Calc. Var. (2009) 36:591–609
DOI 10.1007/s00526-009-0246-1

Calculus of Variations

Linking solutions for quasilinear equations at critical growth involving the "1-Laplace" operator

Marco Degiovanni · Paola Magrone

Received: 6 October 2008 / Accepted: 5 May 2009 / Published online: 3 June 2009

Abstract We show that the problem at critical growth, involving the 1-Laplace operator and obtained by relaxation of $-\Delta_1 u = \lambda |u|^{-1}u + |u|^{1^*-2}\, u$, admits a nontrivial solution $u \in BV(\Omega)$ for any $\lambda \geq \lambda_1$. Nonstandard linking structures, for the associated functional, are recognized.

Mathematics Subject Classification (2000) 58E05 · 35J65

1 Introduction and main result

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary. We are interested in the existence of nontrivial solutions u to the problem which comes from the relaxation of

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{|\nabla u|}\right) = \lambda\, \dfrac{u}{|u|} + |u|^{1^*-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

Communicated by L. Ambrosio.

The research of M. Degiovanni was partially supported by the PRIN project "Variational and topological methods in the study of nonlinear phenomena" and by Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (INdAM).
The research of P. Magrone was partially supported by the PRIN project "Variational methods and nonlinear differential equations".

M. Degiovanni (✉)
Dipartimento di Matematica e Fisica, Università Cattolica, Via dei Musei 41, 25121 Brescia, Italy
e-mail: m.degiovanni@dmf.unicatt.it

P. Magrone
Dipartimento di Matematica, Università di Roma Tre,
Largo San Leonardo Murialdo 1, 00146 Rome, Italy
e-mail: magrone@mat.uniroma3.it

where $\lambda \in \mathbb{R}$ and $1^* = n/(n-1)$ is the critical Sobolev exponent for the embedding of $W_0^{1,1}(\Omega)$ in $L^q(\Omega)$.

Problem (1.1) looks as the formal limit, as $p \to 1^+$, of the problem at critical growth

$$\begin{cases} -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda |u|^{p-2}\, u + |u|^{p^*-2}\, u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $p^* = np/(n-p)$. Let us set, whenever $1 \le p < n$,

$$S = S(n,p) := \inf\left\{ \frac{\int_{\mathbb{R}^n} |\nabla u|^p\, dx}{\left(\int_{\mathbb{R}^n} |u|^{p^*}\, dx\right)^{p/p^*}} \;:\; u \in C_c^\infty(\mathbb{R}^n)\setminus\{0\} \right\}, \tag{1.3}$$

$$\lambda_1 = \lambda_1(\Omega,p) := \inf\left\{ \frac{\int_\Omega |\nabla u|^p\, dx}{\int_\Omega |u|^p\, dx} \;:\; u \in C_c^\infty(\Omega)\setminus\{0\} \right\}. \tag{1.4}$$

Problem (1.2) has received much attention in the last years, starting from the celebrated paper of Brezis and Nirenberg [5], where it was shown that, for $p = 2$, problem (1.2) admits a positive solution u for every $\lambda \in]0, \lambda_1[$ and $n \ge 4$. The result has been extended by Egnell, Garcia Azorero-Peral Alonso, Guedda-Veron [19,22,25], who have proved that (1.2) admits a positive solution u for any $\lambda \in]0, \lambda_1[$, provided that $p > 1$ and $n \ge p^2$. Such a solution u can be obtained via the Mountain pass theorem of Ambrosetti and Rabinowitz [1] applied to the C^1-functional $f : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined as

$$f(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\, dx - \frac{\lambda}{p}\int_\Omega |u|^p\, dx - \frac{1}{p^*}\int_\Omega |u|^{p^*}\, dx$$

and satisfies

$$0 < f(u) < \frac{1}{n}\, S^{n/p}. \tag{1.5}$$

When $\lambda \ge \lambda_1$, it is still meaningful to look for nontrivial solutions u, but the situation is quite different in the two cases $p = 2$ and $p \ne 2$. If $p = 2$, it has been proved by Capozzi et al. [7] that problem (1.2) has a nontrivial solution u for any $\lambda \ge \lambda_1$, provided that $n \ge 5$ (see also Gazzola and Ruf [23, Corollary 1]). Such a solution can be obtained via the Linking theorem of Rabinowitz (see e.g. [31, Theorem 5.3]) applied to the functional f and still satisfies (1.5).

On the other hand, when $p \ne 2$ there is in general no direct sum decomposition of $W_0^{1,p}(\Omega)$, which allows to recognize a linking structure in a standard way, unless λ belongs to a suitable right neighborhood $[\lambda_1, \overline{\lambda}[$ of λ_1, as shown in Arioli and Gazzola [3], where it is proved that, for any $p > 1$, problem (1.2) has a nontrivial solution u for any $\lambda \in [\lambda_1, \overline{\lambda}[$, provided that $\frac{n^2}{n+1} > p^2$. Nevertheless, the result of Capozzi–Fortunato–Palmieri has been recently extended, via a nonstandard linking construction, in Degiovanni and Lancelotti [13], where it is shown that the result of Arioli–Gazzola actually holds for any $\lambda \ge \lambda_1$.

Coming to the case $p = 1$, let us first give a precise relaxed formulation of (1.1). First of all, denote by $\|\ \|_p$ the usual norm in L^p and by $\mathscr{H}^k$ the k-dimensional Hausdorff measure. For every $u \in BV(\Omega)$ (see e.g. [2,24]), let us set

$$|Du|(\Omega) := \sup\left\{ \int_\Omega u \,\mathrm{div}\, v\, dx \;:\; v \in C_c^\infty(\Omega;\mathbb{R}^n),\ \|v\|_\infty \le 1 \right\}.$$

Then, according to Kawohl and Schuricht [28], we mean that we are looking for $u \in BV(\Omega)$ such that

$$\begin{cases} \text{there exist } z \in L^\infty(\Omega; \mathbb{R}^n) \text{ and } \gamma \in L^\infty(\Omega) \text{ such that} \\ \|z\|_\infty \leq 1, \ \operatorname{div} z \in L^n(\Omega), \ -\int_\Omega u \operatorname{div} z \, dx = |Du|(\Omega) + \int_{\partial\Omega} |u| \, d\mathscr{H}^{n-1}, \\ \|\gamma\|_\infty \leq 1, \ \gamma|u| = u \qquad \text{a.e. in } \Omega, \\ -\operatorname{div} z = \lambda\gamma + |u|^{1^*-2} u \qquad \text{a.e. in } \Omega, \end{cases} \tag{1.6}$$

(n is the exponent conjugate to 1^*). Other equivalent formulations can be obtained applying the next Proposition 3.1. Since $u = 0$ is a solution for any λ (take $(z, \gamma) = (0, 0)$), we say that $u = 0$ is the *trivial* solution of (1.6). Let us also define a locally Lipschitz functional $f : BV(\Omega) \longrightarrow \mathbb{R}$ by

$$f(u) = |Du|(\Omega) + \int_{\partial\Omega} |u| \, d\mathscr{H}^{n-1} - \lambda \int_\Omega |u| \, dx - \frac{1}{1^*} \int_\Omega |u|^{1^*} \, dx.$$

The resul of Brezis–Nirenberg has been extended also to this setting by Demengel [17], who has proved that (1.6) admits a nonnegative, nontrivial solution u satisfying

$$0 < f(u) < \frac{1}{n} S^n \tag{1.7}$$

for any $\lambda \in]0, \lambda_1[$. The argument is based on an approximation procedure from the case $p > 1$.

Our purpose is to cover the case $\lambda \geq \lambda_1$, in the line of the result of Capozzi–Fortunato–Palmieri, by a direct approach. Our result is the following

Theorem 1.1 *Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary. Then, for every $\lambda \geq \lambda_1$, problem* (1.6) *admits a nontrivial solution $u \in BV(\Omega) \cap L^\infty(\Omega)$ satisfying* (1.7).

