}T_xX$ denote the tangent directions to lines $\ell = \mathbb{P}^1$ such that $x \in \ell \subset X$. If $deg(X) \leq 2$ and $x, y \in X$ are general points, then $\mathcal{C}_x$ is projectively isomorphic to $\mathcal{C}_y$. The goal of this paper is to answer a question posed by Jun-Muk Hwang:

> Let $deg(X) \geq 3$. How can $\mathcal{C}_x$ vary (modulo projective isomorphism) as x varies over the general points of X?

The question is motivated by Hwang and N. Mok's program to study Fano varieties via the variety of tangent directions to minimal degree rational curves through a point. (See [Hwa01] for an overview.)

The first interesting case is $d = 3$ and $n = 4$. Here $\mathcal{C}_x$ is a curve of degree six in $\mathbb{P}^3$ – i.e., a genus four curve in its canonical embedding. When $d = 3$ and $n = 5$, $\mathcal{C}$ is a $K3$ surface in $\mathbb{P}^4$.

In another direction, this paper is a continuation of a program to understand the relationship between the algebraic geometry of subvarieties of projective space and their local projective differential geometry. This was a project of the classical geometers (e.g. [Fub16, Sev01, Car92]), was revived by Griffiths and Harris in [GH79] and continued in [JM94, Lan99b, Lan03a, Lan99a, Lan99c, Rob08, LR08, AG05, AG04, FP01], for example.

Before addressing Hwang's question, it will be useful to generalize it as follows: Fix $d \leq n$ and let $X^n \subset \mathbb{P}^{n+1}$ be a variety of degree at least d. Given $x \in X$, let $\mathcal{C}_{d,x} \subset \mathbb{P}T_xX$ denote the *tangent directions to lines osculating to order d with X at x*. (See §2 for a precise definition and a discussion of osculation). If $d = deg(X)$, then $\mathcal{C}_{d,x} = \mathcal{C}_x$. For sufficiently general X of degree at least d and general $x \in X$, $\mathcal{C}_{d,x} \subset \mathbb{P}T_xX$ is the transverse intersection of smooth hypersurfaces of degrees $2, ..., d$. In particular it has codimension $d-1$ in $\mathbb{P}T_xX$.

1.2. **The results.** Our results are technical and require substantial notation to state precisely. Here we present central ideas and roughly state the results. The terminology will be made precise in Sections 2 and 3.

Let $X \subset \mathbb{P}^{n+1}$ be a hypersurface of degree at least d. We say $x \in X$ is a *d-general point* if all discrete projective differential invariants up to order d are constant on a Zariski

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open subset of X containing x. Let p denote the Hilbert polynomial of $\mathcal{C}_{d,x} \subset \mathbb{P}T_xX$. Let $U \subset X$ be a neighborhood of x that admits a local framing (cf. §2.2) consisting of d-general points. A choice of a local framing e yields, for all $y \in U$, polynomials $(F_{2,e,y}, ..., F_{d,e,y})$, with $F_{\delta,e,y} \in S^\delta T^*_yX$. While the polynomials depend on our choice of frame, their zero set in $\mathbb{P}T_yX$ does not; that variety is $\mathcal{C}_{d,y} \subset \mathbb{P}T_yX$. The framing enables us to identify the tangent spaces at different points with a fixed complex vector space T of dimension n. Then we may regard these polynomials as elements of $S^\delta T^*$. This determines a map

$$\hat{\phi}_{d,e} : U \to Hilb_p \,,$$

where $Hilb_p$ is the Hilbert scheme of subvarieties of $\mathbb{P}T = \mathbb{P}^{n-1}$ having Hilbert polynomial p.

In order to eliminate the ambiguity in our choice of framing (which determines the identification of T_xX with T) we wish to quotient the image by the action of $GL(T)$. The quotient of $Hilb_p$ by $GL(T)$ is not a manifold or algebraic variety. However, the quotient of $\hat{\phi}_{d,e}(U) \subset Hilb_p$ will usually be a manifold. For example, if X is sufficiently general, the stabilizer in $GL(T)$ of a point in $\hat{\phi}_{d,e}(U)$ will be trivial. For simplicity of discussion, for the moment assume this is the case. Let $Hilb^0_p$ denote the open subset of $Hilb_p$ where the stabilizer in $GL(T)$ is trivial and such that $Hilb^0_p$ contains $\hat{\phi}_{d,e}(U)$ and is a manifold. Set

$$\mathcal{M}^p_d \;=\; Hilb^0_p/GL(T) \,.$$

which by definition is a manifold. We obtain a well defined map

$$\phi_{U,d} : U \to \mathcal{M}^p_d \,.$$

This map is independent of choice of local framing, so it extends to a well-defined map

$$\phi_d : X_{\text{general}} \to \mathcal{M}^p_d \,. \tag{1.1}$$

For most X the Hilbert polynomial p will be that of a generic complete intersection of type $(2, ..., d)$ in $\mathbb{P}^{n-1}$, and the stabilizer G will be trivial. (The Hilbert polynomial is that of a generic complete intersection if and only if the polynomials $F_{2,e,x}, ..., F_{d,e,x}$ have no non-trivial syzygies.) We denote this generic moduli space by $\mathcal{M}_d$. Describing $\mathrm{d}\phi_d$ in the case that d is the degree of X answers Hwang's question. We show

(Theorem 4.1.a) The image of ϕ_d satisfies a first-order system of partial differential equations. The system is expressed by the condition that the Gauss map γ_{ϕ_d} associated to ϕ_d takes image in a proper subvariety $\mathbf{\Xi}^p_d$ of the Grassmann bundle of $\mathcal{M}^p_d$. We give an explicit description of $\mathbf{\Xi}^p_d$ in §4.

(Theorem 4.1.b) When $deg(X) = d$ (so that $\mathcal{C}_{x,d} = \mathcal{C}_x$) the image of ϕ_d satisfies a more restrictive first-order system of partial differential equations: the Gauss image must lie in a smaller subvariety $\mathbf{\Phi}^p_d \subset \mathbf{\Xi}^p_d$. We give an explicit description of $\mathbf{\Phi}^p_d$ in §4.

(Theorem 4.2) Suppose that $deg(X) \geq d$ and that the Gauss image of $\phi_d(X_{\text{general}})$ lies in $\mathbf{\Phi}^p_d$. Then, under mild genericity conditions on X, we may conclude that $deg(X) = d$. (More precisely, it is sufficent to assume that either X is smooth or that for a general line $\ell \subset X$, $\ell \cap X_{\text{sing}} = \emptyset$.)

