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LIOUVILLE PROPERTY FOR GROUPS AND MANIFOLDS

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ABSTRACT. We introduce a new criterion for non-triviality of the Poisson boundary. Using this criterion we prove the existence of compact manifolds with amenable fundamental group such that the universal cover is not Liouville. This gives a positive answer to a question from [9]. We prove that a finitely generated solvable group admits a symmetric measure with non-trivial Poisson boundary if and only if this group is not virtually nilpotent. This gives a partial answer to a conjecture of Vershik and Kaimanovich [13]. Also we give a series of examples of amenable groups such that any non-degenerate measure has non-zero entropy of the random walk. In other words, any non-degenerate measure with trivial Poisson boundary on these groups has infinite entropy.

1. INTRODUCTION

In this paper we study the Poisson boundary for Riemannian manifolds and for random walks on groups. Consider a finitely generated group G and let μ be a probability measure on this group. The measure μ defines a random walk on G with transition probability $p(x|y) = \mu(yx^{-1})$, starting at the identity. We say that the random walk is non-degenerate if μ generates G as a semigroup.

The space G^n of trajectories of length n is equipped with the measure which is the image of the product measure by the map

$$(x_1, x_2, x_3, ..., x_n) \to (x_1, x_1x_2, x_1x_2x_3..., x_1x_2...x_n)$$

and the space of infinite trajectories G^∞ carries the measure which is the image of the infinite product measure by the map

$$(x_1, x_2, x_3...) \to (x_1, x_1x_2, x_1x_2x_3...).$$

Definition. Let A_n^∞ be the σ-algebra of measurable subsets of the trajectory space G^∞ that are determined by the coordinates $y_n, y_{n+1}, ...$ of the trajectory y. The intersection $A_\infty = \cap_n A_n^\infty$ is called exit σ-algebra of the random walk. The corresponding G-space with measure is called *exit boundary* of the random walk.

Another way to describe the boundary is through harmonic functions. A real-valued function f on the group G is called μ-harmonic if $f(g) = \sum_x f(gx)\mu(x)$ for any $g \in G$.

Definition. *Poisson boundary.* Consider the space H_μ^∞ of all μ-harmonic bounded function of G. This space is a commutative Banach algebra with respect

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to the multiplication

$$(f_1 \times f_2)(g) = \lim_n \sum_x f_1(gx) f_2(gx) \mu^{*n}(x).$$

The spectrum of this Banach algebra is called Poisson boundary of the random walk. This space is endowed with a measure which is defined as follows. Let $f \to \hat{f}$ be the Gelfand transform ($f \in H_\mu^\infty$). The measure ν on the Poisson boundary is defined by the equality

$$\int \hat{f}(x)\, d\nu(x) = f(e).$$

As a G-space with measure the Poisson boundary is isomorphic to the exit boundary. Note that the boundary of the group is non-trivial if and only if there exists a non-constant bounded harmonic function of (G, μ).

Now let N be a Riemannian manifold and Δ be the Laplace operator corresponding to the chosen Riemannian structure.

Definition. We say that N has Liouville property if and only if any bounded harmonic function on N is constant.

Basic examples of manifolds with Liouville property are compact manifolds and $\mathbb{R}^n$.

It is known that if μ is a non-degenerate measure on a non-amenable group G then the boundary of (G, μ) is non-trivial. It is also known that if N is a cover of a compact manifold such that the corresponding group is non-amenable, then N is not Liouville.

On the other hand, if G is amenable, than there exists a symmetric non-degenerate measure μ of G with trivial Poisson boundary. This was a conjecture of Furstenberg that was proved by A.M.Vershik and V.A.Kaimanovich in [12], [13] and independently by J.Rosenblatt in [18].

However some amenable groups admit symmetric measures μ with non-trivial boundary. Moreover, for some amenable groups the measure μ can be taken having finite support [13]. The existence of symmetric measures with non-trivial boundaries on amenable groups is a discrete phenomena: note that any symmetric measure absolutely continuous with respect to the Haar measure on a connected amenable Lie group has trivial boundary [4], (see [7] for the definition of the Poisson boundary in the case of Lie groups.).

We will be also interested in Liouville property for universal covers $\tilde{M}$ of compact Riemannian manifolds M. It was already mentioned that if $\pi_1(M)$ is non-amenable, then $\tilde{M}$ is not Liouville. On the other hand, if $\pi_1(M)$ is finite, Abelian, policyclic and in many other cases of amenable $\pi_1(M)$ it is known that $\tilde{M}$ has Liouville property (see [9]). In all previously known examples of Riemannian manifolds N or random walks on amenable groups G non-triviality of the Poisson boundary (absence of the Liouville property) can be checked by finding a non-trivial tail set or by finding a non-constant bounded harmonic function on N or G (see for example [13] and [15]).

In this paper we introduce a new method for proving that the entropy is positive. Using this method together with the entropic criterion for non-triviality of the boundary from [13] and [5] we prove non-triviality of the Poisson boundary for random walks on groups and for compact Riemannian covers.

The paper has following structure. In Section 2 we introduce a method to estimate the entropy of random walk. In Section 3 we apply this method to certain wreath products and free metabelian groups. In particular, we show that any non-degenerate measure on $\mathbb{Z}^d \wr B$ and Met_d ($d \geq 3$) has non-zero entropy. In Section 4 we apply the method to solvable groups. In particular, we show that any solvable group of exponential growth admits a symmetric measure with non-trivial Poisson boundary. In Section 5 we construct first examples of finitely presented amenable groups such that random walks on them have non-trivial Poisson boundary. Then we use these examples to construct universal covers that are not Liouville. In Section 6 we discuss continuity of entropy and list several open questions about boundaries of random walks and Liouville properties of manifolds.

2. Idea of the method.

Definition. Let X be a countable space with a discrete probability measure ν. The entropy is defined as $H(\nu) = -\sum_x \nu(x)\ln(\nu(x))$.

Definition. Let X be a countable space with probability measure ν and η be a countable measurable partition of X. Let $C_1, C_2, ..$ be elements of the partition η that have positive measure. The entropy $H(\nu,\eta) = -\sum_k \nu(C_k)\ln(\nu(C_k))$.

Definition. Entropy of the random walk (G,μ) is the limit

$$h() = \lim_{n\to\infty} H(\mu^{*n})/n.$$

Entropy of the random walk was introduced by A.Avez in [1]. Later, it was proved [13], [5] that for measures with finite entropy ($H(\mu) < \infty$) the entropy of the random walk $h(\mu)$ is positive if and only if the Poisson boundary of the random walk is non-trivial. In the case of infinite entropy ($H(\mu) = \infty$) the entropy of the random walk does not carry any information about Poisson boundary. In fact, in this case $h(\mu) = \infty$, but the boundary can be both trivial and non-trivial.

In this section we introduce a method to estimate the entropy of the random walk.

Take (G,μ) and consider the corresponding random walk. Let G^n be the space of trajectories of length n. Choose $a \neq b$ such that $a, b \in \mathrm{supp}\mu \subset$ G. For $0 \leq i_1 < i_2 < ... < i_k \leq n$ choose a set of $(n-k)$ elements of $w = (x_1, x_2, ...\hat{x}_{i_1}, ..., \hat{x}_{i_j}, ..., x_n)$ of G. Here and it what follows a hat over an element indicates its absence.

