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Linking equations between photon statistics and photocurrent statistics for time-varying stochastic photon rates

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Abstract. Detected photons originating from classical light beams can be described either by means of photon statistics or by means of photocurrent statistics on a semiclassical basis. The statistical parameters of these two descriptions have, up to now, only been related to each other using vague, effective-time-constant arguments. We show that these relations are invalid for the general case of time-varying stochastic photon rates and arbitrary detector impulse responses and derive generally valid linking equations for the ensemble average, the shot noise variance and the excess noise variance of the photon statistics and the photocurrent statistics due to a random optical field. The derivations are based on a general definition of the time average that allows an elegant treatment in the Fourier domain.

1. Introduction

The statistics of detected photons contained in classical light beams have been investigated by several authors using semiclassical methods. The results are either expressed in terms of photon statistics or in terms of photocurrent statistics. Both statistics have a common starting point: the (generally stochastic) rate of photon arrivals‡, $\boldsymbol{\lambda}_{\mathrm{ph}}(t)$. If the detected electromagnetic field possesses an adequate classical description, the photon rate can be shown [1–3] to be proportional to the intensity of the electromagnetic field§. From this photon rate one can calculate‖ the probability $P_{t,T}(n)$ of finding exactly n photons in the time interval $[t, t+T]$, which follows a Poisson process or a doubly stochastic Poisson process whose ensemble average, $\langle \boldsymbol{n}_{t,T} \rangle_e$, and variance, $\sigma^2_{\boldsymbol{n}_{t,T}}$, can be shown to equal

$$\langle \boldsymbol{n}_{t,T} \rangle_e = \langle \boldsymbol{W}^{\mathrm{ph}}_{t,T} \rangle_e \tag{1}$$

and

$$\sigma^2_{\boldsymbol{n}_{t,T}} = \langle \boldsymbol{W}^{\mathrm{ph}}_{t,T} \rangle_e + \sigma^2_{\boldsymbol{W}^{\mathrm{ph}}_{t,T}} \tag{2}$$

using the abbreviation

$$\langle \boldsymbol{W}^{\mathrm{ph}}_{t,T} \rangle_e = \int_t^{t+T} \langle \boldsymbol{\lambda}_{\mathrm{ph}}(\tau) \rangle_e \,\mathrm{d}\tau. \tag{3}$$

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‡ Symbols appearing in bold denote stochastic processes throughout this work.

§ With 'intensity' we denote the squared magnitude of the complex envelope of any field quantity fulfilling the wave equation and being of dimension $\mathrm{W}^{1/2}\ \mathrm{m}^{-1}$.

‖ For a detailed derivation the reader is referred to appendix A or [4].

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The first term on the right-hand side of (2) is known as *shot noise* and the second is called *excess noise*; it is due to the stochastic electromagnetic field [4–6]. If the field is deterministic, the light intensity (and therefore also the photon rate) has a variance of zero, which in turn makes the excess noise term in (2) disappear.

Instead of calculating the counting statistics of detected photons, one can also calculate the statistics of the random current $\boldsymbol{i}(t)$ produced in a photodetector. This is done using a linear superposition of elementary impulse responses†,

$$\boldsymbol{i}(t) = \lim_{\Delta t \to 0} \sum_{k=-\infty}^{\infty} \boldsymbol{X}_k \, h(t - k\Delta t) \tag{4}$$

where $h(t)$ is the detector's response to a single photoelectron event and $\boldsymbol{X}_k$ is a random variable that can either—with a well defined probability—take the value one, meaning that a photon is detected in $[kt, kt + \Delta t]$, or zero, meaning that there is no detection in that interval.

Using this model, the ensemble average of the current can be expressed as

$$\langle \boldsymbol{i}(t) \rangle_e = \langle \boldsymbol{\lambda}(t) \rangle_e * h(t) \tag{5}$$

where the symbol $*$ denotes a convolution,

$$x(t) * y(t) \equiv \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, \mathrm{d}\tau$$

and

$$\boldsymbol{\lambda}(t) = \eta \boldsymbol{\lambda}_{\mathrm{ph}}(t) \tag{6}$$

is the photoelectron rate (η is the detector's quantum efficiency). The variance of the current then reads (cf [2, 8])

$$\sigma_{\boldsymbol{i}}^2(t) = \langle \boldsymbol{\lambda}(t) \rangle_e * h^2(t) + \iint_{-\infty}^{\infty} C_{\boldsymbol{\lambda}}(\tau, \tau')\, h(t-\tau)\, h(t-\tau')\, \mathrm{d}\tau\, \mathrm{d}\tau' \tag{7}$$

where $C_{\boldsymbol{\lambda}}(\tau, \tau')$ is the autocovariance of $\boldsymbol{\lambda}(t)$,

$$C_{\boldsymbol{\lambda}}(\tau, \tau') \equiv \langle \boldsymbol{\lambda}(\tau)\, \boldsymbol{\lambda}(\tau') \rangle_e - \langle \boldsymbol{\lambda}(\tau) \rangle_e \, \langle \boldsymbol{\lambda}(\tau') \rangle_e .$$

Note that both the ensemble average and variance are time dependent in the general case, i.e. we have to deal with non-stationary stochastic processes. (The widely used formula for shot noise, $\sigma_{\boldsymbol{i}}^2 = 2e\langle \boldsymbol{i} \rangle_e B_h$ is *only* obtained if the photon rate is stationary and additionally has a bandwidth that is much smaller than the detector bandwidth (cf [9]).) For a precise treatment and to avoid any confusion, it is thus important to write $\sigma_{\boldsymbol{i}}^2(t)$ and not just $\sigma_{\boldsymbol{i}}^2$, as often found in the literature.

As in equation (2), the first term of the right-hand side of (7) represents shot noise, whereas the second term is excess noise due to the randomness of $\boldsymbol{\lambda}(t)$. (If the field is deterministic, the autocovariance of the photon rate vanishes and the shot noise term is the only one remaining in (7).)