For the proof, we will apply (nonsmooth) variational methods to the functional f. A first idea could be to apply the approach of Chang [8] to the locally Lipschitz functional f defined on $BV(\Omega)$. However, it has been already observed that, in such a setting, the Palais–Smale condition fails even in the subcritical case, as the norm-convergence of BV cannot be usually obtained for a Palais–Smale sequence (see Marzocchi [29] and Degiovanni et al. [15]). For this reason, it is more convenient to extend the functional f to $L^{1^*}(\Omega)$ with value $+\infty$ outside $BV(\Omega)$. In this setting, the nonsmoothness increases, as f is only lower semicontinuous, but the techniques of Corvellec–Degiovanni–Marzocchi, Ioffe–Schwartzman, Katriel [11,26,27] can be applied, in particular as specified in Degiovanni and Schuricht [16]. On the other hand, we have more compactness and in Theorem 5.3 we will show that f satisfies $(PS)_c$ whenever $c < (1/n)S^n$, as one may expect from the case $p > 1$ (see [25, Theorem 3.4]).

A second difficulty, typical in the case $p \neq 2$ when $\lambda \geq \lambda_1$, is that there is no direct sum decomposition which allows to recognize a linking structure in a standard way. Therefore, as in [13], we will apply the Linking theorem of [12], in which linear subspaces are substituted by cones.

In the next section we recall mainly from [16] some tools of nonsmooth analysis. In Sect. 3 we specify our functional framework, taking advantage of the results of [28]. In Sect. 4 we build the cones which have to substitute linear subspaces in the linking structure. Sect. 5 is devoted to the Palais–Smale condition, while in the last section we prove the main result.

2 Tools of nonsmooth analysis

Let Y be a metric space endowed with the distance d and let $f : Y \to [-\infty, +\infty]$ be a function. We set

$$\text{dom}(f) = \{u \in Y : \ |f(u)| < +\infty\}$$

and consider

$$\text{epi}\,(f) = \{(u, s) \in Y \times \mathbb{R} : \ f(u) \le s\}$$

endowed with the topology induced by $Y \times \mathbb{R}$. The next definition, equivalent to that of [14], is taken from [6].

Definition 2.1 For every $u \in \text{dom}(f)$, we denote by $|df|\,(u)$ the supremum of the σ's in $[0, +\infty[$ such that there exist a neighborhood W of $(u, f(u))$ in epi (f), $\delta > 0$ and a continuous map $\mathscr{H} : W \times [0, \delta] \to Y$ satisfying

$$d(\mathscr{H}((v, s), t), v) \le t, \quad f(\mathscr{H}((v, s), t)) \le s - \sigma t,$$

whenever $(v, s) \in W$ and $t \in [0, \delta]$.

The extended real number $|df|\,(u)$ is called the *weak slope* of f at u.

The idea is to look for local deformations $\mathscr{H}$, along which the function f can be decreased with a certain rate σ with respect to the displacement $d(\mathscr{H}((v, s), t), v)$, and then optimize σ.

In particular, if Y is an open subset of a normed space and f is of class C^1, then $|df|\,(u) = \|f'(u)\|$ for every $u \in Y$ (see [14, Corollary 2.12]).

Moreover, it is easily seen that $|df|$ is lower semicontinuous with respect to the graph topology: if (u_k) is a sequence convergent to u in $\text{dom}(f)$ with $f(u_k) \to f(u)$, then

$$\liminf_k \ |df|\,(u_k) \ge |df|\,(u).$$

Definition 2.2 An element $u \in Y$ is said to be a (*lower*) *critical point* of f, if $|f(u)| < +\infty$ and $|df|\,(u) = 0$. A real number c is said to be a (*lower*) *critical value* of f, if there exists a (lower) critical point u of f with $f(u) = c$.

Definition 2.3 A *Palais–Smale sequence* (*(PS)-sequence*, for short) for f is a sequence (u_k) in Y such that

$$\sup_k |f(u_k)| < +\infty$$

and such that $|df|\,(u_k) \to 0$.

Given a real number c, a *Palais–Smale sequence at level c* (*$(PS)_c$-sequence*, for short) is a (PS)-sequence (u_k) such that $f(u_k) \to c$.

The function f is said to satisfy $(PS)_c$, if every $(PS)_c$-sequence admits a convergent subsequence in Y.

Assume now that X is a real Banach space, whose dual space will be denoted by X'. In the following, $\partial f(u)$ will denote the Clarke–Rockafellar subdifferential and $f^0\,(u; v)$ the associated generalized directional derivative [10,32].

Let $f_0 : X \longrightarrow]-\infty, +\infty]$ be a convex, lower semicontinuous function and $f_1, g : X \longrightarrow \mathbb{R}$ two locally Lipschitz continuous functions. Let also $f = f_0 + f_1$ and

$$M = \{u \in X : \ g(u) = 0\}.$$

In such a case, according to the results of [16], we have that the functions

$$|df| : \mathrm{dom}(f) \longrightarrow [0, +\infty], \quad \left|d\left(f\big|_M\right)\right| : \mathrm{dom}(f) \cap M \longrightarrow [0, +\infty]$$

are lower semicontinuous with respect to the topology induced by X.

We are first interested in a (nonsmooth) extension of the Linking theorem, in which linear subspaces are substituted by symmetric cones. If $A \subseteq X\setminus\{0\}$ is symmetric, we denote by Index (A) the $\mathbb{Z}_2$-cohomological index of Fadell and Rabinowitz [20,21]. Let us recall that $\gamma^+(A) \leq \mathrm{Index}\,(A) \leq \gamma^-(A)$, where, according to [9],

$$\gamma^+(A) = \sup\left\{m \in \mathbb{N} : \text{ there exists an odd continuous map } \psi : \mathbb{R}^m\setminus\{0\} \longrightarrow A\right\},$$
$$\gamma^-(A) = \inf\left\{m \in \mathbb{N} : \text{ there exists an odd continuous map } \psi : A \longrightarrow \mathbb{R}^m\setminus\{0\}\right\}.$$

Theorem 2.4 *Let X_-, X_+ be two symmetric cones in X such that X_+ is closed in X,*

$$X_- \cap X_+ = \{0\},$$
$$\mathrm{Index}\,(X_-\setminus\{0\}) = \mathrm{Index}\,(X\setminus X_+) < \infty.$$

Let also $e \in X\setminus X_-$, $0 < r_+ < r_-$,

$$S_+ = \{v \in X_+ : \ \|v\| = r_+\},$$
$$Q = \{te + u : \ t \geq 0, \ u \in X_-, \ \|te + u\| \leq r_-\},$$
$$P = \{u \in X_- : \ \|u\| \leq r_-\} \cup \{te + u : \ t \geq 0, \ u \in X_-, \ \|te + u\| = r_-\}$$

be such that

$$\sup_P f < \inf_{S_+} f, \quad \sup_Q f < +\infty.$$

Then f admits a $(PS)_c$-sequence with

$$\inf_{S_+} f \leq c \leq \sup_Q f.$$

In particular, if f satisfies $(PS)_c$, then c is a critical value of f.

Proof If $f : X \longrightarrow \mathbb{R}$ is of class C^1, by [12, Corollary 2.9] the assertion is a particular case of [12, Theorem 2.2]. If $f : X \longrightarrow \mathbb{R}$ is continuous, the proof is exactly the same, by the Deformation theorem of [11]. The case we are treating can be reduced to the continuous one arguing, as in [16], on the continuous function $\mathscr{G}_f : \mathrm{epi}\,(f) \to \mathbb{R}$ defined by $\mathscr{G}_f(u, s) = s$. □

We also need an information in the constrained case.

Theorem 2.5 *Assume that f and g are even with $g(0) \neq 0$ and that*

$$\mathrm{Index}\,(\{u \in M : \ f(u) < +\infty\}) = \infty.$$

Suppose also that $f_{|M}$ is bounded from below, satisfies $(PS)_c$ for any $c \in \mathbb{R}$ and that, for every $u \in M$ with $f(u) < +\infty$, there exist $u_\pm \in X$ such that $f(u_\pm) < +\infty$ and

$$g^0\,(u; u_- - u) < 0, \quad g^0\,(u; u - u_+) < 0.$$

For every $m \geq 1$, let

$$c_m = \inf\left\{\sup_A f : \ A \subseteq M, \ A \ \textit{ is symmetric and } \mathrm{Index}\,(A) \geq m\right\}.$$

Then $c_m \to +\infty$ and, for every $m \geq 1$ and c with $c_m \leq c < c_{m+1}$, we have

$$\text{Index}\,(\{u \in M :\; f(u) \leq c\}) = m.$$

Proof In the C^1 setting, the assertion follows from the Deformation theorem (see e.g. [12, Theorem 3.2]). For the extension to the nonsmooth case we are treating, we may argue as in the previous proof. □

Finally, let us recall from [16, Theorem 3.5] two results which connect the metric notion of weak slope with that of subdifferential.