(Theorem 4.5) There is a non-empty Zariski open subset A_d of $\mathcal{M}_d$, such that for any X with $deg(X) \geq d \geq 4$ and $x \in X_{\text{general}}$ with $\phi(x) \in A_d$, $\text{rank}\,(\mathrm{d}\phi_d|_x) = n$.

Theorem 4.5 is suprising as it gives conditions that imply that $\mathcal{C}_x$ "must move as much as possible" without any hypotheses on X.

In the special case that $d = 3$, Theorem 4.1.b is related to a classical result of Fubini and Cartan. Fubini [Fub16] stated (and Cartan [Car20] gave a rigorous proof) that under the hypotheses of the theorem, X is projectively determined by $\phi_3(U)$, $U \subset X$ any open subset in the analytic topology. For a short, modern proof, see [JM94].

Remark 1.1. In [JM94] they also extend the Cartan-Fubini theorem when $n \geq 3$ to the case that the second fundamental form is degenerate. The result fails in the case of surfaces, see [Car20], though it still holds "generically."

The results above are consequences of Theorems 3.2 and 3.3 on the Fubini forms (cf. §2.3) of the hypersurface X. More precisely the $\mathcal{C}_{d,x}$ do not contain all the geometric information of X accessible by d derivatives at a point. This information is contained in the Fubini forms $(F_{2,e,x}, ..., F_{d,e,x})$ – cf. §3, with $F_{\delta,e,x} \in S^\delta T^*$. (The exception is the case $d = 3$, where $\mathcal{C}_x$ contains all the information of the Fubini forms.) The collection of Fubini forms are not well-defined at $x \in X$, but do define an equivalence class under the action of a group H of dimension $n^2 + 2n + 3$, see §4.1.

We will first establish results in a fixed local framing e which fixes a choice of Fubini forms. In particular, we have a map

$$\tilde{\phi}_{d,e} : U \ \rightarrow \ \oplus_{\delta=2}^{d} S^\delta T^* \, . \tag{1.2}$$

We determine a first-order system of partial differential equations that $\tilde{\phi}_{d,e}$ must satisfy (Theorem 3.2). We quotient to the Hilbert scheme and then by the action of $GL(T)$ to obtain the choice-free results listed above. This is carried out in Section 4.

Several examples of cubic hypersurfaces are considered in §5.

2. Local frames and Fubini forms

2.1. **Notational conventions.** For subsets $X \subset \mathbb{P}V$, $\widehat{X} \subset V$ denotes the corresponding cone. For a submanifold $X \subset \mathbb{P}V$ and $x \in X$, $\widehat{T}_x X \subset V$ denotes its *affine tangent space*. The *tangent space* and *normal space* at x are

$$T_x X \ = \ \hat{x}^* \otimes (\widehat{T}_x X / \hat{x}) \quad \text{and} \quad N_x X = T_x \mathbb{P}V / T_x X = \hat{x}^* \otimes (V / \widehat{T}_x X) \, ,$$

respectively.

We will use the following index ranges:

$$\begin{aligned} & 0 \leq j, k, \ell \leq n+1 \, , \\ & 1 \leq a, b, a_j \leq n \, , \\ & \mathsf{N} = n+1 \, . \end{aligned}$$

The linear span of vectors $\{v_1, ..., v_k\}$ is denoted $\langle v_1, ..., v_k \rangle$.

2.2. **Adapted frames.** Let $\pi : \mathbb{C}^{n+2}\backslash 0 \to \mathbb{P}^{n+1}$ denote the natural projection $v \mapsto [v]$. Let $X^n \subset \mathbb{P}^{n+1}$ be a submanifold and let $\mathcal{F}^1 \to X$ denote the bundle of first-order adapted frames. Elements of $\mathcal{F}^1 \subset \mathrm{GL}(V)$ are frames (or bases) $e = (e_0, e_1, \ldots, e_{\mathsf{N}})$ of $V = \mathbb{C}^{n+2}$, such that

$$\begin{aligned} \pi(e_0) &\in X\,, \\ \widehat{T}_{[e_0]}X &= \langle e_0, \ldots, e_n \rangle\,. \end{aligned}$$

A *local, first-order adapted framing* is a section $e : U \to \mathcal{F}^1$, $U \subset X$ open in the analytic topology. Every smooth point on a projective variety admits a neighborhood U with local framing (§2.4).

Since X is a hypersurface, N_xX is a line bundle spanned by

$$\underline{e}_{\mathsf{N}} := e^0 \otimes (e_{\mathsf{N}} \bmod \widehat{T}_{e_0}X)\,.$$

Here $(e^0, \ldots, e^{\mathsf{N}})$ is the basis of V^* dual to e.

Define the $\mathfrak{gl}(V)$-valued Maurer-Cartan form $\omega = \omega^j_k e_j \otimes e^k$ on $GL(V)$ by $\mathrm{d}e_j = \omega^k_j\, e_k$. Recall the Maurer-Cartan equation:

$$\mathrm{d}\,\omega^j_k = -\omega^j_\ell \wedge \omega^\ell_k\,.$$

We abuse notation and denote the pullback of ω to $\mathcal{F}^1$ by ω as well.

2.3. **Fubini forms.** The variety $\mathcal{C}_{k,x}$ will be defined (Definition 2.2) as the zero set of the Fubini forms $(F_{2,e,x}, \ldots, F_{k,e,x})$. We review Fubini forms here; see [IL03, Ch. 3] for details.

Since $\widehat{T}_{[e_0]}X = \langle e_0, e_1, \ldots, e_n \rangle$ we have $\omega^{\mathsf{N}}_0 = 0$ on $\mathcal{F}^1$. Differentiating this equation, and an application of the Cartan lemma (see, e.g. [IL03, p. 314]), yields functions $r_{ab} = r_{ba} : \mathcal{F}^1 \to \mathbb{C}$ such that

$$\omega^{\mathsf{N}}_a = r_{ab}\,\omega^b_0\,, \qquad 1 \le a, b \le n\,. \tag{2.1}$$

The coefficients r_{ab} define the Fubini quadric (also known as the projective second fundamental form) $F_2 = r_{ab}\,\underline{e}^a \underline{e}^b \otimes \underline{e}_{\mathsf{N}} \in \Gamma(\mathcal{F}^1, \pi^*(S^2T^*X \otimes NX))$. Here the $\underline{e}^a \in T^*_xX$ are dual to the basis

$$\underline{e}_a = e^0 \otimes (e_a \bmod e_0)$$

of T_xX and $x = \pi(e_0)$.