The striking similarity between the statistical parameters of the two different statistical descriptions of detected photons invokes the question of whether there are linking equations between the means and variances of the two descriptions, i.e. between equations (1) and (5) on the one hand and (2) and (7) on the other. Many authors (see e.g. [7, 10–12]) have established such linking equations using a vague, effective-time-constant argument, neglecting the fact that $\boldsymbol{\lambda}_{\mathrm{ph}}(t)$ may be either a deterministic and time-varying or even a

† For a detailed derivation the reader is referred to appendix B or [2, 7].

stochastic (and not necessarily stationary) quantity. The equations used by the cited authors are of the form

$$\langle \boldsymbol{i}(t)\rangle_e = \frac{e}{T_h}\eta\langle \boldsymbol{n}_{t,T_h}\rangle_e \tag{8}$$

and†

$$\sigma_{\boldsymbol{i}}^2(t) = \frac{e^2}{T_h^2}\eta\langle \boldsymbol{n}_{t,T_h}\rangle_e \tag{9}$$

where e denotes the elementary charge and T_h stands for the effective duration of the detector's impulse response.

In this work we establish very general equations linking the ensemble average and the variance of the photon statistics to the ensemble average and the variance of the photocurrent statistics. To accomplish this task, we use a solid definition of the time average, making use of some properties of the Fourier transform. All our derivations will take into account arbitrary stochastic photon rates. We show that (8) and (9) are wrong in general; they only hold in the limit of a stationary stochastic or of a constant deterministic optical intensity. For equation (9) to be valid, the time–bandwidth product of the photodetector, defined below, additionally has to be minimum, which puts some restrictions on the detector's impulse response. An example at the end of the paper, which assumes a stochastic photon rate, will demonstrate the validity of the relations obtained.

To clearly differentiate between ensemble average, time average and realization of a stochastic process, the notation is as follows: a stochastic process $\boldsymbol{x}(t)$ appears as a bold character, $x(t)$ being any realization of it. The ensemble average is expressed as $\langle \boldsymbol{x}(t)\rangle_e$, whereas the time average is either written as $\overline{x(t)}$ or, equivalently, $\langle x(t)\rangle_{t,\infty}$.

2. Definition of time averages

In this section we will give a solid definition of time averages, valid for all signals of physical interest for which temporal averaging makes sense. In other words, we consider either time-limited signals $\tilde{x}(t)$ with duration T or time-unlimited signals $x(t)$ possessing an infinite amount of energy‡. Time-unlimited signals containing a finite amount of energy (e.g. a Gaussian pulse) will not be considered here, as the concept of time averaging is meaningless in that case; in order to include such signals into our theory, they have to be made time limited by means of windowing and are then of the same type as $\tilde{x}(t)$.

It is important to note that there are, in fact, two independent definitions of the time average operator, namely

$$\langle x(t)\rangle_{t,\infty} = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)\,\mathrm{d}t \tag{10}$$

for time-unlimited signals and

$$\langle \tilde{x}(t)\rangle_{t,T} = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\,\mathrm{d}t \tag{11}$$

† The photocurrent's variance is sometimes even *derived* using (9).
‡ As we use the Fourier transform extensively, it has to be emphasized that time-unlimited signals can be Fourier transformed using the theory of distributions.

for time-limited ones. Let us first consider the time average for time-unlimited signals: The weight of the Dirac impulse in the spectrum of $x(t)$ at $\omega = 0$, $\hat{X}(0)$, is equal to (10). To see this, we write (10) in the form

$$\langle x(t)\rangle_{t,\infty} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\mathrm{j}\omega)\, \mathrm{e}^{\mathrm{j}\omega t}\, \mathrm{d}\omega\, \mathrm{d}t \tag{12}$$

where $X(\mathrm{j}\omega)$ denotes the Fourier transform of $x(t)$,

$$X(\mathrm{j}\omega) = \int_{-\infty}^{\infty} x(t)\, \mathrm{e}^{-\mathrm{j}\omega t}\, \mathrm{d}t. \tag{13}$$

Equation (12) can then be shown to equal

$$\langle x(t)\rangle_{t,\infty} = \frac{1}{2\pi} \lim_{T\to\infty} \int_{-\infty}^{\infty} X(\mathrm{j}\omega) \frac{\sin(\omega T/2)}{\omega T/2}\, \mathrm{d}\omega \tag{14}$$

where it is assumed that the limit and the integral expressions can be interchanged. The $\sin(\omega T/2)/(\omega T/2)$ function equals 1 at $\omega = 0$, whereas it converges to zero at all other frequencies if T tends to infinity. Thus the only term surviving the integration is the weight of a Dirac impulse centred at $\omega = 0$, $\hat{X}(0)$: decomposing $X(\mathrm{j}\omega)$ into $\hat{X}(0)\,\delta(\omega) + \tilde{X}(\mathrm{j}\omega)$ we get

$$\begin{aligned}\langle x(t)\rangle_{t,\infty} &= \frac{1}{2\pi} \lim_{T\to\infty} \int_{-\infty}^{\infty} \hat{X}(0)\,\delta(\omega) \frac{\sin(\omega T/2)}{\omega T/2}\, \mathrm{d}\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{X}(\mathrm{j}\omega) \lim_{T\to\infty} \frac{\sin(\omega T/2)}{\omega T/2}\, \mathrm{d}\omega \\ &= \frac{1}{2\pi} \hat{X}(0).\end{aligned} \tag{15}$$

Let us now consider the case of a time-limited signal $\tilde{x}(t)$. If we make $\tilde{x}(t)$ periodic with period T we get a new, time-unlimited signal $x(t)$; its time average then reads

$$\begin{aligned}\langle x(t)\rangle_{t,\infty} = \frac{1}{2\pi}\hat{X}(0) &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, \mathrm{d}t \\ &= \lim_{N\to\infty} \frac{1}{(2N+1)T} \sum_{k=-N}^{N} \int_{kT}^{(k+1)T} \tilde{x}(t)\, \mathrm{d}t = \langle \tilde{x}(t)\rangle_{t,T}\end{aligned} \tag{16}$$

where N is a positive integer. Thus $(1/2\pi)\hat{X}(0)$ can be interpreted as the time average of the time-limited signal $\tilde{x}(t)$†.