Theorem 2.6 *Let $u \in X$ with $f(u) < +\infty$ and $|df|(u) < +\infty$. Then there exist $w \in X'$ with $\|w\| \leq |df|(u)$ and $\alpha \in \partial f_1(u)$ such that $-\alpha + w \in \partial f_0(u)$, i.e.*

$$f_0(v) \geq f_0(u) - \langle \alpha, v - u\rangle + \langle w, v - u\rangle, \quad \forall v \in X.$$

Theorem 2.7 *Let $u \in M$ with $f(u) < +\infty$ and $\left|d\left(f_{|M}\right)\right|(u) < +\infty$. Assume also that there exist $u_\pm \in X$ such that $f(u_\pm) < +\infty$ and*

$$g^0\,(u; u_- - u) < 0, \quad g^0\,(u; u - u_+) < 0.$$

Then there exist $w \in X'$ with $\|w\| \leq \left|d\left(f_{|M}\right)\right|(u)$ and $\alpha \in \partial f_1(u)$, $\beta \in \partial g(u)$, $\lambda \in \mathbb{R}$ such that $-\alpha + \lambda\beta + w \in \partial f_0(u)$, i.e.

$$f_0(v) \geq f_0(u) - \langle \alpha, v - u\rangle + \lambda\langle \beta, v - u\rangle + \langle w, v - u\rangle, \quad \forall v \in X.$$

3 The functional framework

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary and let $\lambda \in \mathbb{R}$. According to [28], let us define a convex, lower semicontinuous functional $f_0 : L^{1^*}(\Omega) \longrightarrow [0, +\infty]$ by

$$f_0(u) = \begin{cases} |Du|\,(\Omega) + \int_{\partial\Omega} |u|\, d\mathscr{H}^{n-1} & \text{if } u \in BV(\Omega), \\ \\ +\infty & \text{if } u \in L^{1^*}(\Omega) \backslash BV(\Omega), \end{cases}$$

and two locally Lipschitz continuous functionals $f_1, g : L^{1^*}(\Omega) \longrightarrow \mathbb{R}$ by

$$f_1(u) = -\lambda \int_\Omega |u|\, dx - \frac{1}{1^*} \int_\Omega |u|^{1^*}\, dx,$$

$$g(u) = \int_\Omega |u|\, dx - 1.$$

As usual, the dual of $L^{1^*}(\Omega)$ will be identified with $L^{(1^*)'}(\Omega) = L^n(\Omega)$. Moreover, f_0 is a norm on $BV(\Omega)$ equivalent to the canonical one. According to [17,28], we have

$$S = S(n, 1) = \min\left\{\frac{f_0(u)}{\|u\|_{1^*}} :\; u \in BV(\Omega)\backslash\{0\}\right\}, \tag{3.1}$$

$$\lambda_1 = \lambda_1(\Omega, 1) = \min\left\{\frac{f_0(u)}{\|u\|_1} :\; u \in BV(\Omega)\backslash\{0\}\right\}, \tag{3.2}$$

where S, λ_1 are defined in (1.3), (1.4). In particular, contrary to the case $p > 1$, the constant S is achieved in (3.1), for instance on characteristic functions of balls contained in Ω (see [4]).

We are interested in the application of variational methods to $f = f_0 + f_1$ on the whole space $L^{1^*}(\Omega)$ and to f_0 constrained on

$$M = \left\{ u \in L^{1^*}(\Omega) : \; g(u) = 0 \right\}.$$

In order to apply the results of the previous section, let us first recall from [28] the next

Proposition 3.1 *Let $u \in BV(\Omega)$ and $w \in L^n(\Omega)$. Then the following facts are equivalent*:

(a) *we have $w \in \partial f_0(u)$;*
(b) *we have*

$$\int_\Omega uw \, dx = |Du|(\Omega) + \int_{\partial\Omega} |u| \, d\mathscr{H}^{n-1}$$

and there exists $z \in L^\infty(\Omega; \mathbb{R}^n)$ such that $\|z\|_\infty \le 1$ and $-\mathrm{div}\, z = w$;
(c) *there exists $z \in L^\infty(\Omega; \mathbb{R}^n)$ such that $\|z\|_\infty \le 1$, $-\mathrm{div}\, z = w$ and*

$$\int_\Omega uw\varphi \, dx - \int_\Omega uz \cdot \nabla\varphi \, dx = \sup\left\{ \left| \int_\Omega u \, \mathrm{div}\, \psi \, dx \right| : \; \psi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n), \; |\psi| \le \varphi \right\}$$

for every $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi \ge 0$.

Proof It is enough to combine [28, Proposition 4.23] with [28, Proposition A.12] and recall that the function defined as

$$\begin{cases} u & \text{on } \Omega, \\ 0 & \text{on } \mathbb{R}^n \setminus \Omega, \end{cases}$$

belongs to $BV(\mathbb{R}^n)$. □

In general, the graph of the subdifferential of a convex, lower semicontinuous functional is strong-weak* closed. In our case, we have a better property which will be useful later.

Proposition 3.2 *Let (u_k) be a sequence in $BV(\Omega)$ and (w_k) a sequence in $L^n(\Omega)$ such that (u_k) is weakly convergent to u in $L^{1^*}(\Omega)$, (w_k) is weakly convergent to w in $L^n(\Omega)$ and $w_k \in \partial f_0(u_k)$ for every $k \in \mathbb{N}$.*

Then $u \in BV(\Omega)$ and $w \in \partial f_0(u)$.

Proof For every $h > 0$, define $T_h, R_h : \mathbb{R} \longrightarrow \mathbb{R}$ by $T_h(s) = \min\{\max\{s, -h\}, h\}$, $R_h(s) = s - T_h(s)$. By [2, Theorem 3.99] we have

$$|Du|(\Omega) = |D(T_h(u))|(\Omega) + |D(R_h(u))|(\Omega),$$

hence

$$f_0(u) = f_0(T_h(u)) + f_0(R_h(u)), \quad \forall u \in BV(\Omega). \tag{3.3}$$

First of all, from the inequality

$$0 = f_0(0) \ge f_0(u_k) - \int_\Omega w_k u_k \, dx$$

we see that (u_k) is bounded in $BV(\Omega)$. It follows that $u \in BV(\Omega)$ and that $(T_h(u_k))$ is strongly convergent to $T_h(u)$ in $L^{1^*}(\Omega)$, for every $h > 0$.

We also have

$$\begin{aligned} f_0(v) + f_0(R_h(u_k)) \geq f_0(v + R_h(u_k)) &\geq f_0(u_k) + \int_\Omega w_k(v + R_h(u_k) - u_k)\, dx \\ &= f_0(T_h(u_k)) + f_0(R_h(u_k)) + \int_\Omega w_k(v - T_h(u_k))\, dx, \end{aligned}$$

whence

$$f_0(v) \geq f_0(T_h(u_k)) + \int_\Omega w_k(v - T_h(u_k))\, dx.$$

Passing to the limit as $k \to \infty$ and taking into account the lower semicontinuity of f_0, we get

$$f_0(v) \geq f_0(T_h(u)) + \int_\Omega w(v - T_h(u))\, dx.$$

Passing to the limit as $h \to \infty$, the assertion follows. □

Let us also prove a simple regularity property. A related result is contained in [18, Proposition 7].