Denote by $T^{a,b}(w, i_1, ..., i_k)$ the set of trajectories $y = (y_1, y_2, ..., y_n)$ of length n that have the following property. For each $s \neq i_j$ it holds $y_s = y_{s-1}x_s$ and for $s = i_j$ one has $y_s = y_{s-1}a$ or $y_s = y_{s-1}b$. Note that

$$\mu^n(T^{a,b}(w, i_1, ..., i_k)) = \prod_{s\neq i_j} \mu(x_s)(\mu(a)+\mu(b))^k.$$

In the sequel, we assume that $\mu(x_s) \neq 0$ for $s \neq i_j$. Then

$$\mu^n(T^{a,b}(w, i_1, ..., i_k)) \neq 0$$

and $T^{a,b}(w, i_1, ..., i_k)$ consists of 2^k trajectories of length n.

Definition. We say that a subset T of the space of trajectories of length n is *satisfactory* if all the trajectories in T hit different elements of the group G at the moment n .

Theorem 1. *Suppose that there exist $p, c > 0$ such that for each n there exists a set A_n of collections $a = (w, i_1, .., i_k)$ with the following properties.*

1). For each $a = (w, i_1, .., i_k) \in A_n$ one has $k \geq cn$.

2). For each $a = (w, i_1, .., i_k) \in A_n$ the set $T^{a,b}(a)$ is satisfactory.

3). For each $a_1 \neq a_2 \in A_n$ *one has*

$$T^{a,b}(a_1) \cap T^{a,b}(a_2) = \emptyset$$

.

4) For each n *it holds*

$$\mu^n(\bigsqcup_{a \in A_n} T^{a,b}) \geq p.$$

Then the entropy $h(\mu) \neq 0$. *In particular, if* $H(\mu) < \infty$, *then* μ *has non-trivial Poisson boundary.*

Proof. First note that $h(\mu^{*n}) = H(\mu^n, \eta_n)$, where η_n is the partition of the space of trajectories of length n with the following property. Two trajectories belong to the same element of the partition η_n if and only if they hit the same element of the group G at the moment n.

Let $\nu(a,b)$ be the probability measure concentrated on two points which have measures $\mu(a)/(\mu(a)+\mu(b))$ and $\mu(b)/(\mu(a)+\mu(b))$.

For each $a = (w, i_1, i_2, ..., i_k) \in A_n$ consider the conditional event $\tilde{a}$ that the trajectory of length n belongs to $T^{a,b}(a)$. The second condition of the theorem implies that the restriction of μ^n to this conditional event is isomorphic to the product measure $\nu(a,b)^k$.

Consequently, the conditional entropy $H(\mu^n, \eta_n|\tilde{a})$ is equal to

$$H(\nu(a,b)^k) = kH(\nu(a,b)) \geq H(\nu(a,b))cn \qquad (\star).$$

A well-known property of the conditional entropy ([17]) states that the mean conditional entropy is not greater than the entropy. Hence,

$$H(\mu^n, \eta_n) \geq \sum_a H(\mu^n, \eta_n|\tilde{a})\mu^n(a) + H(\mu^n, \eta_n|b)\mu^n(b).$$

Here b is the event that the trajectory does not belong to any $a \in A_n$ and the sum is taken over all collections $a \in A_n$.

Combining this with $(\star)$ and with the condition 4). of the theorem, we obtain that

$$h(\mu^{*n}) = H(\mu^n, \eta_n) \geq pcnH(a,b).$$

Therefore,

$$h(\mu) \geq pcH(\nu(a,b)) = pc(\ln(\nu(a,b)(a))\nu(a,b)(a) + \ln(\nu(a,b)(b))\nu(a,b)(b) > 0.$$

This completes the proof of the theorem.

3. Wreath products and free metabelian groups.

Definition. The wreath product of A and B is a semidirect product of A and $\sum_A B$, where A acts on $\sum_A B$ by shifts: if $a \in A$ and $f : A \to B$, $f \in \sum_A B$, then ${}^a f(x) = f(xa^{-1})$, $x \in A$. Let $A \wr B$ denotes the wreath product.

Note that if A and B are finitely generated, then the wreath product $A \wr B$ is also finitely generated.

Let Met_d denote the free metabelian group on d generators.

In the following remark we describe a normal form for the free metabelian group. For more on the normal form for Met_d see [21].

Remark. Let $E(\mathbb{Z}^d)$ be the set of edges of $\mathbb{Z}^d$. To any g in Met^d we can assign a pair (z, f), where $z \in \mathbb{Z}^d$ and $f : E(\mathbb{Z}^d) \to \mathbb{Z}$ in the following way. Let $g_1, ..., g_d$ be the canonical set of generators of Met_d and $o_1, ..., o_d$ be their projections on

$\mathbb{Z}^d$. Clearly, $o_1, ..., o_d$ generate $\mathbb{Z}^d$. Choose an orientation on the edges $E(\mathbb{Z}^d)$ of $\mathbb{Z}^d$ in such a way that an edge z_1z_2 is oriented from z_1 to z_2 if $z_2 = z_1 + o_j$ for some $1 \leq j \leq d$. Consider $g \in Met_d$ and take any word $w(g_1, g_2, .., g_d)$ representing g. This word defines a path in $E(\mathbb{Z}^d)$. For each $e_\alpha = z_1z_2 \in E(\mathbb{Z}^d)$ let $N^+(e_\alpha)$ be the number of times such that this path passes through e_α going from z_1 to z_2 and N^- be the number of times such that this path passes through e_α in the opposite direction. We put $f(e_\alpha) = N^+(e_\alpha) - N^-(e_\alpha)$ and let z be the projection of g onto $\mathbb{Z}^d$. We must prove that (z, f) does not depend on the choice of the word $w(g_1, g_2, .., g_d)$ representing g.

Proof. By construction, z depends only on g. So we have to prove that f depends only on g. Take two different words w_1 and w_2 representing g. That is, $g = w_1(g_1, g_2, ..., g_d) = w_2(g_1, g_2, ..., g_d)$ in Met_d. Let ρ_1 and ρ_2 be the paths in $E(\mathbb{Z}^d)$ corresponding to w_1 and w_2 respectively. Note that both ρ_1 and ρ_2 start with 0 and end with z. We want to prove that for each $e_\alpha \in E(\mathbb{Z}^d)$ the number of oriented passes of ρ_1 through e_α is the same as this number for ρ_2. Consider the path $\rho_1\rho_2^{-1}$. It is sufficient to prove that this path crosses e_α in the positive direction the same number of times as in the opposite direction. This path corresponds to the word $w_1w_2^{-1}$ and $w_1w_2^{-1}$ represents the identity element e in Met_d. Hence in the free group on d generators it holds

$$w_1w_2^{-1} = \prod_j x_j r_j x_j^{-1},$$

where r_j are defining relators of Met_d, that is $r_j = [[t_j, u_j], [v_j, w_j]]$ for some t_j, u_j, v_j, w_j. Let ρ_j^1 be the path corresponding to $[t_j, u_j]$ and ρ_j^2 be the path corresponding to $[v_j, w_j]$. Note that both ρ_j^1 and ρ_j^2 are loops starting and ending in the identity element 0 of $\mathbb{Z}^d$. Hence $\rho_j^1\rho_j^2(\rho_j^1)^{-1}(\rho_j^2)^{-1}$ passes through each e_α the same number of times in the positive and in the negative direction. And this implies the same about $\rho_1\rho_2^{-1}$.

Note that each pair (z, f) is assigned to at most one element of Met_d.