When speaking about time averages we will therefore solely use the unified definition

$$\overline{x(t)} \equiv \langle x(t)\rangle_{t,\infty} = \frac{1}{2\pi}\hat{X}(0) \tag{17}$$

and assume tacitly that time-limited signals are made periodic, as explained above.

3. The linking equations

It was sketched in the introduction and is explained in detail in appendices A and B how the ensemble averages and variances of the photon statistics and the photocurrent statistics

† The procedure of making the signal $\tilde{x}(t)$ periodic for the definition of a time average may look strange at first glance. However, it will be seen below that it in fact unifies the derivations for time-limited and time-unlimited signals.

can be expressed for arbitrary stochastic optical fields. The results are

$$\langle \boldsymbol{n}_{t,T} \rangle_e = \langle \boldsymbol{W}^{\text{ph}}_{t,T} \rangle_e \tag{18}$$

$$\sigma^2_{\boldsymbol{n}_{t,T}} = \underbrace{\langle \boldsymbol{W}^{\text{ph}}_{t,T} \rangle_e}_{\sigma^2_{\boldsymbol{n}_{t,T},\text{shot}}} + \underbrace{\sigma^2_{\boldsymbol{W}^{\text{ph}}_{t,T}}}_{\sigma^2_{\boldsymbol{n}_{t,T},\text{excess}}} \tag{19}$$

for the photon statistics and

$$\langle \boldsymbol{i}(t) \rangle_e = \langle \boldsymbol{\lambda}(t) \rangle_e * h(t) \tag{20}$$

$$\sigma^2_{\boldsymbol{i}}(t) = \underbrace{\langle \boldsymbol{\lambda}(t) \rangle_e * h^2(t)}_{\sigma^2_{\boldsymbol{i},\text{shot}}(t)} + \underbrace{\iint_{-\infty}^{\infty} C_{\boldsymbol{\lambda}}(\tau, \tau')\, h(t-\tau)\, h(t-\tau')\, \mathrm{d}\tau\, \mathrm{d}\tau'}_{\sigma^2_{\boldsymbol{i},\text{excess}}(t)} \tag{21}$$

for the photocurrent statistics.

It is apparent that the *instantaneous* values of the statistical parameters, as expressed in the above equations, cannot be directly related to each other. Their time averages, however, can, as will now be shown. After having derived general formulae valid for arbitrary stochastic fields, we will specialize for the case where the field is random and stationary or deterministic and constant with respect to time; these cases will be found to be in agreement with the formulae found in the literature, equations (8) and (9), if we additionally restrict our attention to photodetectors whose time–bandwidth product is minimum.

3.1. The ensemble average

We will start establishing a link between the ensemble averages of the two statistical descriptions. The time average of equation (20) is evaluated easily using the results of the previous section,

$$\overline{\langle \boldsymbol{i}(t) \rangle_e} = \frac{1}{2\pi} \hat{\Lambda}(0)\, H(0) \tag{22}$$

if $\Lambda(\mathrm{j}\omega)$ and $H(\mathrm{j}\omega)$ are the spectra of $\langle \boldsymbol{\lambda}(t) \rangle_e$ and $h(t)$, respectively. Equation (18), on the other hand, can be written in the form

$$\begin{aligned} \eta \langle \boldsymbol{n}_{t,T} \rangle_e = \eta \langle \boldsymbol{W}^{\text{ph}}_{t,T} \rangle_e &= \eta \int_t^{t+T} \langle \boldsymbol{\lambda}_{\text{ph}}(\tau) \rangle_e \, \mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} f(t-\tau) \langle \boldsymbol{\lambda}(\tau) \rangle_e \, \mathrm{d}\tau = f(t) * \langle \boldsymbol{\lambda}(t) \rangle_e \end{aligned} \tag{23}$$

where $f(t)$ is a window function defined as

$$f(t) = \begin{cases} 1 & -T < t < 0 \\ 0 & \text{elsewhere.} \end{cases} \tag{24}$$

The temporal mean of the convolution in (23) can easily be evaluated using the Fourier transform. If $F(\mathrm{j}\omega)$ denotes the spectrum of $f(t)$, we obtain

$$\eta \overline{\langle \boldsymbol{n}_{t,T} \rangle_e} = \frac{1}{2\pi} \hat{\Lambda}(0)\, F(0) = \frac{1}{2\pi} \hat{\Lambda}(0)\, T. \tag{25}$$

Combining (25) and (22) yields the desired link,

$$\overline{\langle \boldsymbol{i}(t)\rangle_e} = \frac{\eta H(0)}{T}\overline{\langle \boldsymbol{n}_{t,T}\rangle_e} \tag{26}$$

which is a general result for the ensemble averages of the two statistical descriptions. Note that only the *time averages* of the two statistics can be related to each other. This is the most important difference between our equation (26) and (8). In the case of a random but stationary optical field, the ensemble averages are time independent and, observing $H(0) = e$, which holds for real, non-multiplying photodetectors, we arrive at (8). If the optical field is deterministic and constant we, too, arrive at (8).