Proposition 3.3 *Let $u \in BV(\Omega)$ with $\partial f_0(u) \neq \emptyset$. Then $u \in L^\infty(\Omega)$.*

Proof Let $w \in L^n(\Omega)$ with $w \in \partial f_0(u)$. For every $h > 0$, we have

$$f_0(T_h(u)) \geq f_0(u) + \int_\Omega w(T_h(u) - u)\, dx.$$

By (3.1), (3.3) and Hölder's inequality, it follows

$$S\|R_h(u)\|_{1^*} \leq f_0(R_h(u)) \leq \int_\Omega w R_h(u)\, dx \leq \left(\int_{\{|u|>h\}} |w|^n\, dx \right)^{1/n} \|R_h(u)\|_{1^*}.$$

If h is large enough to guarantee that

$$\left(\int_{\{|u|>h\}} |w|^n\, dx \right)^{1/n} < S,$$

we infer that $\|R_h(u)\|_{1^*} = 0$ and the assertion follows. □

Finally, from [28] we have the

Proposition 3.4 *Let $u \in BV(\Omega)$ with $|df|(u) < +\infty$. Then $u \in L^\infty(\Omega)$ and there exist $\gamma \in L^\infty(\Omega)$ and $w \in L^n(\Omega)$ such that $\|\gamma\|_\infty \le 1$, $\gamma|u| = u$ a.e. in Ω, $\|w\|_n \le |df|(u)$ and*

$$f_0(v) \ge f_0(u) + \lambda \int_\Omega \gamma(v-u)\,dx + \int_\Omega |u|^{1^*-2}u(v-u)\,dx + \int_\Omega w(v-u)\,dx, \quad \forall v \in BV(\Omega).$$

Proof It is enough to combine Theorem 2.6 with Proposition 3.3 and [28, Proposition 4.23]. □

Corollary 3.5 *If $u \in L^{1^*}(\Omega)$ is a critical point of f, then $u \in BV(\Omega) \cap L^\infty(\Omega)$ and u is a solution of* (1.6).

Proof It is enough to combine Proposition 3.1 with Proposition 3.4. □

4 Symmetric cones related to the 1-Laplace operator

In this section we show how to build, for the 1-Laplace operator, two cones X_-, X_+ with the properties required in Theorem 2.4. The construction is based on a sequence of eigenvalues for the 1-Laplace operator. We refer the reader to Milbers and Schuricht [30] for a slightly different construction of such a sequence.

Proposition 4.1 *The following facts hold:*

(a) *for every $u \in BV(\Omega) \cap M$, there exist $u_\pm \in BV(\Omega)$ such that*

$$g^0(u; u_- - u) < 0, \quad g^0(u; u - u_+) < 0;$$

(b) *for every $u \in BV(\Omega) \cap M$ with $\left|d\left(f_{0|M}\right)\right|(u) < +\infty$, we have $u \in L^\infty(\Omega)$ and there exist $\lambda \in \mathbb{R}$, $\gamma \in L^\infty(\Omega)$ and $w \in L^n(\Omega)$ such that $\|\gamma\|_\infty \le 1$, $\gamma|u| = u$ a.e. in Ω, $\|w\|_n \le \left|d\left(f_{0|M}\right)\right|(u)$ and*

$$f_0(v) \ge f_0(u) + \lambda \int_\Omega \gamma(v-u)\,dx + \int_\Omega w(v-u)\,dx, \quad \forall v \in BV(\Omega);$$

(c) *the functionals f_0 and g are even with $g(0) \ne 0$ and* Index $(BV(\Omega) \cap M) = \infty$ *with respect to the topology of $L^{1^*}(\Omega)$; moreover, $f_{0|M}$ is bounded from below and satisfies $(PS)_c$ for any $c \in \mathbb{R}$.*

Proof In the proof of [28, Theorem 4.6] it is shown that (a) holds. Then assertion (b) follows from Theorem 2.7, Proposition 3.3 and [28, Proposition 4.23]. Since $BV(\Omega)$ has infinite dimension, it is obvious that $\gamma^+(BV(\Omega) \cap M) = \infty$, also with respect to the topology of $L^{1^*}(\Omega)$. Therefore Index $(BV(\Omega) \cap M) = \infty$.

If (u_k) is a (PS)-sequence for $f_{0|M}$, by (b) we have

$$f_0(v) \ge f_0(u_k) + \lambda_k \int_\Omega \gamma_k(v-u_k)\,dx + \int_\Omega w_k(v-u_k)\,dx, \quad \forall v \in BV(\Omega)$$

with $\lambda_k \in \mathbb{R}$, $\gamma_k \in L^\infty(\Omega)$ and $w_k \in L^n(\Omega)$ satisfying $\|\gamma_k\|_\infty \le 1$, $\gamma_k|u_k| = u_k$ a.e. in Ω and $\|w_k\|_n \to 0$. Since f_0 is an equivalent norm in $BV(\Omega)$, up to a subsequence (u_k) is convergent to $u \in BV(\Omega)$ weakly in $L^{1^*}(\Omega)$ and strongly in $L^1(\Omega)$, while (γ_k) is convergent to γ in the weak* topology of $L^\infty(\Omega)$. Moreover, by Proposition 3.1 we have

$$\begin{aligned} f_0(u_k) &= \lambda_k \int_\Omega \gamma_k u_k\,dx + \int_\Omega w_k u_k\,dx \\ &= \lambda_k \int_\Omega |u_k|\,dx + \int_\Omega w_k u_k\,dx = \lambda_k + \int_\Omega w_k u_k\,dx. \end{aligned}$$

Therefore, also (λ_k) is bounded, hence convergent, up to a subsequence, to some λ. From Proposition 3.2 it follows that $\lambda\gamma \in \partial f_0(u)$, whence, by Proposition 3.1,

$$\lim_k f_0(u_k) = \lim_k \left(\lambda_k \int_\Omega \gamma_k u_k\,dx + \int_\Omega w_k u_k\,dx \right) = \lambda \int_\Omega \gamma u\,dx = f_0(u).$$

From [15, Theorem 4.10] we conclude that (u_k) is strongly convergent to u in $L^{1^*}(\Omega)$.
The other assertions contained in (c) are obvious. □

For every $m \geq 1$, let

$$\lambda_m = \inf \left\{ \sup_A f_0 : \; A \subseteq M, A \text{ is symmetric and Index}\,(A) \geq m \right\}.$$

Since Index $(A) = 0$ only for $A = \emptyset$, the definition of λ_1 agrees with (3.2).

Theorem 4.2 *We have that $\lambda_m \to +\infty$. Moreover, for every $m \geq 1$ and μ with $\lambda_m \leq \mu < \lambda_{m+1}$, we have*

$$\text{Index}\left(\left\{ u \in BV(\Omega)\setminus\{0\} : \; |Du|(\Omega) + \int_{\partial\Omega} |u|\,d\mathscr{H}^{n-1} \leq \mu \int_\Omega |u|\,dx \right\} \right) = m$$

with respect to the topology of $L^{1^*}(\Omega)$.

Proof Since f_0 and $\| \|_1$ are both positively homogeneous of degree 1, it is enough to combine Theorem 2.5 with Proposition 4.1. □

In view of the application of Theorem 2.4, let us see a first possible choice of X_-, X_+.

Theorem 4.3 *Let $m \geq 1$ and let $\lambda_m < \mu < \lambda_{m+1}$. Then there exist a symmetric cone X_- in $BV(\Omega)$ and a symmetric cone X_+ in $L^{1^*}(\Omega)$ such that X_- is closed in $L^1(\Omega)$, X_+ is closed in $L^{1^*}(\Omega)$ and*:

(a) *we have*

$$X_- \subseteq \left\{ u \in BV(\Omega) : \; |Du|(\Omega) + \int_{\partial\Omega} |u|\,d\mathscr{H}^{n-1} \leq \lambda_m \int_\Omega |u|\,dx \right\} \cap L^\infty(\Omega);$$

(b) *$X_- \cap M$ is bounded in $L^\infty(\Omega)$ and strongly compact in $L^1(\Omega)$;*
(c) *we have*

$$X_+ \cap BV(\Omega) \subseteq \left\{ u \in BV(\Omega) : \; |Du|(\Omega) + \int_{\partial\Omega} |u|\,d\mathscr{H}^{n-1} \geq \mu \int_\Omega |u|\,dx \right\};$$

(d) *we have* Index $(X_-\setminus\{0\}) =$ Index $\left(L^{1^*}(\Omega)\setminus X_+\right) = m$ *with respect to the topology of* $L^{1^*}(\Omega)$.