Theorem 2. *1). Let $G = A \wr B$ and $\#B \neq 1$. Let μ be a non-degenerate measure on G such that its projection on A is transient. Then $h(\mu) \neq 0$.*

2). In particular, if A is infinite and does not contain $\mathbb{Z}$ or $\mathbb{Z}^2$ as a subgroup of finite index, then any non-degenerate random walk on $G = A \wr B$ has non-zero entropy.

3).For $d \geq 3$ any non-degenerate random walk on Met_d has non-zero entropy.

Remark. Under the additional assumption that the first moment of μ is finite the first statement of the theorem was already known for $\mathbb{Z}^d \wr B$ $(d \geq 3)$ (see [13] for the case of finite support and [10] for the case of the finite first moment.) The existence of some measure with non-trivial boundary on Met_d $(d \geq 3)$ can be deduced from a result of T.Lyons and D.Sullivan [15] and non-triviality of the boundary for any measure with a finite support on Met_d $(d \geq 3)$ was proved by A.M.Vershik in [21].

Corollary. Let $d \geq 3$. Then any non-degenerate measure on $G = \mathbb{Z}^d \wr B$ or on Met_d with trivial Poisson boundary has infinite entropy.

Remark. However symmetric non-degenerate measures with trivial Poisson boundary do exist on each of these groups, since they are amenable ([13], [18]).

Proof of the theorem. Let $b = (e, f)$ be such that $f(e_A) \neq e_B$ and $f(x) = e_B$ for any $x \neq e_A$. Put $a = e$. It is sufficient to consider the case when a and b belong to the support of μ. In fact, since μ is non-degenerate, there exists N such that a and b belong to supp$\mu^{*\mathrm{N}}$ and also note that $h(\mu^{*N}) = Nh(\mu)$.

Let us introduce a a set of collections for the space of trajectories of length n in the following way. Consider a collection $a = (i_1, i_2, ..., i_k, x_1, x_2, ..., \hat{x}_{i_1}, ..., \hat{x}_{i_j}, ..., x_n)$, where $0 \le i_1 < i_2 < ... < i_k \le n$ and $x_s \in$ suppμ for any $s \ne i_j$. We say that the collection a is good if the two following conditions hold.

1). All the elements x_1, x_1x_2,..., $x_1x_2...\hat{x}_{i_1}$, $x_1x_2...\hat{x}_{i_1}x_{i_1+1}$, $x_1x_2...\hat{x}_{i_j}$, ...x_n have different projections on A.

2). If for some s it holds $x_s = a$ or $x_s = b$ then there exists j such that $i_j \ge s$ and $x_1x_2...\hat{x}_{i_j}$ has the same projection on A as $x_1x_2...x_s$.

Lemma 3.1. *For any different good collections a_1 and a_2 one has*

$$T^{a,b}(a_1) \cap T^{a,b}(a_2) = \emptyset$$

and for each n the union $\bigsqcup_a T^{a,b}$ is the space of all the trajectories of length n (here the union is taken over all good collections a.)

Proof. In fact, take any trajectory $y = \bar{x}_1, \bar{x}_1\bar{x}_2, ..., \bar{x}_1\bar{x}_2, ...\bar{x}_n$ of length n. For any point $z \in A$ look whether multiplications on a or b have ever occurred at this point and consider the last moment i_z of such multiplications. Let $z_1, z_2, .., z_m$ be different points of A in which there were multiplication by a or b. Put $a = (i_{z_1}, i_{z_2}, ..., i_{z_m}, x_1, x_2, ..., \hat{x}_{i_{z_j}} ..., x_n)$, where $x_s = \bar{x}_s$ for any $s \ne i_{z_j}$. Then the trajectory y belongs to $T^{a,b}(a)$ and does not belong to any other $T^{a,b}(a')$.

We say that a collection $a = (i_1, i_2, ..., i_k, x_1, x_2, ..., \hat{x}_{i_1}, ..., \hat{x}_{i_j}, ..., x_n)$ is c-good, if it is good and $k \ge cn$.

Lemma 3.2. *There exist $c, p > 0$ such that for any n*

$$\mu^n(\bigsqcup_a T^{a,b}) > p,$$

where the union is taken over all c-good collections a.

Proof. Recall that a range of the random walk is the number of different elements visited up to the moment n. Let $R(n)$ be the range of the projection of the random walk on A. Since this projection is transient,

$$R(n)/n \to K > 0. \qquad (*)$$

with probability one (see, e.g., [5])

Note also that $K < 1$.

Let $M^{a,b}(n)$ be the number of multiplications by a or b up to the moment n and let $\tilde{M}^{a,b}(n)$ be the number of different points of A where such multiplications have occurred. Obviously, for some $K_2 > 0$

$$M^{a,b}(n)/n \to K_2$$

with probability one. Combining this with $(*)$ we get that there exists $c, p > 0$ such that for any n

$$\Pr(\tilde{M}^{a,b}(n) \ge cn) \ge p,$$

and this implies the statement of the lemma.

Lemma 3.3. *If a is a good collection, then $T^{a,b}$ is a satisfactory subset of the space of trajectories of length n.*

Proof. Let $a = (i_1, i_2, ..., i_k, x_1, x_2, ..., \hat{x}_{i_1}, ..., \hat{x}_{i_j}, ..., x_n)$. Consider two different trajectories from $T^{a,b}(a)$

$$y = (x_1, x_1x_2, ..., x_1x_2...x_{i_1-1}g_1, ..., x_1x_2...x_{i_j-1}g_jx_{i_j+1}...g_k...x_n)$$

and

$$y' = (x_1, x_1x_2, ..., x_1x_2...x_{i_1-1}g'_1, ..., x_1x_2...x_{i_j-1}g'_jx_{i_j+1}...g'_k...x_n),$$

where for each j it holds $g_j = a$ or $g_j = b$ and $g'_j = a$ or $g'_j = b$.

Let

$$x_1x_2...x_{i_j-1}g_jx_{i_j+1}...g_k...x_n = (c, f)$$

and

$$x_1x_2...x_{i_j-1}g'_jx_{i_j+1}...g'_k...x_n = (c', f').$$

Since the trajectories are different, there exists j such that $g_j \neq g'_j$. That is, $g_j = a = e$ and $g'_j = b$ or vice versa. Let z be the projection of $x_1x_2...\hat{x}_j$ onto A. Note that $f(z) \neq f(z')$. Hence, $(c, f) \neq (c', f')$ and this completes the proof of the lemma.

Lemma 2 and Lemma 3 ensure that we can apply Theorem 1 to c-good collections and get that $h(\mu) \neq 0$. Hence, we have proved the first statement of the theorem. The second statement follows from the first one, since if an infinite group admits a non-degenerate recurrent random walk, then this group is contains $\mathbb{Z}$ or $\mathbb{Z}^2$ as a subgroup of finite index [20].

It remains to prove the third statement. Let $g_1, g_2, .., g_d$ be the canonical generators of Met_d. Put $a = e$ and $b = g_1g_2g_1^{-1}g_2^{-1}$. As before, without loss of generality we can assume that a and b belong to suppμ.

Let $(0, f)$, $f : \mathbb{Z}^d \to \mathbb{Z}$ be the pair that is assigned to b. Note that

$$f(e_\alpha) = 1, \ f(e_\beta) = -1, \ f(e_\gamma) = -1, \ f(e_\delta) = 1,$$

where e_α is the edge joining 0 to o_1, e_β is the edge joining o_2 to $o_2 + o_1$, e_γ is the edge joining 0 to o_2, and e_δ is the edge joining o_1 to $o_1 + o_2$.