3.2. The shot noise variance

As the variance of both stochastic descriptions splits additively into a shot noise part and an excess noise part, we will treat the two terms separately. Assuming a real-valued function $h(t)$ and applying basic Fourier transform relations, the time average of the shot noise part of (21) can be expressed as

$$\overline{\sigma^2_{\boldsymbol{i},\mathrm{shot}}(t)} = \frac{1}{2\pi}\hat{\Lambda}(0)\, H'(0) = \frac{\hat{\Lambda}(0)}{2\pi}\int_{-\infty}^{\infty} |H(\mathrm{j}\omega)|^2\,\mathrm{d}\omega \tag{27}$$

where $H'(\mathrm{j}\omega)$ stands for the Fourier transform of $h^2(t)$. As the shot noise part of (19) equals the ensemble average, we can use (23) directly; this yields the desired linking equation,

$$\overline{\sigma^2_{\boldsymbol{i},\mathrm{shot}}(t)} = \frac{\eta}{2T\pi}\int_{-\infty}^{\infty} |H(\mathrm{j}\omega)|^2\,\mathrm{d}\omega\ \overline{\sigma^2_{\boldsymbol{n}_{t,T},\mathrm{shot}}}. \tag{28}$$

In order to arrive at a relation similar to (9), we have to introduce a bandwidth definition called the power equivalent width (cf [7]),

$$2\pi B_x = \frac{\int_{-\infty}^{\infty} |X(\mathrm{j}\omega)|^2\,\mathrm{d}\omega}{2|X(\mathrm{j}\omega)|^2_{\max}} \tag{29}$$

and an appropriate definition of the time duration [13, 14],

$$T_x = \frac{\left(\int_{-\infty}^{\infty} |x(t)|\,\mathrm{d}t\right)^2}{\int_{-\infty}^{\infty} |x(t)|^2\,\mathrm{d}t} \tag{30}$$

which satisfy the time–bandwidth product relation

$$B_x T_x \geqslant \tfrac{1}{2}. \tag{31}$$

The equality sign can easily be shown to hold for the important special case where $|H(\mathrm{j}\omega)|_{\max} = |H(0)|$ and $h(t) > 0$ for all t, regardless of the pulse shape.

Using the above definitions together with the fact that the variance of the photon statistics equals its mean, we arrive at

$$\overline{\sigma^2_{\boldsymbol{i},\mathrm{shot}}(t)} \geqslant \frac{\eta|H(\mathrm{j}\omega)|^2_{\max}}{T_h^2}\overline{\langle \boldsymbol{n}_{t,T_h}\rangle_e}. \tag{32}$$

Note that we had to calculate the variance of the photon statistics for the effective temporal duration of the detector's impulse response, T_h, for the sake of a fair comparison.

Inequality (32) *only* reduces to an equality similar in form to (9) if we assume the photodetector's impulse response to be always positive *and* to have a spectrum that satisfies $|H(\mathrm{j}\omega)|_{\max} = |H(0)| = e$. If we specialize further, assuming either a stochastic and stationary or a deterministic and constant optical intensity, the time average operators disappear, too, and we arrive at equation (9).

3.3. The excess noise variance

To combine the excess noise components of the two statistical descriptions we use the generally valid relation

$$\sigma_{\boldsymbol{x}}^2 = \langle \boldsymbol{x}^2 \rangle_e - \langle \boldsymbol{x} \rangle_e^2 \tag{33}$$

and write $\sigma^2_{\boldsymbol{W}^{\mathrm{ph}}_{t,T}}$ as

$$\begin{aligned}
\eta^2 \sigma^2_{\boldsymbol{W}^{\mathrm{ph}}_{t,T}} &= \left\langle \left(\int_t^{t+T} \boldsymbol{\lambda}(\tau)\,\mathrm{d}\tau \right)^2 \right\rangle_e - \left\langle \int_t^{t+T} \boldsymbol{\lambda}(\tau)\,\mathrm{d}\tau \right\rangle_e^2 \\
&= \left\langle \int_t^{t+T} \int_t^{t+T} \boldsymbol{\lambda}(\tau)\,\boldsymbol{\lambda}(\tau')\,\mathrm{d}\tau\,\mathrm{d}\tau' - \int_t^{t+T} \int_t^{t+T} \langle \boldsymbol{\lambda}(\tau) \rangle_e \,\langle \boldsymbol{\lambda}(\tau') \rangle_e \,\mathrm{d}\tau\,\mathrm{d}\tau' \right\rangle_e \\
&= \int_t^{t+T} \int_t^{t+T} C_{\boldsymbol{\lambda}}(\tau,\tau')\,\mathrm{d}\tau\,\mathrm{d}\tau' = \iint_{-\infty}^{\infty} C_{\boldsymbol{\lambda}}(\tau,\tau') f(t-\tau) f(t-\tau')\,\mathrm{d}\tau\,\mathrm{d}\tau'.
\end{aligned} \tag{34}$$

As before, $f(t)$ denotes the time-window function (24). This equation can be interpreted as a two-dimensional convolution (see appendix C).

Using (19) and the results of appendix C we obtain for the temporal mean of $\sigma^2_{\boldsymbol{W}^{\mathrm{ph}}_{t,T}}$,

$$\begin{aligned}
\eta^2\, \overline{\sigma^2_{\boldsymbol{n}_{t,T},\mathrm{excess}}} = \eta^2\, \overline{\sigma^2_{\boldsymbol{W}^{\mathrm{ph}}_{t,T}}} &= \sum_i \hat{S}_{C_{\boldsymbol{\lambda}}}(\mathrm{j}\omega_1^{(i)}, -\mathrm{j}\omega_1^{(i)}) |F(\mathrm{j}\omega_1^{(i)})|^2 \\
&= \sum_i \hat{S}_{C_{\boldsymbol{\lambda}}}(\mathrm{j}\omega_1^{(i)}, -\mathrm{j}\omega_1^{(i)}) \frac{4\sin^2(\omega_1^{(i)} T/2)}{\omega_1^{(i)}}
\end{aligned} \tag{35}$$

if $\hat{S}_{C_{\boldsymbol{\lambda}}}(\mathrm{j}\omega_1, \mathrm{j}\omega_2)$ denotes the weight of the Dirac impulse at (ω_1, ω_2) in the two-dimensional spectrum of $C_{\boldsymbol{\lambda}}(t_1, t_2)$. Similarly, we get for the excess noise part of (21),

$$\overline{\sigma^2_{\boldsymbol{i},\mathrm{excess}}(t)} = \sum_i \hat{S}_{C_{\boldsymbol{\lambda}}}(\mathrm{j}\omega_1^{(i)}, -\mathrm{j}\omega_1^{(i)}) |H(\mathrm{j}\omega_1^{(i)})|^2. \tag{36}$$

Combining (35) and (36) yields the desired link between the excess noise parts of (19) and (21). A factor of proportionality, however, cannot be given in the general case. Only in special cases, where $S_{C_{\boldsymbol{\lambda}}}(\mathrm{j}\omega_1, \mathrm{j}\omega_2)$ has a single Dirac impulse or only impulses with the same weight on the straight line $\mathrm{j}\omega_1 = -\mathrm{j}\omega_2$, can such a factor be given. This is the case in the example addressed in the next section.

