of $\lambda$ is of finite dimension, it is a complemented subspace of the Banach space $\boldsymbol{g}$ and $\mu$ has a continuous linear section. Now the cohomology of $\boldsymbol{gl}(H, C)$ with continuous cochains and trivial scalar coefficients reduces to zero in degree two; see [3], page IV.8. The usual argument (sketched in [1], § 3, exercise 12i) shows that te extension is inessential. Hence $\boldsymbol{g}$ is a semi-direct product of $\boldsymbol{gl}(H, C)$ and $\boldsymbol{a}$, relative to some morphism from $\boldsymbol{gl}(H, C)$ to the algebra of derivations of $\boldsymbol{a}$ (see [1], § 1 no 8). This morphism is trivial, again by the theorem above, and the product is direct. ■

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# Lipschitz classes and Poisson integrals on stratified groups*

by

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**Abstract.** It is shown that the Lipschitz classes $\Gamma_\alpha$ on stratified groups can be characterized in terms of Poisson integrals, and some interpolation and approximation theorems are proved.

**Introduction.** It is well known that the classical Lipschitz classes $\Lambda_\alpha$ $(\alpha > 0)$ on $\boldsymbol{R}^n$ can be characterized in terms of Poisson integrals; see [7]. In this paper we generalize this result to the Lipschitz classes $\Gamma_\alpha$ $(\alpha > 0)$ on stratified groups studied in [3]. To some extent our arguments are adaptations of those in [7], but the non-commutativity and non-ellipticity in the general situation present a number of difficulties which do not occur in the classical case. From the Poisson integral characterization we obtain a simple proof that the classes $\Gamma_\alpha$ form a scale of interpolation spaces, a result which has been proved with different techniques by Krantz [6]. Actually, the logical order of the paper is somewhat different; we prove the interpolation theorems for the spaces defined by Poisson integrals and then use them in showing that these spaces coincide with the spaces $\Gamma_\alpha$.

The plan of the paper is as follows. In Section 1 we recall the basic facts about stratified groups and the spaces $\Gamma_\alpha$. (For proofs and further details the reader is referred to [3].) In Section 2 we construct the Poisson kernel and derive its fundamental properties. In Section 3 we define spaces $\Gamma_\alpha^*$ in terms of the Poisson integral and prove the interpolation and approximation theorems. Sections 4 and 5 are devoted to the proof that $\Gamma_\alpha = \Gamma_\alpha^*$.

**1.** Let $\mathfrak{g}$ be a stratified Lie algebra in the sense of [3]; that is, $\mathfrak{g}$ is a finite-dimensional nilpotent Lie algebra over $\boldsymbol{R}$ together with a vector space decomposition $\mathfrak{g} = \bigoplus_1^m V_j$ such that $[V_1, V_j] = V_{j+1}$ for $j < m$ and $[V_1, V_m] = \{0\}$. We define a one-parameter family $\{\gamma_r\colon r > 0\}$ of

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automorphisms of g, called *dilations*, by the formula

$$\gamma_r\Big(\sum_1^m Y_j\Big) = \sum_1^m r^j Y_j \quad (Y_j \in V_j).$$

Let $G$ be the corresponding simply connected Lie group, which will also be called "stratified". Since g is nilpotent, the exponential map is a diffeomorphism from g onto $G$ which takes Lebesgue measure on g to a bi-invariant Haar measure $dx$ on $G$. The group identity of $G$ will be referred to as the origin and denoted by 0.

The dilations $\{\gamma_r\}$ on g induce automorphisms of $G$, still called dilations and denoted simply by $x \to rx$, by the formula

$$rx = \exp\big(\gamma_r(\exp^{-1}x)\big) \quad (x \in G, r > 0).$$

A function $f$ on $G-\{0\}$ will be called *homogeneous of degree* $\lambda$ ($\lambda \in \boldsymbol{R}$) *if* $f(rx) = r^\lambda f(x)$. The number

$$Q = \sum_1^m j(\dim V_j)$$

is called the *homogeneous dimension* of $G$, since $d(rx) = r^Q dx$ for $r > 0$.