Consider a collection $a = (i_1, i_2, ..., i_k, x_1, x_2, ..., \hat{x}_{i_1}, ..., \hat{x}_{i_j}, ..., x_n)$, where $0 \leq i_1 < i_2 < ... < i_k \leq n$ and $x_s \in$ suppμ for any $s \neq i_j$.

Similarly to the proof of the first part of the theorem we say that this collection is good if the two following conditions hold.

1). All the elements x_1, x_1x_2,..., $x_1x_2...\hat{x}_{i_1}$, $x_1x_2...\hat{x}_{i_1}x_{i_1+1}$, $x_1x_2...\hat{x}_{i_j}$, $...x_n$ have different even projections on $\mathbb{Z}^d$.

2). If for some s it holds $x_s = a$ or $x_s = b$ and $x_1x_2...\hat{x}_{i_j}$ has even projection on $\mathbb{Z}^d$, then there exists j such that $i_j \geq s$ and $x_1x_2...\hat{x}_{i_j}$ has the same projection on $\mathbb{Z}^d$ as $x_1x_2...x_s$.

(Apart from the proof of the first part of the theorem we fix last multiplications on a and b not in all points of $\mathbb{Z}^d$ but only in all even points of $\mathbb{Z}^d$. We say that a point of $\mathbb{Z}^d$ is even, if all its coordinates are even.)

Similarly to the first part of the theorem , we say that a is c-good if it is good and $k \geq cn$. With the same arguments as in 1). we can check that the statements, which are analogous to Lemma 1, Lemma 2 and Lemma 3, are valid for good collections and c-good collections. Hence, we can apply Theorem 1 to c-good collections and see that $h(\mu) \neq 0$. This completes the proof of the theorem.

4. Solvable groups.

Recall that any finite set of generators S of G gives rise to a word length l_S, and growth function of the group G is

$$v_{G,S}(n) = \#\{g \in G : l_S(g) \leq n\}.$$

Theorem 3. *Let G be a solvable group. Then G admits a symmetric measure μ with non-trivial Poisson boundary if and only if G is not virtually nilpotent. Moreover, if G is not virtually nilpotent, then this measure μ can be chosen having finite entropy $H(\mu)$.*

Note that we do not claim in this theorem that the measure μ is non-degenerate. However it seem plausible that the measure μ can be taken being non-degenerate in this situation.

Proof. "Only if" part of the theorem is well-known. In fact, let $H \subset G$ be a nilpotent subgroup of finite index. Then any (symmetric) measure μ of G gives rise to a (symmetric) measure μ' on H. If the Poisson boundary of μ is non-trivial, then the boundary of μ' is also non-trivial. But any measure on a nilpotent group has trivial Poisson boundary [6].

So we must prove the "if" part. Since G is solvable and not virtually nilpotent, G has exponential growth ([16], [22]).

Notation. Let $M \in Mat(m, \mathbb{Q})$ be an invertible matrix, $m \geq 1$. Then M defines an automorphism of $\mathbb{Q}^m$. Take the corresponding extension $(M, \mathbb{Q}^m)$. Consider its subgroup generated by M and $\mathbb{Z}^m$. Denote this subgroup by $G(M)$. Let $\pi_1 : G(M) \to \mathbb{Z}$ be the homomorphism such that $\pi_1(M) = 1$ and $\pi_1(\mathbb{Z}^m) = 0$. Note that each element $g \in G(M)$ can be written in a unique way in the form $g = q_g M^{\pi_1(g)}$, where $q_g \in \mathbb{Q}^n$. This defines a map $s_2 : G(M) \to \mathbb{Q}^n$, such that $s_2(g) = q_g$,

In the sequel, $||M||$ denotes the absolute value of a largest eigenvalue of M.

Remark. 1). $G(M) \simeq G(M^{-1})$.

2). The group $G(M)$ has exponential growth if and only if $||M|| \neq 1$.

3). If $M \in SL(n, \mathbb{Z})$, then $G(M)$ is polycyclic.

Proposition 1. *Take an invertible matrix $M \in Mat(m, \mathbb{Q})$ such that $||M|| > 1$ and consider a non-degenerate probability measure on $G = G(M)$. Assume either that*

1. $\pi_1(\mu)$ is symmetric and transient

or that

2. $\pi_1(\mu)$ has finite support and that

$$0 < m_1(\pi_1(M)) = \sum_{i \in \mathbb{Z}} i\pi_1(i).$$

(Clearly, in the second case $\pi_1(M)$ is also transient.)

Then $h(\mu) \neq 0$.

Corollary. Take M such that $||M|| \neq 1$.

1). There exists a non-degenerate symmetric measure μ on $G(M)$ such that $H(\mu) < \infty$ and μ has non-trivial Poisson boundary.

2). There exists a non-degenerate finitely supported (not symmetric) measure μ on $G(M)$ such that $H(\mu) < \infty$ and μ has non-trivial Poisson boundary.

Proof of the corollary. For the proof of the first part it is sufficient to note that there exist symmetric transient measures μ on $\mathbb{Z}$ having finite entropy $H(\mu)$.

For example, any symmetric μ such that

$$\mu(N) \sim \frac{C}{|N|^{(1+\epsilon)}}$$

is transient for $0 < \epsilon < 1$ [19]. And the entropy

$$H(\mu) \sim \sum_N \frac{(1+\epsilon)\ln(|N|)}{|N|^{(1+\epsilon)}}$$

is obviously finite.

The proof of the second statement of the corollary is straightforward.

Proof of the proposition. Consider an eigenvalue λ of $M \otimes \mathbb{C}$ such that $|\lambda| = \|M\|$. It holds $|\lambda| > 1$, since $\|M\| > 1$. Consider a basis $v_1, .., v_d$ of $\mathbb{C}^m$ consisting of eigenvectors and generalised eigenvectors of $M \otimes \mathbb{C}$ with the following properties. First, for any real eigenvalue all the corresponding eigenvectors and generalised eigenvectors have real coefficient. Secondly, for any pair of conjugated complex eigenvalues all the corresponding eigenvectors and generalised eigenvectors are conjugated. Thirdly, v_1 is an eigenvector corresponding to λ and v_1 has non-zero projection onto $\mathbb{R}^m$.

Note that there exists a vector $(z_1, .., z_m) \in \mathbb{R}^m$ such that its coordinates $(\beta_1, \beta_2, ..., \beta_m)$ in the eigenbasis under consideration satisfy $|\beta_1| > |\beta_j|$ for each j such that v_j is a (generalised) eigenvector corresponding to λ.

Consequently, there exists a vector $(z_1, .., z_m) \in \mathbb{Z}^m$ such that its coordinates $(\alpha_1, \alpha_2, ..., \alpha_m)$ in the eigenbasis under consideration satisfy $|\alpha_1| > |\alpha_j|$ for each j such that v_j is a (generalised) eigenvector corresponding to λ.

Now we introduce a set of collections for the space of trajectories of length n. Put $a = e_G$ and $b = (z_1, ..., z_m) \in \mathbb{Z}^m \subset G(M) = G$, where $(z_1, ..., z_m)$ is the vector defined above. As before, we can assume that $a, b \subset \mathrm{supp}\mu$.

Choose $K \in \mathbb{N}$.

We say that a collection $a = (i_1, i_2, ..., i_k, x_1, x_2, ..., \hat{x}_{i_1}, ..., \hat{x}_{i_j}, ..., x_n)$ ($0 \leq i_1 < i_2 < ... < i_k \leq n$ and $x_s \in \mathrm{supp}\mu$ for any $s \neq i_j$) is good if the two following conditions hold.

