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LINES OF PRINCIPAL CURVATURE AROUND UMBILICS AND WHITNEY UMBRELLAS

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Abstract. In this paper is studied the configuration of lines of curvature near a Whitney umbrella which is the unique stable singularity for maps of surfaces into $\boldsymbol{R}^3$. The pattern of such configuration is established and characterized in terms of the 3-jet of the map. The result is used to establish an expression for the Euler-Poincaré characteristic in terms of the number of umbilics and umbrellas.

1. Introduction. The bending or curvature pattern of a smooth mapping $\alpha : M \to \boldsymbol{R}^3$, where M is a compact oriented two dimensional manifold, will be represented here by singular points, $\mathcal{S}_\alpha$, at which the mapping has rank less than 2 and the bending can be regarded to be infinite; the umbilic points, $\mathcal{U}_\alpha$, at which the bending is finite but equal in all directions: and by the family of lines of principal curvature $\mathcal{F}_{1,\alpha}$ and $\mathcal{F}_{2,\alpha}$ defined on $M \setminus (\mathcal{U}_\alpha \cup \mathcal{S}_\alpha)$, which represent the directions along which the bending, quantitatively expressed by the *normal curvature*, is extremal (maximal along $\mathcal{F}_{1,\alpha}$ and minimal along $\mathcal{F}_{2,\alpha}$). These four objects will be assembled into the *principal configuration* of the mapping, denoted by $\mathcal{P}_\alpha = (\mathcal{S}_\alpha, \mathcal{U}_\alpha, \mathcal{F}_{1,\alpha}, \mathcal{F}_{2,\alpha})$. The points of $\mathcal{S}_\alpha$ and $\mathcal{U}_\alpha$ are regarded as the singularities of the foliations $\mathcal{F}_{1,\alpha}$ and $\mathcal{F}_{2,\alpha}$.

The study of these foliations near umbilic singularities was started by Darboux [Dar], in the class of analytic surfaces. Under generic conditions on the third derivatives, he found three types, D_1, D_2 and D_3, called here *Darbouxian Umbilics*. These points are illustrated in Figure 3.

This result of Darboux was rediscovered and reproved by Gutierrez and Sotomayor, [GS1]–[GS3], in the context of structural stability of principal lines on regularly immersed surfaces of class C^r, $r \geq 4$. They showed that Darbouxian umbilic points characterize those with local structurally stable configuration, under small C^3 deformations of the surface. See also the work of Bruce and Fidal [B-F].

A study of principal foliations near the set $\mathcal{S}_\alpha$ of singular points, aiming to characterize their local stability, was carried out by Gutierrez and Sotomayor [GS2]. To this end they gave two sufficient conditions, the first of which is the Whitney singularity condition for stability of mappings in the sense of Singularity Theory [G-G]. However, in this paper, an erroneous

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conclusion slipped into the description of principal foliations around these singularities. The correct configuration is established in Theorem 1 below, and illustrated in Figure 2.a.

The configuration exhibits one hyperbolic and one parabolic sector for each principal foliation. In fact, it is topologically equivalent to the Darbouxian D_2 principal configuration. See Figures 2.a and 3.

Theorem 2 states that stability of principal configuration at a singular point, in the sense described above, is equivalent to the Whitney umbrella singularity condition. A well known result in Singularity Theory establishes that this is the condition which characterizes the stability of singularities of maps of two dimensional into three dimensional manifolds [G-G].

Theorem 3 expresses the Euler-Poincaré characteristic of M in terms of the cardinality of Darbouxian umbilics and Whitney umbrellas.

Theorem 4 formulates the global stability result for principal configurations, taking into account the local structure around the umbrella obtained in Theorem 1. This corrects the statement of Theorem 2 in [GS2].

Below are mentioned two papers pertinent to the subject of lines of curvature near singularities. In [SG1], Garcia and Sotomayor established the stable patterns of lines of curvature near a conic singularity of an implicit surface. This situation corresponds to mappings of zero rank, which are degenerate of codimension 6 in the space of mappings. In [SG2], the stable patterns of lines of curvature at generic ends of algebraic surfaces was determined by the same authors. This amounts to the consideration of singular points at the origin, obtained after the inversion, $I(p) = p/|p|^2$, of the ends of the algebraic surface.