Let $Y \to \|Y\|$ be a Euclidean norm on g. If $x \in G$, we set $\|x\| = \|\exp^{-1}x\|$. We also define a *homogeneous norm* $x \to |x|$ on $G$ by

$$\Big|\exp\sum_1^m Y_j\Big| = \Big(\sum_1^m \|Y_j\|^{2m!/j}\Big)^{1/2m!} \quad (Y_j \in V_j). \tag{1.1}$$

The homogeneous norm is continuous on $G$, $C^\infty$ on $G-\{0\}$, homogeneous of degree 1, and satisfies (a) $|x| > 0$ if $x \neq 0$, (b) $|x| = |x^{-1}|$. We recall from [3] that there is a constant $C \geq 1$ such that

$$\big||xy|-|x|\big| \leq C|y| \quad \text{if} \quad |y| \leq |x|/2, \tag{1.2}$$

$$C^{-1}\|x\| \leq |x| \leq C\|x\|^{1/m} \quad \text{if} \quad |x| \leq 1, \tag{1.3}$$

where $m$ is the number of steps in the stratification of g. We also have the following "integration in polar coordinates" formula, which will be used without comment in the sequel: there is a constant $C > 0$ such that for every nonnegative measurable function $f$ on $(0, \infty)$,

$$\int_G f(|x|)\,dx = C\int_0^\infty r^{Q-1} f(r)\,dr.$$

The elements of g will be considered as left-invariant vector fields on $G$. We fix once and for all a basis $X_1, \ldots, X_n$ for $V_1 \subset$ g. The operator

$$\mathcal{J} = -\sum_1^n X_j^2$$



is called the *sub-Laplacian* of $G$. We also introduce the following multi-index notation for derivatives: if $I = (i_1, \ldots, i_k)$, where $k = 1, 2, 3, \ldots$ and $1 \leq i_j \leq n$, we set $|I| = k$ and

$$X_I = X_{i_1} X_{i_2} \ldots X_{i_k}.$$

We shall also allow the empty multi-index $\emptyset$: by convention, $|\emptyset| = 0$ and $X_\emptyset =$ identity. Since $V_1$ generates g, every left-invariant differential operator on $G$ is a linear combination of $X_I$'s. Moreover, if $f$ is smooth and homogeneous of degree $\lambda$, $X_I f$ is homogeneous of degree $\lambda - |I|$.

Next, some function spaces. If $1 \leq p \leq \infty$, $L^p$ is the usual Lebesgue space on $G$ with respect to the Haar measure $dx$, with norm $\|\ \|_p$. $C_0^\infty$ is the space of compactly supported $C^\infty$ functions on $G$. $\mathcal{D}'$ and $\mathcal{E}'$ are the spaces of distributions and compactly supported distributions on $G$. In particular, $\delta \in \mathcal{E}'$ is the Dirac distribution at 0. $\mathcal{C}$ is the space of bounded left uniformly continuous functions on $G$, and if $k$ is a positive integer, $\mathcal{C}^k$ is the space of all $f \in \mathcal{C}$ whose (distribution) derivatives $X_I f$ are in $\mathcal{C}$ for $|I| \leq k$.

Finally, we define the Lipschitz classes $\Gamma_a$. If $0 < a < 1$,

$$\Gamma_a = \{f \in \mathcal{C}\colon |f|_a = \sup_{x,y} |f(xy)-f(x)|/|y|^a < \infty\}.$$

If $a = 1$, $\Gamma_a$ is the "Zygmund class":

$$\Gamma_1 = \{f \in \mathcal{C}\colon |f|_1 = \sup_{x,y} |f(xy)+f(xy^{-1})-2f(x)|/|y| < \infty\}.$$

For $0 < a \leq 1$, $\Gamma_a$ is a Banach space with norm

$$\|f\|_{(a)} = |f|_a + \|f\|_\infty.$$

If $k = 1, 2, 3, \ldots$, and $k < a \leq k+1$,

$$\Gamma_a = \{f \in \mathcal{C}^k\colon X_I f \in \Gamma_{a-k} \text{ for } |I| \leq k\},$$

which is a Banach space with norm

$$\|f\|_{(a)} = \sum_{0 \leq |I| \leq k} \|X_I f\|_{(a-k)}.$$

For $k < a \leq k+1$ we also set

$$|f|_a = \sum_{0 \leq |I| \leq k} |X_I f|_{a-k}.$$

We remark that in the definition of $\Gamma_a$, $0 < a \leq 1$, we could have replaced the supremum over all $x, y \in G$ by the supremum over $x \in G$ and $|y| \leq 1$, since for $|y| > 1$ the boundedness of $f$ is already a stronger condition.

