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Liouville theory, $\mathcal{N} = 2$ gauge theories and accessory parameters

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Abstract: The correspondence between the semiclassical limit of the DOZZ quantum Liouville theory and the Nekrasov-Shatashvili limit of the $\mathcal{N} = 2$ (Ω-deformed) U(2) super-Yang-Mills theories is used to calculate the unknown accessory parameter of the Fuchsian uniformization of the 4-punctured sphere. The computation is based on the saddle point method. This allows to find an analytic expression for the $N_f = 4$, U(2) instanton twisted superpotential and, in turn, to sum up the 4-point classical block. It is well known that the critical value of the Liouville action functional is the generating function of the accessory parameters. This statement and the factorization property of the 4-point action allow to express the unknown accessory parameter as the derivative of the 4-point classical block with respect to the modular parameter of the 4-punctured sphere. It has been found that this accessory parameter is related to the sum of all rescaled column lengths of the so-called 'critical' Young diagram extremizing the instanton 'free energy'. It is shown that the sum over the 'critical' column lengths can be rewritten in terms of a contour integral in which the integrand is built out of certain special functions closely related to the ordinary Gamma function.

Keywords: Supersymmetric gauge theory, Field Theories in Lower Dimensions, Solitons Monopoles and Instantons

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JHEP05(2012)025

Contents

1 Introduction

The studies of the interrelationships between conformal field theory in two dimensions, supersymmetric $\mathcal{N} = 2$ quiver gauge theories and integrable systems are recently attracting a great attention in the scientific community [1–12]. This is mainly due to the discovery of the so-called AGT [13] and Bethe/gauge [14–16] correspondences.

The AGT conjecture states that the Liouville field theory (LFT) correlators on the Riemann surface $C_{g,n}$ with genus g and n punctures can be identified with the partition functions of a class $T_{g,n}$ of four-dimensional $\mathcal{N} = 2$ supersymmetric $SU(2)$ quiver gauge theories. A significant part of the AGT conjecture is an exact correspondence between the Virasoro blocks on $C_{g,n}$ and the instanton sectors of the Nekrasov partition functions of the gauge theories $T_{g,n}$. Soon after its discovery, the AGT hypothesis has been extended to the SU(N)-gauge theories/conformal Toda correspondence [17–19].

The AGT correspondence works at the level of the quantum Liouville field theory. It arises at this point the question of what happens if we proceed to the classical limit of the Liouville theory. It turns out that the semiclassical limit of the LFT correlation functions [20], i. e. the limit in which the central charge and the external and intermediate conformal weights tend to infinity while their ratios are fixed, corresponds to the Nekrasov-Shatashvili limit of the Nekrasov partition functions [14]. In particular, a consequence of that correspondence is that the *classical conformal block* can be identified with the

JHEP05(2012)025

instanton sector of the *effective twisted superpotential* [12].[1] The latter quantity determines the low energy effective dynamics of the two-dimensional gauge theories restricted to the Ω-background. The twisted superpotential plays also a pivotal role in the already mentioned Bethe/gauge correspondence that maps supersymmetric vacua of the $\mathcal{N} = 2$ theories to Bethe states of quantum integrable systems. A result of that duality is that the twisted superpotential is identified with the Yang's functional [21] which describes the spectrum of the corresponding quantum integrable system. Joining together the AGT duality and the Bethe/gauge correspondence it is thus possible to link the classical blocks (or more in general the classical Liouville actions) to the Yang's functionals.

The motivations to study the classical block were until now mainly confined to applications in pure mathematics, in particular to the celebrated uniformization problem, which roughly speaking is related to the construction of conformal mappings between Riemann surfaces (RS) admitting a simply connected universal covering and the three existing simply connected RS, the sphere, the complex plane and the upper half plane. The uniformization problem is well illustrated by the example of the uniformization of the Riemann sphere with four punctures [22]. Its uniformization may be associated to a Fuchsian equation whose form is known up to some constants that are called *accessory parameters*. Their computation is an open longstanding problem, which can however be solved if we succeed to derive an analytical expression of the classical block obtained by performing the classical limit of the four-point correlation function of the DOZZ quantum Liouville field theory [20, 23]. The importance of the classical blocks is not only limited to the uniformization theorem, but gives also information about the solution of the Liouville equation on surfaces with punctures. For instance, if the accessory parameters for $C_{0,4}$ are available, it is then possible to construct the solution of the Liouville equation and the hyperbolic metric on $C_{0,4}$. Due to the recent discoveries mentioned above the classical blocks have become relevant also in mathematical and theoretical physics, since they are related to quantum integrable systems[2] and to the instantonic sector of certain $\mathcal{N} = 2$ supersymmetric gauge field theories.[3]

The link between classical blocks and Yang's functionals has been exploited in [12] to conjecture a novel representation of the 4-point classical block in terms of the elliptic Calogero-Moser (eCM) Yang's functional found in [14]. As an application of that result the relation between the accessory parameter of the Fuchsian uniformization of the 4-punctured Riemann sphere and the eCM Yang's functional has been proposed [12]. However, the results described above have an important limitation. They are not general, i.e. they hold only for certain classes of classical block parameters or, in other words, for restricted families of the 4-punctured spheres.

[1] One can see that a slightly more general statement holds, i.e. the so-called *classical Liouville action* [20] can be connected with the full twisted superpotential.

[2] The classical conformal block and the accessory parameters have fascinating interpretations in that context. It has been recently found in [1] that the classical conformal block corresponds to the generating function of the so-called variety of opers which has been introduced to define the Yang's functional. In [3] the accessory parameters have been identified with the Hitchin Hamiltonians.

[3] Surprising relationships between the LFT and the $\mathcal{N} = 2$ SYM theories were also observed before the discovery of the AGT correspondence, see for instance [24].

The purpose of the present paper is to find an analytical expression of the *generic* classical 4-point block and apply it to compute the unknown accessory prameter appearing in the Fuchsian differential equation with four elliptic/parabolic singularities. In order to accomplish this task we will employ the correspondence mentioned above between the classical limit of the Liouville theory and the Nekrasov-Shatashvili limit of the $\mathcal{N} = 2$ (Ω-deformed) U(2) super-Yang-Mills theories. The relevant technical problem of this strategy consists in the summation of the series defining the twisted superpotential (and/or the classical block). This problem will be tackled hereusing the saddle point method [9, 25–27].

The structure of the paper is as follows. In section 2 we formulate the problem of the accessory parameters of the Fuchsian uniformization of the n-punctured sphere and describe its connection with the classical Liouville theory. Afterwards, we briefly review the so-called *geometric approach* to quantum Liouville theory originally proposed by Polyakov (as reported in refs. [28–32]) and further developed by Takhtajan [29, 31, 33, 34] (see also [35]). Some of the predictions derived from the path integral representation of the geometric approach can be proved rigorously and lead to deep geometrical results. One of these results has been the suggestion that the classical Liouville action is the generating function for the accessory parameters of the Fuchsian uniformization of the punctured sphere. This statement yields an essentially new insight into the problem of accessory parameters. However, its usefulness is restricted by our ability to calculate the classical Liouville action for more than three singularities. We focus on the case of the sphere with four punctures and show that there is only one unknown accessory parameter whose computation is equivalent to the calculation of the 4-point classical block.

In section 3 we compute the Nekrasov-Shatashvili limit of the Nekrasov instanton partition function of the U(2), $N_f = 4$, $\mathcal{N} = 2$ SYM theory closely following [9] (see also [27]). On the basis of arguments worked out by Nekrasov and Okounkov [26] it is found that the effective twisted superpotential is equal to the critical value of the instanton 'free energy'. The critical (or classical) instanton configuration is determined by a saddle point equation that can be solved recursively order by order in the instanton parameter q. In the language of Young diagrams the solution of the saddle point equation describes the shape of the most relevant 'critical' Young diagram contributing to the instanton partition function. We check that the instanton 'free energy' evaluated at the critical configuration (i.e. the twisted superpotential) gives the correct q-expansion of the classical 4-point block provided that certain relations between the parameters are holding. Taking these relations into account, we are able to express the 4-point classical block in terms of the twisted superpotential and apply this representation to calculate the unknown accessory parameter. We find in this way that the accessory parameter is related to the sum of all column lengths of the 'critical' Young diagram. As it has been shown in [9], this sum can be rewritten as a contour integral where the integrand contains as an essential ingredient certain special functions closely related to the ordinary Gamma function (cf. [27]). In section 4 we present our conclusions. The problems that are still open and the possible extensions of the present work are discussed. Finally, in the appendix we define the quantum and classical four-point conformal blocks.

2 Liouville theory and accessory parameters

2.1 Monodromy problem and uniformization

Let us choose a set of complex coordinates $z_1, \ldots, z_n$ on the n-punctured Riemann sphere $C_{0,n}$ in such a way that $z_n = \infty$. The so-called problem of accessory parameters can be formulated as follows. Consider the ordinary linear differential equation:

$$\partial_z^2 \psi(z) + T^{\,\rm cl}(z)\,\psi(z) \;=\; 0, \tag{2.1}$$

where $T^{\,\rm cl}(z)$ is a meromorphic function on the Riemann sphere of the form:

$$T^{\,\rm cl}(z) \;=\; \sum_{k=1}^{n-1}\left[\frac{\delta_k}{(z-z_k)^2} + \frac{c_k}{z-z_k}\right] \tag{2.2}$$

and

$$T^{\,\rm cl}(z) = \frac{\delta_n}{z^2} + \frac{c_n}{z^3} + \mathcal{O}\left(z^{-4}\right) \qquad\qquad \text{for } \; z\to\infty \tag{2.3}$$

with

$$\delta_i = \frac{1}{4}(1-\xi_i^2), \qquad \xi_i \in \mathbb{R}_{\geq 0}, \qquad i = 1,\ldots,n.$$

The asymptotic behaviour (2.3) of $T^{\,\rm cl}(z)$ implies that the coefficients $c_1, \ldots, c_n$, known as the accessory parameters, obey the relations

$$\sum_{k=1}^{n-1} c_k = 0, \qquad\qquad \sum_{k=1}^{n-1}(\delta_k + c_k z_k) = \delta_n, \qquad\qquad \sum_{k=1}^{n-1}\left(2\delta_k z_k + c_k z_k^2\right) = c_n. \tag{2.4}$$

The problem is to tune these parameters in such a way that the eq. (2.1) admits a fundamental system of solutions with monodromy in $P\mathrm{SL}(2,\mathbb{R})$.[4] Note that for $n > 3$ the equations (2.4) do not provide enough constraints in order to calculate all the c_k's. The computation of the accessory parameters in that case is difficult and in general still unsolved problem.