2. Statement of main results. In order to formulate the main results of the present work, we will review the elements involved in the principal configuration $\mathcal{P}_\alpha = (\mathcal{S}_\alpha, \mathcal{U}_\alpha, \mathcal{F}_{1,\alpha}, \mathcal{F}_{2,\alpha})$ associated to a mapping $\alpha : M \to \boldsymbol{R}^3$.

Denote by $\mathcal{M}^k = \mathcal{M}^k(M, \boldsymbol{R}^3)$ the space of C^k mappings of M into $\boldsymbol{R}^3$. When endowed with the C^s topology $s \leq k$, this space will be denoted by $\mathcal{M}^{k,s}$.

Denote by $\mathcal{S}_\alpha$ the set of singular points of α, that is, those points p where $\mathcal{D}\alpha_p$ (the derivative of α at p) has rank ≤ 1. Denote also by $\mathcal{U}_\alpha$ the set of umbilic points of α, that is, those points p where the *second fundamental form* $II_\alpha(p) = -\langle DN_\alpha(p), D\alpha(p)\rangle$ is proportional to the *first fundamental form* $I_\alpha(p) = -\langle D\alpha(p), D\alpha(p)\rangle$. Here $\langle\, ,\, \rangle$ is the Euclidean metric on $\boldsymbol{R}^3$ and $N_\alpha : M \setminus \mathcal{S}_\alpha \to \boldsymbol{S}^2$ is the normal map of α defined by $N_\alpha(p) = \alpha_u \wedge \alpha_v / \|\alpha_u \wedge \alpha_v\|$, where $(u, v) : M \to \boldsymbol{R}^2$ is a positive local chart of M around p, $\wedge$ denotes the exterior product of vectors in $\boldsymbol{R}^3$, determined by a fixed (once for all) orientation of $\boldsymbol{R}^3$, $\alpha_u = \partial\alpha/\partial u$, $\alpha_v = \partial\alpha/\partial v$ and $\|\ \| = \langle\, ,\, \rangle^{1/2}$ is the Euclidean norm of $\boldsymbol{R}^3$.

Finally, $\mathcal{F}_{1,\alpha}$ (resp. $\mathcal{F}_{2,\alpha}$) denotes the foliation on $M \setminus (\mathcal{U}_\alpha \cup \mathcal{S}_\alpha)$ defined by the family of curves of maximal (resp. minimal) principal curvature of α. This means that, at each point $p \in M \setminus (\mathcal{U}_\alpha \cup \mathcal{S}_\alpha)$, any vector v which spans the line $\mathcal{L}_{1,\alpha}$ (resp. $\mathcal{L}_{2,\alpha}$) tangent to $\mathcal{F}_{1,\alpha}$ (resp. $\mathcal{F}_{2,\alpha}$) provides the maximum $k_{1,\alpha}$ (resp. minimum $k_{2,\alpha}$), among all possible directions $u \in T_pM$, of the normal curvature k_n at p, $k_n(p, u) = II_a(p)(u, u)/I_a(u, u)$.

The function $k_{1,\alpha}$ (resp. $k_{2,\alpha}$) on $M \setminus \mathcal{S}_\alpha$ is called the maximal (resp. minimal) principal curvature of α. It is to be of class C^{k-2} on $M \setminus (\mathcal{U}_\alpha \cup \mathcal{S}_\alpha)$.

In a local chart (u, v), the principal line fields $\mathcal{L}_{1,\alpha}$ and $\mathcal{L}_{2,\alpha}$ are expressed implicitly by the following quadratic differential equation ([Spi]):

$$(Fg - Gf)dv^2 + (Eg - Ge)dudv + (Ef - Fe)du^2 = 0,$$

where $I_\alpha = Edu^2 + 2Fdudv + Gdv^2$ and $II_\alpha = edu^2 + 2fdudv + gdv^2$ are respectively the first and the second fundamental forms of α.