1). All the elements x_1, x_1x_2,..., $x_1x_2...\hat{x}_{i_1}$, $x_1x_2...\hat{x}_{i_1}x_{i_1+1}$, $x_1x_2...\hat{x}_{i_j}$, ...x_n have different positive divisible by K projections on $\mathbb{Z}$.

2). If for some s it holds $x_s = a$ or $x_s = b$ and $x_1x_2...\hat{x}_{i_j}$ has positive divisible by K projection on $\mathbb{Z}$, then there exists j such that $i_j \geq s$ and $x_1x_2...\hat{x}_{i_j}$ has the same projection on $\mathbb{Z}$ as $x_1x_2...x_s$.

Analogously to Lemma 3.1 we have

Lemma 4.1. *For any different good collections a_1 and a_2 one has*

$$T^{a,b}(a_1) \cap T^{a,b}(a_2) = \emptyset$$

and for each n the union $\bigsqcup_a T^{a,b}$ is the space of all the trajectories of length n (here the union is taken over all good collections a.)

Proof. Take a trajectory of length n and look at positive divided by K points of $\mathbb{Z}$ where multiplication by a or b have ever occurred. For each such point look at the last moment of multiplication by a or b. Analogously to Lemma 1 this shows to which $T^{a,b}(a)$ the trajectory belongs.

As before, we say that a collection $a = (i_1, i_2, ..., i_k, x_1, x_2, ..., \hat{x}_{i_1}, ..., \hat{x}_{i_j}, ..., x_n)$ is c-good, if it is good and $k \geq cn$.

Lemma 4.2. *There exist $c, p > 0$ such that for any n*

$$\mu^n(\bigsqcup_a T^{a,b}) > p,$$

where the union is taken over all c-good collections a.

Proof. Let $R(n)$ be the range of the projection of the random walk on $\mathbb{Z}$, $R_+(n)$ ($R_-(n)$) be the number of positive (negative) points, visited by this projection up to the moment n, and $R_{+,K}(n)$ be the number of positive divisible by K points, visited up to the moment n.

Also let $R^{a,b}_{+,K}(n)$ be the number of different positive divisible by K points, in which multiplication by a or b have occurred up to the moment n. Since the projection is transient, we have

$$R(n)/n \to K_1 > 0.$$

Note that either the assumption 1 or the assumption 2 of the proposition imply that

$$\Pr(R_+(n) \geq R_-(n)) \geq \frac{1}{2}.$$

Hence, there exists $C_1, C_2 > 0$ such that

$$\Pr(R_+(n) \geq C_1 n) \geq C_2.$$

Therefore, for each positive integer K there exist $C_3, C_4 > 0$ such that

$$\Pr(R_{+,K}(n) \geq C_3 n) \geq C_4.$$

On the other hand, the number of multiplications by a or b is linear in n with probability close to one. Similarly to Lemma 3.2, this implies that

$$\Pr(R^{a,b}_{+,K}(n) \geq Cn) \geq p$$

for some positive C and p. This completes the proof of the lemma.

Lemma 4.3. *Take K such that $\|M\|^K > 5$. If a is a good collection for K, then $T^{a,b}$ is a satisfactory subset of the space of trajectories of length n.*

Proof. Put $A = M^K$. We have $\|A| = \|M\|^K > 5$. Take δ_i, δ_i' such that $\delta_i = 0$ or $\delta_i = 1$ and $\delta_i' = 0$ or $\delta_i' = 1$ for any $1 \leq i \leq N$. Assume that for some j such that $1 \leq j \leq N$ it holds $\delta_j' \neq \delta_j$. Then

1).

$$\sum_{i=1}^{N} \delta_i A^i \neq \sum_{i=1}^{N} \delta_i' A^i.$$

2). Let v be such that its coordinates in the chosen eigenbasis satisfy $|\alpha_1| > |\alpha_j|$ for any j such that v_j is a (generalised) eigenvector corresponding to λ. (Recall that v_1 is an eigenvector corresponding to one of the maximal eigenvalues of M and hence of A). Then

$$\sum_{i=1}^{N} \delta_i A^i(v) \neq \sum_{i=1}^{N} \delta_i' A^i(v).$$

The proof of the both statements is straightforward and left to the reader. The statements imply that different trajectories of length n in $T^{a,b}(a)$ have different projections onto $\mathbb{Q}^n$ at the moment n. Therefore, $T^{a,b}(a)$ is satisfactory.

Lemma 4.2 and Lemma 4.3 allow us to choose K large enough and to apply Theorem 1 to c-good collections. Hence $h(\mu) \neq 0$ and this completes the proof of the proposition.

Definition. SQ-closure of a group G is the set of quotients of its subgroups.

Note that if we can get H starting from G and taking several times subgroups and quotient groups in any order, then H belongs to the SQ-closure of G.

Proposition 2. *SQ-closure of any solvable group G of exponential growth contains $\mathbb{Z} \wr \mathbb{Z}/q\mathbb{Z}$ for some $q > 1$ or $G(M)$ for some M such that $\|M\| \neq 1$.*

For the proof of the proposition we need the two following lemmas.

Lemma 4.4. *SQ-closure of any solvable group G of exponential growth contains a group G_1 of exponential growth such that G is an extension of an Abelian group by a nilpotent group. That is, there is an exact sequence*

$$1 \to A \to G_1 \to N \to 1,$$

where A is Abelian and N is nilpotent.

Proof. We prove the statement of the lemma by induction on solvability length of G. Consider a series of commutators $G_0 = G$, $G_1 = [G,G]$,..., $G_{i+1} = [G_i, G_i]$. Take maximal k such that $G_k \neq e$. Since $G_{k+1} = [G_k, G_k] = e$, the group G_k is Abelian. Note that G_k is a normal subgroup of G and that solvability length of G/G_k is less than that of G. Hence it is sufficient to consider the case when G/G_k has subexponential growth (otherwise we can apply the induction hypothesis to G/G_k.) Since G/G_k is solvable and has subexponential growth, G/G_k is virtually nilpotent ([16], [22]), that is there exists a finite index nilpotent subgroup in G/G_k. Hence G contains a finite index subgroup G_1 which is Abelian-by-nilpotent. Since G_1 is a finite index subgroup in G, the group G_1 has exponential growth.

Lemma 4.5. *Let G_2 be a group of exponential growth that is a quotient group of $\mathbb{Z} \wr \mathbb{Z}$, $G_2 = (\mathbb{Z} \wr \mathbb{Z})/H$, where H is a normal subgroup of $\mathbb{Z} \wr \mathbb{Z}$. Then SQ-closure of G_2 contains $\mathbb{Z} \wr \mathbb{Z}/q\mathbb{Z}$ for some $q > 1$ or $G(M)$ for some M such that $\|M\| \neq 1$*

Proof. Let

$$A = \sum_{\mathbb{Z}} \mathbb{Z} \subset \mathbb{Z} \wr \mathbb{Z}$$

and let g and a be the canonical generators of $\mathbb{Z} \wr \mathbb{Z}$. First note that $H \subset A$. In fact, otherwise $g^L \in A_2$ for some positive L and hence G_2/A_2 is finite. The G_2 is Abelian-by-finite and hence G_2 is not of exponential growth.

Let A_2 be the subgroup of G_2 generated by a. Clearly, A_2 is Abelian and there is an exact sequence

$$1 \to A_2 \to G_2 \to \mathbb{Z} \to 1.$$

Let T be the torsion of A_2.

We consider two cases.