One can shed some light on the form of the c_k's by relating the problem of finding the accessory parameters to the Liouville field theory on $C_{0,n}$. Indeed, let us consider the quotient:

$$\rho = \frac{\psi_1}{\psi_2} \tag{2.5}$$

of the fundamental solutions (ψ_1, ψ_2) of the eq. (2.1) with Wronskian $\psi_1\psi_2' - \psi_1'\psi_2 = 1$ and $\mathrm{SL}(2,\mathbb{R})$ monodromy with respect to all punctures. It is a well known fact [28, 36] that

$$\rho\colon C_{0,n} \ni z \longrightarrow \tau(z) \in \mathbb{H}$$

is a multi-valued map from the n-punctured Riemann sphere to the upper half plane $\mathbb{H} = \{\tau \in \mathbb{C} \,:\, \mathfrak{Im}\,\tau > 0\}$ with branch points $z_1, \ldots, z_n$. The connection with Liouville theory

[4] Equivalently, one could employ here the group $P\mathrm{SU}(1,1)$ which is isomorphic to $P\mathrm{SL}(2,\mathbb{R})$.

JHEP05(2012)025

comes out from the existence of the Poincaré metric $ds^2_{\mathbb{H}} = d\tau\, d\bar{\tau}/\,(\mathfrak{Im}\,\tau)^2$ on the upper half plane $\mathbb{H}$. The pull back

$$\rho^* ds^2_{\mathbb{H}} = \frac{1}{(\mathfrak{Im}\,\tau)^2}\left|\frac{\partial\tau}{\partial z}\right|^2 dz d\bar{z} = \mathrm{e}^{\phi(z,\bar{z})} dz d\bar{z} \tag{2.6}$$

is a regular hyperbolic metric on $C_{0,n}$, conformal to the standard flat matric $dzd\bar{z}$ on $\mathbb{C}$. The conformal factor $\phi(z,\bar{z})$ of that metric satisfies the Liouville equation:

$$\partial_z\partial_{\bar{z}}\phi(z,\bar{z}) = \frac{1}{2}\,\mathrm{e}^{\phi(z,\bar{z})} \tag{2.7}$$

and has one of the following asymptotic behaviors near the punctures:

1. case of *elliptic singularities*:

$$\phi(z,\bar{z}) = \begin{cases} -2\,(1-\xi_j)\log|z-z_j| + O(1) & \text{as } z\to z_j, \quad j=1,\ldots,n-1, \\ -2\,(1+\xi_n)\log|z| + O(1) & \text{as } z\to\infty, \end{cases} \tag{2.8}$$

$$\xi_i \in \mathbb{R}_{>0} \quad \text{for all} \quad i=1,\ldots,n \quad \text{and} \quad \sum_{i=1}^{n}\xi_i < n-2;$$

2. case of *parabolic singularities* $(\xi_i \to 0)$:

$$\phi(z,\bar{z}) = \begin{cases} -2\log|z-z_j| - 2\log|\log|z-z_j|| + O(1) & \text{as } z\to z_j, \\ -2\log|z| - 2\log|\log|z|| + O(1) & \text{as } z\to\infty. \end{cases} \tag{2.9}$$

It is known that it exists a unique solution of eq. (2.7) if one of the conditions (2.8) [37–39] or (2.9) [40] is satisfied. One may show that the meromorphic function $T^{\,\mathrm{cl}}(z)$ introduced in eqs. (2.2), (2.3) is the holomorphic component of the energy-momentum tensor:

$$T(z) \equiv -\frac{1}{4}\,(\partial_z\phi)^2 + \frac{1}{2}\,\partial_z^2\phi \tag{2.10}$$

evaluated at the solution $\phi(z,\bar{z})$ of the Liouville equation with one of the asymptotic conditions[5] (2.8) or (2.9). Once the classical solution is known, it is possible to calculate all the accessory parameters.

The monodromy problem for the Fuchs equation (2.1) formulated above has been proposed by Poincaré in order to construct the so-called *uniformization map* in the case of the n-punctured sphere with parabolic singularities. To derive the uniformization map for the n-punctured sphere $C_{0,n}$ it is necessary to compute a meromorphic map λ from the upper half-plane $\mathbb{H}$ to $C_{0,n}$ such that λ is the covering map

$$\lambda : \mathbb{H} \longrightarrow \mathbb{H}/G \simeq C_{0,n}$$

with G being a discrete subgroup of $PSL(2,\mathbb{R})$. The computation of λ is the main problem of the Fuchsian uniformization scheme. If λ is the uniformization map of $C_{0,n}$, then the

[5]In the case of parabolic singularities the *classical conformal weights* in (2.2), (2.3) are $\delta_i = \frac{1}{4} \iff \xi_i \to 0$.

multi-valued function ρ defined in eq. (2.5) coincides with the inverse map λ^{-1}, i. e. $\rho = \lambda^{-1}$. The branches of ρ are related by the elements of the group G. It is possible to show that the classical energy-momentum tensor $T^{\,\mathrm{cl}}(z)$ is equal to one half of the Schwarzian derivative of the map ρ:

$$T^{\,\mathrm{cl}}(z) = \frac{1}{2}\,\{\rho, z\}. \tag{2.11}$$

Thus, the inverse map ρ can be computed if the appropriate solution of the Liouville equation is available or, equivalently, if the accessory parameters in the Fuchs equation (2.1) are known and one may select fundamental solutions of it with a suitable monodromy.

In the case of elliptic singularities the multi-valued function ρ is also of interest. It is no longer the inverse to the covering map of $C_{0,n}$, but it can still be used to construct solutions of the Liouville equation with the asymptotic behavior (2.8) according to the formula (2.6). Thus, the result (2.11) holds also when the singularities are elliptic and, as before, the problem of calculating the map ρ is equivalent to that of finding the solution of the Liouville equation or solving the monodromy problem for the Fuchs equation (2.1).

2.2 Liouville action and accessory parameters

For almost a century the problem of accessory parameters has remained unsolved until the appearance of its solution proposed by Polyakov (as reported in refs. [28–30]). Polyakov observed that the (properly defined and normalized) Liouville action functional evaluated at the classical solution $\phi(z,\bar z)$ is the generating functional for the accessory parameters:

$$c_j = -\frac{\partial S_{\mathrm{L}}^{\mathrm{cl}}[\phi]}{\partial z_j}. \tag{2.12}$$

This formula was derived within the so-called *geometric path integral approach* to the quantum Liouville theory by analyzing the quasi-classical limit of the conformal Ward identity [29].

In the geometric approach the correlators of the LFT are expressed in terms of path integrals over the conformal class of Riemannian metrics with prescribed singularities at the punctures. In particular, in the case of the quantum Liouville theory on the sphere the central objects in the geometric approach are[6]

1. the 'partition functions' on $C_{0,n}$:

$$\langle\, C_{0,n}\,\rangle = \int\limits_{\mathcal{M}} \mathcal{D}\phi\; \mathrm{e}^{-Q^2 S_{\mathrm{L}}[\phi]}, \tag{2.13}$$

where $\mathcal{M}$ is the space of conformal factors appearing in the metrics on $C_{0,n}$ with either the asymptotic behavior of eq. (2.8) or that of eq. (2.9);

2. the correlation functions of the energy-momentum tensor:

$$\begin{aligned}\Big\langle \widehat{T}(u_1)\ldots\widehat{T}(u_k)\widehat{\overline{T}}(\bar w_1)\ldots\widehat{\overline{T}}(\bar w_l)\, C_{0,n}\Big\rangle &= \\ = \int\limits_{\mathcal{M}}\mathcal{D}\phi\; \mathrm{e}^{-Q^2 S_{\mathrm{L}}[\phi]}\; \widehat{T}(u_1)\ldots\widehat{T}(u_k)\widehat{\overline{T}}(\bar w_1)\ldots\widehat{\overline{T}}(\bar w_l)\end{aligned} \tag{2.14}$$

[6] In this subsection we closely follow [31].

with

$$\widehat{T}(u) = Q^2\left[-\frac{1}{4}\left(\partial_u\phi(u,\bar{u})\right)^2 + \frac{1}{2}\,\partial_u^2\phi(u,\bar{u})\right]. \tag{2.15}$$

The singular nature of the Liouville field at the punctures requires regularizing terms in the Louville action:

$$S_{\rm L}[\phi] \;=\; \frac{1}{4\pi}\lim_{\epsilon\to 0} S_{\rm L}^{\epsilon}[\phi] \tag{2.16}$$

where the regularized action $S_{\rm L}^{\epsilon}[\phi]$ is given by

$$\begin{aligned} S_{\rm L}^{\epsilon}[\phi] = \int\limits_{X_\epsilon} d^2z\left[|\partial\phi|^2 + {\rm e}^{\phi}\right] + \sum_{j=1}^{n-1}(1-\xi_j)\int\limits_{|z-z_j|=\epsilon}|dz|\;\kappa_z\phi + (1+\xi_n)\int\limits_{|z|=\frac{1}{\epsilon}}|dz|\;\kappa_z\phi \\ -2\pi\sum_{j=1}^{n-1}(1-\xi_j)^2\log\epsilon - 2\pi\,(1+\xi_n)^2\log\epsilon, \end{aligned} \tag{2.17}$$

and $X_\epsilon = \mathbb{C}\setminus\left\{\left(\bigcup_{j=1}^{n}|z-z_j|<\epsilon\right)\cup\left(|z|>\frac{1}{\epsilon}\right)\right\}$. The prescription given in eqs. (2.16) and (2.17) is valid for parabolic singularities (corresponding to $\xi_j = 0$) as well.