4. Example

In this section we demonstrate the validity of the linking equations between the ensemble averages, the shot noise and the excess noise according to equations (26), (28) and (35) + (36) considering an example. We will study the case where the sum of two identically polarized and transversally homogenous optical fields, given by their (scalar) analytic signals† $V_1 \exp(\mathrm{j}\omega_1 t)$ and $V_2 \exp(\mathrm{j}\omega_2 t) \exp(\mathrm{j}\phi)$, impinges on a photodetector (this is the

† The physical quantities represented by these signals are proportional to $\mathrm{W}^{1/2}\ \mathrm{m}^{-1}$, as mentioned in the introduction.

case in a heterodyne set-up, for instance); ω_1 and ω_2 are the optical frequencies and ϕ is a random phase, equally distributed in $[0, 2\pi]$. The photon rate $\boldsymbol{\lambda}_{\rm ph}(t)$ then reads

$$\boldsymbol{\lambda}_{\rm ph}(t) = \frac{A}{\hbar\overline{\omega}}\left|V_1 \exp(\mathrm{j}\omega_1 t) + V_2 \exp(\mathrm{j}\omega_2 t)\exp(\mathrm{j}\phi)\right|^2$$
$$= \frac{A}{\hbar\overline{\omega}}\left[V_1^2 + V_2^2 + 2V_1V_2\cos(\Delta\omega t + \phi)\right] \tag{37}$$

according to (A3). The variable A denotes the area of detection, $\overline{\omega}$ represents the arithmetic mean of the two beating frequencies making up the photons' energies, and $\Delta\omega = \omega_1 - \omega_2$.

4.1. Photon statistics

Using (A2) and (37), the integrated photon rate $\boldsymbol{W}_{t,T}^{\rm ph}$ can be written in the form

$$\boldsymbol{W}_{t,T}^{\rm ph} = \frac{A}{\hbar\,\overline{\omega}}\left[V_1^2 T + V_2^2 T + \frac{4V_1V_2}{\Delta\omega}\cos\left(\tfrac{1}{2}\Delta\omega(2t+T)+\phi\right)\sin\left(\tfrac{1}{2}\Delta\omega T\right)\right]. \tag{38}$$

As ϕ is equally distributed in $[0, 2\pi]$, the ensemble average and variance of $\boldsymbol{W}_{t,T}^{\rm ph}$ are

$$\langle \boldsymbol{W}_{t,T}^{\rm ph}\rangle_e = \frac{A}{\hbar\,\overline{\omega}}\left[V_1^2 T + V_2^2 T\right] \tag{39}$$

and

$$\sigma^2_{\boldsymbol{W}_{t,T}^{\rm ph}} = \frac{A^2}{\hbar^2\,\overline{\omega}^2}\frac{8V_1^2V_2^2}{\Delta\omega^2}\sin^2\left(\tfrac{1}{2}\Delta\omega T\right). \tag{40}$$

This leads to a photon counting distribution of variance

$$\sigma^2_{\boldsymbol{n}_{t,T}} = \langle \boldsymbol{W}_{t,T}^{\rm ph}\rangle_e + \sigma^2_{\boldsymbol{W}_{t,T}^{\rm ph}} = \frac{A}{\hbar\,\overline{\omega}}\left[V_1^2 T + V_2^2 T\right] + \frac{A^2}{\hbar^2\,\overline{\omega}^2}\frac{8V_1^2V_2^2}{\Delta\omega^2}\sin^2\left(\tfrac{1}{2}\Delta\omega T\right) \tag{41}$$

and mean

$$\langle \boldsymbol{n}_{t,T}\rangle_e = \langle \boldsymbol{W}_{t,T}^{\rm ph}\rangle_e = \frac{A}{\hbar\,\overline{\omega}}\left[V_1^2 T + V_2^2 T\right] \tag{42}$$

according to (18) and (19). As usual, the first term of (41) represents shot noise and the second is excess noise.

4.2. Current statistics

From (37) we get for the photoelectron rate

$$\boldsymbol{\lambda}(t) = \eta\boldsymbol{\lambda}_{\rm ph}(t) = \frac{\eta A}{\hbar\,\overline{\omega}}\left[V_1^2 + V_2^2 + 2V_1V_2\cos(\Delta\omega t + \phi)\right]. \tag{43}$$

As we are dealing with a wide-sense stationary stochastic process in this example, the ensemble average and variance are time independent, and the easiest way to determine the variance of the photocurrent is to first calculate its spectrum†, which, integrated over ω, gives the power of the stochastic process. Subtracting the squared ensemble average then yields the variance.

† The spectrum of the photocurrent is given by $S_{\boldsymbol{i}}(\mathrm{j}\omega) = [\langle\boldsymbol{\lambda}(t)\rangle_e + S_{\boldsymbol{\lambda}}(\mathrm{j}\omega)]|H(\mathrm{j}\omega)|^2$, where $S_{\boldsymbol{\lambda}}(\mathrm{j}\omega)$ is the spectrum of $\boldsymbol{\lambda}(t)$ and $|H(\mathrm{j}\omega)|$ is the Fourier transform of $h(t)$. A derivation of this formula can be found in [2, 8].