The coefficients of the Fubini cubic are obtained by differentiating (2.1) and another application of Cartan's Lemma. The coefficients $r_{a_1a_2\cdots a_p}$ of the p-th Fubini form $F_p = r_{a_1\cdots a_p}\,\underline{e}^{a_1}\cdots\underline{e}^{a_p} \otimes \underline{e}_{\mathsf{N}} \in \Gamma(\mathcal{F}^1, \pi^*(S^pT^*X \otimes NX))$ are defined inductively. The defining formula is as follows. Let $\mathfrak{S}_{p+q}$ denote the symmetric group on $p+q$ letters. Given two tensors $T_{a_1\ldots a_p}$ and $U_{a_{p+1}\ldots a_{p+q}}$, let

$$T_{(a_1\ldots a_p}U_{a_{p+1}\ldots a_{p+q})} = \frac{1}{(p+q)!}\sum_{\sigma\in\mathfrak{S}_{p+q}} T_{\sigma(a_1)\ldots\sigma(a_p)}U_{\sigma(a_{p+1})\ldots\sigma(a_{p+q})}$$

denote the *symmetrization* of their product. For example, $T_{(a_1}U_{a_2)} = \frac{1}{2}(T_{a_1}U_{a_2} + T_{a_2}U_{a_1})$. We exclude from the symmetrization operation any index that is outside the parethenses. For example, in $r_{b(a_1\ldots a_{p-1}}\,\omega^b_{a_p)}$ we symmetrize over only the a_i, excluding the b index. Define

$$r_a = 0\,.$$

Proposition ([Rob08]). *The coefficients of F_{p+1}, $p > 1$, fully symmetric in their lower indices, are defined by*

$$\begin{aligned} r_{a_1\ldots a_p b}\,\omega^b_0 \;&=\; -\,\mathrm{d}r_{a_1\ldots a_p} \,-\, r_{a_1\ldots a_p}\,\left\{(p-1)\,\omega^0_0 \,+\, \omega^{\mathsf{N}}_{\mathsf{N}}\right\} \\ &\quad +\, p\,\left\{(p-2)\,r_{(a_1\ldots a_{p-1}}\omega^0_{a_p)} \,+\, r_{b(a_1\ldots a_{p-1}}\,\omega^b_{a_p)}\right\} \\ &\quad -\,\sum_{j=1}^{p-2}\binom{p}{j}\left\{\, r_{b(a_1\ldots a_j}\,r_{a_{j+1}\ldots a_p)}\,\omega^b_{\mathsf{N}} \,+\, (j-1)\,r_{(a_1\ldots a_j}\,r_{a_{j+1}\ldots a_p)}\,\omega^0_{\mathsf{N}}\right\}. \end{aligned}$$

The Fubini forms F_p are defined on $\mathcal{F}^1$ and do not descend to well-defined sections of $S^pT^*X \otimes NX$ over X. However, the ideal generated by $\{F_2, F_3, \ldots, F_k\}$ is independent of our choice of first-order adapted frame over $x \in X$.

Proposition 2.1. *Given $X^n \subset \mathbb{P}^{n+1}$ and a smooth point $x \in X$, there exists a local framing about x such that all entries of the Maurer-Cartan form (pulled back via this framing) vanish with the exception of ω^a_0 and ω^N_a, $1 \le a \le n$.*

Proof. One such local framing is given in §2.4 below. □

Definition 2.2. Given a smooth point $x \in X$, $\mathcal{C}_{k,x} \subset \mathbb{P}T_xX$ is the zero locus of $F_{2,x}, \ldots, F_{k,x}$.

2.4. **Fubini forms in a coordinate framing.** Given a smooth point $x \in X$, fix homogeneous coordinates $[z_0 : z_1 : \cdots : z_{\mathsf{N}}]$ of $\mathbb{P}V$ so that $x = [1 : 0 : \cdots : 0]$, and $\widehat{X}$ is tangent to $\{z_{\mathsf{N}} = 0\}$ at $(1, 0, \ldots, 0) \in \hat{x} \subset V$. Setting $z_0 = 1$ yields a coordinate neighborhood on $\mathbb{P}^{\mathsf{N}}$ with local coordinates $(z^1, \ldots, z^{\mathsf{N}})$ centered at x. Shrinking the coordinate neighborhood U if necessary, we may assume that $U \cap X$ is a graph $z^{\mathsf{N}} = f(z^1, \ldots, z^n)$ over its embedded tangent space at x. A local, first-order adapted frame $e : U \to \mathcal{F}^1$ over U is defined by

$$\begin{aligned} e_0 &= \frac{\partial}{\partial z^0} + z^a\frac{\partial}{\partial z^a} + f\frac{\partial}{\partial z^{\mathsf{N}}} \\ e_a &= \frac{\partial}{\partial z^a} + f_{z^a}\frac{\partial}{\partial z^{\mathsf{N}}} \\ e_{\mathsf{N}} &= \frac{\partial}{\partial z^{\mathsf{N}}}\,; \end{aligned}$$

here, f_{z^a} denotes partial differentiation, and $1 \le a \le n$. With respect to this frame, the only nonzero entries in the Maurer-Cartan form $\omega = \omega^j_k e_j \otimes e^k$ are

$$\omega^a_0 \;=\; \mathrm{d}z^a \qquad \text{and} \qquad \omega^{\mathsf{N}}_a \;=\; f_{z^a z^b}\,\mathrm{d}z^b\,.$$

Regard $z = (z^1, \ldots, z^n)$ as local coordinates on X. It is immediate from the expression for ω^{N}_a above that the coefficients of the second Fubini form at z are $r_{ab}(z) = f_{z^a z^b}(z)$. More generally, Proposition 2.3 implies the Fubini forms at z are given by

$$F_{k,e}(z) = (-1)^k f_{z^{a_1}\ldots z^{a_k}}(z)\,\mathrm{d}z^{a_1}\cdots\mathrm{d}z^{a_k} \otimes \underline{e_{\mathsf{N}}}\,.$$

See [IL03, §3.3.7] for details.