One can check by perturbative calculations of the correlators (2.14) [31] that the central charge reads

$$c = 1 + 6Q^2. \tag{2.18}$$

The transformation properties of (2.13) with respect to global conformal transformations show [31] that the punctures behave as primary fields with dimensions

$$\Delta_j = \bar{\Delta}_j = \frac{Q^2}{4}\left(1-\xi_j^2\right). \tag{2.19}$$

For fixed ξ_j, the dimensions scale like Q^2 and the punctures correspond to *heavy* fields of the operator approach [20]. In the classical limit $Q^2\to\infty$ with all classical weights

$$\delta_i \stackrel{\rm def}{=} \frac{\Delta_i}{Q^2} = \frac{1-\xi_j^2}{4} \tag{2.20}$$

kept fixed, we expect the path integral to be dominated by the classical action $S_{\rm L}^{\rm cl}(\delta_i\,;\,z_i)$,

$$\langle\; C_{0,n}\;\rangle \sim {\rm e}^{-Q^2 S_{\rm L}^{\rm cl}(\delta_i\,;\,z_i)}. \tag{2.21}$$

In the above equation $S_{\rm L}^{\rm cl}(\delta_i\,;\,z_i)$ denotes the functional $S_{\rm L}[\,\cdot\,]$ of eq. (2.16) evaluated at the classical solution ϕ of (2.7) with the asymptotics (2.8) or (2.9). A similar result holds for the correlation function $\left\langle\widehat{T}(z)\;C_{0,n}\right\rangle$:

$$\left\langle\widehat{T}(z)\;C_{0,n}\right\rangle \sim \widehat{T}^{\,\rm cl}(z)\;{\rm e}^{-Q^2 S_{\rm L}^{\rm cl}(\delta_i\,;\,z_i)}, \tag{2.22}$$

where $\widehat{T}^{\,\rm cl}(z)$ is the classical energy-momentum tensor.

From (2.15) and (2.8) or (2.9) it follows that

$$\widehat{T}^{\,\rm cl}(z) \sim \frac{\Delta_j}{(z-z_j)^2} \qquad \text{for } \ z \to z_j,$$
$$\widehat{T}^{\,\rm cl}(z) \sim \frac{\Delta_n}{z^2} \qquad \text{for } \ z \to \infty, \tag{2.23}$$

and consequently

$$\widehat{T}^{\,\rm cl}(z) = Q^2 \sum_{j=1}^{n-1} \left[\frac{\delta_j}{(z-z_j)^2} + \frac{c_j}{z-z_j} \right]. \tag{2.24}$$

Combining now (2.21), (2.22) and (2.24) with the conformal Ward identity [41]

$$\left\langle \widehat{T}(z)\, C_{0,n} \right\rangle = \sum_{j=1}^{n-1} \left[\frac{\Delta_j}{(z-z_j)^2} + \frac{1}{z-z_j}\frac{\partial}{\partial z_j} \right] \langle\, C_{0,n}\, \rangle\,, \tag{2.25}$$

we get the relation (2.12). Amazingly, this relation obtained by general heuristic path integral arguments turns out to provide an exact solution of the problem of the accessory parameters. Indeed, it can be rigorously proved[7] that the formula (2.12) yields the accessory parameters c_j for which the Fuchsian equation

$$\partial_z^2 \psi(z) + \frac{1}{Q^2}\widehat{T}^{\,\rm cl}(z)\psi(z) = 0, \tag{2.26}$$

admits a fundamental system of solutions with $P\mathrm{SU}(1,1)$ monodromies around all singularities. Note that if $\{\chi_1(z), \chi_2(z)\}$ is such a system, then the function $\phi(z,\bar z)$ determined by the relation

$$\mathrm{e}^{\phi(z,\bar z)} = \frac{4\,|w'|^2}{(1-|w|^2)^2}, \qquad w(z) = \frac{\chi_1(z)}{\chi_2(z)}, \tag{2.27}$$

satisfies (2.7) and (2.8) (or (2.9)). The $P\mathrm{SU}(1,1)$ monodromy condition is then equivalent to the existence of the well defined hyperbolic metric on $C_{0,n}$.

2.3 Fuchs equation with four elliptic/parabolic singularities

Let us consider the case $n = 4$ in which the four elliptic/parabolic[8] singularities are at the standard locations $z_4 = \infty, z_3 = 1, z_2 = q, z_1 = 0$. Accordingly, the expression of the classical energy-momentum tensor is given by:

$$T^{\,\rm cl}(z) = \frac{\delta_1}{z^2} + \frac{\delta_2}{(z-q)^2} + \frac{\delta_3}{(z-1)^2} + \frac{c_1(q)}{z} + \frac{c_2(q)}{z-q} + \frac{c_3(q)}{z-1}\,.$$

and the first two relations of eq. (2.4) can be written as follows

$$c_1(q) = \delta_1 + \delta_2 + \delta_3 - \delta_4 + (q-1)c_2(q), \qquad c_3(q) = \delta_4 - \delta_1 - \delta_2 - \delta_3 - qc_2(q).$$

[7] For parabolic singularities formula (2.12) has been proved by Takhtajan and Zograf. The details can be found in [30]. In ref. [32] the extension of [30] to compact Riemann surfaces has been presented. For general elliptic singularities eq. (2.12) has been proved in [28] and non rigorously derived in [42]. It is also possible to construct the Liouville action satisfying (2.12) for the so-called hyperbolic singularities on the Riemann sphere, see [43].

[8] In that case $\delta_1 = \ldots = \delta_4 = \frac{1}{4}$.

JHEP05(2012)025

The Fuchsian differential equation (2.1) has the following form:

$$\partial_z^2\psi(z)+\left[\frac{\delta_1}{z^2}+\frac{\delta_2}{(z-q)^2}+\frac{\delta_3}{(1-z)^2}+\frac{\delta_1+\delta_2+\delta_3-\delta_4}{z(1-z)}+\frac{q(1-q)c_2(q)}{z(z-q)(1-z)}\right]\psi(z)=0. \tag{2.28}$$

There is only one undetermined accessory parameter, namely $c_2(q)$. This parameter can be computed using the Polyakov conjecture once the classical four-point Liouville action is known. Before considering that problem it is important to stress that equation (2.28) appears also in the context of the classical limit of DOZZ Liouville theory.

The partition function (2.13) corresponds in the operator formulation to the correlation function of the primary fields $V_{\alpha_j}(z_j,\bar z_j)$,

$$\langle X\rangle=\Big\langle V_{\alpha_n}(\infty,\infty)\ldots V_{\alpha_1}(z_1,\bar z_1)\Big\rangle, \tag{2.29}$$

with conformal weights

$$\Delta_j=\alpha_j(Q-\alpha_j)$$

where

$$\alpha_j=\frac{Q}{2}\left(1+\xi_j\right),\qquad Q=b+\frac{1}{b}.$$

The DOZZ four-point correlation function for the standard locations $z_4=\infty, z_3=1, z_2=q, z_1=0$ is expressed as an integral over the continuous spectrum

$$\begin{aligned}&\Big\langle V_{\alpha_4}(\infty,\infty)V_{\alpha_3}(1,1)V_{\alpha_2}(q,\bar q)V_{\alpha_1}(0,0)\Big\rangle= \\ &\qquad\int\limits_{\frac{Q}{2}+i\mathbb{R}^+}d\alpha\; C(\alpha_4,\alpha_3,\alpha)C(Q-\alpha,\alpha_2,\alpha_1)\left|\mathcal{F}_{1+6Q^2,\Delta}\left[\begin{smallmatrix}\Delta_3&\Delta_2\\\Delta_4&\Delta_1\end{smallmatrix}\right](q)\right|^2.\end{aligned} \tag{2.30}$$

Let

$$\mathbf{1}_{\Delta,\Delta}=\sum_I(|\,\xi_{\Delta,I}\,\rangle\otimes|\xi_{\Delta,I}\rangle)(\langle\,\xi_{\Delta,I}\,|\otimes\langle\,\xi_{\Delta,I}\,|)$$

be an operator that projects onto the space spanned by the states belonging to the conformal family with the highest weight Δ. The correlation function with the $\mathbf{1}_{\Delta,\Delta}$ insertion factorizes into the product of the holomorphic and anti-holomorphic factors,

$$\begin{aligned}&\Big\langle V_4(\infty,\infty)V_3(1,1)\mathbf{1}_{\Delta,\Delta}V_2(q,\bar q)V_1(0,0)\Big\rangle= \\ &\qquad C(\alpha_4,\alpha_3,\alpha)\,C(Q-\alpha,\alpha_2,\alpha_1)\,\mathcal{F}_{1+6Q^2,\Delta}\left[\begin{smallmatrix}\Delta_3&\Delta_2\\\Delta_4&\Delta_1\end{smallmatrix}\right](q)\,\mathcal{F}_{1+6Q^2,\Delta}\left[\begin{smallmatrix}\Delta_3&\Delta_2\\\Delta_4&\Delta_1\end{smallmatrix}\right](\bar q).\end{aligned} \tag{2.31}$$

Assuming a path integral representation of the left hand side, one should expect in the limit $b\to 0$, with all the weights being heavy, i. e. $\Delta,\Delta_i\sim\frac{1}{b^2}$, the following asymptotic behavior

$$\Big\langle V_4(\infty,\infty)V_3(1,1)\mathbf{1}_{\Delta,\Delta}V_2(q,\bar q)V_1(0,0)\Big\rangle\sim \mathrm{e}^{-\frac{1}{b^2}S_{\rm L}^{\rm cl}(\delta_i,q;\delta)}. \tag{2.32}$$

On the other hand, the $b\to 0$ limit of the DOZZ coupling constants [20, 44] gives as a result[9]

$$C(\alpha_4,\alpha_3,\alpha)C(Q-\alpha,\alpha_2,\alpha_1)\sim \mathrm{e}^{-\frac{1}{b^2}\left(S_{\rm L}^{\rm cl}(\delta_4,\delta_3,\delta)+S_{\rm L}^{\rm cl}(\delta,\delta_2,\delta_1)\right)}. \tag{2.33}$$

[9]The quantity $S_{\rm L}^{\rm cl}(\delta_3,\delta_2,\delta_1)$ is the classical three-point Liouville action whose form is known for various types of singularities, see: [20, 44].