Using the expression

$$R_{\boldsymbol{\lambda}}(\tau) = \left(\frac{\eta A}{\hbar\,\varpi}\right)^2 \left(\left[V_1^2 + V_2^2\right]^2 + 2(V_1V_2)^2\cos(\Delta\omega\tau)\right) \tag{44}$$

for the autocorrelation of $\boldsymbol{\lambda}(t)$ and

$$\langle \boldsymbol{\lambda}(t)\rangle_e = \frac{\eta A}{\hbar\,\varpi}\left[V_1^2 + V_2^2\right] \tag{45}$$

for its mean, we get

$$S_{\boldsymbol{i}}(\mathrm{j}\omega) = \frac{\eta A}{\hbar\,\varpi}\Bigg[V_1^2 + V_2^2 + \frac{\eta A}{\hbar\,\varpi}\Big(2\pi(V_1^2 + V_2^2)^2\delta(\omega) + 2\pi(V_1V_2)^2[\delta(\omega - \Delta\omega) + \delta(\omega + \Delta\omega)]\Big)\Bigg]|H(\mathrm{j}\omega)|^2 \tag{46}$$

for the spectrum of the photocurrent. The symbol $\delta(\omega)$ denotes the Dirac impulse.

The mean of the stochastic process $\boldsymbol{i}(t)$ can be obtained directly from (20) using (45). This yields†

$$\overline{\langle \boldsymbol{i}(t)\rangle_e} = \langle \boldsymbol{i}(t)\rangle_e = \frac{\eta A}{\hbar\,\varpi}\left[V_1^2 + V_2^2\right]\int_{-\infty}^{\infty} h(\tau)\,\mathrm{d}\tau = \frac{\eta A|H(0)|}{\hbar\,\varpi}\left[V_1^2 + V_2^2\right]. \tag{47}$$

The variance of $\boldsymbol{i}(t)$ can then be calculated to be

$$\begin{aligned}\overline{\sigma_{\boldsymbol{i}}^2(t)} = \sigma_{\boldsymbol{i}}^2(t) &= \langle \boldsymbol{i}^2(t)\rangle_e - \langle \boldsymbol{i}(t)\rangle_e^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{\boldsymbol{i}}(\mathrm{j}\omega)\,\mathrm{d}\omega - \langle \boldsymbol{i}(t)\rangle_e^2 \\ &= 2|H(\mathrm{j}\omega)|_{\max} I_0 B_h + 2(V_1V_2)^2\left(\frac{\eta A}{\hbar\,\varpi}\right)^2 |H(\mathrm{j}\Delta\omega)|^2\end{aligned} \tag{48}$$

where B_h is defined in (29) and I_0 stands for $(\eta A/\hbar\,\varpi)[V_1^2 + V_2^2]|H(0)|$. The first term of the right-hand side of (48) is attributable to shot noise only, which can be seen considering the situation where the phase ϕ is deterministic‡. Under the realistic assumption $|H(0)| = |H(\mathrm{j}\omega)|_{\max}$ we thus have the relations

$$\overline{\sigma_{\boldsymbol{i},\mathrm{shot}}^2(t)} = 2|H(0)|I_0B_h \tag{49}$$

and

$$\overline{\sigma_{\boldsymbol{i},\mathrm{excess}}^2(t)} = 2(V_1V_2)^2\left(\frac{\eta A}{\hbar\,\varpi}\right)^2 |H(\mathrm{j}\Delta\omega)|^2. \tag{50}$$

Applying the linking equations (26) and (28) to the pairs (42) + (47), (41) + (49) shows that they are valid for this example. In order to show the validity of (35) + (36), we calculate the spectrum of the autocovariance of $\boldsymbol{\lambda}(t)$. This yields

$$S_{C_{\boldsymbol{\lambda}}}(\mathrm{j}\omega_1, \mathrm{j}\omega_2) = \left(\frac{\eta A}{\hbar\,\varpi}\right)^2 4\pi^2 V_1^2V_2^2[\delta(\omega_1 + \Delta\omega) + \delta(\omega_1 - \Delta\omega)]\delta(\omega_1 + \omega_2). \tag{51}$$

† In order to demonstrate the generality of the linking equations, we do not substitute $|H(0)| = e$, which would be true if a detector without any filtering in the electrical regime were employed.

‡ In this case we get $\overline{\sigma_{\boldsymbol{i}}^2(t)} = 2|H(0)|I_0B_h$ for the variance of the process, which is pure shot noise, as no randomness is introduced by the optical field.

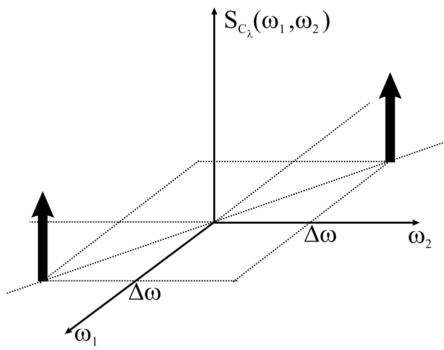

Figure 1. The two-dimensional Fourier transform $S_{C_\lambda}(\mathrm{j}\omega_1, \mathrm{j}\omega_2)$ has two Dirac impulses on the line $\omega_1 = -\omega_2$.