The concept of C^s*-principal structural stability* at a point $p \in M$ can be formulated as follows: For every neighborhood V_p of p in M, there must be a neighborhood $\mathcal{V}_\alpha$ of α in $\mathcal{M}^{k,s}$ such that for every map $\beta \in \mathcal{V}_\alpha$ there exist a point q_β in V_p and a local homeomorphism h_β on the domain such that $h_\beta : W_p \to W_{q_\beta}$ between neighborhoods of p and q_β, which maps p to q_β and maps $\mathcal{F}_{1,\alpha}|W_p$ and $\mathcal{F}_{2,\alpha}|W_p$ respectively onto $\mathcal{F}_{1,\beta}|W_{q_\beta}$ and $\mathcal{F}_{2,\beta}|W_{q_\beta}$.

A smooth map $\alpha : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ sending the origin to the origin is said to be *regular* if the first jet extension $j^1\alpha$ of α has rank 2; otherwise it is called *singular*.

The mapping α is said to have a Whitney umbrella at 0 provided it has rank 1 and its first jet extension $j^1\alpha$ is transversal to the codimension 2 submanifold $S^1(2, 3)$ of 1-jets of rank 1 in the space $J^1(2, 3)$ of 1-jets of smooth mappings of $(\boldsymbol{R}^2, 0)$ to $(\boldsymbol{R}^3, 0)$. In coordinates this means that there exist a local chart (u, v) such that $\alpha_u(0) \neq 0$, $\alpha_v(0) = 0$ and $[\alpha_u, \alpha_{uv}, \alpha_{vv}] \neq 0$. Here [., ., .] means the determinant of three vectors.

The structure of a smooth map near such point is illustrated in Figure 1. It follows from the work of Whitney [Whi] that these points are isolated and in fact have the following normal form under diffeomorphic changes of coordinates in the source and target ($\mathcal{A}$-equivalence):

$$x = u, \quad y = uv, \quad z = v^2.$$

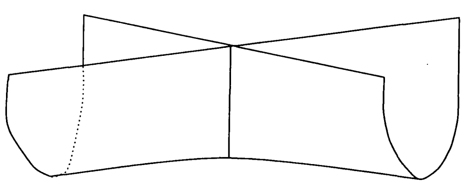

FIGURE 1. Whitney umbrella singularity.

THEOREM 1. *Let p be a Whitney umbrella singularity of a map $\alpha : M \to \boldsymbol{R}^3$ of class C^k, $k \geq 4$. Then the principal configuration near p has the following structure: Each principal foliation $\mathcal{F}_{i,\alpha}$ of α has exactly two sectors at p; one parabolic and the other hyperbolic. Also, the separatrices of these sectors are tangent to the kernel of $D\alpha_p$.*

Figure 2.a illustrates the behavior of curvature lines in the domain of a map α with a Whitney umbrella singularity. This corrects an erroneous assertion on Theorem 1 of [GS2, page 553]. The proof is given in Section 3.

THEOREM 2. *A mapping of class C^k, $k \geq 4$, is C^3-principal structurally stable at a singular point if and only if it has at this point a Whitney umbrella.*

The proof is given in Section 4.

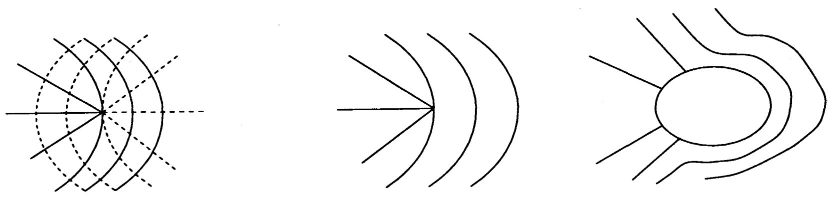

FIGURE 2.a. FIGURE 2.b. FIGURE 2.c.

Now we will recall the result of [GS1,3] about the local behavior of curvature lines near an umbilic point.

THEOREM ([GS1]–[GS3]). *Let p be an umbilic point of a mapping α given in a Monge chart (u, v) by*

$$\alpha(u, v) = \left(u, v, \frac{k}{2}(u^2 + v^2) + \frac{a}{6}u^3 + \frac{b}{2}u^2 v + \frac{c}{6}v^3 + O(4)\right).$$

Suppose $b(b - a) \neq 0$ and that either:

$$(D_1) \quad \left(\frac{c}{2b}\right)^2 - \frac{a}{b} + 2 < 0, \quad \textit{or}$$

$$(D_2) \quad \left(\frac{c}{2b}\right)^2 + 2 > \frac{a}{b} > 1, \quad a \neq 2b, \quad \textit{or else}$$

$$(D_3) \quad \frac{a}{b} < 1.$$

Then the behavior of lines of curvature near the umbilic point p, in the cases D_1, D_2 and D_3, called Darbouxian umbilics, is as in Figure 3.