It follows that the conformal block should have the following asymptotic behavior when $b \to 0$

$$\mathcal{F}_{1+6Q^2,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](q) \;\sim\; \exp\left\{\frac{1}{b^2}\, f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](q)\right\}, \tag{2.34}$$

so that

$$S_{\rm L}^{\rm cl}(\delta_i, q; \delta) = S_{\rm L}^{\rm cl}(\delta_4, \delta_3, \delta) + S_{\rm L}^{\rm cl}(\delta, \delta_2, \delta_1) - f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](q) - \bar{f}_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](\bar{q}). \tag{2.35}$$

It should be stressed that the asymptotic behavior (2.34) is a nontrivial statement concerning the quantum conformal block. Although there is no proof of this property, it seems to be well confirmed together with its consequences by sample numerical calculations [45, 46]. One of such consequences is a relation between the classical four-point block $f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](q)$ and the accessory parameter $c_2(q)$ in eq. (2.28) as it will be shown below.

Consider the null field corresponding to the null vector on the second level of the Verma module. This is given by

$$\chi_{-\frac{b}{2}}(z) = \left[L_{-2}(z) - \frac{3}{2(2\Delta_{-\frac{b}{2}}+1)}\, L_{-1}^2(z)\right] V_{-\frac{b}{2}}(z,\bar{z}), \tag{2.36}$$

where $V_{\alpha=-\frac{b}{2}}$ is the degenerate primary field with the degenerate weight $\Delta_{-\frac{b}{2}} = -\frac{3}{4}\, b^2 - \frac{1}{2}$. It turns out that the projected five-point correlation function on a sphere with the null field (2.36) must vanish:

$$\left\langle \chi_{-\frac{b}{2}}(z) X\right\rangle_\Delta \;\equiv\; \left\langle V_4(\infty,\infty)V_3(1,1)\mathbf{1}_{\Delta,\Delta}\chi_{-\frac{b}{2}}(z)V_2(q,\bar{q})V_1(0,0)\right\rangle \;=\; 0. \tag{2.37}$$

The above condition and the conformal Ward identities on the sphere [41] imply that the five-point function with the degenerate operator $V_{-\frac{b}{2}}(z)$ satisfies the equation:

$$\begin{aligned}&\left[\frac{\partial^2}{\partial z^2} - b^2\left(\frac{1}{z} - \frac{1}{1-z}\right)\frac{\partial}{\partial z}\right]\left\langle V_{-\frac{b}{2}}(z)X\right\rangle_\Delta = \\ &-b^2\left[\frac{\Delta_1}{z^2} + \frac{\Delta_2}{(z-q)^2} + \frac{\Delta_3}{(1-z)^2} + \frac{\Lambda}{z(1-z)} + \frac{q(1-q)}{z(z-q)(1-z)}\frac{\partial}{\partial q}\right]\left\langle V_{-\frac{b}{2}}(z)X\right\rangle_\Delta,\end{aligned} \tag{2.38}$$

where $\Lambda \equiv \Delta_1 + \Delta_2 + \Delta_3 + \Delta_{-\frac{b}{2}} - \Delta_4$. Let us assume that all the weights Δ, Δ_i, $i = 1, \ldots, 4$ in (2.37) and then in (2.38) are heavy, i.e. $\Delta = \frac{1}{b^2}\,\delta$, $\Delta_i = \frac{1}{b^2}\,\delta_i$, $\delta, \delta_i = \mathcal{O}(1)$. In the limit $b \to 0$ only the operator with weight $\Delta_{-\frac{b}{2}}$ remains light ($\Delta_{-\frac{b}{2}} = \mathcal{O}(1)$) and its presence in the correlation function has no influence on the classical dynamics. Then, for $b \to 0$

$$\left\langle V_{-\frac{b}{2}}(z)X\right\rangle_\Delta \sim \psi(z)\,{\rm e}^{-\frac{1}{b^2}\left(S_{\rm L}^{\rm cl}(\delta_4,\delta_3,\delta)+S_{\rm L}^{\rm cl}(\delta,\delta_2,\delta_1)-f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](q)-\bar{f}_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](\bar{q})\right)} \tag{2.39}$$

and from (2.38) and (2.39) we get eq. (2.28) where the unknown accessory parameter is given by

$$c_2(q) = \frac{\partial}{\partial q}\, f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](q). \tag{2.40}$$

The relation (2.40) is nothing but the Polyakov conjecture in the case under consideration.

Indeed, in the semiclassical limit $b \to 0$ the left hand side of formula (2.30) takes the form $\mathrm{e}^{-\frac{1}{b^2}S_{\mathrm{L}}^{\mathrm{cl}}(\delta_4,\delta_3,\delta_2,\delta_1;q)}$, where we have used the shorthand notation

$$S_{\mathrm{L}}^{\mathrm{cl}}(\delta_4,\delta_3,\delta_2,\delta_1;q) \equiv S_{\mathrm{L}}^{\mathrm{cl}}(\delta_4,\delta_3,\delta_2,\delta_1;\infty,1,q,0).$$

The right hand side of (2.30) is in this limit determined by the saddle point approximation

$$\mathrm{e}^{-\frac{1}{b^2}S_{\mathrm{L}}^{\mathrm{cl}}(\delta_i,q)} \;\approx\; \int\limits_0^\infty dp\; \mathrm{e}^{-\frac{1}{b^2}S_{\mathrm{L}}^{\mathrm{cl}}(\delta_i,q;\delta)}$$

where $\delta \equiv \delta_s(q) = \frac{1}{4} + p_s(q)^2$ and the s-channel saddle point Liouville momentum $p_s(q)$, is determined by the condition

$$\frac{\partial}{\partial p} S_{\mathrm{L}}^{\mathrm{cl}}(\delta_i, q; \tfrac{1}{4} + p^2)_{|p=p_s} = 0\;. \tag{2.41}$$

One gets thus the factorization

$$\begin{aligned} S_{\mathrm{L}}^{\mathrm{cl}}(\delta_4,\delta_3,\delta_2,\delta_1;q) \;=\; & S_{\mathrm{L}}^{\mathrm{cl}}(\delta_4,\delta_3,\delta_s(q)) + S_{\mathrm{L}}^{\mathrm{cl}}(\delta_s(q),\delta_2,\delta_1) \\ & - f_{\delta_s(q)}\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](q) - \bar{f}_{\delta_s(q)}\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](\bar{q}) \end{aligned} \tag{2.42}$$

first obtained in [20]. Having the classical four-point Liouville action (2.35), one can apply the Polyakov conjecture and calculate the accessory parameter:

$$\begin{aligned} c_2(q) \;&=\; -\frac{\partial}{\partial q} S_{\mathrm{L}}^{\mathrm{cl}}(\delta_i, q) \\ &=\; -\frac{\partial}{\partial p} S_{\mathrm{L}}^{\mathrm{cl}}(\delta_i, q, \tfrac{1}{4}+p^2)\Big|_{p=p_s(q)} \frac{\partial p_s(q)}{\partial q} - \frac{\partial}{\partial q} S_{\mathrm{L}}^{\mathrm{cl}}(\delta_i, q, \tfrac{1}{4}+p^2)\Big|_{p=p_s(q)} \\ &=\; -\frac{\partial}{\partial q} S_{\mathrm{L}}^{\mathrm{cl}}(\delta_i, q, \tfrac{1}{4}+p^2)\Big|_{p=p_s(q)} \;=\; \frac{\partial}{\partial q}\, f_{\frac{1}{4}+p^2}\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](q)\Big|_{p=p_s(q)}. \end{aligned} \tag{2.43}$$

Here the saddle point equation (2.41) and the factorization (2.42) have been used.

Hence, the problem of computing the accessory parameter $c_2(q)$ is equivalent to the problem of calculating the classical four-point block. The function $f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](q)$ is known in general only as a formal power series with coefficients calculated exploiting the asymptotic behavior (2.34) and the expansion of the quantum conformal block (see the appendix). However, one can sum up the series defining the classical four-point block by applying the 'chiral' part of the AGT correspondence. More concretely, one should apply its 'classical version', which relates the classical limit of conformal blocks to the Nekrasov-Shatashvili limit of the Nekrasov instanton partition functions. The derivation of the analytic expression for the four-point classical block will be one of our main tasks in the next section.

JHEP05(2012)025

3 Accessory parameters from gauge theory

3.1 Nekrasov-Shatashvili limit

Consider the instanton part of the Nekrasov partition function of the $\mathcal{N}=2$ supersymmetric U(2) gauge theory with four hypermultiplets in the fundamental representation [25]:

$$\begin{aligned}\mathcal{Z}_{\rm inst}^{\mathrm{U}(2),N_f=4} &= 1+\sum_{k=1}^{\infty}\frac{q^k}{k!}\left(\frac{\epsilon_1+\epsilon_2}{\epsilon_1\epsilon_2}\right)^k \mathcal{Z}_k \\ &= 1+\sum_{k=1}^{\infty}\frac{q^k}{k!}\left(\frac{\epsilon_1+\epsilon_2}{\epsilon_1\epsilon_2}\right)^k \oint\frac{d\phi_1}{2\pi i}\cdots\oint\frac{d\phi_k}{2\pi i}\,\Omega_k,\end{aligned}\tag{3.1}$$

where

$$\begin{aligned}\Omega_k &= \prod_{I=1}^{k}\frac{\prod\limits_{\alpha=1}^{4}(\phi_I+m_\alpha)}{\prod\limits_{u=1}^{2}(\phi_I-a_u-i0)\,(\phi_I-a_u+\epsilon_1+\epsilon_2+i0)} \\ &\times\prod_{\substack{I,J=1\\ I\neq J}}^{k}\frac{(\phi_I-\phi_J)\,(\phi_I-\phi_J+\epsilon_1+\epsilon_2)}{(\phi_I-\phi_J+\epsilon_1+i0)\,(\phi_I-\phi_J+\epsilon_2+i0)}.\end{aligned}$$

We will assume that $a_u,\epsilon_1,\epsilon_2\in\mathbb{R}$. The contours in (3.1) go over the real axis and close in the upper half-plane. Recall, that the poles which contribute to (3.1) are in correspondence with pairs of Young diagrams $Y=\{Y_1,Y_2\}$:

$$Y\longrightarrow\phi_I=\phi_{u,r,s}=a_u+(r-1)\epsilon_1+(s-1)\epsilon_2,\qquad u=1,2.\tag{3.2}$$

The index r labels the columns while s runs over the rows of the diagram Y_u. The parameters ϵ_1 and ϵ_2 describe the size of the box $(r,s)\in Y_u$ in the horizontal and vertical direction respectively. The total number of boxes $|Y|=|Y_1|+|Y_2|$ is equal to the instanton number k. The instanton sum over k in (3.1) can be rewritten as a sum over a pairs of Young diagrams as follows:

$$\mathcal{Z}_k=\sum_{\substack{Y\\|Y|=k}}\mathcal{Z}_Y.$$

The contributions $\mathcal{Z}_Y$ to the instanton sum correspond to those obtained by performing (in some specific order) the contour integrals in (3.1).