If the $y = (y^1, \ldots, y^n)$ are linear coordinates on T_zX induced by $\{e_1, \ldots, e_n\}$, it follows that $\mathcal{C}_{k,z}$ is the zero set of the polynomials

$$f_{z^{a_1}\ldots z^{a_\ell}}(z)\,y^{a_1}\cdots y^{a_\ell}\,, \quad 2 \le \ell \le k\,.$$

In particular,

$$\mathcal{C}_{k,x} = \{\ell \in \mathbb{P}T_xX \mid \exists L = \mathbb{P}^1 \subset \mathbb{P}^n,\ \mathbb{P}T_xL = \ell,\ \mathrm{mult}_x(X \cap L) \geq k+1\}.$$

Lemma 2.3. *Given $\mathcal{C} \subset \mathbb{P}T$, a complete intersection of hypersurfaces of degrees $2, ..., d$ and $d' \geq d$, there exists a hypersurface X of degree d', a point $x \in X$, and a local framing e such that Zeros$(\tilde{\phi}_{d,e}(x)) = \mathcal{C}$.*

Remark. Thus given $\mathbf{P} \in \oplus_{\delta=2}^d S^\delta T^*$ there exist X and $x \in X$ such that $\phi_d(x) = \mathbf{P}$. It is not clear which points of $\oplus_{\delta=2}^d S^\delta T^*$ can be the image of a *general* point of some X.

Proof. Pick $\mathbf{P} = (P_2, ..., P_d)$, with $P_j \in S^jT^*$ so that $\mathcal{C} = [Zeros(\mathbf{P})]$. Let $f(z) = f(z^1, \ldots, z^n)$ be a polynomial of degree d' and let $f_k(z)$ denote the degree k homogeneous component of $f(z)$. Specify $f_0 = 0$, $f_1 = 0$ and $f_\delta = P_\delta$ for $2 \leq \delta \leq d$. Take X to be the closure of the graph of f, and $x = [1 : 0 : \cdots : 0]$. □

3. Results in terms of Fubini forms

3.1. **Gauss maps.** We will consider two types of Gauss map: the Gauss map of a hypersurface in $\mathbb{P}V$, and the Gauss map associated to a differentiable map between manifolds. Given a vector space V, let $G(m, V)$ denote the Grassmannian of m-planes through the origin in V.

The *Gauss map* γ_X *of a hypersurface* $X \subset \mathbb{P}^{\mathsf{N}} = \mathbb{P}V$ is

$$\begin{array}{rcl} \gamma_X : X_{\text{smooth}} & \to & G(n+1, V)\,, \\ x & \mapsto & \gamma_X(x) := \widehat{T}_xX\,. \end{array}$$

The tangent space $T_EG(n+1, V)$ may be identified with $(V/E) \otimes E^*$. Making use of a frame e we may identify $T_{\gamma_X(x)}G(n+1, V)$ with $N_xX \otimes \hat{x} \otimes (\widehat{T}_xX)^*$. Under the identification the differential is given by $\mathrm{d}\gamma_{X,x}(v^a \underline{e}_a) = v^a r_{ab}\, \underline{e}_{\mathsf{N}} \otimes e^b \in N_xX \otimes \hat{x} \otimes (\widehat{T}_xX)^*$.

The Gauss map γ_X is *nondegenerate* if $\mathrm{d}\gamma_{X,x} : T_xX \to T_{\gamma_X(x)}G(n+1, V)$ is of maximial rank n. The Gauss map is nondegenerate if and only if the quadratic polynomial $F_{2,e}$ is of full rank; equivalently, the quadric hypersurface $\{F_{2,e}(x) = 0\} \subset \mathbb{P}T_xX$ is smooth.

Remark. The second Fubini form F_2 and (projective) second fundamental form (defined by $\mathrm{d}\gamma_X$) agree. See [IL03, Ch. 3].

Given a manifold Σ, let $\mathbf{G}(n, T\Sigma) \to \Sigma$ denote the Grassmann bundle, whose fiber over $\sigma \in \Sigma$ is $G(n, T_\sigma\Sigma)$. Let $f : Z \to \Sigma$ be a C^1 map of manifolds of generic rank r and set $Z' = \{z \in Z \mid \mathrm{rank}\,(df_z) = r\} \subset Z$. The *Gauss map* γ_f *associated to* f is

$$\begin{array}{l} \gamma_f : Z' \to \mathbf{G}(r, T\Sigma) \\ \qquad x \mapsto T_{f(x)}f(Z). \end{array}$$

3.2. **Results.** Given $S = (S_3, ..., S_{d+1}) \in \oplus_{\delta=3}^{d+1} S^\delta T^*$, such that the common zero locus of the S_j is not a cone, we obtain an n-plane $E_S \in G(n, \oplus_{\delta=2}^d S^\delta T^*)$ by

$$E_S = \langle (v \lrcorner S_3, ..., v \lrcorner S_{d+1}) \mid v \in T \rangle\,.$$

Here $v\lrcorner : S^\delta T^* \to S^{\delta-1}T^*$ denotes the interior product, i.e., $(v\lrcorner P)(w_1,...,w_{\delta-1}) = P(v,w_1,...,w_{\delta-1})$. Fix $\mathbf{P} = (P_2,...,P_d) \in \oplus_{\delta=2}^d S^\delta T^*$. Define a map

$$\begin{aligned} \mu_{\mathbf{P}} : S^{d+1}T^* &\to \oplus_{\delta=3}^{d+1} S^\delta T^*\,, \\ \alpha &\mapsto (P_3,...,P_d,\alpha)\,. \end{aligned}$$

Let $I_\delta(\mathbf{P})$ denote the degree δ homogeneous component of the ideal generated by $(P_2,...,P_d)$.

Definition 3.1. Using the notations $E_S, \mu_{\mathbf{P}}$ above, define the varieties

$$\begin{aligned} \boldsymbol{\Xi}_{\mathbf{P}} &:= \{E_{\mu_{\mathbf{P}}(\alpha)} \mid \alpha \in S^{d+1}T^*\} \subset G(n, \oplus_{\delta=2}^d S^\delta T^*) \\ \boldsymbol{\Phi}_{\mathbf{P}} &:= \{E_{\mu_{\mathbf{P}}(\beta)} \mid \beta \in I_{d+1}(\mathbf{P})\} \subset G(n, \oplus_{\delta=2}^d S^\delta T^*)\,. \end{aligned}$$

Since $\oplus_{\delta=2}^d S^\delta T^*$ is a vector space, we may identify the tangent space $T_{\mathbf{P}}(\oplus_{\delta=2}^d S^\delta T^*)$ with $\oplus_{\delta=2}^d S^\delta T^*$, and define sub-bundles of the Grassmann bundle: $\boldsymbol{\Phi}_d \subset \boldsymbol{\Xi}_d \subset \mathbf{G}(n, T(\oplus_{\delta=2}^d S^\delta T^*))$ by $(\boldsymbol{\Xi}_d)_{\mathbf{P}} = \boldsymbol{\Xi}_{\mathbf{P}}$, and $(\boldsymbol{\Phi}_d)_{\mathbf{P}} = \boldsymbol{\Phi}_{\mathbf{P}}$.