A mapping $\alpha \in \mathcal{M}^r$, $r \geq 4$, is C^3-principally structurally stable at a point $p \in \mathcal{U}_\alpha$ if and only if p is a Darbouxian umbilic point.

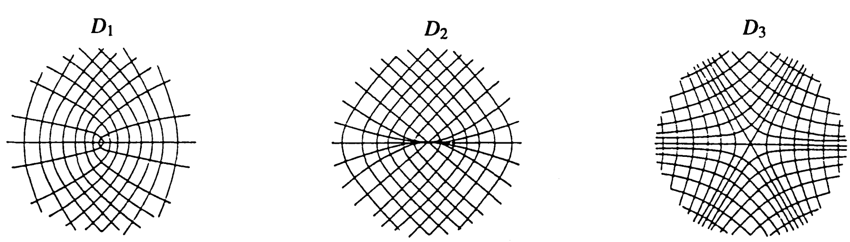

FIGURE 3. Lines of curvature near Darbouxian umbilic points.

THEOREM 3. *Let $\alpha : M \to \boldsymbol{R}^3$ be a mapping of class C^k, $k \geq 4$, of a compact and oriented two dimensional manifold M into $\boldsymbol{R}^3$. Suppose that all the umbilic points of α*

are Darbouxian and all the singular points of α are Whitney umbrellas. Then the following expression for the Euler-Poincaré characteristic of M holds:

$$\chi(M) = \frac{1}{2}[\#(D_1) + \#(D_2) + \#(W) - \#(D_3)],$$

where $\#(D_i)$, $i = 1, 2, 3$, *is the number of Darbouxian umbilic points of type* D_i *and* $\#(W)$ *is the number of Whitney umbrella points.*

The proof is given in Section 5. In Section 6 is formulated Theorem 4 on global principal stability for mappings, extending that for immersions ([GS1]–[GS3]).

REMARK 1. An anonymous referee has kindly pointed out to us a connection between the Euler characteristic of $\alpha(M)$ and umbilic points.

Under conditions of Theorem 3 concerning Whitney umbrella singularities together with the assumption that the map has only normal crossings (i.e., generic double and triple points), Izumiya and Marar proved the following formula

$$\chi(\alpha(M)) = \chi(M) + \#(\tau) + \#(W)/2,$$

where $\#(\tau)$ is the number of triple points of $\alpha(M)$ ([I-M]).

This and Theorem 3, assuming furthermore that the double and triple points are disjoint from $\mathcal{U}_\alpha$, lead to the following expression:

$$\chi(\alpha(M)) = \#(W) + \#(\tau) + \frac{1}{2}[\#(D_1) + \#(D_2) - \#(D_3)].$$

3. Proof of Theorem 1. The proof of Theorem 1 follows from Propositions 1 through 3 below.

Denote by $J^k(2,3)$ the space of k-jets of smooth mappings of $\boldsymbol{R}^2$ to $\boldsymbol{R}^3$, sending the origin to the origin. On this space consider the action of the group $\mathcal{G}^k$ generated by the k-jets of smooth diffeomorphisms in the domain and that of the group of positive isometries and homotheties in the target. The proof below is inspired by West's thesis [Wes].