First case. The torsion T of A_2 is finite. Note that T is a normal subgroup of G_2 and that G_2/T is torsion-free. Note also, that G/T is of exponential growth since T is finite. Applying the second step to G_2 we get a torsion-free group of exponential growth that is a quotient of $\mathbb{Z} \wr \mathbb{Z}$. Hence, we can assume without loss of generality that A_2 (and hence G_2) is torsion-free. If all the elements a, gag^{-1}, g^2ag^{-2}, g^3ag^{-3}... are linearly independent in $A_2 \otimes \mathbb{Q}$, then $H = 0$ and $G = \mathbb{Z} \wr \mathbb{Z}$.

Hence it is sufficient to consider the case when there is a linear dependence between a, gag^{-1}, g^2ag^{-2}, g^3ag^{-3}...

Take maximal N such that a, $a^g = gag^{-1}$, $a^{g^2} = g^2ag^{-2}$, g^3ag^{-3},..., g^Nag^{-N} are linearly independent. There exist $x_0, x_1, ..., x_{N+1} \subset \mathbb{Z}$ such that

$$x_0a_o + x_1a^g + ... + x_{N+1}a^{g^{N+1}} = 0.$$

Obviously, a, a^g,...a^{g^N} generate $\mathbb{Q}^N$ inside $A_2 \otimes \mathbb{Q}$. Note that g acts on this space $\mathbb{Q}^N$ in the following way.

$$a \to a^g,\ a^g \to a^{g^2},\ ...a^{g^{N-1}} \to a^{g^N},$$

and

$$a^{g^N} \to a^{g^{N+1}} = -\frac{x_0a_o + x_1a^g + ... + x_Na^{g^N}}{x_{N+1}}$$

It is clear that g is an invertible operator, since g^{-1} acts as inverse to g. Let $M \in Mat(N, \mathbb{Q})$ be the corresponding invertible matrix. Note that $G_2 \simeq G(M)$. In fact, obviously G_2 is a quotient of $G(M)$. But any additional relation in G_2 contradict to the fact that a, a^g,...a^{g^N} are linearly independent.

Note also that $||M|| \neq 1$ since G_2 has exponential growth. We can assume that $||M|| > 1$, because $G(M) = G(M^{-1})$.

Second case. The torsion T of A_2 is infinite. In this case we will show that SQ-closure of G_2 contains $\mathbb{Z} \wr \mathbb{Z}/q\mathbb{Z}$ for some $\mathbb{Q}$. (Note that in this case we do not use the fact that G_2 is of exponential growth.)

Note that since G_2 is Abelian-by-cyclic, and, in particular, Abelian-by-nilpotent, any normal subgroup of G_2 is finitely generated as a normal subgroup. Therefore, there exist $t_1, t_2, .., t_M$ that generate T as a normal subgroup. Since T is infinite and torsion, there exists i such that the orbit of t_i under g

$$\{g^m t_i g^{-m} : m \in \mathbb{Z}\}$$

is infinite. Take minimal s such that $t_i^s = e$. Suppose that $s = pr$. Consider two groups

$$G_2' = \{a, t_i^p, t_j : j \neq i\}$$

and

$$G_2'' = G_2/t_i^p.$$

Note that at least one of the orbits of t_i^p in G_2' or of t_i in G_2'' under g is infinite. Hence, without loss of generality we can assume that s is a prime number. Let us show that in this case a subgroup G_3 of G_2 that is generated by g and t_j is isomorphic to $\mathbb{Z} \wr \mathbb{Z}/s\mathbb{Z}$. In fact, it is clear that G_3 is a quotient of $\mathbb{Z}/ \wr \mathbb{Z}/s\mathbb{Z}$. To complete the proof note that if s is prime, then any proper quotient group of $\mathbb{Z} \wr (\mathbb{Z}/s\mathbb{Z})$ has finite torsion. This completes the proof of the lemma.

Proof of the Proposition. By Lemma 3.4 we can assume that G is a group of exponential growth such that

$$1 \to A \to G \to N \to 1,$$

where A is Abelian and N is nilpotent.

We consider two cases.

First case. The group A is finitely generated. Then A has a torsion-free subgroup which is isomorphic to $\mathbb{Z}^m$ for some $m \geq 1$. Similarly to the proof of Lemma 3.5 we can assume that $A = \mathbb{Z}^m$. Recall that N acts on Z^m by conjugations. There exists $n \in N$ such that the norm of the corresponding operator on $\mathbb{Z}^m$ is greater then 1 (See [22] for more details). Hence G has a subgroup that is isomorphic to $G(M)$ for some matrix M such that $\|M\| > 1$.

Second case. The A is not finitely generated. Since G is Abelian-by-Nilpotent, then any normal subgroup of this group is finitely generated as a normal subgroup.

Since A is finitely generated as a normal subgroup in G, but not finitely generated as a group, there exist $g \in G$ and $a \in A$ such that for any $N \in \mathbb{N}$ the element $g^N a g^{-N}$ does not belong to the subgroup generated by a, gag^{-1}, g^2ag^{-2},..., $g^{N-1}ag^{N-1}$.

Take δ_i, δ_i' such that δ_i is equal to 0 or 1 and δ_i' is equal to 0 or 1. For any $N > 0$ it holds

$$\sum_{i=1}^{N} g^i a g^{-i} \delta_i \neq \sum_{i=1}^{N} g^i a g^{-i} \delta_i'.$$

Note that then g and a generate a group of superpolynomial growth. Since this subgroup is solvable, we get that it has exponential growth. Hence we can apply Lemma 3.5 to this group, and this completes the proof of the proposition.

Now let G be a solvable group of exponential growth. By Proposition 2 we know that the SQ-closure of G contains G', where $G' = G(M)$ for some M such that $\|M\| > 1$ or $G' = \mathbb{Z} \wr \mathbb{Z}/q\mathbb{Z}$ for $q > 1$. Note that in both cases there exists a symmetric measure with finite entropy μ' on G' having non-trivial Poisson boundary. In the case $G' = G(M)$ it follows from the corollary of Proposition 1 and in the case $G' = \mathbb{Z} \wr \mathbb{Z}/q\mathbb{Z}$ it follows from Theorem 2. Let H be a subgroup of G such that G' is a quotient of H. Clearly, there exists a symmetric measure with finite entropy μ on G such that supp$\mu \subset$ H and μ' is the projection of μ by the quotient map $H \to G'$. Obviously, $h(\mu) \geq h(\mu')$ and this completes the proof of the theorem.

5. Groups of Baumslag type and Liuoville property for universal covers.

In this section we give a series of examples of random walks on finitely presented amenable groups with non-trivial Poisson boundary. Then we use these examples to construct a series of quasi-homogeneous regularly exhausted simply connected manifolds that are not Liouville.

Let $B_d \subset GL_2(\mathbb{Q}(X_1, X_2, ..., X_d))$ be the group generated by the elements

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$s_i = \begin{pmatrix} 1 & 0 \\ 0 & X_i \end{pmatrix}, \; 1 \leq i \leq d,$$

and

$$t_i = \begin{pmatrix} 1 & 0 \\ 0 & X_i + 1 \end{pmatrix}, \; 1 \leq i \leq d.$$

These groups have the following properties.

Lemma 5.1. *1). $B_d \subset GL_2(\mathbb{Z}(X_1, X_2, ..., X_d))$.*

2). B_d is finitely presented. It has the presentation

$$\langle a, s_j, t_j | a^{t_i} = a a^{s_i}, [s_i, t_j] = [s_i, s_j] = [t_i, t_j] = 1, [a^u, a^v] = 1 \rangle,$$

where $1 \leq i, j \leq d$ and $u = \prod s_i^{\varepsilon_i} \prod t_j^{\varepsilon_j'}$, $v = \prod s_i^{\delta_i} \prod t_j^{\delta_j'}$, ε_i , ε_j' , δ_i, δ_j' are equal to 0 or 1.