Proof Let

$$\widetilde{X}_- = \left\{u \in BV(\Omega) :\ |Du|(\Omega) + \int_{\partial\Omega} |u|\, d\mathscr{H}^{n-1} \le \lambda_m \int_\Omega |u|\, dx\right\}.$$

Since $\widetilde{X}_- \cap M$ is an odd deformation retract of $\widetilde{X}_-\setminus\{0\}$, by Theorem 4.2 we have that Index $\left(\widetilde{X}_- \cap M\right) = m$. Moreover, $\widetilde{X}_- \cap M$ is strongly compact in $L^1(\Omega)$.

Let T_h, R_h be defined as before. First of all, we claim that there exists $h > 0$ such that

$$f_0\left(T_h(u)\right) \le \lambda_m \int_\Omega |T_h(u)|\, dx, \quad \forall u \in \widetilde{X}_- \cap M; \tag{4.1}$$

$$\int_\Omega |T_h(u)|\, dx \ge \frac{1}{2}, \quad \forall u \in \widetilde{X}_- \cap M. \tag{4.2}$$

Actually, for every $u \in BV(\Omega)$ Hölder's inequality and (3.1) yield

$$\int_\Omega |u|\, dx \le \mathscr{L}^n\left(\{u \ne 0\}\right)^{\frac{1}{n}} \left(\int_\Omega |u|^{1^*}\, dx\right)^{\frac{1}{1^*}} \le \frac{1}{S}\mathscr{L}^n\left(\{u \ne 0\}\right)^{\frac{1}{n}}\ f_0(u).$$

Since for every $u \in BV(\Omega) \cap M$ we have $R_h(u) \in BV(\Omega)$ and

$$1 = \int_\Omega |u|\, dx \ge \int_{\{R_h(u)\ne 0\}} |u|\, dx \ge h\mathscr{L}^n\left(\{R_h(u) \ne 0\}\right),$$

it follows

$$S\, h^{\frac{1}{n}} \int_\Omega |R_h(u)|\, dx \le f_0(R_h(u)) \quad \forall u \in BV(\Omega) \cap M.$$

Then, if h is large enough, we have

$$\lambda_m \int_\Omega |R_h(u)|\, dx \le f_0(R_h(u)) \quad \forall u \in BV(\Omega) \cap M$$

and (4.1) follows from (3.3). Moreover, if $u \in \widetilde{X}_- \cap M$, we also have

$$S\, h^{\frac{1}{n}} \int_\Omega |R_h(u)|\, dx \le f_0(R_h(u)) \le f_0(u) \le \lambda_m.$$

Then (4.2) also follows, provided that h is large enough.

With this choice of h, let

$$X_- = \left\{t\, T_h(u) :\ t \ge 0,\ u \in \widetilde{X}_- \cap M\right\}.$$

Then X_- is a symmetric cone in $BV(\Omega) \cap L^\infty(\Omega)$. From (4.1) it follows that $X_- \subseteq \widetilde{X}_-$, while (4.2) implies that

$$\|v\|_\infty \le 2h\, \|v\|_1, \quad \forall v \in X_-.$$

In particular, $X_- \cap M$ is bounded in $L^\infty(\Omega)$. Since the surjective map

$$\begin{aligned} \widetilde{X}_- \cap M &\longrightarrow X_- \cap M \\ u &\mapsto \frac{T_h(u)}{\|T_h(u)\|_1} \end{aligned}$$

is odd and continuous with respect to the topology of $L^{1^*}(\Omega)$, we have

$$\text{Index}\,(X_-\backslash\{0\}) \geq \text{Index}\,(X_- \cap M) \geq \text{Index}\,\left(\widetilde{X}_- \cap M\right) = m.$$

Actually, equality holds, as $X_- \subseteq \widetilde{X}_-$. Finally, the above map is also continuous with respect to the topology of $L^1(\Omega)$. Therefore $X_- \cap M$ is strongly compact in $L^1(\Omega)$ and X_- is closed in $L^1(\Omega)$.

Again from Theorem 4.2 we know that

$$\text{Index}\left(\left\{u \in BV(\Omega) \cap M : |Du|(\Omega) + \int_{\partial\Omega} |u|\, d\mathscr{H}^{n-1} \leq \mu\right\}\right) = m.$$

Let U be a symmetric open neighborhood of such a set satisfying Index $(U) = m$. Then

$$X_+ = L^{1^*}(\Omega)\backslash \{tu : t > 0,\ u \in U\}$$

has the required properties. □

5 The Palais–Smale condition

Lemma 5.1 *Let (u_k) be a (PS) sequence for f and let $u \in BV(\Omega)$. Assume that (u_k) is bounded in $BV(\Omega)$ and weakly convergent to u in $L^{1^*}(\Omega)$.*

Then we have

$$\lim_k \left(f_0(u_k) - \|u_k\|_{1^*}^{1^*}\right) = f_0(u) - \|u\|_{1^*}^{1^*},$$

$$\limsup_k \left(f_0(R_h(u_k)) - \|R_h(u_k)\|_{1^*}^{1^*}\right) \leq f_0(R_h(u)) - \|R_h(u)\|_{1^*}^{1^*}, \quad \forall h > 0.$$

Proof By Proposition 3.4, there exist (γ_k) in $L^\infty(\Omega)$ and (w_k) in $L^n(\Omega)$ such that $\|\gamma_k\|_\infty \leq 1$, $\gamma_k|u_k| = u_k$ a.e. in Ω, $\|w_k\|_n \to 0$ and $\lambda\gamma_k + |u_k|^{1^*-2}u_k + w_k \in \partial f_0(u_k)$. Moreover, (u_k) is also strongly convergent to u in $L^1(\Omega)$ and, up to a subsequence, (γ_k) is convergent to some γ in the weak* topology of $L^\infty(\Omega)$. By Proposition 3.2 it follows $\lambda\gamma + |u|^{1^*-2}u \in \partial f_0(u)$. Then by Proposition 3.1 we have

$$\begin{aligned} f_0(u_k) &= \lambda \int_\Omega \gamma_k u_k\, dx + \int_\Omega |u_k|^{1^*}\, dx + \int_\Omega w_k u_k\, dx \\ &= \lambda \int_\Omega |u_k|\, dx + \int_\Omega |u_k|^{1^*}\, dx + \int_\Omega w_k u_k\, dx, \qquad (5.1) \\ f_0(u) &= \lambda \int_\Omega \gamma u\, dx + \int_\Omega |u|^{1^*}\, dx, \end{aligned}$$

whence

$$\lim_k \left(f_0(u_k) - \int_\Omega |u_k|^{1^*} dx \right) = \lim_k \left(\lambda \int_\Omega \gamma_k u_k \, dx + \int_\Omega w_k u_k \, dx \right)$$
$$= \lambda \int_\Omega \gamma u \, dx = f_0(u) - \int_\Omega |u|^{1^*} dx.$$

By (3.3) we also have

$$f_0(R_h(u_k)) - \|R_h(u_k)\|_{1^*}^{1^*}$$
$$= \left(f_0(u_k) - \|u_k\|_{1^*}^{1^*} \right) - f_0(T_h(u_k)) + \left(\|u_k\|_{1^*}^{1^*} - \|R_h(u_k)\|_{1^*}^{1^*} \right).$$

On the other hand, $(T_h(u_k))$ is convergent to $T_h(u)$ in $L^{1^*}(\Omega)$ and we have that

$$0 \le |s|^{1^*} - |R_h(s)|^{1^*} \le \varepsilon |s|^{1^*} + C_{h,\varepsilon}, \quad \forall \varepsilon > 0.$$

From [12, Lemma 4.2] it follows that

$$\lim_k \left(\|u_k\|_{1^*}^{1^*} - \|R_h(u_k)\|_{1^*}^{1^*} \right) = \left(\|u\|_{1^*}^{1^*} - \|R_h(u)\|_{1^*}^{1^*} \right).$$