Now we want to calculate the limit $\epsilon_2\to0$ of the instanton partition function (3.1). Based on the arguments developed in [26], it is reasonable to expect that for vanishingly small values of ϵ_2 the dominant contribution to the instanton partition function (3.1) will

JHEP05(2012)025

occur when $k \sim \frac{1}{\epsilon_2}$.[10] For future purposes it will be necessary to compute the leading behavior of $\log\left|q^k\Omega_k\right|$ for large k (i. e. small values of ϵ_2 and finite ϵ_1). After simple calculations[11] we find:

$$\log\left|q^k\Omega_k\right| \sim \frac{1}{\epsilon_2}\left[\epsilon_2 k\log|\,q|+\epsilon_2\sum_{I=1}^{k}\left[\sum_{\alpha=1}^{4}\log|\phi_I+m_\alpha|-\sum_{u=1}^{2}\log\left(|\phi_I-a_u|\,|\phi_I-a_u+\epsilon_1|\right)\right]\right.$$
$$\left.+\,\epsilon_2^2\sum_{\substack{I,J=1\\ I\neq J}}^{k}\left[\frac{1}{\phi_I-\phi_J+\epsilon_1}-\frac{1}{\phi_I-\phi_J}\right]\right]. \tag{3.3}$$

In eq. (3.3) it is implicitly understood that the poles ϕ_I are obtained from eq. (3.2) in the limit $\epsilon_2 \to 0$. It turns out that the right hand side of eq. (3.3) is equal up to the factor $\frac{1}{\epsilon_2}$ to the instantonic free energy $\mathcal{H}_{\text{inst}}^{\mathrm{U}(2),N_f=4}$. Note that in the limit $\epsilon_2 \to 0$ the poles form a continuous distribution:

$$\phi_I = \phi_{u,r} \in \left[x_{u,r}^0, x_{u,r}\right] \tag{3.4}$$

where

$$\begin{aligned} x_{u,r}^0 &= a_u + (r-1)\epsilon_1, \qquad u = 1,2, \qquad r = 1,\ldots,\infty,\\ x_{u,r} &= a_u + (r-1)\epsilon_1 + \omega_{u,r}. \end{aligned}$$

In the language of Young diagrams the two formulas above can be explained as follows. When ϵ_2 is very small, the number of boxes $k_{u,r}$ in the vertical direction (the number of rows) is very large, while the quantity $\omega_{u,r} = \epsilon_2 k_{u,r}$ is expected to be finite. In other words, we obtain a continuous distribution of rows in the limit under consideration. As a consequence, in order to evaluate the instanton free energy, the sums 'over the instantons' in (3.3) may be replaced by continuous integrals in the row index, with the range of integration specified by eq. (3.4). It is thus possible to write:

$$\epsilon_2\sum_I \longrightarrow \sum_{u,r}\int_{x_{u,r}^0}^{x_{u,r}} d\phi_{u,r}. \tag{3.5}$$

[10]This statement becomes evident in the trivial case in which $\mathcal{Z}_k = 1\ \forall k = 1,2,\ldots$. For $\epsilon_2 \to 0$ and $x = q/\epsilon_2 \in \mathbb{R}_{>0}$ we have then from eq. (3.1)

$$\mathcal{Z}_{\text{inst}} = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{q}{\epsilon_2}\right)^k = \sum_{k=0}^{\infty}\frac{x^k}{k!} = \mathrm{e}^x = \frac{\mathrm{e}^{x\log x}}{\mathrm{e}^{x\log x - x}} \sim \frac{\mathrm{e}^{x\log x}}{\mathrm{e}^{\log x!}} = \frac{x^x}{x!}.$$

This means that the whole sum is dominated by a single term with $k \sim x \to \infty$. Unfortunately, we have found no proof of that mechanism in the general case.

[11]The computations which lead to (3.3) are elementary and rely on the Taylor expansion of $\log(x \pm \epsilon_2)$ for small ϵ_2, i.e.: $\log(x \pm \epsilon_2) = \log(x) \pm \frac{\epsilon_2}{x} + O(\epsilon_2^2)$.

The integration limits $x^0_{u,r}$ and $x_{u,r}$ represent the bottom and the top ends of the r-th column in Y_u respectively. Applying eq. (3.5) to eq. (3.3) one gets

$$\begin{aligned}
\mathcal{H}^{\mathrm{U}(2),N_f=4}_{\text{inst}}(x_{u,r}) = & \sum_{u,v=1}^{2}\sum_{r,l=1}^{\infty}\Big[-F\left(x_{u,r}-x_{v,l}+\epsilon_1\right)+F\left(x_{u,r}-x^0_{v,l}+\epsilon_1\right) \\
& +F\left(x^0_{u,r}-x_{v,l}+\epsilon_1\right)-F\left(x^0_{u,r}-x^0_{v,l}+\epsilon_1\right)+F\left(x_{u,r}-x_{v,l}\right) \\
& -F\left(x_{u,r}-x^0_{v,l}\right)-F\left(x^0_{u,r}-x_{v,l}\right)+F\left(x^0_{u,r}-x^0_{v,l}\right)\Big] \\
& +\sum_{u,v=1}^{2}\sum_{r=1}^{\infty}\Big[-F\left(x_{u,r}-a_v\right)+F\left(x^0_{u,r}-a_v\right)-F\left(x_{u,r}-a_v+\epsilon_1\right)+F\left(x^0_{u,r}-a_v+\epsilon_1\right)\Big] \\
& +\sum_{u=1}^{2}\sum_{r=1}^{\infty}\sum_{\alpha=1}^{4}\Big[F\left(x_{u,r}+m_\alpha\right)-F\left(x^0_{u,r}+m_\alpha\right)\Big]+\sum_{u=1}^{2}\sum_{r=1}^{\infty}\left(x_{u,r}-(r-1)\epsilon_1-a_u\right)\log|q|\,,
\end{aligned} \tag{3.6}$$

where $F(x) = x(\log|x|-1)$.

Let us turn to the main problem of our interest. According to the ideology of [26] the Nekrasov instanton partition function in the limit $\epsilon_2 \to 0$ can be represented as follows:

$$\mathcal{Z}^{\mathrm{U}(2),N_f=4}_{\text{inst}} \sim \int \left[\prod_{u,r} dx_{u,r}\right] \exp\left\{\frac{1}{\epsilon_2}\mathcal{H}^{\mathrm{U}(2),N_f=4}_{\text{inst}}(x_{u,r})\right\}, \tag{3.7}$$

where the integral is over the infinite set of variables $\{x_{u,r} : u = 1,2; r = 1,\dots,\infty\}$. As a consequence, the Nekrasov-Shatashvili limit of $\mathcal{Z}^{\mathrm{U}(2),N_f=4}_{\text{inst}}$ is nothing but the critical value of $\mathcal{H}^{\mathrm{U}(2),N_f=4}_{\text{inst}}$:

$$\mathcal{W}^{\mathrm{U}(2),N_f=4}_{\text{inst}} \equiv \lim_{\epsilon_2\to 0}\epsilon_2 \log \mathcal{Z}^{\mathrm{U}(2),N_f=4}_{\text{inst}} = \mathcal{H}^{\mathrm{U}(2),N_f=4}_{\text{inst}}(x^*_{u,r}), \tag{3.8}$$

where $x^*_{u,r}$ denotes the 'critical configuration' extremizing the 'free energy' (3.6).

3.2 Saddle point equation

The extremality condition for the 'action' $\mathcal{H}^{\mathrm{U}(2),N_f=4}_{\text{inst}}$ given by (3.6) reads as follows:

$$\left| q\left(\prod_{v=1}^{2}\prod_{l=1}^{\infty}\frac{(x_{u,r}-x_{v,l}-\epsilon_1)(x_{u,r}-x^0_{v,l}+\epsilon_1)}{(x_{u,r}-x_{v,l}+\epsilon_1)(x_{u,r}-x^0_{v,l}-\epsilon_1)}\right)\left(\frac{\prod_{\alpha=1}^{4}(x_{u,r}+m_\alpha)}{\prod_{v=1}^{2}(x_{u,r}-a_v)(x_{u,r}-a_v+\epsilon_1)}\right)\right| = 1.$$

This implies that either the following identity:

$$-q\left(\prod_{v=1}^{2}\prod_{l=1}^{\infty}\frac{(x_{u,r}-x_{v,l}-\epsilon_1)(x_{u,r}-x^0_{v,l}+\epsilon_1)}{(x_{u,r}-x_{v,l}+\epsilon_1)(x_{u,r}-x^0_{v,l}-\epsilon_1)}\right)\left(\frac{\prod_{\alpha=1}^{4}(x_{u,r}+m_\alpha)}{\prod_{v=1}^{2}(x_{u,r}-a_v)(x_{u,r}-a_v+\epsilon_1)}\right) = 1 \tag{3.9}$$

or its analog in which $-q$ is replaced by $+q$ are holding. To find the solution of eq. (3.9) will be the main task of this subsection. Eq. (3.9) can be regularized assuming that there is an integer L such that the length of the column $\omega_{u,r}$ is equal to zero for $r > L$. Analyzing eq. (3.9) in such a case, i.e. when $l = 1, \ldots, L$, one can observe that the column lengths extremizing the 'free energy' are of the order $\omega_{u,r} \sim \mathcal{O}(q^r)$. For example, at order q^L one can write

$$x^*_{u,r} \equiv x_{u,r} = a_u + (r-1)\epsilon_1 + \omega_{u,r}(q) = a_u + (r-1)\epsilon_1 + \sum_{n=r}^{L} \omega_{u,r,n}\, q^n. \tag{3.10}$$

Here the symbols $\omega_{u,r,n}$ denote the contributions to the coefficients $\omega_{u,r}$ at the $n-$th order in q. Now it is possible to solve equation (2.41) starting from $L = 1$ and deriving recursively the $\omega_{u,r,n}$'s step by step up to desired order. The calculation of the first few coefficients $\omega_{u,r,n}$ is presented below.