Theorem 3.2. *Let $X^n \subset \mathbb{P}^{n+1}$ be a hypersurface of degree at least d, with nondegenerate Gauss map. Let $U \subset X$ be a d-general open subset admitting a local framing e as in §2.4, and consider the map*

$$\begin{aligned} \tilde\phi_{d,e} : U &\to \oplus_{\delta=2}^d S^\delta T^* \\ x &\mapsto (F_{2,e}(x),...,F_{d,e}(x))\,. \end{aligned} \tag{3.1}$$

Assume further that $\gamma_{\tilde\phi_{d,e}}$ is defined on U; that is, $\operatorname{rank}(\mathrm{d}\tilde\phi_{d,e}|_x) = n$ *for all $x \in U$. Then, recalling the varieties $\boldsymbol{\Xi}_d$ and $\boldsymbol{\Phi}_d$ of Definition 3.1, we have the following:*

(a) *The Gauss image $\gamma_{\tilde\phi_{d,e}}(U)$ is contained in $\boldsymbol{\Xi}_d$.*
(b) *If $deg(X) = d$, then $\gamma_{\tilde\phi_{d,e}}(U) \subset \boldsymbol{\Phi}_d$.*

Remark. The assumption on the rank of $\mathrm{d}\tilde\phi_{d,e}$ is generic; see Proposition 3.3 below.

Proof. Choose a local framing as in Proposition 2.1, then

$$\begin{aligned} \mathrm{d}\tilde\phi_{d,e}(z) &= (\mathrm{d}F_{2,e}(z),\ldots,\mathrm{d}F_{d,e}(z)) \\ &= -(F_{3,e}(z),\ldots,F_{d+1,e}(z))\,. \end{aligned}$$

Given $z \in U$, $\gamma_{\tilde\phi_{d,e}}(z)$ is the n-plane

$$\langle -(v\lrcorner F_{3,e}(z),\ldots,v\lrcorner F_{d+1,e}(z)) \mid v \in T\rangle\,. \tag{3.2}$$

This proves Part (a).

If $deg(X) = d$, then $F_{d+1,e}(x) \in I_{d+1}(\mathcal{C}_{d,x})$, establishing Part (b). □

Proposition 3.3. *Let $X^n \subset \mathbb{P}^{n+1}$ be a general hypersurface of degree $d' \geq d$ with nondegenerate Gauss map. Let $U \subset X$ be a d-general open subset admitting a local first-order adapted framing e of X as in Proposition 2.1, and consider the map* (3.1)*. Then* $\operatorname{rank}(\mathrm{d}\tilde\phi_{d,e}|_x) = n$ *for all $x \in U$.*

Proof. By lower semi-continuity it suffices to show that $\operatorname{rank}(\mathrm{d}\tilde\phi_{d,e}|_x) = n$ for just one X. A family of such X is given in §3.3. □

3.3. **Example.** Fix linear coordinates $\overline{z} = (z^0, \ldots, z^{\mathsf{N}}) \in \mathbb{C}^{\mathsf{N}+1}$. Set $z = (z^1, \ldots, z^n)$ and let $p_j(z)$ be a homogeneous polynomial of degree $j = 2, ..., d$ with $d > 2$. Consider the degree d hypersurface X_F given by the zero set of:

$$F(\overline{z}) \ = \ -(z^0)^{d-1}\, z^{\mathsf{N}} + (z^0)^{d-2}\, p_2(z) + (z^0)^{d-3}\, p_3(z) \, + \cdots + p_d(z).$$

Note that

$$(X_F)_{\mathrm{sing}} = \{z^0 = 0\} \cap \mathit{Zeros}(p_{d-1}) \cap \mathit{Zeros}(p_d)_{\mathrm{sing}}\,.$$

In particular, X_F will usually be smooth. The point $x = [1 : 0 : \cdots : 0] \in \mathbb{P}^{\mathsf{N}}$ lies on X_F. In the affine coordinate neighborhood $U = \{z^0 = 1\} \subset \mathbb{P}^{\mathsf{N}}$, X may be expressed as a graph

$$z^{\mathsf{N}} = f(z) = p_2(z) + \cdots + p_d(z)\,.$$

Let $e = (e_0, \ldots, e_{\mathsf{N}})$ be a first-order framing, constructed as in §2.4.

Let $y = (y^1, \ldots, y^n)$ be linear coordinates on T^* induced by the frame e. Then

$$F_{\delta,e}(z) \ = \ (-1)^\delta \left(\frac{\partial^\delta f}{\partial z^{a_1} \cdots \partial x^{a_\delta}}(z) \right) y^{a_1} \cdots y^{a_\delta}\,.$$

If $p_\delta(z) = p_{a_1,\ldots,a_\delta}\, z^{a_1} \cdots z^{a_\delta}$, with $p_{a_1,\ldots,a_\delta}$ symmetric, then the second Fubini quadric is given by $F_{2,e}(z) = r_{ab}(z) y^a y^b$ with

$$r_{ab}(z) \ = \ 2\, p_{ab} + \sum_{\delta=3}^{d} j(j-1)\, p_{abc_3\cdots c_\delta} z^{c_3} \cdots z^{c_\delta}\,.$$

It follows that if p_2 is nondegenerate, then the Fubini quadric will be nondegenerate in a neighborhood of $z = 0$.

Additionally, note that $F_{\delta,e}(0) = (-1)^\delta \delta!\, p_{a_1\cdots a_\delta} y^{a_1} \cdots y^{a_\delta}$, $2 \le \delta \le d$, and $F_{\delta,e} = 0$ for all $\delta > d$, so that

$$\mathrm{d}\tilde{\phi}_{d,e}(0)(T) \ = \ \left\langle \left(3 \cdot 3!\, p_{ab_1b_2}\, y^{b_1} y^{b_2}\,,\ \cdots\,, (-1)^{d-1} d \cdot d!\, p_{ab_1\cdots b_{d-1}} y^{b_1} \cdots y^{b_{d-1}}\,,\, 0 \right) \mid 1 \le a \le n \right\rangle.$$

In particular, for a generic choice of p_3, the differential $\mathrm{d}\tilde{\phi}_{d,e}(0)$ will have maximal rank n.