By the lower semicontinuity of f_0, the second assertion also follows. □

Lemma 5.2 *Each (PS) sequence for f is bounded in $BV(\Omega)$.*

Proof Let (u_k) be a (PS) sequence for f. Assume, for a contradiction, that $f_0(u_k) \to +\infty$. If we set

$$v_k = \frac{u_k}{f_0(u_k)},$$

up to a subsequence (v_k) is strongly convergent in $L^1(\Omega)$ to some $v \in BV(\Omega)$. Since

$$\frac{f(u_k)}{f_0(u_k)} = 1 - \lambda \|v_k\|_1 - \frac{1}{1^*} (f_0(u_k))^{1^*-1} \|v_k\|_{1^*}^{1^*},$$

from the boundedness of $(f(u_k))$ we deduce that (v_k) is strongly convergent to 0 in $L^{1^*}(\Omega)$. On the other hand, as before it holds (5.1) with $\|w_k\|_n \to 0$. It follows

$$f(u_k) = \frac{1}{n} [f_0(u_k) - \lambda \|u_k\|_1] + \frac{1}{1^*} \int_\Omega w_k u_k \, dx,$$

namely

$$\frac{f(u_k)}{f_0(u_k)} = \frac{1}{n} [1 - \lambda \|v_k\|_1] + \frac{1}{1^*} \int_\Omega w_k v_k \, dx.$$

Passing to the limit as $k \to \infty$, we get $0 = 1/n$ and a contradiction follows. □

Theorem 5.3 *For any $\lambda \in \mathbb{R}$, the functional f satisfies $(PS)_c$ whenever $c < (1/n)S^n$.*

Proof Let (u_k) be a $(PS)_c$ sequence with $c < (1/n)S^n$. We already know that (u_k) is bounded in $BV(\Omega)$, hence convergent, up to a subsequence, to some $u \in BV(\Omega)$ weakly in $L^{1^*}(\Omega)$ and strongly in $L^1(\Omega)$. From (5.1) it also follows that

$$f(u_k) = \frac{1}{n}\, \|u_k\|_{1^*}^{1^*} + \int_\Omega w_k u_k\, dx,$$

with $\|w_k\|_n \to 0$, whence

$$\lim_k \|u_k\|_{1^*}^{1^*-1} = (nc)^{1/n} < S.$$

Given $\varepsilon > 0$, let $h > 0$ be such that

$$f_0(R_h(u)) - \|R_h(u)\|_{1^*}^{1^*} < \varepsilon\left(S - (nc)^{1/n}\right).$$

Then we have

$$\limsup_k \|R_h(u_k)\|_{1^*}^{1^*-1} \le (nc)^{1/n}$$

and, by (3.1),

$$\left(S - \|R_h(u_k)\|_{1^*}^{1^*-1}\right) \|R_h(u_k)\|_{1^*} \le f_0(R_h(u_k)) - \|R_h(u_k)\|_{1^*}^{1^*}.$$

From Lemma 5.1 it follows

$$\limsup_k \|R_h(u_k)\|_{1^*} < \varepsilon,$$

whence $\|R_h(u)\|_{1^*} < \varepsilon$. Since $(T_h(u_k))$ is strongly convergent to $T_h(u)$ in $L^{1^*}(\Omega)$, we have

$$\limsup_k \|u_k - u\|_{1^*} \le \limsup_k \|T_h(u_k) - T_h(u)\|_{1^*} + \limsup_k \|R_h(u_k)\|_{1^*} + \|R_h(u)\|_{1^*} \le 2\varepsilon$$

and the assertion follows by the arbitrariness of ε. □

6 Proof of the main result

Let $x_0 \in \Omega$ and let

$$e_\rho = n^{n-1}\, \rho^{1-n}\, \chi_{\mathrm{B}_\rho(x_0)}.$$

Then it is well known (see [4]) that $e_\rho \in BV(\mathbb{R}^n)$ and

$$|De_\rho|(\mathbb{R}^n) = \int_{\mathbb{R}^n} \left|e_\rho\right|^{1^*} dx = S^n, \tag{6.1}$$

$$\int_{\mathbb{R}^n} \left|u_\rho\right| dx = n^{n-1}\, \mathscr{L}^n\left(\mathrm{B}_1(0)\right) \rho. \tag{6.2}$$

Let $\lambda \ge \lambda_1$, let $m \ge 1$ be such that $\lambda_m \le \lambda < \lambda_{m+1}$ and let $\lambda < \mu < \lambda_{m+1}$. Let X_-, X_+ be as in Theorem 4.3. Let also

$$v_\rho = \chi_{\mathbb{R}^n \setminus \mathrm{B}_{2\rho}(x_0)}\, v, \quad \forall v \in X_-;$$
$$X_-^\rho = \left\{v_\rho :\ v \in X_-\right\}.$$

Lemma 6.1 *There exist* $C, \overline{\rho} > 0$ *such that* $\overline{\mathrm{B}_{2\overline{\rho}}(x_0)} \subseteq \Omega$ *and*

$$f_0(v_\rho) \le f_0(v) + C\rho^{n-1} \left(\int_\Omega |v|^{1^*} dx \right)^{1/1^*}, \tag{6.3}$$

$$\int_\Omega |v_\rho|^{1^*} dx \ge \int_\Omega |v|^{1^*} dx - C\rho^n \int_\Omega |v|^{1^*} dx, \tag{6.4}$$

$$\int_\Omega |v_\rho|\, dx \ge \int_\Omega |v|\, dx - C\rho^n \left(\int_\Omega |v|^{1^*} dx \right)^{1/1^*}, \tag{6.5}$$

$$e_\rho \notin X_-^\rho \text{ and } X_-^\rho \text{ is closed in } L^1(\Omega), \tag{6.6}$$

$$X_-^\rho \cap X_+ = \{0\}, \quad \mathrm{Index}\left(X_-^\rho \setminus \{0\}\right) = \mathrm{Index}\left(L^{1^*}(\Omega) \setminus X_+\right) = m, \tag{6.7}$$

for every $v \in X_-$ *and* $\rho \in]0, \overline{\rho}]$.

Proof Let first $\overline{\rho} > 0$ be such that $\overline{\mathrm{B}_{2\overline{\rho}}(x_0)} \subseteq \Omega$ and let $0 < \rho \le \overline{\rho}$. According to [2] and Theorem 4.3, we have

$$f_0(v_\rho) \le f_0(v) + \|v\|_\infty |D\chi_{\mathrm{B}_{2\rho}(x_0)}|(\Omega) \le f_0(v) + C\,\rho^{n-1}\,\|v\|_{1^*},$$

whence (6.3). The proof of (6.4) and (6.5) is similar and even simpler.

It is clear that $e_\rho \notin X_-^\rho$. From (6.3), (6.5) and Theorem 4.3 it also follows that

$$f_0(v_\rho) \le \frac{1}{2}(\lambda_m + \mu) \int_\Omega |v_\rho|\, dx, \quad \forall v \in X_-,$$

provided that ρ is small enough. Therefore $X_-^\rho \cap X_+ = \{0\}$. Moreover, for every $v \in X_-$ we have

$$\begin{aligned}
\int_\Omega |v|\, dx &\le \mathscr{L}^n\left(\mathrm{B}_{2\rho}(x_0)\right)^{\frac{1}{n}} \left(\int_\Omega |v|^{1^*} dx \right)^{\frac{1}{1^*}} + \int_{\Omega \setminus \mathrm{B}_{2\rho}(x_0)} |v|\, dx \\
&\le S^{-1}\,\mathscr{L}^n\left(\mathrm{B}_{2\rho}(x_0)\right)^{\frac{1}{n}} f_0(v) + \int_{\Omega \setminus \mathrm{B}_{2\rho}(x_0)} |v|\, dx \\
&\le S^{-1}\lambda_m\,\mathscr{L}^n\left(\mathrm{B}_{2\rho}(x_0)\right)^{\frac{1}{n}} \int_\Omega |v|\, dx + \int_{\Omega \setminus \mathrm{B}_{2\rho}(x_0)} |v|\, dx.
\end{aligned}$$

If ρ is small enough, we get

$$\int_\Omega |v|\, dx \le C \int_{\Omega \setminus \mathrm{B}_{2\rho}(x_0)} |v|\, dx \quad \text{for every } v \in X_-.$$