1. For $L = 1$ the equation (3.9) becomes

$$-q \prod_{v=1}^{2} \left(\frac{(x_{u,r} - x_{v,1} - \epsilon_1)(x_{u,r} - x^0_{v,1} + \epsilon_1)}{(x_{u,r} - x^0_{v,1} - \epsilon_1)(x_{u,r} - x_{v,1} + \epsilon_1)(x_{u,r} - a_v)(x_{u,r} - a_v + \epsilon_1)} \right) \prod_{\alpha=1}^{4} (x_{u,r} + m_\alpha) = 1$$

   or equivalently

$$\begin{aligned} &q(x_{u,r} - x_{1,1} - \epsilon_1)(x_{u,r} - x_{2,1} - \epsilon_1)(x_{u,r} + m_1)(x_{u,r} + m_2)(x_{u,r} + m_3)(x_{u,r} + m_4) \\ &\quad +(x_{u,r} - a_1 - \epsilon_1)(x_{u,r} - a_2 - \epsilon_1)(x_{u,r} - x_{1,1} + \epsilon_1)(x_{u,r} - x_{2,1} + \epsilon_1) \\ &\quad \times (x_{u,r} - a_1)(x_{u,r} - a_2) \;=\; 0. \end{aligned} \tag{3.11}$$

   Hereafter we fix the freedom in the choice of the parameters a_1 and a_2 by setting $(a_1, a_2) = (a, -a)$. Thus, expanding (3.10) up to the first order in q and substituting the result into (3.11) one finds that

$$\omega_{1,1,1} = -\frac{\prod_{\alpha=1}^{4} (a + m_\alpha)}{\epsilon_1 2a(2a + \epsilon_1)}, \qquad \omega_{2,1,1} = -\frac{\prod_{\alpha=1}^{4} (a - m_\alpha)}{\epsilon_1 2a(2a - \epsilon_1)}. \tag{3.12}$$

2. For $L = 2$ the system of linear equations obtained from (3.9) yields the second order corrections to the length of the first column. The coefficients are

$$\begin{aligned} \omega_{1,1,2} = \Big\{ &(a + m_1)(a + m_2)(a + m_3)(a + m_4)\,(a - \epsilon_1) \\ &\times \Big(8a^5 (a + m_1)(a + m_2)(a + m_3)(a + m_4) - a^2 \epsilon_1^6 \,(2a + \mathrm{m}) - a^2 \epsilon_1^7 \\ &+ \epsilon_1^5 \left[13a^4 + 4a^3 \mathrm{m} + 2a^2 \mu + a\hat{\mu} - \mathfrak{m} \right] \\ &+ a\epsilon_1^4 \left[5a^4 + 6a^3 \mathrm{m} - 7a^2 \mu + 2a\hat{\mu} - 3\mathfrak{m} \right] \\ &- a^2 \epsilon_1^3 \left[51a^4 + 5a^3 \mathrm{m} + 19a^2 \mu - 11a\hat{\mu} + 3\mathfrak{m} \right] \\ &+ 2a^3 \epsilon_1^2 \left[-11a^4 + 3a^3 \mathrm{m} - 3a^2 \mu + 3a\hat{\mu} - 11\mathfrak{m} \right] \\ &+ 4a^4 \epsilon_1 \left[7a^4 + 5a^3 \mathrm{m} + 3a^2 \mu + a\hat{\mu} - \mathfrak{m} \right] \Big) \Big\} \\ \times \Big\{ &8a^3 \epsilon_1^3 \,(2a + \epsilon_1)^2 \,(2a^2 + a\epsilon_1 - \epsilon_1^2)\,(4a^3 - 4a^2 \epsilon_1 - a\epsilon_1^2 + \epsilon_1^3) \Big\}^{-1}, \end{aligned}$$

and

$$\omega_{2,1,2} = \Big\{(a-m_1)(a-m_2)(a-m_3)(a-m_4)\,(a+\epsilon_1)$$
$$\times\Big(8a^5(a-m_1)(a-m_2)(a-m_3)(a-m_4) + a^2\epsilon_1^6\,(-2a+\mathrm{m}) + a^2\epsilon_1^7$$
$$+\epsilon_1^5\left[-13a^4+4a^3\mathrm{m}-2a^2\mu+a\hat{\mu}+\mathfrak{m}\right]$$
$$+a\epsilon_1^4\left[5a^4-6a^3\mathrm{m}-7a^2\mu+2a\hat{\mu}-3\mathfrak{m}\right]$$
$$+a^2\epsilon_1^3\left[51a^4-5a^3\mathrm{m}+19a^2\mu+11a\hat{\mu}+3\mathfrak{m}\right]$$
$$-2a^3\epsilon_1^2\left[11a^4+3a^3\mathrm{m}+3a^2\mu+3a\hat{\mu}+11\mathfrak{m}\right]$$
$$-4a^4\epsilon_1\left[7a^4-5a^3\mathrm{m}+3a^2\mu-a\hat{\mu}-\mathfrak{m}\right]\Big)\Big\}$$
$$\times\Big\{8a^3\epsilon_1^3\,(\epsilon_1-2a)^2\,\left(-2a^2+a\epsilon_1+\epsilon_1^2\right)\left(-4a^3-4a^2\epsilon_1+a\epsilon_1^2+\epsilon_1^3\right)\Big\}^{-1},$$

where

$$\mathrm{m}\equiv\sum_{i=1}^{4}m_i,\qquad \mathfrak{m}\equiv\prod_{i=1}^{4}m_i,\qquad \mu\equiv\sum_{1\le i<j\le 4}m_im_j,\qquad \hat{\mu}\equiv\sum_{1\le i<j<k\le 4}m_im_jm_k.$$

Moreover, from eq. (3.9) with $L=2$ it is also possible to determine the length of the second column at the leading order in q^2. Indeed, one can derive the following coefficients:

$$\omega_{1,2,2} = -\frac{\prod_{\alpha=1}^{4}(a+m_\alpha)(a+\epsilon_1+m_\alpha)}{8a\epsilon_1^3(a+\epsilon_1)(2a+\epsilon_1)^2},$$
$$\omega_{2,2,2} = -\frac{\prod_{\alpha=1}^{4}(a-m_\alpha)(a-\epsilon_1-m_\alpha)}{8a\epsilon_1^3(a-\epsilon_1)(\epsilon_1-2a)^2}.$$

3.3 Twisted superpotential, classical block and accessory parameter

Knowing the extremal lengths of the columns one can calculate the critical value of the 'free energy' (3.8), i.e. the so-called twisted superpotential. In order to compute this critical value it is convenient first to calculate the derivative of $\mathcal{W}_{\rm inst}^{{\rm U}(2),N_f=4}(\,q,a,m_i;\epsilon_1)$ with respect to q:

$$\frac{\partial}{\partial q}\mathcal{W}_{\rm inst}^{{\rm U}(2),N_f=4}(\,q,a,m_i;\epsilon_1) = \frac{\partial\mathcal{H}_{\rm inst}}{\partial x_{u,r}}\frac{\partial x_{u,r}}{\partial q}+\frac{\partial\mathcal{H}_{\rm inst}}{\partial q} = \frac{1}{q}\sum_{u,r}\omega_{u,r}. \tag{3.13}$$

In the above calculation we have used the fact that $\partial\mathcal{H}_{\rm inst}/\partial x_{u,r}=0$. It is easy to realize that the last term in (3.13) coincides with the sum over the column lengths of the 'critical'

Young diagram. Performing this sum one obtains the correct expansion of the twisted superpotential. Indeed, using (3.13) one gets:

$$
\begin{aligned}
q\frac{d}{dq}\mathcal{W}_{\rm inst}^{\mathrm{U}(2),N_f=4} &= \sum_r\left(\omega_{1,r}(q)+\omega_{2,r}(q)\right)=\sum_r\left[\sum_{n=r}\left(\omega_{1,r,n}+\omega_{2,r,n}\right)q^n\right]\\
&=\left[\left(\omega_{1,1,1}+\omega_{2,1,1}\right)q+\left(\omega_{1,1,2}+\omega_{2,1,2}\right)q^2+\ldots\right]\\
&\quad+\left[\left(\omega_{1,2,2}+\omega_{2,2,2}\right)q^2+\left(\omega_{1,2,3}+\omega_{2,2,3}\right)q^3+\ldots\right]+\ldots\,. \qquad (3.14)
\end{aligned}
$$

Then,

$$
\begin{aligned}
\mathcal{W}_{\rm inst}^{\mathrm{U}(2),N_f=4} &= \left(\omega_{1,1,1}+\omega_{2,1,1}\right)q+\left(\omega_{1,1,2}+\omega_{2,1,2}+\omega_{1,2,2}+\omega_{2,2,2}\right)\frac{q^2}{2}+\ldots\\
&=\mathcal{W}_1^{\mathrm{U}(2),N_f=4}\,q+\mathcal{W}_2^{\mathrm{U}(2),N_f=4}\,q^2+\ldots\,. \qquad (3.15)
\end{aligned}
$$