As depicted in figure 1, there are two Dirac impulses with equal weights on the straight line $\omega_1 = -\omega_2$. This means that the sums in (41) and (49) can be reduced to a factor of proportionality and the final link for this example reads

$$\overline{\sigma^2_{i,\mathrm{excess}}(t)} = \eta^2 |H(\mathrm{j}\Delta\omega)|^2 \frac{\Delta\omega^2}{4\sin^2(\Delta\omega T/2)} \overline{\sigma^2_{n_{t,T},\mathrm{excess}}}.$$

5. Conclusion

We have shown that the traditional linking equations between the ensemble averages and shot noise variances of the photon statistics and the photocurrent statistics, equations (8) and (9), are *only* valid in the special case where the optical field is either random and stationary or deterministic and constant with respect to time. For equation (9) to apply, we additionally have to demand that the impulse response of the detector fulfils the minimum time–bandwidth product $B_h T_h = \frac{1}{2}$, which is the case for impulse responses that are always positive and possess a spectral maximum at $\omega = 0$. We have derived linking equations that are valid for arbitrary time-varying stochastic photon rates and arbitrary detector impulse responses. The traditional formulae are reproduced if the above restrictions are met.

The major point in our results is that it is only possible to relate the *time averages* of the statistical parameters but not their instantaneous values in the general case. A careful definition of the time average in this context allows an elegant treatment in the Fourier domain.

Apart from a generalization of the linking equations for the ensemble averages and the shot noise variances, a relation has been found between the excess noise terms of the photon statistics and the photocurrent statistics. Under certain restrictions imposed on the autocovariance function of the photon rate, even a multiplicative factor linking the two variances can be given.

The validity of the equations obtained has been demonstrated in an example related to optical heterodyning.

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Appendix A. Photon statistics

In this appendix we will give a compact derivation of the ensemble average and variance of the photon counting statistics for a stochastic, but classical, optical field.

It can be shown (see e.g. [4]) that the number of photons contained by a coherent optical field, i.e. a field with deterministic classical field quantities, form a random point process that is Poisson distributed. The probability of detecting exactly n photoelectrons in the time interval $[t, t+T]$, $P_{t,T}(n)$, is therefore given by

$$P_{t,T}(n) = \frac{(W_{t,T}^{\mathrm{ph}})^n}{n!} \exp(-W_{t,T}^{\mathrm{ph}}) \tag{A1}$$

$$W_{t,T}^{\mathrm{ph}} = \int_t^{t+T} \lambda_{\mathrm{ph}}(\tau)\,\mathrm{d}\tau \tag{A2}$$

where $\lambda_{\mathrm{ph}}(t)$ is the photon rate of the coherent electromagnetic field†. It is related to the real part of the complex Poynting vector of the optical field, $\vec{I}(\vec{r}, t)$, by

$$\lambda_{\mathrm{ph}}(t) = \frac{1}{\hbar\omega} \int_{\mathcal{A}} \vec{I}(\vec{r}, t)\,\mathrm{d}\vec{r} \tag{A3}$$

where $\mathcal{A}$ denotes the area of detection and ω stands for the optical frequency.

Using the total probability theorem, also known as Mandel's formula (see e.g. [4]), the total probability of detecting exactly n photons in the time interval $[t, t+\Delta t]$ becomes

$$P_{t,T}(n) = \int_0^{\infty} \frac{(W_{t,T}^{\mathrm{ph}})^n \exp(-W_{t,T}^{\mathrm{ph}})}{n!}\, p_{\boldsymbol{W}_{t,T}^{\mathrm{ph}}}(W_{t,T}^{\mathrm{ph}})\,\mathrm{d}W_{t,T}^{\mathrm{ph}} \tag{A4}$$

where $p_{\boldsymbol{W}_{t,T}^{\mathrm{ph}}}(W_{t,T}^{\mathrm{ph}})$ is the probability density of $\boldsymbol{W}_{t,T}^{\mathrm{ph}}$. The result of this integration is known as a doubly stochastic Poisson distribution [4, 5]. The mean and variance of the doubly stochastic distribution can now be calculated to be

$$\langle \boldsymbol{n}_{t,T} \rangle_e = \langle \boldsymbol{W}_{t,T}^{\mathrm{ph}} \rangle_e \tag{A5}$$

and

$$\sigma^2_{\boldsymbol{n}_{t,T}} = \langle \boldsymbol{W}_{t,T}^{\mathrm{ph}} \rangle_e + \sigma^2_{\boldsymbol{W}_{t,T}^{\mathrm{ph}}}. \tag{A6}$$

The first term of this equation is the fundamental shot noise also encountered in the case of a deterministic electromagnetic field and the second term, the excess noise, is due to the randomness of the optical field.

Appendix B. Photocurrent statistics

Expanding the Poisson distribution (A1) in a Taylor series for small values of $T = \Delta t$, the probability of finding exactly one photon in a time interval $[t, t+\Delta t]$ can be written as

$$P\big[\text{one photon in } [t, t+\Delta t]\big] = \lambda_{\mathrm{ph}}(t)\,\Delta t + \mathcal{O}(\Delta t) \tag{B1}$$

where $\mathcal{O}$ denotes the Landau symbol.

† Any realization $x(t)$ of a stochastic process $\boldsymbol{x}(t)$ can be regarded as a deterministic function. Thus, $\lambda_{\mathrm{ph}}(t)$ can be considered to be the photon rate associated with either a coherent optical field or with one particular realization of a stochastic field.

If the photons impinging on a detector obey Poisson or doubly stochastic Poisson statistics, it can be shown [7, 15] that the photoelectrons produced in a detector with quantum efficiency η, too, obey Poisson or doubly stochastic Poisson statistics; only the photon rate $\lambda_{\mathrm{ph}}(t)$ has to be substituted by the photoelectron rate $\lambda(t) = \eta\lambda_{\mathrm{ph}}(t)$. From (B1) we thus get for the probability of observing a single photoelectron in $[t, t + \Delta t]$,

$$P\big[\text{one photoelectron in } [t, t + \Delta t]\big] = \lambda(t)\,\Delta t \tag{B2}$$

if Δt is small enough to neglect $\mathcal{O}(\Delta t)$.