**2.** In this section we construct the Poisson kernel for $G$. We shall denote the canonical coordinate on $\boldsymbol{R}$ by $t$ and the coordinate vector field by $\partial_t$. Consider the group $G\times\boldsymbol{R}$, whose Lie algebra has a natural stratification $\bigoplus_1^m W_j$, where $W_1$ is the span of $V_1$ and $\partial_t$ and $W_j = V_j$ for $j>1$. The corresponding dilations are given by

$$r(x,t) = (rx, rt),$$

the second factor being ordinary multiplication, and the homogeneous dimension of $G\times\boldsymbol{R}$ is $Q+1$. Also, the operator

$$\mathscr{L} = \mathscr{J} - \partial_t^2$$

is a sub-Laplacian on $G\times\boldsymbol{R}$. We shall need the following two facts about $\mathscr{L}$, due respectively to Bony [1] and Folland [3]:

(2.1) $\mathscr{L}$ satisfies the strong maximum principle: if $f$ is a real-valued solution of $\mathscr{L}f = 0$ on a connected open set $U$ which attains its supremum or infimum on $U$ at some point in $U$, then $f$ is constant on $U$.

(2.2) There is a unique $C^\infty$ function $K$ on $G\times\boldsymbol{R}-\{(0,0)\}$ which satisfies (a) $K(rx, rt) = r^{1-Q}K(x,t)$, (b) $\mathscr{L}K$ is the Dirac distribution at $(0,0)$. (This result holds only if $Q>1$. If $Q=1$, then $G=\boldsymbol{R}$ and $\mathscr{L}$ is minus the classical Laplacian on $\boldsymbol{R}^2$, and we take $K$ to be the usual logarithmic potential.) Since $\mathscr{L}$ is real, self-adjoint, and invariant under the transformation $(x,t)\to(x,-t)$, $K$ is real and satisfies $K(x,t) = K(x^{-1},-t)$ and $K(x,t) = K(x,-t)$, hence also $K(x,t) = K(x^{-1},t)$.

Let $q(x,t) = \partial_t K(x,t)$. Then $q(rx, rt) = r^{-Q}q(x,t)$, and $q$ satisfies $\mathscr{L}q = 0$ away from the origin. Also, $q(x,t) = q(x^{-1},t)$, and since $q$ is odd in $t$, $X_I q(x,t) = -X_I q(x,-t)$ for any $I$. In particular, $X_I q(x,0) = 0$ for $x\neq 0$, so since $q(x,t)$ is smooth for $|x|=1$, we have

(2.3) $$\sup_{|x|=1} |X_I q(x,t)| = O(|t|) \quad \text{as} \quad t\to 0.$$

Henceforth we restrict attention to the half-space $t>0$. For each fixed $t>0$, set $q_t(x) = q(x,t)$. If $x\neq 0$ and $y = x/|x|$, we have

$$X_I q_t(x) = X_I q(x,t) = |x|^{-Q-|I|} X_I q(y, |x|^{-1}t),$$

so by (2.3),

$$\begin{aligned}|X_I q_t(x)| &\leqslant |x|^{-Q-|I|}\sup_{|y|=1}|X_I q(y,|x|^{-1}t)|\\ &= O(|x|^{-Q-|I|-1}) \quad \text{as} \quad x\to\infty.\end{aligned}$$

It follows that $X_I q_t\in L^1$ for all $I$, $t$: in particular, $q_t\in L^1$. Also,

$$q_t(x) = q(x,t) = t^{-Q}q(t^{-1}x, 1) = t^{-Q}q_1(t^{-1}x).$$

Thus

$$\int q_t(x)\,dx = \int q_1(t^{-1}x)t^{-Q}\,dx = \int q_1(x)\,dx = A$$



is independent of $t$. By a standard argument, it follows that $q_t\to A\delta$ as $t\to 0$, and, more precisely, that $f*q_t\to Af$ uniformly as $t\to 0$ for any $f\in\mathscr{C}$.