3). Let W_d be the subgroups of B_d generated by a and s_i $(1 \leq i \leq n)$. The group W_d is isomorphic to $\mathbb{Z}^d \wr \mathbb{Z}$.

Proof. 1). It suffices to prove that all generator of B_d and their inverses lie in $GL_2(\mathbb{Z}(X_1, X_2, ..., X_d))$. Note that

$$a^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

$$s_i^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & X_i^{-1} \end{pmatrix}, \ 1 \leq i \leq d,$$

and

$$t_i^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & (X_i+1)^{-1} \end{pmatrix}, \ 1 \leq i \leq d.$$

2). Let F_d be the subgroup of B_d generated by s_i, t_j. First let us prove that the group B_d has the presentation

$$\langle a, s_j, t_j | a^{t_i} = aa^{s_i}, [s_i, t_j] = [s_i, s_j] = [t_i, t_j] = 1, [a^u, a^v] = 1 \rangle,$$

where $1 \leq i, j \leq d$ and $u, v \in F_d$. It is easy to check that elements of B_d satisfy these relations. On the other hand let B'_d be the group that has this presentation. We know that B_d is a quotient of B'_d. Let F'_d be the subgroup of B'_d generated by s_i, t_j and N'_d be the normal subgroup generated by a. Any element of B'_d can be written as $n'f'$, $f' \in F'_d$, $n' \in N'_d$. Consider its image in B_d. If $f' \neq e$ then in the image the element $*$

$$\begin{pmatrix} 1 & \cdot \\ 0 & * \end{pmatrix}, \ 1 \leq i \leq d.$$

is not equal to 1.

If $n' \neq e$ then in the image the element $**$

$$\begin{pmatrix} 1 & ** \\ 0 & \cdot \end{pmatrix}, \ 1 \leq i \leq d.$$

is not equal to 0. Hence $B_d = B'_d$.

Now using the relation $a^{t_i} = aa^{s_i}$ we can prove by induction that all previous relations follow from that mentioned in the lemma. (See also [2] for more details).

3). From the proof of 2. we know that $[a^u, a^v] = 1 = [u, v]$ for any u, v that lie in the subgroup generated by s_i. Similarly to the proof of 2). we deduce that there are no other relations is W_d.

Remark. The group $W_d = \mathbb{Z}^d \wr \mathbb{Z}$ is a finitely generated metabelian group. By a theorem of Baumslag ([2]) it can be embedded into finitely presented metabelian group. The embedding $W_d \subset B_d$ is an example of that construction.

However, there is no known general relation between Liouville property of a group and its subgroups (see Section 6 for more on this). Because of that the embedding $W_d \subset B_d$ does not help us to prove Theorem 4 below.

Now let us describe a normal form for the group B_d.

Normal form. Let H be the subgroup of B_d consisting of matrixes that have the following form

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

Let F be the subgroup of B_d, generated by s_i and t_j $(1 \leq i, j \leq d)$. Note that F is isomorphic to $\mathbb{Z}^{2d}$.

Any element of $g \subset B_d$ has unique presentation

$$g = hf,$$

where $f \subset F$ and $h \subset H$ (this follows from the form of the defining relators of B_d). Let $f = (S_1, ..., S_m, T_1, ..., T_m)$, where $S_i, T_j \in \mathbb{Z}$. Let $s : B_d \to H$ be the map that sends g to h.

We can consider h as a rational function in $X_1, ..., X_m$. Moreover, h is a polynomial in

$$X_1, ..., X_d; X_1^{-1}, ..., X_d^{-1}; (1 + X_1)^{-1}, ..., (1 + X_d)^{-1}.$$

We will consider this polynomial as a series in $X_1, ..., X_d$. Taking the coefficients of this series, we get a function from $\mathbb{Z}^d$ to $\mathbb{Z}$. The space of such functions we call the configuration space.

To each element $g \in B_d$ we have assigned a vector $f \in \mathbb{Z}^{2d}$ and a configuration in the configuration space.

Suppose that $x = s_i, t_j, s_i^{-1}$ or t_j^{-1}. Then gx has normal form $h(fx)$. Note also that ga has normal form $hfaf^{-1}(f)$.

5.1. Random walks on B_d.

Theorem 4. *Let $d \geq 3$. Then any non-degenerate random walk on B_d has non-zero entropy.*

Proof. The beginning of the proof is similar to that of Theorem 2. We consider a as in the definition of B_d and we put $b = e_{B_d}$. As before, we can assume that $a, b \in \text{supp}\mu$. We introduce a set of collections for the space of trajectories of length n in the following way.

Let $\pi_1 : B^d \to \mathbb{Z}^d$ be the projection of the $F = \mathbb{Z}^{2d}$ component onto the first d coordinates and $\pi_2 : B^d \to \mathbb{Z}^d$ be the projection of the $F = \mathbb{Z}^{2d}$ component onto the last d coordinates.

Consider a collection $a = (i_1, i_2, ..., i_k, x_1, x_2, ..., \hat{x}_{i_1}, ..., \hat{x}_{i_j}, ..., x_n)$, where $0 \leq i_1 < i_2 < ... < i_k \leq n$ and $x_s \in \text{supp}\mu$ for any $s \neq i_j$. We say that the collection a is good if the two following conditions hold.

1). All the elements x_1, x_1x_2,..., $x_1x_2...\hat{x}_{i_1}$, $x_1x_2...\hat{x}_{i_1}x_{i_1+1}$, $x_1x_2...\hat{x}_{i_j}$, ...x_n have different projections underπ_1.

2). If for some s it holds $x_s = a$ or $x_s = b$ then there exists j such that $i_j \geq s$ and $x_1x_2...\hat{x}_{i_j}$ has the same projection under π_1 as $x_1x_2...x_s$.

Similarly to the proof of Theorem 2 one can check that the statement that are analogous to Lemma 3.1 and Lemma 3.2 are valid for good and c-good collections a.

The main difference between the case of wreath products and B_d is in the proof of the following lemma.

Lemma 5.2. *If a is a good collection, then $T^{a,b}$ is a satisfactory subset of the space of trajectories of length n.*

Proof. Let $a = (i_1, i_2, ..., i_k, x_1, x_2, ..., \hat{x}_{i_1}, ..., \hat{x}_{i_j}, ..., x_n)$. Consider two different trajectories from $T^{a,b}(a)$

$$y = (x_1, x_1x_2, ..., x_1x_2...x_{i_1-1}g_1, ..., x_1x_2...x_{i_j-1}g_jx_{i_j+1}...g_k...x_n)$$

and

$$y' = (x_1, x_1x_2, ..., x_1x_2...x_{i_1-1}g_1', ..., x_1x_2...x_{i_j-1}g_j'x_{i_j+1}...g_k'...x_n),$$

where for each j it holds $g_j = a$ or $g_j = b$ and $g'_j = a$ or $g'_j = b$.

The difference from the case of wreath products (Theorem 2) is that the multiplication of g by a changes g not only at the point $\pi(g)$.

To solve this difficulty, let us introduce a partial order on $\mathbb{Z}^d$. For $z_1, z_2 \in \mathbb{Z}^d$ we say that $z_1 \preceq z_2$, if each coordinate of z_1 is less or equal to the corresponding coordinate of z_2.