First of all, it follows that we have $v_\rho = 0$ only for $v = 0$. Since $\{v \mapsto v_\rho\}$ is continuous and odd with respect to the topology of $L^{1^*}(\Omega)$ from $X_- \setminus \{0\}$ to $X_-^\rho \setminus \{0\}$, we get

$$\mathrm{Index}\left(X_-^\rho \setminus \{0\}\right) \ge \mathrm{Index}\left(X_- \setminus \{0\}\right) = \mathrm{Index}\left(L^{1^*}(\Omega) \setminus X_+\right) = m.$$

Actually, equality holds, as $X_-^\rho \setminus \{0\} \subseteq L^{1^*}(\Omega) \setminus X_+$. Finally, let $(v^{(k)})$ be a sequence in X_- with $(v_\rho^{(k)})$ convergent to some u in $L^1(\Omega)$. Then $(v^{(k)})$ is bounded in $L^1(\Omega \setminus \mathrm{B}_{2\rho}(x_0))$, hence in $L^1(\Omega)$, hence in $BV(\Omega)$. Up to a subsequence, $(v^{(k)})$ is $L^1(\Omega)$-convergent to some element of X_-, whence $u \in X_-^\rho$. Therefore, X_-^ρ is closed in $L^1(\Omega)$. □

Lemma 6.2 *There exist* $\overline{\rho}, \delta > 0$ *such that*

$$\sup\left\{f(te_\rho + u) :\ t \geq 0,\ u \in X_-^\rho\right\} \leq \frac{1}{n} S^n (1 - \delta\rho)^n, \quad \forall \rho \in]0, \overline{\rho}]. \tag{6.8}$$

Proof Let $\overline{\rho} > 0$ be first such that the assertion of Lemma 6.1 holds and let $0 < \rho \leq \overline{\rho}$. Since X_-^ρ is a cone, it is easily seen that

$$\begin{aligned}
&\sup\left\{f\left(te_\rho + u\right) :\ t \geq 0,\ u \in X_-^\rho\right\}\\
&= \frac{1}{n}\left[\sup\left\{\frac{f_0(e_\rho + u) - \lambda\|e_\rho + u\|_1}{\|e_\rho + u\|_{1^*}} :\ u \in X_-^\rho\right\}\right]^n\\
&= \frac{1}{n}\left[\sup\left\{\frac{\left(f_0(e_\rho) - \lambda\|e_\rho\|_1\right) + \left(f_0(u) - \lambda\|u\|_1\right)}{\left(\|e_\rho\|_{1^*}^{1^*} + \|u\|_{1^*}^{1^*}\right)^{1/1^*}} :\ u \in X_-^\rho\right\}\right]^n,
\end{aligned}$$

as e_ρ and u have disjoint supports. Writing $u = v_\rho$ with $v \in X_-$, the assertion we need to prove takes the form

$$\sup\left\{\frac{\left(f_0(e_\rho) - \lambda\|e_\rho\|_1\right) + \left(f_0(v_\rho) - \lambda\|v_\rho\|_1\right)}{\left(\|e_\rho\|_{1^*}^{1^*} + \|v_\rho\|_{1^*}^{1^*}\right)^{1/1^*}} :\ v \in X_-\right\} \leq S(1 - \delta\rho).$$

If we set $\sigma = n^{n-1}\mathscr{L}^n(\mathrm{B}_1(0))$, by (6.1), (6.2), Lemma 6.1 and the fact that $\lambda_m \leq \lambda$, we have

$$\begin{aligned}
&\frac{\left(f_0(e_\rho) - \lambda\|e_\rho\|_1\right) + \left(f_0(v_\rho) - \lambda\|v_\rho\|_1\right)}{\left(\|e_\rho\|_{1^*}^{1^*} + \|v_\rho\|_{1^*}^{1^*}\right)^{1/1^*}}\\
&\qquad \leq \frac{(S^n - \sigma\rho) + \left(C\rho^{n-1}\|v\|_{1^*} + \lambda C\rho^n\|v\|_{1^*}\right)}{\left(S^n + \|v\|_{1^*}^{1^*} - C\rho^n\|v\|_{1^*}^{1^*}\right)^{1/1^*}}.
\end{aligned}$$

Now, arguing by contradiction, let $\delta = 1/k$, let $\rho_k \to 0^+$ and let $v^{(k)} \in X_-$ be such that

$$\frac{\left(f_0(e_{\rho_k}) - \lambda\|e_{\rho_k}\|_1\right) + \left(f_0(v_{\rho_k}^{(k)}) - \lambda\|v_{\rho_k}^{(k)}\|_1\right)}{\left(\|e_{\rho_k}\|_{1^*}^{1^*} + \|v_{\rho_k}^{(k)}\|_{1^*}^{1^*}\right)^{1/1^*}} > S\left(1 - \frac{\rho_k}{k}\right).$$

It follows

$$\frac{(S^n - \sigma\rho_k) + \left(C\rho_k^{n-1}\|v_k\|_{1^*} + \lambda C\rho_k^n\|v_k\|_{1^*}\right)}{\left(S^n + \|v_k\|_{1^*}^{1^*} - C\rho_k^n\|v_k\|_{1^*}^{1^*}\right)^{1/1^*}} > S\left(1 - \frac{\rho_k}{k}\right).$$

Up to subsequences, it is enough to consider the three cases:

(i) $\|v_k\|_{1^*} \to +\infty$,
(ii) $\|v_k\|_{1^*} \to \ell \in]0, +\infty[$,
(iii) $\|v_k\|_{1^*} \to 0$.

In case (i) we get

$$\frac{(S^n - \sigma\rho_k) + \left(C\rho_k^{n-1}\|v_k\|_{1^*} + \lambda C\rho_k^n\|v_k\|_{1^*}\right)}{\left(S^n + \|v_k\|_{1^*}^{1^*} - C\rho_k^n\|v_k\|_{1^*}^{1^*}\right)^{1/1^*}} \to 0$$

while in case (ii) we obtain

$$\frac{(S^n - \sigma\rho_k) + \left(C\rho_k^{n-1}\|v_k\|_{1^*} + \lambda C\rho_k^n\|v_k\|_{1^*}\right)}{\left(S^n + \|v_k\|_{1^*}^{1^*} - C\rho_k^n\|v_k\|_{1^*}^{1^*}\right)^{1/1^*}} \to \frac{S^n}{\left(S^n + \ell^{1^*}\right)^{1/1^*}} < S.$$

In both cases, a contradiction follows. In case (iii) we have, eventually as $k \to \infty$,

$$\begin{aligned}
&\frac{(S^n - \sigma\rho_k) + \left(C\rho_k^{n-1}\|v_k\|_{1^*} + \lambda C\rho_k^n\|v_k\|_{1^*}\right)}{\left(S^n + \|v_k\|_{1^*}^{1^*} - C\rho_k^n\|v_k\|_{1^*}^{1^*}\right)^{1/1^*}} \\
&\quad \le \frac{(S^n - \sigma\rho_k) + \left(C\rho_k^{n-1}\|v_k\|_{1^*} + \lambda C\rho_k^n\|v_k\|_{1^*}\right)}{S^{n-1}} \\
&\quad = S - S^{1-n}\rho_k\left(\sigma - C\rho_k^{n-2}\|v_k\|_{1^*} - \lambda C\rho_k^{n-1}\|v_k\|_{1^*}\right).
\end{aligned}$$

Then a contradiction follows also in this case. □

Proof of Theorem 1.1 Let $\lambda \ge \lambda_1$, let $m \ge 1$ be such that $\lambda_m \le \lambda < \lambda_{m+1}$ and let $\lambda < \mu < \lambda_{m+1}$. Let X_-, X_+ be as in Theorem 4.3 and let $\overline{\rho} > 0$ be small enough to guarantee that the assertions of Lemmata 6.1 and 6.2 hold.