The expansion (3.15) with the coefficients calculated from the saddle point equation exactly agrees with that obtained directly form the expansion of the instanton partition function. Moreover, assuming the following relations between parameters:

$$
\begin{aligned}
m_1 &= \epsilon_1\left(\eta_1+\eta_2-\frac{1}{2}\right),\quad m_2=\epsilon_1\left(\eta_2-\eta_1+\frac{1}{2}\right),\\
m_3 &= \epsilon_1\left(\eta_3+\eta_4-\frac{1}{2}\right),\quad m_4=\epsilon_1\left(\eta_3-\eta_4+\frac{1}{2}\right),\qquad a=\epsilon_1\left(\eta-\frac{1}{2}\right) \qquad (3.16)
\end{aligned}
$$

and using the expression of the coefficients $\omega_{u,r,n}$'s calculated in the previous paragraph, one can check that

$$
\begin{aligned}
\frac{1}{\epsilon_1}\mathcal{W}_1^{\mathrm{U}(2),N_f=4} &= \frac{1}{\epsilon_1}\left(\omega_{1,1,1}+\omega_{2,1,1}\right)=\frac{(\delta+\delta_2-\delta_1)(\delta+\delta_3-\delta_4)-4\delta\eta_2\eta_3}{2\delta}\\
&= \mathsf{f}_\delta^1\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right]-2\eta_2\eta_3=\mathsf{f}_\delta^1\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right]-\frac{(m_1+m_2)(m_3+m_4)}{2\epsilon_1^2}, \qquad (3.17)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{\epsilon_1}\mathcal{W}_2^{\mathrm{U}(2),N_f=4} &= \frac{1}{\epsilon_1}\frac{1}{2}\left(\omega_{1,1,2}+\omega_{2,1,2}+\omega_{1,2,2}+\omega_{2,2,2}\right)\\
&= \mathsf{f}_\delta^2\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right]-\eta_2\eta_3=\mathsf{f}_\delta^2\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right]-\frac{1}{2}\left(2\eta_2\eta_3\right), \qquad (3.18)
\end{aligned}
$$

where

$$
\delta=\eta\,(1-\eta)\,,\qquad \delta_i=\eta_i\,(1-\eta_i)\,,\qquad i=1,\ldots,4. \qquad (3.19)
$$

In eqs. (3.17) and (3.18) the symbols $\mathsf{f}_\delta^n\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right]$'s for $n=1,2$ are the first two coefficients of the classical four-point block introduced in (2.34) (see appendix):

$$
\begin{aligned}
f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](q) &= (\delta-\delta_1-\delta_2)\log q+\mathsf{f}_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](q)\\
&= (\delta-\delta_1-\delta_2)\log q+\sum_{n=1}^{\infty}\mathsf{f}_\delta^n\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right]\,q^n. \qquad (3.20)
\end{aligned}
$$

It is thus reasonable to expect that

$$\frac{1}{\epsilon_1}\left(\mathcal{W}_1^{\mathrm{U}(2),N_f=4}\,q+\mathcal{W}_2^{\mathrm{U}(2),N_f=4}\,q^2+\ldots\right)=\left(\mathsf{f}^1_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right]-2\eta_2\eta_3\right)q$$
$$+\left(\mathsf{f}^2_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right]-\frac{1}{2}\left(2\eta_2\eta_3\right)\right)q^2+\ldots\,. \tag{3.21}$$

The conjectured identity (3.21) is nothing but the expansion of both sides of the relation:

$$\frac{1}{\epsilon_1}\,\mathcal{W}_{\mathrm{inst}}^{\mathrm{U}(2),N_f=4}(q,a,m_i;\epsilon_1)=\mathrm{f}_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](q)+\frac{(m_1+m_2)(m_3+m_4)}{2\epsilon_1^2}\,\log(1-q). \tag{3.22}$$

The identities (3.21) or (3.22) are justified not only by the above calculations. Note that the relation (3.22) is nothing else but the classical/Nekrasov-Shatashvili limit of the AGT relation:[12]

$$q^{\Delta_1+\Delta_2-\Delta}\,\mathcal{F}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](\,q)=(1-q)^{-\frac{(m_1+m_2)(m_3+m_4)}{2\epsilon_1\epsilon_2}}\,\mathcal{Z}_{\mathrm{inst}}^{\mathrm{U}(2),N_f=4}(q,a,m_i;\epsilon_1,\epsilon_2),$$

where

$$\begin{aligned}
c&=1+6\frac{(\epsilon_1+\epsilon_2)^2}{\epsilon_1\epsilon_2}\equiv 1+6Q^2, & \Delta&=\frac{(\epsilon_1+\epsilon_2)^2-4a^2}{4\epsilon_1\epsilon_2},\\
\Delta_1&=\frac{\frac{1}{4}(\epsilon_1+\epsilon_2)^2-\frac{1}{4}(m_1-m_2)^2}{\epsilon_1\epsilon_2}, & \Delta_2&=\frac{\frac{1}{2}(m_1+m_2)(\epsilon_1+\epsilon_2-\frac{1}{2}(m_1+m_2))}{\epsilon_1\epsilon_2},\\
\Delta_3&=\frac{\frac{1}{2}(m_3+m_4)(\epsilon_1+\epsilon_2-\frac{1}{2}(m_3+m_4))}{\epsilon_1\epsilon_2}, & \Delta_4&=\frac{\frac{1}{4}(\epsilon_1+\epsilon_2)^2-\frac{1}{4}(m_3-m_4)^2}{\epsilon_1\epsilon_2}
\end{aligned}$$

and

$$Q=b+\frac{1}{b}\equiv\sqrt{\frac{\epsilon_2}{\epsilon_1}}+\sqrt{\frac{\epsilon_1}{\epsilon_2}}\qquad\Longleftrightarrow\qquad b=\sqrt{\frac{\epsilon_2}{\epsilon_1}}.$$

As a final conclusion of this subsection let us write down the two main results of the present work. The first one is a novel representation for the four-point classical block with four elliptic/parabolic external classical weights and a hyperbolic intermediate classical weight. Indeed, from eqs. (3.8), (3.20) and (3.22) we have

$$\begin{aligned}
f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](\,q)&=(\delta-\delta_1-\delta_2)\log q-\frac{(m_1+m_2)(m_3+m_4)}{2\epsilon_1^2}\,\log(1-q)\\
&\quad+\frac{1}{\epsilon_1}\,\mathcal{H}_{\mathrm{inst}}^{\mathrm{U}(2),N_f=4}(x^*_{u,r}(\,q))\\
&=(\delta-\delta_1-\delta_2)\log q-\frac{(m_1+m_2)(m_3+m_4)}{2\epsilon_1^2}\,\log(1-q)\\
&\quad+\frac{1}{\epsilon_1}\,\mathcal{W}_{\mathrm{inst}}^{\mathrm{U}(2),N_f=4}(\,q,a,m_i;\epsilon_1),
\end{aligned}\tag{3.23}$$

[12] For a proof of the AGT relation on $C_{0,n}$ see [6].

JHEP05(2012)025

where the classical conformal weights are parameterized as in (3.19) with η's given by (3.16). Knowing the classical four-point block from (3.23) and applying eqs. (2.43) and (3.13), one arrives at the following expression of the accessory parameter $c_2(q)$:

$$c_2(q) \;=\; \frac{\delta-\delta_1-\delta_2+\frac{1}{\epsilon_1}\sum_{u,r}\omega_{u,r}(q,a,m_i;\epsilon_1)}{q}+\frac{2\eta_2\eta_3}{1-q}, \tag{3.24}$$

The four masses m_i appearing in eq. (3.24) are given by eq. (3.16) and the vacuum expectation value $a=-i\epsilon_1 p_s(q)$ is proportional to the s-channel momentum $p_s(q)$. Hence, we have found that the accessory parameter $c_2(q)$ is related to the sum of column lengths of the 'critical' Young diagram. The latter can be rewritten using the contour integral representation. Following [9, 27] let us define the functions:

$$\begin{aligned} Y(z) &= \prod_{u=1}^{2}\exp\left\{\frac{z}{\epsilon_1}\,\psi\left(\frac{a_u}{\epsilon_1}\right)\right\}\prod_{r=1}^{\infty}\left(1-\frac{z}{x_{u,r}}\right)\exp\left\{\frac{z}{x^0_{u,r}}\right\},\\ Y_0(z) &= \prod_{u=1}^{2}\exp\left\{\frac{z}{\epsilon_1}\,\psi\left(\frac{a_u}{\epsilon_1}\right)\right\}\prod_{r=1}^{\infty}\left(1-\frac{z}{x^0_{u,r}}\right)\exp\left\{\frac{z}{x^0_{u,r}}\right\}, \end{aligned}$$

where $\psi(z)=\partial_z\log\Gamma(z)$. The functions $Y(z)$, $Y_0(z)$ are holomorphic with zeros located at $x_{u,r}$ and $x^0_{u,r}$ respectively. Then,

$$\sum_{u,r}\omega_{u,r} \;=\; \left[\sum_{u,r}\left(x_{u,r}-x^0_{u,r}\right)\right]\Big|_{x_{u,r}=x^*_{u,r}} \;=\; \left(\oint_\gamma\frac{dz}{2\pi i}\,z\partial_z\log\frac{Y(z)}{Y_0(z)}\right)\Big|_{x_{u,r}=x^*_{u,r}},$$

where γ encloses all the points $x_{u,r}$, $x^0_{u,r}$, $u=1,2$, $r=1,\ldots,\infty$.

4 Conclusions

The original results of the present paper are:

- The derivation of the generic classical four-point block provided in eq. (3.23), where the classical four-point block has been written in terms of the critical value of the instanton 'free energy' $\mathcal{H}^{\mathrm{U}(2),N_f=4}_{\mathrm{inst}}$.

- The derivation in closed form of the accessory parameter $c_2(q)$ appearing in the Fuchs equation with four parabolic/elliptic singularities. So far the expression of this accessory parameter was unknown. From eq. (3.24) $c_2(q)$ can be interpreted as the sum of all column lengths of the 'critical' Young diagram which extramizes $\mathcal{H}^{\mathrm{U}(2),N_f=4}_{\mathrm{inst}}$.