Note that the configuration of ga can differ from that of g only at the points z such that $\pi_1(g) \preceq z$. In fact, let $f = s(g) \in \mathbb{Z}[X_i^{\pm 1}, (X_i + 1)^{\pm 1}]$ be the rational function corresponding to g. Let $\pi_1(g) = (v_1, v_2, ..., v_d)$ and $\pi_2(g) = (w_1, w_2, ..., w_d)$. Then the rational function corresponding to ga is

$$f + X_1^{v_1} X_2^{v_2} ... X_d^{v_d} (1 + X_1)^{w_1} (1 + X_2)^{w_2} ... (1 + X_d)^{w_d}.$$

It suffices to note that

$$\prod_{i=1}^{d} (1 + X_i)^{w_i} = 1 + \sum c_\alpha X^\alpha,$$

where the sum is taken over multi-indexes $\alpha \in \mathbb{Z}^d$ such that $\alpha > 0$.

Since y and y' are different trajectories of length n, there exists j such that $g_j \neq g'_j$. Consider all such j and look at the corresponding projection $\pi_1(x_1 x_2 ... \hat{x}_{i_j})$. By the definition of good collections all these projections are different. Take the minimal (with respect to the chosen partial order on $\mathbb{Z}^d$) projection $z = \pi_1(x_1 x_2 ... \hat{x}_{i_m})$. It holds $g_m \neq g'_m$. For any $M \neq m$ such that $g_M \neq g'_M$ it holds

$$\pi_1(x_1 x_2 ... \hat{x}_{i_M}) > \pi_1(x_1 x_2 ... \hat{x}_{i_m}).$$

Note that the coefficient of $s(x_1 x_2 ... x_{i_k - 1} g_k ... x_n)$ in z differs from the coefficient of $s(x_1 x_2 ... x_{i_k - 1} g'_k ... x_n)$ in z. In fact, all the changes of this coefficient that correspond to $z' < z$ are the same for both trajectories y and y'. At the moment m these trajectories hit z and exactly one of them is multiplied by a. This changes the coefficient at z for exactly one of these trajectories. It was already mentioned that any changes at points $z" > z$ do not change this coefficient. Hence, at the moment n the trajectories hit different points of G and this completes the proof of the lemma.

We have verified that the assumption of Theorem 1 hold for good collections. Therefore, we can apply this theorem and get that $h(\mu) \neq 0$.

5.2. **Harmonic functions on universal covers. Definition.** A Riemannian manifold N is regularly exhausted if there exists a sequence of open sets $N_i \subset N$ such that $\cup N_i = N$ and

$$\lim_{i \to \infty} \frac{vol_{D-1} \partial N_i}{vol_D(N_i)} = 0,$$

where D is the dimension of N.

Now let M be a compact manifold and N be a regular cover of M with a group G. It is known (see e.g. [8]) that in this case N is regularly exhausted if and only if G is amenable (in particular, this does not depend on the choice of Riemannian structure on M).

If G is non-amenable (that is, N is not regularly exhausted, "open at infinity" in the terminology of [8]), then N is not Liouville. On the other hand, for many examples of amenable G the cover N is Liouville [9]. D.Sullivan and T.Lyons provided the following example [15]. There is a regular cover N of the surface S_g of genus g such that the corresponding group G is solvable, but N is not Liouville. This example as well as examples of random walks walks on amenable groups with non-trivial boundary led V.A.Kaimanovich [9] to ask the following question. Does there exist a universal cover $\tilde{M}$ of a compact manifold M such that $G = \pi_1(M)$ is amenable, but $\tilde{M}$ is not Liouville? The following theorem gives a positive answer to this question.

Theorem 5. *There exists a compact Riemannian manifold M such that the universal cover $\tilde{M}$ is regularly exhausted, but for any choice of Riemannian structure on M the universal cover admits non-constant harmonic functions.*

Proof. Put $G = B_d$ for some $d \geq 3$. Since G is finitely presentable, there exists a compact manifold M such that $\pi_1(M) = G$. Note that $G = \pi_1(M)$ is amenable, moreover, it is solvable of length 2. Choose any Riemannian structure on M and lift it to the universal cover $\tilde{M}$. This Riemannian structure defines a probability measure μ_{FLS} on $G = \pi_1(M)$, which is the Furstenberg-Lyons-Sullivan discretisation of the Brownian Motion on $\tilde{M}$. This measure is non-degenerate, and hence by Theorem 4 we know that $h(\mu_{FLS}) \neq 0$. It is known that FLS discretisation has finite entropy (moreover, all the moments of this measure are finite.) Hence the Poisson boundary of (G, μ_{FLS}) is non-trivial.

It is known ([9], [11]) that the Poisson boundary of (G, μ_{FLS}) is isomorphic to the Poisson boundary of $\tilde{M}$. Consequently, $\tilde{M}$ admits non-constant bounded harmonic functions and this completes the proof of the theorem.

6. Concluding remarks.

Now we list several open questions about Poisson boundary and Liouville property.

- Which groups admit a measure with non-trivial Poisson boundary? For which groups any non-degenerate measure with finite entropy has non-trivial Poisson boundary? It would be interesting to answer this question at least for certain classes of groups (e.g., metabelian, solvable).
- Can non-triviality of Poisson boundary depend on the non-degenerate symmetric measure with finite support? Can Liouville property of a compact cover depend on the choice of Riemannian structure on the compact manifold?

Note that for graphs and non-quasi-homogeneous manifolds the analogous question has negative answer, since examples of T.Lyons [14] and Y.Benjamini [3] show the existence of quasi-isometric graphs such that one is Liouville and the other is not. The same examples can be reformulated for manifolds.

- **Conjecture of Vershik and Kaimanovich.** Does any group of exponential growth admits a symmetric measure with non-trivial Poisson boundary? Does any group of exponential growth admits a symmetric measure with positive entropy?
- Which groups admit a non-degenerate measure with zero entropy? Obviously, any group of subexponential growth admits such a measure, but examples of sections 3 and 5 show that there exist amenable groups which do not.
- Is entropy of the random walk continuous in the following sense. Suppose that $\mu_i \to \mu$ point-wise, and supports of all these measures lie inside the same finite set. Does $h(\mu_i) \to h(\mu)$? Is there a connection between boundary of a group and boundary of its subgroup? Is it possible that some (all) non-degenerate finitely supported symmetric measure on a group has trivial boundary, but

some (all) non-degenerate finitely supported measure on a subgroup has non-trivial boundary?

Remark. Note that in a stronger sense the entropy of the random walk is not continuous. In fact, V.A.Kaimanovich provided an example of symmetric measure μ with finite entropy on the group of finite permutation of countable set such that the corresponding random walk has non-trivial boundary. On the other hand, the entropy of the random walk is equal to zero for any finitely supported measure on this group, and hence there exists no approximation of μ by finitely supported measures μ_i such that $h(\mu_i)$ converges to $h(\mu)$. In this example the group is not finitely generated, but it is possible to produce an example of non-degenerate measure on a finitely generated group of the similar kind. In fact, take $M \in SL(n, \mathbb{Z})$ such that $\|M\| \neq 1$. In Section 4 we proved that there exists a symmetric non-degenerate measure of finite entropy on $G(M)$ that has non-trivial boundary. On the other, $G(M)$ is polycyclic and hence any finitely supported symmetric measure has trivial boundary and zero entropy [10]. Clearly, the same kind of example can be constructed for $\mathbb{Z} \wr \mathbb{Z}$.

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