We claim that $A\neq 0$. Indeed, $q$ is clearly not identically zero (otherwise $K$ would be constant in $t$, hence zero by homogeneity), so we can choose $f\in C_0^\infty$ such that

$$\int f(x^{-1})q_{t_0}(x)\,dx \neq 0$$

for some $t_0>0$. Set $u(x,t) = (f*q_t)(x)$. Then $u(x,t)\to Af(x)$ as $t\to 0$, $\mathscr{L}u = 0$ for $t>0$, and $u(0,t_0)\neq 0$. Moreover, since $q(x,t)\to 0$ as $x,t\to\infty$, the same is true of $u$. If $A$ were zero, we could apply the maximum principle (2.1) to $u$ on a rectangle $|x|<R$, $0<t<T$ and let $T,R\to\infty$ to conclude that $u\equiv 0$. This not being the case, $A\neq 0$.

We now define the *Poisson kernel* $p(x,t) = p_t(x)$ by

$$p(x,t) = A^{-1}q(x,t) \qquad (t>0,\ x\in G).$$

Moreover, we define the operator $P_t$ on $\mathscr{C}$ $(t>0)$ by

$$P_t f = f*p_t.$$

We summarize the properties of the Poisson kernel in a theorem:

(2.4) THEOREM. (a) *If* $k\geqslant 0$, $|I|\geqslant 0$, *and* $r>0$,

$$\partial_t^k X_I p(rx, rt) = r^{-Q-k-|I|}\partial_t^k X_I p(x,t).$$

*In particular,*

$$|\partial_t^k X_I p(x,t)| = O\big((|x|+t)^{-Q-k-|I|}\big) \quad \textit{as} \quad x,t\to\infty.$$

(b) *For each* $t>0$ *and multi-index* $I$,

$$|X_I p(x,t)| = O(|x|^{-Q-|I|-1}) \quad \textit{as} \quad x\to\infty.$$

(c) $p(x,t) = p(x^{-1},t)$.

(d) *For each* $t>0$, $\int p_t(x)\,dx = 1$.

(e) *If* $f\in\mathscr{C}$, $P_t f\to f$ *uniformly as* $t\to 0$. *Moreover,* $u(x,t) = P_t f(x)$ *satisfies* $\mathscr{L}u = 0$ *for* $t>0$.

(f) *For each* $k\geqslant 1$ *and* $t>0$, $\int\partial_t^k p_t(x)\,dx = 0$.

(g) *For each* $k\geqslant 0$ *and* $|I|\geqslant 0$, *there is a constant* $C>0$ *such that* $\int|\partial_t^k X_I p_t(x)|\,dx\leqslant Ct^{-|I|-k}$.

(h) $p(x,t)>0$ *for all* $x\in G$, $t>0$.

(i) $p_t*p_s = p_{t+s}$ *(and hence* $P_sP_t = P_{s+t}$*) for all* $s,t>0$.

(j) $\partial_t p_t = (\partial_t p_{t/2})*p_{t/2} = p_{t/2}*(\partial_t p_{t/2})$. *(By* $\partial_t p_{t/2}$ *we mean* $\partial_s p_s|_{s=t/2}$.*)*

Proof. (a), (b), (c), (d), and (e) follow from the corresponding properties of $q$. (f) follows from (d):

$$\int\partial_t^k p_t(x)\,dx = \partial_t^k\int p_t(x)\,dx = \partial_t^k 1 = 0.$$

(g) follows from (a):

$$\int |\partial_t^k X_I p_t(x)|\,dx \leqslant C_1\Big[\int_{|x|\leqslant t} t^{-Q-|I|-k}dx + \int_{|x|>t} |x|^{-Q-|I|-k}dx\Big]$$
$$= C_1[C_2 t^{-Q-|I|-k}t^Q + C_3 t^{-|I|-k}] = Ct^{-|I|-k}.$$