3.4. **Exterior differential systems interpretation.** We may rephrase Theorem 3.2 in the language of exterior differential systems (EDS) as follows. (See [IL03, p. 177] or [BCG$^+$91].) Let Σ be a manifold and let $\pi : \mathbf{G}(n, T\Sigma) \to \Sigma$ denote the Grassmann bundle. The Grassmannian $\mathbf{G}(n, T\Sigma)$ carries a canonical linear Pfaffian system $(\mathcal{I}, \mathcal{J})$. Let $p \in \Sigma$, $E \in \mathbf{G}(n, T_p\Sigma)$, and $E^\perp \subset T_p^*\Sigma$ be the forms vanishing on E. The canonical EDS on $\mathbf{G}(n, T\Sigma)$ is generated by the subspace $\mathcal{I} \subset T^*\mathbf{G}(n, T\Sigma)$, defined fiber-wise by $\mathcal{I}_{p,E} = \pi^*(E^\perp)$. The independence condition $\mathcal{J} \supset \mathcal{I}$ is given by $\mathcal{J}_{p,E} = \pi^*(T_p^*\Sigma)$. Integral manifolds of this tautological system are immersed n-dimensional submanifolds $i : M \to \mathbf{G}(n, T\Sigma)$ such that $i^*(\mathcal{I}) \equiv 0$ and $i^*(\Lambda^n(\mathcal{J}/\mathcal{I}))$ is non-vanishing. They are characterized as the Gauss images of immersed n-dimensional submanifolds N of Σ. That is, there exists N such that $M = \gamma(N)$, where γ is the Gauss map $p \mapsto (p, T_pN) \in G(n, T_p\Sigma)$. Theorems 3.2.a and 3.2.b are rephrased as

> *The image $\tilde{\phi}_d(X)$ is an integral manifold of the pull-back of the tautological system $(\mathcal{I}, \mathcal{J})$ on $\mathbf{G}(n, T(\oplus_{\delta=2}^{d} S^\delta T^*))$ to $\mathbf{\Xi}_d$ (respectively, $\mathbf{\Phi}_d$).*

Other results of this paper can similarly be rephrased in the language of EDS.

4. Descent to $\mathcal{M}_d$

Fix a polynomial p that occurs as a Hilbert polynomial of a codimension $d-1$ complete intersection of hypersurfaces of degrees $2, 3, ..., d$ in $\mathbb{P}^{n-1}$ such that the degree two hypersurface is smooth. (Recall that at a general point x on a generic X, $\mathcal{C}_x$ with be such a complete intersection.)

Let

$$\mathcal{U}_p = \{(P_2, ..., P_d) \in \oplus_{\delta=2}^d S^\delta T^* \mid \mathit{Hilb}(\mathit{Zeros}(P_2, ..., P_d)) = p\}$$

and let

$$\pi_p : \mathcal{U}_p \to \mathit{Hilb}_p$$

denote the projection map to the Hilbert scheme. Let $X^n \subset \mathbb{P}^{n+1}$ be a hypersurface with a nondegenerate Gauss map (§3.1); equivalently the Fubini quadric is of maximal rank at general points. Let $x \in X_{\text{general}}$, and choose a local first-order adapted framing e on an open set $U \subset X$, containing x as in Proposition 2.1. Let p denote the Hilbert polynomial of $\mathit{Zeros}(\tilde{\phi}_{d,e}(x))$. The polynomial p is independent of our choices of x and e. Consider the map

$$\hat{\phi}_{d,e} = \pi_p \circ \tilde{\phi}_{d,e} : U \to \mathit{Hilb}_p \, .$$

The map $\hat{\phi}_{d,e}$ depends on our choice of framing. (Recall that our identification of T_xX with the fixed vector space T is made via the frame e.) To remove this ambiguity, we would like to quotient the image in Hilb_p by the action of $GL(T)$. Unfortunately the quotient of Hilb_p by the action of $GL(T)$ is not a manifold. As mentioned in the introduction, if X is sufficiently general, there is a well-defined quotient of Hilb_p^0. In the case there is a nontrivial stabilizer G of $\hat{\phi}_{d,e}(x) \in \mathit{Hilb}_p$, let $\mathcal{H}^G \subset \mathit{Hilb}_p$ denote the connected component of Hilb_p with stabilizer G containing $\mathit{Zeros}(\tilde{\phi}_{d,e}(x))$. In particular $\mathcal{H}^{\text{Id}} = \mathit{Hilb}_p^0$ and when $G = \text{Id}$, we drop the superscript. Note that we are making the implicit assumption that G does not vary with the point x.

Set

$$\mathcal{M}_d^{p,G} \; = \; \mathcal{H}^G / GL(T)$$

and let

$$\phi_{d,U} : U \to \mathcal{M}_d^{p,G}$$

be the quotient map induced by $\hat{\phi}_{d,e}$. This extends to a well defined map

$$\phi_d : X_{\text{general}} \to \mathcal{M}_d^{p,G}. \tag{4.1}$$

Let $\mathcal{S}_d^{p,G} \subset \oplus_{\delta=2}^d S^\delta T^*$ denote the inverse image $\hat{\phi}_{d,e}^{-1}(\mathcal{H}^G)$ and let $\mathbf{\Xi}_d^{p,G}$ and $\mathbf{\Phi}_d^{p,G}$ denote the images of the bundles $\mathbf{\Xi}_d$ and $\mathbf{\Phi}_d$ (Definition 3.1), restricted to $\mathcal{S}_d^{p,G}$, and pushed down to $\mathcal{M}_d^{p,G}$. Theorem 3.2 implies

Theorem 4.1. *Let $X^n \subset \mathbb{P}^{n+1}$ be a hypersurface of degree at least d, with nondegenerate Gauss map. Let $x \in X_{\text{general}}$ and let p denote the Hilbert polynomial determined by the Fubini forms of degree $\leq d$. Let $G \subseteq GL(T)$ denote the stabilizer of $\mathit{Zeros}(\phi_{d,e}(x))$ in the Hilbert scheme Hilb_p, which we assume to be independent of $x \in X_{\text{general}}$, so the map* (4.1) *is well defined.*