Since $\lambda < \mu$, for every $u \in X_+$ we have

$$f(u) \ge \left(1 - \frac{\lambda}{\mu}\right) S\,\|u\|_{1^*} - \frac{1}{1^*}\,\|u\|_{1^*}^{1^*}.$$

Therefore, there exist $r_+, \alpha > 0$ such that $f(u) \ge \alpha$ for every $u \in X_+$ with $\|u\|_{1^*} = r_+$. On the other hand, since $\lambda \ge \lambda_m$, by Lemma 6.1 we also have, for every $v \in X_-$,

$$f(v_\rho) \le C\rho^{n-1}\|v\|_{1^*} + \lambda C\rho^n\|v\|_{1^*} - \frac{1}{1^*}\,\|v\|_{1^*}^{1^*} + \frac{C}{1^*}\,\rho^n\|v\|_{1^*}^{1^*} \le \frac{\alpha}{2} - \frac{1}{2\cdot 1^*}\,\|v\|_{1^*}^{1^*},$$

provided that $\rho > 0$ is small enough. Combining this fact with Lemmata 6.1 and 6.2, we see that there exists $\rho > 0$ such that $e_\rho \notin X_-^\rho$, X_-^ρ is closed in $L^1(\Omega)$ and

$$\begin{aligned}
&X_-^\rho \cap X_+ = \{0\}, \qquad \mathrm{Index}\left(X_-^\rho\setminus\{0\}\right) = \mathrm{Index}\left(L^{1^*}(\Omega)\setminus X_+\right) = m, \\
&\sup\left\{f\left(te_\rho + u\right) :\ t \ge 0,\ u \in X_-^\rho\right\} < \frac{1}{n}\,S^n, \\
&\sup\left\{f(u) :\ u \in X_-^\rho\right\} \le \frac{\alpha}{2}.
\end{aligned}$$

Since X_-^ρ is closed in $L^1(\Omega)$, hence in $L^{1^*}(\Omega)$, there exists $b > 0$ such that

$$\|te_\rho\|_{1^*} + \|u\|_{1^*} \le b\|te_\rho + u\|_{1^*} \qquad \text{for every } t \in \mathbb{R} \text{ and } u \in X_-^\rho$$

(see also [12]). Consequently, there exists $b' > 0$ such that

$$f_0(u) \le b'\|u\|_{1^*} \quad \text{for every } u \in \mathbb{R}e_\rho + X_-^\rho,$$

whence

$$f(u) \to -\infty \qquad \text{whenever } \|u\|_{1^*} \to \infty \text{ with } u \in \mathbb{R}e_\rho + X_-^\rho.$$

In particular, there exists $r_- > r_+$ such that $f(u) \leq 0$ whenever $u \in \mathbb{R}e_\rho + X_-^\rho$ with $\|u\|_{1^*} = r_-$.

From Theorems 2.4 and Theorem 5.3 we deduce that f admits a critical value c with $0 < c < \frac{1}{n} S^n$. By Corollary 3.5, there exists a solution $u \in BV(\Omega) \cap L^\infty(\Omega)$ of (1.6) with

$$0 < f(u) < \frac{1}{n} S^n.$$

Of course, u is a nontrivial solution. □

References

1. Ambrosetti, A., Rabinowitz, P.H.: Dual variational methods in critical point theory and applications. J. Funct. Anal. **14**, 349–381 (1973)
2. Ambrosio, L., Fusco, N., Pallara, D.: Functions of Bounded Variation and Free Discontinuity Problems. Oxford University Press, New York (2000)
3. Arioli, G., Gazzola, F.: Some results on p-Laplace equations with a critical growth term. Differ. Int. Equ. **11**, 311–326 (1998)
4. Aubin, T.: Problèmes isopérimétriques et espaces de Sobolev. J. Differ. Geom. **11**, 573–598 (1976)
5. Brezis, H., Nirenberg, L.: Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. Comm. Pure Appl. Math. **36**, 437–477 (1983)
6. Campa, I., Degiovanni, M.: Subdifferential calculus and nonsmooth critical point theory. SIAM J. Optim. **10**, 1020–1048 (2000)
7. Capozzi, A., Fortunato, D., Palmieri, G.: An existence result for nonlinear elliptic problems involving critical Sobolev exponent. Ann. Inst. H. Poincaré Anal. Non Linéaire **2**, 463–470 (1985)
8. Chang, K.-C.: Variational methods for nondifferentiable functionals and their applications to partial differential equations. J. Math. Anal. Appl. **80**, 102–129 (1981)
9. Chang, K.-C.: Infinite-dimensional Morse Theory and Multiple Solution Problems. Birkhäuser, Boston (1993)
10. Clarke, F.H.: Optimization and Nonsmooth Analysis. Wiley, New York (1983)
11. Corvellec, J.-N., Degiovanni, M., Marzocchi, M.: Deformation properties for continuous functionals and critical point theory. Topol. Methods Nonlinear Anal. **1**, 151–171 (1993)
12. Degiovanni, M., Lancelotti, S.: Linking over cones and nontrivial solutions for p-Laplace equations with p-superlinear nonlinearity. Ann. Inst. H. Poincaré Anal. Non Linéaire **24**, 907–919 (2007)
13. Degiovanni, M., Lancelotti, S.: Linking solutions for p-Laplace equations with nonlinearity at critical growth. J. Funct. Anal. **256**, 3643–3659 (2009)
14. Degiovanni, M., Marzocchi, M.: A critical point theory for nonsmooth functionals. Ann. Mat. Pura Appl. **167**, 73–100 (1994)
15. Degiovanni, M., Marzocchi, M., Rădulescu, V.D.: Multiple solutions of hemivariational inequalities with area-type term. Calc. Var. Partial Differ. Equ. **10**, 355–387 (2000)
16. Degiovanni, M., Schuricht, F.: Buckling of nonlinearly elastic rods in the presence of obstacles treated by nonsmooth critical point theory. Math. Ann. **311**, 675–728 (1998)
17. Demengel, F.: On some nonlinear partial differential equations involving the “1”-Laplacian and critical Sobolev exponent. ESAIM Control Optim. Calc. Var. **4**, 667–686 (1999)
18. Demengel, F.: Some existence’s results for noncoercive “1-Laplacian” operator. Asymptot. Anal. **43**, 287–322 (2005)
19. Egnell, H.: Existence and nonexistence results for m-Laplace equations involving critical Sobolev exponents. Arch. Ration. Mech. Anal. **104**, 57–77 (1988)
20. Fadell, E.R., Rabinowitz, P.H.: Bifurcation for odd potential operators and an alternative topological index. J. Funct. Anal. **26**, 48–67 (1977)
21. Fadell, E.R., Rabinowitz, P.H.: Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems. Invent. Math. **45**, 139–174 (1978)
22. García Azorero, J., Peral Alonso, I.: Existence and nonuniqueness for the p-Laplacian: nonlinear eigenvalues. Comm. Partial Differ. Equ. **12**, 1389–1430 (1987)

23. Gazzola, F., Ruf, B.: Lower-order perturbations of critical growth nonlinearities in semilinear elliptic equations. Adv. Differ. Equ. **2**, 555–572 (1997)
24. Giusti, E.: Minimal Surfaces and Functions of Bounded Variation. Birkhäuser Verlag, Basel (1984)
25. Guedda, M., Véron, L.: Quasilinear elliptic equations involving critical Sobolev exponents. Nonlinear Anal. **13**, 879–902 (1989)
26. Ioffe, A., Schwartzman, E.: Metric critical point theory. I. Morse regularity and homotopic stability of a minimum. J. Math. Pures Appl. **75**, 125–153 (1996)
27. Katriel, G.: Mountain pass theorems and global homeomorphism theorems. Ann. Inst. H. Poincaré Anal. Non Linéaire **11**, 189–209 (1994)
28. Kawohl, B., Schuricht, F.: Dirichlet problems for the 1-Laplace operator, including the eigenvalue problem. Commun. Contemp. Math. **9**, 515–543 (2007)
29. Marzocchi, M.: Multiple solutions of quasilinear equations involving an area-type term. J. Math. Anal. Appl. **196**, 1093–1104 (1995)
30. Milbers, Z., Schuricht, F.: Existence of a sequence of eigensolutions for the 1-Laplace operator. Technische Universität Dresden. MATH-AN-04-2008 (2008. preprint)
31. Rabinowitz, P.H.: Minimax Methods in Critical Point Theory with Applications to Differential Equations. American Mathematical Society, Providence (1986)
32. Rockafellar, R.T.: Generalized directional derivatives and subgradients of nonconvex functions. Can. J. Math. **32**, 257–280 (1980)