For (h), given $\varepsilon > 0$ choose a nonnegative $f \in C_0^\infty$ so that $\|f*p_1 - p_1\|_\infty < \varepsilon$. If $u(x,t) = P_tf(x)$, then, by (a) and (e) we have $u(x,0) = f(x) \geqslant 0$, $u(x,t) \to 0$ as $x, t \to \infty$, and $\mathcal{L}u = 0$ for $t > 0$, so by the maximum principle, $u \geqslant 0$ everywhere. Hence $p_1 \geqslant -\varepsilon$, and $\varepsilon$ being arbitrary, $p_1 \geqslant 0$. By (a), $p(x,t) \geqslant 0$ for all $x, t$. But $p$ cannot achieve its infimum, namely zero, on the region $t > 0$, so $p(x,t) > 0$.

To see (i), let $s > 0$ be fixed, and set $u(x,t) = p_s * p_t(x) - p_{s+t}(x)$. Then $u$ is continuous for $t \geqslant 0$, $\mathcal{L}u = 0$ for $t > 0$, $u(x,0) = p_s(x) - p_s(x) = 0$, and $u(x,t) \to 0$ as $x, t \to \infty$. By the maximum principle, $u \equiv 0$.

Finally, by (i) we have

$$\partial_t p_t = \partial_t(p_{t/2} * p_{t/2}) = 1/2\,[(\partial_t p_{t/2}) * p_{t/2} + p_{t/2} * (\partial_t p_{t/2})].$$

(j) then follows since (by (i) again) $p_{t/2}$ and $\partial_t p_{t/2}$ commute.

**3.** Suppose $a > 0$, and let $[a]$ be the greatest integer in $a$. We define

$$\Gamma_a^* = \{f \in \mathscr{C}:\ |f|_a^* = \sup_{t>0} t^{k-a}\|\partial_t^k P_t f\|_\infty < \infty\} \qquad (k = [a]+1).$$

$\Gamma_a^*$ is a Banach space with norm

$$\|f\|_{(a)}^* = |f|_a^* + \|f\|_\infty.$$

We note that $f \in \Gamma_a^*$ if and only if $f \in \mathscr{C}$ and

$$\sup_{0<t<1} t^{k-a}\|\partial_t^k P_t f\|_\infty < \infty,$$

since for $t \geqslant 1$, by Theorem 2.4 (g), the mere boundedness of $f$ implies that

(3.1) $$\|\partial_t^k P_t f\|_\infty = \|f * \partial_t^k p_t\|_\infty \leqslant C\|f\|_\infty t^{-k} \leqslant C\|f\|_\infty t^{a-k}.$$

Moreover, in the definition of $\Gamma_a^*$ we could replace $k$ by any integer greater than $a$, as the following proposition shows.

(3.2) PROPOSITION. *If $j, k$ are any integers greater than $a$, the conditions*

$$\|\partial_t^j P_t f\|_\infty \leqslant Ct^{a-j}, \qquad \|\partial_t^k P_t f\|_\infty \leqslant C' t^{a-k} \qquad (0 < t < \infty)$$

*are equivalent for $f \in \mathscr{C}$, and the smallest constants $C$, $C'$ satisfying these inequalities are bounded by multiples of each other, independent of $f$.*

Proof. We may assume that $k < j$, and by induction it suffices to assume that $j = k+1$. On the one hand, since by (3.1)

$$\|\partial_t^k P_t f\|_\infty \to 0 \quad \text{as} \quad t \to \infty,$$



we have

$$\partial_t^k P_t f = -\int_t^\infty \partial_s^{k+1} P_s f\,ds,$$

so if $\|\partial_t^{k+1} P_t f\|_\infty \leqslant Ct^{a-k-1}$,

$$\|\partial_t^k P_t f\|_\infty \leqslant \int_t^\infty \|\partial_s^{k+1} P_s f\|_\infty\,ds \leqslant C\int_t^\infty s^{a-k-1}ds = C(k-a)^{-1}t^{a-k}.$$

On the other hand, by Theorem 2.4 (j),

$$\partial_t^{k+1} P_t f = f * (\partial_t^k p_{t/2}) * \partial_t p_{t/2} = \partial_t^k P_{t/2} f * \partial_t p_{t/2},$$

so by Theorem 2.4 (g), if $\|\partial_t^k P_t f\|_\infty \leqslant C' t^{a-k}$,

$$\|\partial_t^{k+1} P_t f\|_\infty \leqslant \|\partial_t^k P_{t/2} f\|_\infty \|\partial_t p_{t/2}\|_1$$
$$\leqslant C'(t/2)^{a-k}\cdot C_1(t/2)^{-1} = C' C_1 2^{k+1-a} t^{a-k-1}.$$

This completes the proof.