(a) *Then $\gamma_{\phi_d}(X_{\text{general}}) \subset \mathbf{\Xi}_d^{p,G}$.*

(b) *If $deg(X) = d$, then $\gamma_{\phi_d}(X_{\text{general}}) \subset \mathbf{\Phi}_d^{p,G}$.*

Theorem 4.2. *Let $X^n \subset \mathbb{P}^{n+1}$ be either smooth or such that X is $\mathbb{P}^1$-uniruled and for a general line $\ell \subset X$, $\ell \cap X_{\text{sing}} = \emptyset$. If for some $d \leq n$, $\gamma_{\phi_d}(X_{\text{general}}) \subset \mathbf{\Phi}_d^{p,G}$, then $deg(X) = d$.*

Proof. If X satisfies the hypotheses of the theorem, then $F_{d+1} \subset I(\mathcal{C}_{d,x})$ by Definition 3.1. Therefore $\mathcal{C}_{d,x} = \mathcal{C}_{d+1,x}$. By Theorem 2 of [Lan03b], such X are $\mathbb{P}^1$-uniruled. But if $deg(X) > d$, and B is an irreducible variety of lines on X that covers X, then every line in B intersects X_{sing}; see, e.g., [LT]. □

Remark. Compare Theorem 4.2 with Proposition 3.23 of [Lan96]. There, instead of imposing smoothness conditions on X, one requires that the generators of $I(\mathcal{C}_{d,x})$ have no nontrivial syzygies, and concludes that X has degree d after $2d+1$ derivatives, as opposed to the $d+1$ derivatives of Theorem 4.2.

Let $\pi = \pi_{p,G} : \mathcal{S}_d^{p,G} \to \mathcal{M}_d^{p,G}$ denote the projection. Set $\mathfrak{m} = \{\mathbf{m} = (m_{\tau,\delta}) \mid m_{\tau,\delta} \in S^{\tau-\delta}T^*\}$. Then $\mathfrak{m}$ acts on $\oplus_{\delta=2}^d S^\delta T^*$ by

$$\mathbf{m}.(P_2, \ldots, P_d) \ := \ \left(P_2\, ,\, P_3 + m_{3,2}P_2\, ,\, \ldots\, ,\, P_d + \textstyle\sum_{j=2}^{d-1} m_{d,j}P_j\right) .$$

Additionally, the action of $\mathfrak{gl}(T)$ on T induces an action on $S^\delta T^*$ and thence on $\oplus_{\delta=2}^d S^\delta T^*$. We denote the action by $X.(P_2, \ldots, P_d) := (X.P_2, \ldots, X.P_d)$.

Proposition 4.3. *With the notation as introduced above, we have*

$$ker\, \mathrm{d}\pi|_{\mathbf{P}} = (\mathfrak{gl}(T) + \mathfrak{m}).\mathbf{P}$$

Proof. The fiber over $\pi(\mathbf{P})$ of the projection $\mathcal{H}^G \to \mathcal{M}_d^{p,G}$ is $PGL(T) \cdot I(\mathbf{P})$. The fiber over $\pi_p(\mathbf{P}) = I(\mathbf{P})$ of the projection from $\mathcal{S}_d^{p,G} \to \mathcal{H}^G$ is $M \cdot \mathbf{P}$, where M is the Lie group associated to the Lie algebra $\mathfrak{m}$.

Next, note that given $X \in \mathfrak{gl}(T)$ and $\mathbf{m} \in \mathfrak{m}$ there exists $X' \in \mathfrak{gl}(T)$, $\mathbf{m}' \in \mathfrak{m}$ such that $X.\mathbf{m}.\mathbf{P} = \mathbf{m}'.X'.\mathbf{P}$. It now follows that the fiber of the differential $\mathrm{d}\pi$ is the sum of the $\mathfrak{gl}(T)$ and $\mathfrak{m}$ actions. □

Proposition 4.4. *If $d \geq 4$ and $\mathbf{P} \in \oplus_{\delta=2}^d S^\delta T^*$ is sufficiently general as described in the proof, then for all $E \in \Xi_{\mathbf{P}}$,*

$$E \cap ker\,(\mathrm{d}\pi|_{\mathbf{P}}) = 0. \tag{4.2}$$

Proof. Let $\mathbf{P} = (P_2, ..., P_d)$ be such that (P_2, P_3, P_4) is a generic triple. Any point of the linear space E is of the form $(v\lrcorner P_3, v\lrcorner P_4, ..., v\lrcorner P_d, v\lrcorner \alpha)$ for some $v \in T$ and $\alpha \in S^{d+1}T^*$. A point of $\ker(\mathrm{d}\pi|_{\mathbf{P}})$ is of the form

$$\left(X.P_2\, ,\, X.P_3 + m_{3,2}P_2\, ,\, \ldots\, ,\, X.P_d + \textstyle\sum_{j=2}^{d} m_{d,j}P_j\right) .$$

We can find X such that $X.P_2 = v\lrcorner P_3$, but for generic triples (P_2, P_3, P_4) the vector spaces $\{v\lrcorner P_4 \mid v \in T\}$, which is n-dimensional and $X.P_3 + m_{3,2}P_2$ which is less than (n^2+n)-dimensional will have zero intersection in the $\binom{n+2}{3}$-dimensional S^3T^*. □

Remark. Note that assuming genericity of any of the P_δ for $\delta \geq 4$ would have been enough to conclude (4.2).

Combining Propositions 4.3 and 4.4, we obtain

Theorem 4.5. *For $d \geq 4$, there is a non-empty Zariski open subset $A_d \subset \mathcal{M}_d$ such that any $X^n \subset \mathbb{P}^{n+1}$ of degree at least d with $x \in X_{\text{general}}$ such that $\phi_d(x) \in A_d$, must have* $\operatorname{rank}(\mathrm{d}\phi_d|_x) = n$.

Remark. More generally, we have

$$\operatorname{rank} \mathrm{d}\tilde{\phi}_{d,e}(x) \; - \; \operatorname{rank} \mathrm{d}\phi_d(x) \; = \; \dim \left(\mathrm{d}\tilde{\phi}_{d,e}(x)(T) \cap (\mathfrak{gl}(T) + \mathfrak{m}) . \tilde{\phi}_{d,e}(x) \right) . \tag{4.3}$$

Example 3.3 continued. Generically, the stabilizer of p_d in $GL(T)$ will be trivial so that in (4.2) we need only consider the intersection of $\langle p_{abc}\, y^b y^c \rangle_{a=1}^n$ with $\langle (p_{ab} - p_{abc} z^c)\, y^a y^b \rangle$. Generically, the intersection will be trivial, so that the rank of $\mathrm{d}\phi_3$ is maximal.