The above results possess interesting further applications. According to the formula (2.6) the solution of the accessory parameters problem for the Fuchs equation on $C_{0,4}$ offers the possibility of constructing the solution of the Liouville equation on $C_{0,4}$. In the case of a sphere with four parabolic singularities the above results pave the way for the construction of the uniformization map and the computation of the so-called geodesic length functions [45].

As a next point let us discuss the possible extensions of this work. One of its aims was to find an analytic expression of the classical four-point block exploiting the 'chiral' sector of the AGT correspondence on $C_{0,4}$. It has been found that the classical block can be expressed in terms of the U(2), $N_f = 4$ instanton twisted superpotential. If the classical four-point block and the three-point classical Liouville action are available one can construct the four-point classical action (cf. (2.35)). The three-point Liouville action can be recovered from the DOZZ structure constant in the classical limit. As a consequence, due to the AGT duality the four-point classical action should correspond to the full effective twisted superpotential $\mathcal{W}^{\mathrm{U}(2),N_f=4} = \mathcal{W}_{\mathrm{pert}}^{\mathrm{U}(2),N_f=4} + \mathcal{W}_{\mathrm{inst}}^{\mathrm{U}(2),N_f=4}$. Based on this example it is reasonable to expect that:

1. the U(2) Nekrasov partition function with $N_f = n$ flavors encodes an information on the n-point classical Liouville action $S_{\mathrm{L}}^{\mathrm{cl},(n)}$ on the sphere;

2. the classical Liouville action $S_{\mathrm{L}}^{\mathrm{cl},(n)}$ can be recovered from the Nekrasov partition function in the Nekrasov-Shatashvili limit;

3. the classical action $S_{\mathrm{L}}^{\mathrm{cl},(n)}$ can be expressed as the critical value of the gauge theory 'free energy'.

If these conjectures are true they provide a direct way to calculate $S_{\mathrm{L}}^{\mathrm{cl},(n)}$.

In this work we have studied a version of the problem of accessory parameters which is related to the classical Liouville theory on the sphere. Let us stress that there exists also an analogous problem in the case of the torus topology [47, 48]. It would be interesting to investigate whether the AGT correspondence can be applied to solve the problem of accessory parameters also in that case.

Finally, it seems to be an interesting task to study possible overlaps of our results and those in papers [1, 3].

A Quantum and classical four-point conformal blocks

Let

$$
\begin{aligned}
\mathcal{V}_{\Delta_j} &= \bigoplus_{n=0}^{\infty} \mathcal{V}_{\Delta_j}^n, \\
\mathcal{V}_{\Delta_j}^n &= \mathrm{Span}\Big\{ \nu_{\Delta_j,I}^n = L_{-I}\nu_{\Delta_j} = L_{-i_k}\ldots L_{-i_2}L_{-i_1}\nu_{\Delta_j} \\
&\qquad\qquad : I = (i_k \geq \ldots \geq i_1 \geq 1) \text{ an ordered set of positive integers} \\
&\qquad\qquad\qquad\qquad \text{of the length } |I| \equiv i_1 + \ldots + i_k = n \Big\}
\end{aligned}
$$

be the Verma module with the highest weight state ν_{Δ_j}. The chiral vertex operator is the linear map

$$
V_{\infty\ z\ 0}^{\Delta_3\Delta_2\Delta_1} : \mathcal{V}_{\Delta_2} \otimes \mathcal{V}_{\Delta_1} \longrightarrow \mathcal{V}_{\Delta_3}
$$

such that for all $\xi_2 \in \mathcal{V}_{\Delta_2}$ the operator

$$V(\xi_2|z) \equiv V^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\; z\;\; 0}(\xi_2 \otimes \cdot\,) : \mathcal{V}_{\Delta_1} \longrightarrow \mathcal{V}_{\Delta_3}$$

satisfies the following conditions

$$\begin{aligned}
[L_n, V(\nu_2|z)] &= z^n \left(z\frac{d}{dz} + (n+1)\Delta_2 \right) V(\nu_2|z)\,, \qquad n \in \mathbb{Z} \qquad\qquad (A.1)\\
V(L_{-1}\xi_2|z) &= \frac{d}{dz} V(\xi_2|z)\,,\\
V(L_n\xi_2|z) &= \sum_{k=0}^{n+1} \binom{n+1}{k} (-z)^k \,[L_{n-k}, V(\xi_2|z)]\,, \qquad n > -1,\\
V(L_{-n}\xi_2|z) &= \sum_{k=0}^{\infty} \binom{n-2+k}{n-2} z^k \, L_{-n-k}\, V(\xi_2|z)\\
&\quad + (-1)^n \sum_{k=0}^{\infty} \binom{n-2+k}{n-2} z^{-n+1-k}\; V(\xi_2|z)\; L_{k-1}, \qquad n > 1
\end{aligned}$$

and

$$\langle \nu_{\Delta_3}, V(\nu_2|z)\, \nu_{\Delta_1}\rangle \;=\; z^{\Delta_3-\Delta_2-\Delta_1}.$$

Let q be the moduli of the 4-punctured sphere. The quantum four-point conformal block is defined as the formal power series:

$$\mathcal{F}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](\,q) \;=\; q^{\Delta-\Delta_2-\Delta_1}\left(1 + \sum_{n=1}^{\infty} \mathcal{F}^{\,n}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] q^{\,n}\right) \qquad (A.2)$$

with coefficients given by

$$\mathcal{F}^{\,n}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] \;=\; \sum_{n=|I|=|J|} \langle \nu_{\Delta_4}, V(\nu_3|1)\nu_{\Delta,I}\rangle\; \Big[G_{c,\Delta}\Big]^{IJ}\; \langle \nu_{\Delta,J}, V(\nu_2|1)\nu_{\Delta_1}\rangle\,. \qquad (A.3)$$

Above $\Big[G_{c,\Delta}\Big]^{IJ}$ is the inverse of the Gram matrix $\Big[G_{c,\Delta}\Big]_{IJ} = \langle \nu_{\Delta,I}, \nu_{\Delta,J}\rangle$ of the standard symmetric bilinear form in the Verma module. Taking into account the covariance properties (A.1) of the primary chiral vertex operator with respect to the Virasoro algebra one can calculate the matrix elements in (A.3). Hence, for lower orders of the expansion the coefficients (A.3) can be easily computed directly from definition. For instance,

$$\begin{aligned}
\mathcal{F}^{1}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] &= \frac{(\Delta+\Delta_3-\Delta_4)(\Delta+\Delta_2-\Delta_1)}{2\Delta},\\
\mathcal{F}^{2}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] &= \Big[\,(4\Delta(1+2\Delta))^{-1}\,(\Delta-\Delta_1+\Delta_2)(1+\Delta-\Delta_1+\Delta_2)\\
&\qquad \times(\Delta+\Delta_3-\Delta_4)(1+\Delta+\Delta_3-\Delta_4)\\
&\qquad + \left(\Delta-\Delta^2-\Delta_1-\Delta_2+3(\Delta_2-\Delta_1)^2-2\Delta(\Delta_1+\Delta_2)\right)\\
&\qquad \times \left(\Delta-\Delta^2-\Delta_3-\Delta_4+3(\Delta_3-\Delta_4)^2-2\Delta(\Delta_3+\Delta_4)\right)\Big]\\
&\quad \times \left[2(1+2\Delta)^2\left(c - \frac{4\Delta(5-8\Delta)}{2+4\Delta}\right)\right]^{-1}.
\end{aligned}$$

As the dimension of $\mathcal{V}_\Delta^n$ grows rapidly with n, the calculations of conformal block coefficients by inverting the Gram matrices become very laborious for higher orders. A more efficient method based on recurrence relations for the coefficients can be used [49–51].

Let us assume that all the conformal weights in the conformal block are heavy. Then, the asymptotic behavior (2.34) implies the following expansion of the 4-point classical block:

$$
\begin{aligned}
f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](q) &= (\delta-\delta_1-\delta_2)\log q + \sum_{n=1}^{\infty} q^n\, \mathrm{f}_\delta^n\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right] \\
&= (\delta-\delta_1-\delta_2)\log q + \lim_{b\to 0} b^2 \log\left(1+\sum_{n=1}^{\infty}\mathcal{F}_{c,\Delta}^n\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] q^n\right).
\end{aligned}
\tag{A.4}
$$

The coefficients $\mathrm{f}_\delta^n\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right]$ in (A.4) are calculated directly from the limit (2.34) and the power expansion of the quantum block. For example, expanding the logarithm into power series and then taking the limit of each term separately for $n=1,2$ one finds

$$
\begin{aligned}
\mathrm{f}_\delta^1\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right] &= \frac{(\delta+\delta_3-\delta_4)(\delta+\delta_2-\delta_1)}{2\delta}, \\
\mathrm{f}_\delta^2\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right] &= \Big[16\delta^3(4\delta+3)\Big]^{-1}\Big[13\delta^5+\delta^4\left(18\delta_2-14\delta_1+18\delta_3-14\delta_4+9\right) \\
&\quad +\delta^3\Big(\delta_1^2+\delta_2^2-2\delta_1(\delta_2+6\delta_3-10\delta_4+6) \\
&\quad +4\delta_2(5\delta_3-3\delta_4+3)+(\delta_3-\delta_4)(\delta_3-\delta_4+12)\Big) \\
&\quad -3\delta^2\Big(\delta_1^2(2\delta_3+2\delta_4-1)+2\delta_1(\delta_3^2+\delta_4^2+2\delta_3+\delta_2-2\delta_2\delta_3-2\delta_4(\delta_2+\delta_3+1)) \\
&\quad +\delta_2^2(2\delta_3+2\delta_4-1)+2\delta_2(\delta_3-\delta_4-2)(\delta_3-\delta_4)-(\delta_3-\delta_4)^2\Big) \\
&\quad +5\delta(\delta_1-\delta_2)^2(\delta_3-\delta_4)^2-3(\delta_1-\delta_2)^2(\delta_3-\delta_4)^2\Big].
\end{aligned}
$$

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JHEP05(2012)025

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