PROPOSITION 1. *Let* $\alpha : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ *be a* C^r, $r \geq 4$, *map with a Whitney umbrella at* 0. *Then by the action of the group* $\mathcal{G}^k$ *and that of rotations and homotheties of* $\boldsymbol{R}^3$, *the map* α *can be written in the following form*:

$$\alpha(u, v) = (u, y(u, v), z(u, v)),$$

where,

$$\begin{aligned} y(u,v) &= uv + \left(\frac{a}{6}\right)v^3 + O(4), \\ z(u,v) &= \left(\frac{b}{2}\right)u^2 + cuv + v^2 + \\ &\quad \left(\frac{A}{6}\right)u^3 + \left(\frac{B}{2}\right)u^2v + \left(\frac{C}{2}\right)uv^2 + \left(\frac{D}{6}\right)v^3 + O(4) \end{aligned} \tag{1}$$

and $O(4)$ *means terms of order greater than or equal to four.*

PROOF. By the rank 1 condition imposed at 0, we can find a rotation $R : (\boldsymbol{R}^3, 0) \to (\boldsymbol{R}^3, 0)$ and a diffeomorphism $h : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ such that $\alpha_1(u, v) = (R \circ \alpha \circ h)(u, v) = (u, s(u, v), t(u, v))$ with $Ds(0) = Dt(0) = 0$. Then, using the condition $[\alpha_u, \alpha_{uv}, \alpha_{vv}](0) \neq 0$, we can eliminate the term v^2 of s. More precisely, there exists a rotation R_x of $\boldsymbol{R}^3$ fixing the x axis such that the following holds.

$$\begin{aligned}\alpha_2(u, v) &= (R_x \circ \alpha_1)(u, v)\\ &= (u, a_1 uv + a_2 u^2 + s_3(u, v), b_1 u^2 + b_2 uv + b_3 v^2 + t_3(u, v)),\end{aligned}$$

where s_3 and t_3 are of order three or more.

As $a_1 \neq 0$, $\beta(u, v) = (u, -(a_2/a_1)u + (1/a_1)v)$ is a change of coordinates. Then $\alpha_3 = \alpha_2 \circ \beta$ is of the form:

$$\alpha_3(u, v) = (u, uv + c_1 u^3 + c_2 u^2 v + c_3 uv^2 + Av^3 + S_4(u, v), T_2(u, v)),$$

where $[\alpha_{3u}, \alpha_{3uv}, \alpha_{3vv}] = 2d_3 \neq 0$ and S_4 and T_2 are terms of order 4 and 2, respectively.

By the local diffeomorphism $\gamma(u, v) = (u, v - c_1 u^2 - c_2 uv - c_3 v^2)$, we can reduce α_3 to

$$\alpha_4(u, v) = (u, uv + Av^3 + \bar{S}_4(u, v), d_1 u^2 + d_2 uv + dv^2 + T_3(u, v)).$$

Finally, rescaling the target by the homothety $r(x, y, z) = (1/d)(x, y, z)$ and the domain by the linear map $p(u, v) = (du, v)$, we obtain:

$$\begin{aligned}\alpha_5(u, v) &= (r \circ \alpha_4 \circ p)(u, v)\\ &= (u, uv + (a/6)v^3 + y_4(u, v), (b/2)u^2 + cuv + v^2 + z_3(u, v)).\end{aligned}$$

This completes the proof. □

REMARK 2. The change of coordinates in Proposition 1 above does not modify the geometry of the principal configuration of α.

The differential equation of the lines of curvature of the map α around a Whitney umbrella singular point (0, 0), as in Proposition 1, is given by:

$$\begin{aligned}&[8v^3 + uO(\sqrt{u^2 + v^2}) + O(u^2 + v^2)]dv^2\\ &+ [2u + Cu^2 + (D - ac)uv - av^2 + O((u^2 + v^2)^{3/2})]dudv\\ &+ \Big[-2v + \frac{B}{2}u^2 + \frac{1}{2}(ac - D)v^2 - b^2 cu^3 +\\ &\qquad vO(\sqrt{u^2 + v^2}) + O(u^2 + v^2)\Big]du^2 = 0.\end{aligned} \tag{2}$$

REMARK 3. Equation (2) restricted to the positive u-axis ($\{u \geq 0, v = 0\}$), after dividing by u, has the form:

$$(O(|u|))dv^2 + (2 + O(|u|))dudv + ((B/2)u^2 + O(|u|^2))du^2 = 0.$$

Therefore, close to the origin, one of the foliations, from now on named as $\mathcal{F}_{1,\alpha}$ (resp. named as $\mathcal{F}_{2,\alpha}$) is almost orthogonal (resp. parallel) to the positive u-axis.

The proof of the following proposition is immediate.