As the detected photons form a stochastic process, the photocurrent has to be modelled as a stochastic process, too. Following [7], the current can be put as

$$\boldsymbol{i}(t) = \lim_{\Delta t \to 0} \sum_{k=-\infty}^{\infty} \boldsymbol{X}_k\, h(t - k\Delta t) \tag{B3}$$

where $\boldsymbol{X}_k$ is a discrete random variable that takes the value 1 if a photoelectron is produced in $[kt, kt + \Delta t]$ and 0 if there is no detection in that time interval. The function $h(t)$ denotes the system's impulse response.

As equation (B2) is only valid for a coherent field or for a particular realization of a random field, the total probability theorem again has to be employed,

$$P[\boldsymbol{X}_k = 1] = \int_{-\infty}^{\infty} P[\boldsymbol{X}_k = 1 |\, \boldsymbol{\lambda}_k = \lambda_k]\, p_{\boldsymbol{\lambda}_k}(\lambda_k)\,\mathrm{d}\lambda_k \tag{B4}$$

where $\boldsymbol{\lambda}_k$ stands for $\boldsymbol{\lambda}(k\Delta t)$, $p_{\boldsymbol{\lambda}_k}(\lambda_k)$ is its probability density and $P[A|B]$ is the usual abbreviation for 'the probability of event A under the condition that B is satisfied'.

Performing some rather lengthy calculations on (B3), the ensemble average of the current becomes

$$\langle \boldsymbol{i}(t)\rangle_e = \langle \boldsymbol{\lambda}(t)\rangle_e * h(t) \tag{B5}$$

and the variance follows as

$$\sigma_{\boldsymbol{i}}^2(t) = \langle \boldsymbol{\lambda}(t)\rangle_e * h^2(t) + \iint_{-\infty}^{\infty} C_{\boldsymbol{\lambda}}(\tau, \tau')\, h(t - \tau)\, h(t - \tau')\,\mathrm{d}\tau\,\mathrm{d}\tau' \tag{B6}$$

where the symbol $*$ denotes a convolution,

$$x(t) * y(t) \equiv \int_{-\infty}^{\infty} x(\tau)\, y(t - \tau)\,\mathrm{d}\tau$$

and $C_{\boldsymbol{\lambda}}(\tau, \tau')$ is the autocovariance function of the stochastic process $\boldsymbol{\lambda}(t)$,

$$C_{\boldsymbol{\lambda}}(\tau, \tau') \equiv \langle \boldsymbol{\lambda}(\tau)\, \boldsymbol{\lambda}(\tau')\rangle_e - \langle \boldsymbol{\lambda}(\tau)\rangle_e\, \langle \boldsymbol{\lambda}(\tau')\rangle_e.$$

Like the variance of the photon statistics, the photocurrent statistics' variance splits additively into two parts. The first one, also present in the case of a deterministic electromagnetic field, is solely due to the quantized nature of light; it is called shot noise. The second term can be ascribed to the random optical field; it is excess noise.

Appendix C. The two-dimensional Fourier transform

We have to evaluate the time average of a signal $a(t)$ given in the form

$$a(t) = \iint_{-\infty}^{\infty} b(\tau, \tau')\, f(t - \tau)\, f(t - \tau')\,\mathrm{d}\tau\,\mathrm{d}\tau'. \tag{C1}$$

This equation can be treated using the two-dimensional Fourier transform

$$X(\mathrm{j}\omega_1, \mathrm{j}\omega_2) = \iint_{-\infty}^{\infty} x(t_1, t_2)\, \mathrm{e}^{-\mathrm{j}\omega_1 t_1}\, \mathrm{e}^{-\mathrm{j}\omega_2 t_2}\, \mathrm{d}t_1\, \mathrm{d}t_2 \tag{C2}$$

and its inverse

$$x(t_1, t_2) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} X(\mathrm{j}\omega_1, \mathrm{j}\omega_2)\, \mathrm{e}^{\mathrm{j}\omega_1 t_1}\, \mathrm{e}^{\mathrm{j}\omega_2 t_2}\, \mathrm{d}\omega_1\, \mathrm{d}\omega_2 \tag{C3}$$

using the following procedure: first, we define the function $a'(t_1, t_2)$ as

$$a'(t_1, t_2) = \iint_{-\infty}^{\infty} b(\tau, \tau')\, f(t_1 - \tau)\, f(t_2 - \tau')\, \mathrm{d}\tau\, \mathrm{d}\tau'. \tag{C4}$$

It can be shown that the two-dimensional Fourier transform (C2) of $a'(t_1, t_2)$ can be written as

$$A'(\mathrm{j}\omega_1, \mathrm{j}\omega_2) = B(\mathrm{j}\omega_1, \mathrm{j}\omega_2)\, F(\mathrm{j}\omega_1)\, F(\mathrm{j}\omega_2) \tag{C5}$$

where $B(\mathrm{j}\omega_1, \mathrm{j}\omega_2)$ denotes the two-dimensional Fourier transform of $b(t_1, t_2)$ and $F(\mathrm{j}\omega)$ stands for the one-dimensional Fourier transform of $f(t)$. This is a generalization of the (one-dimensional) convolution. The time average of $a(t) \equiv a'(t, t)$ then follows to be

$$\begin{aligned} \overline{a(t)} &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} a(t, t)\, \mathrm{d}t \\ &= \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} A(\mathrm{j}\omega_1, \mathrm{j}\omega_2) \lim_{T\to\infty} \frac{\sin[(\omega_1 + \omega_2)T/2]}{(\omega_1 + \omega_2)T/2}\, \mathrm{d}\omega_1\, \mathrm{d}\omega_2 \end{aligned}$$

which, using (C5) and the arguments leading to (14), can be simplified to yield

$$\overline{a(t)} = \sum_i \hat{B}(\mathrm{j}\omega_1^{(i)}, -\mathrm{j}\omega_1^{(i)}) |F(\mathrm{j}\omega_1^{(i)})|^2 \tag{C6}$$

if $f(t)$ is real. The sum has to be taken over the weights of all Dirac impulses in the two-dimensional spectrum of $b(t_1, t_2)$ lying on the straight line $\omega_1 = -\omega_2$.

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