Remark. Had we normalized F_2 to be constant, F_3 would have been forced to vary. See §4.2.

4.1. **Sharper theorems.** Instead of descending all the way to $\mathcal{M}_d^{p,G}$, one may obtain similar results by descending just to $(\oplus_{\delta=2}^d S^\delta T^*)/H$ as we have a canonical map $\overline{\phi}_d : X_{\text{general}} \to (\oplus_{\delta=2}^d S^\delta T^*)/H$. Here H is the unipotent group preserving the flag $\hat{x} \subset \hat{T}_x X \subset V$.

4.2. **Second order adapted frames.** One could choose to work with second order adapted frames where $F_{2,X}$ is normalized to be a fixed quadratic form Q (e.g. $Q = {y_1}^2 + \cdots + {y_n}^2$), and consider the map $\tilde{\phi}'_{d,e}$ from such a framing to $\oplus_{\delta=3}^d S^\delta T^*$. In this case the derivative $d\tilde{\phi}'_{d,e}(z)$ is *shifted* in the following sense. Contract $P_\delta \otimes P_3$ via (the dual quadric to) Q to obtain an element of $S^{\delta-1}T^* \otimes S^2 T^*$; then symmetrize to get an element $\xi_\delta(\mathbf{P}) \in S^{\delta+1}T^*$. We find that

$$\begin{aligned} \mathrm{d}\tilde{\phi}'_{d,e}(z) &= (0, \mathrm{d}F_{3,e}(z), \ldots, \mathrm{d}F_{d,e}(z)) \\ &= -(F_{3,e}(z) + \xi_3(\mathbf{P}), \ldots, F_{d+1,e}(z) + \xi_d(\mathbf{P})) \,. \end{aligned}$$

5. Cubic examples

5.1. **Fermat cubic.** Given linear coordinates $\overline{z} = (z^0, \ldots, z^{\mathsf{N}}) \in \mathbb{C}^{\mathsf{N}+1}$, consider the Fermat cubic

$$F(\overline{z}) \; = \; (z^0)^3 + \cdots + (z^{\mathsf{N}})^3 \,.$$

It is easy to see that $x = [\mathsf{N}^{1/3} : -1 : \cdots : -1]$ is a smooth point of the hypersurface $X = \{F = 0\}$. In the cases $3 \leq n \leq 6$ we used Maple to confirm that the Fubini quadric F_2 is nondegenerate at x, and that $\mathrm{d}\phi_3$ has maximal rank at x.

5.2. **The determinant.** In contrast to the hypersurfaces discussed in this article, if X is quasi-homogeneous, i.e., the Zariski closure in $\mathbb{P}V$ of an orbit of a group G acting linearly on V, then its differential invariants will be constant on a Zariski open subset. More generally, if it is a G-variety for some group G, then its differential invariants will be constant along G-orbits. For example, consider the the cubic hypersurface $\mathrm{Det}(3) \subset \mathbb{P}^8$. Given coordinates

$$w \; = \; \begin{pmatrix} w^0 & w^3 & w^8 \\ w^6 & w^1 & w^4 \\ w^5 & w^7 & w^2 \end{pmatrix} \; \in \; \mathbb{C}^9 \,,$$

Det(3) is given by the equation

$$F(w) \;=\; w^0w^1w^2 \;+\; w^3w^4w^5 \;+\; w^6w^7w^8 \;-\; w^0w^4w^7 \;-\; w^2w^3w^6 \;-\; w^1w^5w^8\,.$$

We will consider an open coordinate neighborhood of the smooth point $x = [1 : 1 : 0 : \cdots : 0] \in \mathrm{Det}(3)$.

Note that Det(3) is preserved by the action of $GL(9)$ on $\mathbb{P}^8$ and that x, corresponding to a rank two matrix, lies in the maximal orbit. Hence we expect ϕ_3 to be constant. Indeed it is possible to construct a local, first-order adapted framing e' in a neighborhood of x with respect to which the Fubini invariants and thus $\tilde\phi_{3,e'}$ are constant.

However, the framing e' is not of the form constructed in §2.4. Given a framing e of the type constructed §2.4, $\mathrm{d}\tilde\phi_{3,e}$ will have maximal rank 7. However, $\mathrm{d}\tilde\phi_{3,e}(x)(T) \subset I^d(x) + \mathfrak{gl}(T).\tilde\phi_{3,e}(x)$, and (4.3) yields $\operatorname{rank} d\phi_{3,e}(x) = 0$.

The first three differential invariants in such a framing are

$$\begin{aligned}
F_{2,e}(x) &= 2\left(y^3y^6 + y^5y^7\right)\,,\\
F_{3,e}(x) &= 6\left(y^1y^3y^6 - y^1y^5y^7 + y^2y^5y^6 + y^3y^4y^7\right)\,,\\
F_{4,e}(x) &= 12\left((y^1)^2 + y^2y^4\right)\,F_{2,e}(z)\,.
\end{aligned}$$

Note also that the quadric $F_{2,e}$ is singular. This requires that we alter the moduli space that ϕ_3 maps into.

5.3. **The permanent.** Maintaining the coordinates of §5.2, the equation of the permanent Perm(3) is

$$F(w) \;=\; w^0w^1w^2 \;+\; w^3w^4w^5 \;+\; w^6w^7w^8 \;+\; w^0w^4w^7 \;+\; w^2w^3w^6 \;+\; w^1w^5w^8\,.$$

While the permanent is not invariant under the action of $GL(9)$ on $\mathbb{P}^8$, it is invariant under left and right multiplication by diagonal matrices (with a one dimensional stabilizer) and a permutation group. So we must have $\operatorname{rank} \mathrm{d}\phi_3 \le 8 - (3+3-1) = 3$.

If the entries w^5, w^6, w^0, w^3, w^8 in the first column and row of the matrix w are nonzero, then they can be normalized to 1 by the group action. We selected eight normalized points on the permanent, and found that that computations with (4.3) (aided by Maple) yield $\operatorname{rank} \mathrm{d}\phi_3(x) = 3$ in each case. Thus $\operatorname{rank} \mathrm{d}\phi_3(x) = 3$ at a general point x on the permanent.

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Department of Mathematics, Texas A&M University, College Station, TX 77843
E-mail address: jml@math.tamu.edu, robles@math.tamu.edu