In view of the remarks following the definition of $\Gamma_a^*$, the following result is immediate:

(3.3) COROLLARY. *$\Gamma_a^* \subset \Gamma_\beta^*$ and $\|\ \|_{(a)}^*$ dominates $\|\ \|_{(\beta)}^*$ whenever $a > \beta$.*

We now derive some more properties of $\Gamma_a^*$.

(3.4) LEMMA. *If $k + |I| > a > 0$, there is a constant $C > 0$ such that for all $f \in \Gamma_a^*$,*

$$\|\partial_t^k X_I P_t f\|_\infty \leqslant C|f|_a^* t^{a-k-|I|}.$$

Proof. By Theorem 2.4 (j),

$$\partial_t^k X_I P_t f = f * (\partial_t^k p_{t/2}) * (X_I p_{t/2}) = (\partial_t^k P_{t/2} f) * (X_I p_{t/2}).$$

If $k > a$, then $\|\partial_t^k P_t f\|_\infty \leqslant C_1 |f|_a^* t^{a-k}$, so by Theorem 2.4 (g),

$$\|\partial_t^k X_I P_t f\|_\infty \leqslant \|\partial_t^k P_{t/2} f\|_\infty \|X_I p_{t/2}\|_1 \leqslant C|f|_a^* t^{a-k-|I|}.$$

This estimate is valid in any event if $k$ is replaced by $[a]+1$. If $a - |I| < k \leqslant a$, the desired result follows by integrating $[a]+1-k$ times as in the proof of Proposition 3.2.

(3.5) LEMMA. *If $f \in \mathscr{C}$, $X_I P_t f \to X_I f$ as $t \to 0$, in the sense of distributions.* (This assertion isn't completely obvious, since $X_I P_t f \neq P_t X_I f$.)

Proof. Choose $\varphi \in C_0^\infty$ with $0 \leqslant \varphi \leqslant 1$ and $\varphi(x) = 1$ for $|x| \leqslant 1$, and write

$$X_I P_t f = f * [\varphi X_I p_t] + f * [(1-\varphi) X_I p_t].$$

On the one hand, $\varphi X_I p_t$ has compact support and converges to $\varphi X_I \delta = X_I \delta$ as $t \to 0$, so since convolution is continuous from $\mathscr{D}' \times \mathscr{E}'$ to $\mathscr{D}'$,

$$f * [\varphi X_I p_t] \to f * X_I \delta = X_I f$$

as distributions when $t\to 0$. On the other hand, by Theorem 2.4 (a, b), $(1-\varphi)X_Ip_t\in L^1$, and

$$\|(1-\varphi)X_Ip_t\|_1\leqslant \int_{|x|\geqslant 1}|X_Ip_t(x)|\,dx = \int_{|x|\geqslant 1/t}|X_Ip_t(tx)|\,t^Q\,dx$$
$$= t^{-|I|}\int_{|x|\geqslant 1/t}|X_Ip_1(x)|\,dx\leqslant Ct^{-|I|}\int_{|x|\geqslant 1/t}|x|^{-Q-|I|-1}dx$$
$$= C't^{-|I|}\cdot t^{|I|+1} = C't\to 0 \quad\text{as}\quad t\to 0.$$

Hence $f*[(1-\varphi)X_Ip_t]\to 0$ uniformly as $t\to 0$.