PROPOSITION 2. *Consider the planar blowing-up $\psi(u,t) = (u, tu)$ around the origin. Then, in (u,t)-coordinates, Equation* (2) *restricted to the t-axis has the form:*

$$du([2 + O(|t|)]dt + [(B/2) + O(|t|)]du) = 0\,.$$

Therefore, the pull-back foliation $\psi^*(\mathcal{F}_{1,\alpha})$ (resp. $\psi^*(\mathcal{F}_{2,\alpha})$) restricted to a small neighborhood of the t-axis, in the half-plane $\{u \geq 0\}$, is as in Figure 4.a (*resp. Figure* 4.b).

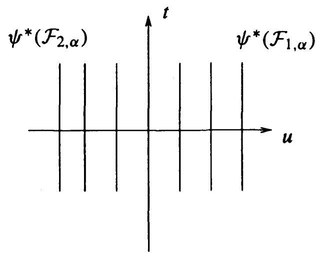

FIGURE 4.a.

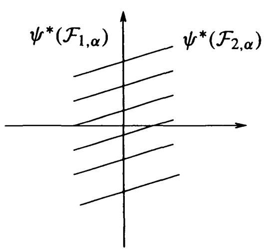

FIGURE 4.b.

PROPOSITION 3. *Consider the planar blowing-up $\varphi(s,v) = (sv, v)$ around the origin. Then, in (s,v)-coordinates, Equation* (2) *has the form:*

(3) $\quad [-2v + v(O(1))]ds^2 - [av + 2s + v(O(1))]dsdv + [-as + 8v + O(2)]dv^2 = 0\,,$

where each $O(i)$ denotes a function (*on* (s,v)) *of order i. Moreover, the pull-back foliation $\varphi^*(\mathcal{F}_{1,\alpha})$ (resp. $\varphi^*(\mathcal{F}_{2,\alpha})$) has a D_3-singular point at the origin and it is as in Figure* 5.a (*resp. Figure* 5.b). *Finally, if $\varphi^*(\mathcal{F}_{1,\alpha})$ and $\psi^*(\mathcal{F}_{1,\alpha})$ are put together, the resulting phase portrait is as in Figure* 2.c, *and therefore, the phase portrait of $\mathcal{F}_{1,\alpha}$ is as in Figure* 2.b.

PROOF. The first statement of the proposition is immediate. By performing the linear change of coordinates $S = 2^{-1/6}s - (a/2)(4^{-1/3})v$ and $V = 4^{-1/3}v$, Equation (3) takes the form

$$(V + O(2))dV^2 + (-U + (a/\sqrt{2})V + O(2))dUdS + (-V + O(2))dU^2 = 0$$

and so the origin is a D_3 singularity of $\varphi^*(\mathcal{F}_{1,\alpha})$ and $\varphi^*(\mathcal{F}_{2,\alpha})$ [Gui], [GS1] (see Figure 3). By restricting Equation (3) to the s-axis, we conclude that $\varphi^*(\mathcal{F}_{1,\alpha})$ (resp. $\varphi^*(\mathcal{F}_{2,\alpha})$) has a separatrix contained in the s-axis and it is, around the origin, either as in Figure 5.a or as in Figure 5.b.

Now, as the foliation $\psi^*(\mathcal{F}_{1,\alpha})$ (see Proposition 2), restricted to the half-plane $\{u \geq 0\}$ is almost vertical, we must have that $\varphi^*(\mathcal{F}_{1,\alpha})$ restricted to the cone $\{v \geq 0, s \geq 0\}$ must be almost horizontal. This is only possible if $\varphi^*(\mathcal{F}_{1,\alpha})$ is as in Figure 5.a (resp. $\varphi^*(\mathcal{F}_{2,\alpha})$ is as in Figure 5.b). Similarly, $\psi^*(\mathcal{F}_{2,\alpha})$ (resp. $\psi^*(\mathcal{F}_{1,\alpha})$) above restricted to a small neighborhood of the t-axis, in the half-plane $\{u \leq 0\}$, is as in Figure 4.a (resp. Figure 4.b). □

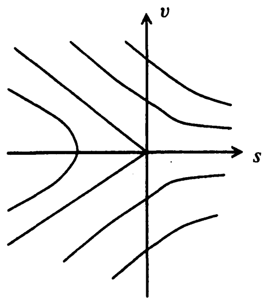

FIGURE 5.a. $\varphi^*(\mathcal{F}_{1,\alpha})$.

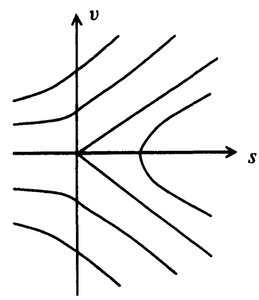

FIGURE 5.b. $\varphi^*(\mathcal{F}_{2,\alpha})$.

4. Proof of Theorem 2. Let p be a Whitney umbrella singular point of α. By the intrinsic transversality characterization of Whitney umbrellas, any map β, C^2-close to α, has a unique Whitney umbrella singular point p_β near p.

The C^s-principal structural stability of α at p follows by using the method of canonical regions of differential equations [GS1]–[GS3].

5. Proof of Theorem 3. Recall that the index of a line field at a singularity is the total number of turns it accomplishes after running once along the boundary of a disk, positively oriented, containing the singularity in its interior.

For line fields defined by principal directions around a Darbouxian umbilic D_3 the index is $-1/2$ while for the points D_1 and D_2 the index is $1/2$. By Poincaré-Hopf Theorem (see [Spi], [Hop]) $\chi(M)$ is equal to the sum of the indices of the singularities of the principal line field $\mathcal{L}_{1,\alpha}$. The theorem follows by noticing that at Whitney umbrella the index is $1/2$.

6. On global principal stability. In this section we show how to formulate the global principal stability result (Theorem 4 below) for mappings, in view of the structure of principal lines around Whitney umbrellas, established in Theorem 1. This corrects Theorem 2 of [GS3, page 555].

A map $\alpha \in \mathcal{M}^r$ is said to be *C^s-principal structurally stable* if there is a neighborhood $\mathcal{V}_\alpha$ of α in $\mathcal{M}^r$ such that for every map $\beta \in \mathcal{V}_\alpha$ there exists a homeomorphism h_β on the domain such that $h_\beta(\mathcal{S}_\alpha) = \mathcal{S}_\beta$, $h_\beta(\mathcal{U}_\alpha) = \mathcal{U}_\beta$ and h_β maps lines of $\mathcal{F}_{1,\alpha}$ (resp. $\mathcal{F}_{2,\alpha}$) on those of $\mathcal{F}_{1,\beta}$ (resp. $\mathcal{F}_{2,\beta}$) and $\alpha = \beta \circ h_\beta$.

PROPOSITION ([GS1]–[GS3]). *A mapping $\alpha \in \mathcal{M}^r$, $r \geq 4$, is C^3-principally structurally stable at a principal cycle c (closed principal line) provided one of the following equivalent conditions,* H_1 *or* H_2, *is satisfied*:

$$(\mathrm{H}_1) \qquad \int_c \frac{dk_1}{k_2 - k_1} = \int_c \frac{dk_2}{k_2 - k_1} \neq 0 .$$

(H_2) *The cycle is a hyperbolic principal cycle of the principal foliation to which it belongs.*

Next we define the set $\mathcal{S}^r(M) \subset \mathcal{M}^r$ by the following properties:

a) All the umbilic points, $\mathcal{U}_\alpha$, of α are Darbouxian.
b) All the singular points, $\mathcal{S}_\alpha$, of α are Whitney umbrellas.
c) All principal cycles of α are hyperbolic.
d) The limit set of every principal line of α is the union of singular points, umbilic points and principal cycles.
e) There is no umbilic or singular separatrix of α which is separatrix of two umbilic or singular points or twice a separatrix of the same umbilic or singular point (i.e. homoclinic loops are not allowed).

THEOREM 4. *Let $r \geq 4$ and M be a compact oriented two manifold. Then the following hold*:

(a) *The set $\mathcal{S}^r(M)$ is open in $\mathcal{M}^{r,3}$ and every $\alpha \in \mathcal{S}^r(M)$ is C^3-principally structurally stable.*
(b) *The set $\mathcal{S}^r(M)$ is dense in $\mathcal{M}^{r,2}$.*

The proof of this theorem is similar to that for immersions [GS1], [GS2].

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