(3.6) PROPOSITION. *If $a>k$, then $\Gamma_a^*\subset\mathscr{C}^k$ and there is a constant $C>0$ such that*

$$\|X_If\|_\infty\leqslant C\|f\|_{(a)}^* \qquad (f\in\Gamma_a^*,\ |I|\leqslant k).$$

Proof. We must show that if $f\in\Gamma_a^*$ and $|I|<a$, then $X_If\in\mathscr{C}$ and $\|X_If\|_\infty\leqslant C\|f\|_{(a)}^*$. By decreasing $a$, we may assume that $a<|I|+1$. Then if $0<t<s$, by Lemma 3.4 we have

$$\text{(3.7)}\quad \|X_IP_sf-X_IP_tf\|_\infty\leqslant\int_t^s\|\partial_rX_IP_rf\|_\infty dr$$
$$\leqslant C_1|f|_a^*\int_t^s r^{a-|I|-1}dr = C_2|f|_a^*(s^{a-|I|}-t^{a-|I|}).$$

Since $a-|I|>0$, $\{X_IP_tf\}$ is Cauchy in the uniform norm as $t\to 0$, so by Lemma 3.5, $X_IP_tf\to X_If$ uniformly as $t\to 0$. Thus $X_If\in\mathscr{C}$, and by taking $s=1$ and letting $t\to 0$ in (3.7), we obtain

$$\|X_If\|_\infty\leqslant\|X_IP_1f\|_\infty+\|X_IP_1f-X_If\|_\infty$$
$$\leqslant\|X_Ip_1\|_1\|f\|_\infty+C_2|f|_a^*\leqslant C\|f\|_{(a)}^*.$$

The same argument, with $X_I$ replaced by $\partial_t$, proves the following:

(3.8) PROPOSITION. *If $a>1$ and $f\in\Gamma_a^*$, then $\partial_tP_tf$ converges uniformly to a limit in $\mathscr{C}$ as $t\to 0$, and there is a constant $C>0$, independent of $t$ and $f$, such that $\|\partial_tP_tf\|_\infty\leqslant C\|f\|_{(a)}^*$:*

The following theorem is related to some well-known approximation and interpolation results for the classical Lipschitz classes: see, for example, [2]. A special case of this theorem (for $\Gamma_a$ rather than $\Gamma_a^*$) was stated in [3], but the proof given there seems to be valid only when $G$ is stratified of step 2.

(3.9) THEOREM. *Suppose $0<a_0<a<a_1<\infty$, and $f\in\mathscr{C}$. Then $f\in\Gamma_a^*$ if and only if there is a constant $B>0$ such that for every $r>0$ there exist $f_r\in\Gamma_{a_0}^*$, $f^r\in\Gamma_{a_1}^*$ with $|f_r|_{a_0}^*\leqslant Br^{a-a_0}$, $|f^r|_{a_1}^*\leqslant Br^{a-a_1}$, and $f=f_r+f^r$. In this case, the smallest such $B$ is comparable to $|f|_a^*$. The same conclusions hold if $|\ |_\beta^*$ is replaced by $\|\ \|_{(\beta)}^*$ ($\beta=a_0,a,a_1$).*



Proof. The "if" part is easy: suppose we can find $B, f_r, f^r$ as above. Then if $k>a_1$ we have, for every $r>0$ and $t>0$,

$$\|\partial_t^kP_tf\|_\infty\leqslant\|\partial_t^kP_tf_r\|_\infty+\|\partial_t^kP_tf^r\|_\infty$$
$$\leqslant C_1B(r^{a-a_0}t^{a_0-k}+r^{a-a_1}t^{a_1-k}).$$

Take $r=t$; it follows that $f\in\Gamma_a^*$ and $|f|_a^*\leqslant 2C_1B$. Also, if $\|f_r\|_\infty\leqslant Br^{a-a_0}$ and $\|f^r\|_\infty\leqslant Br^{a-a_1}$, taking $r=1$ we have

$$\|f\|_\infty\leqslant\|f_r\|_\infty+\|f^r\|_\infty\leqslant 2B,$$

so that $\|f\|_{(a)}^*\leqslant 2\max(C_1,1)B$.

To prove the converse, note first that it suffices to consider $r\leqslant 1$, since for $r>1$ we can simply take $f_r=f$, $f^r=0$. Suppose first that $a-a_0\leqslant 1$. Given $f\in\Gamma_a^*$ and $r\leqslant 1$, set $f^r=P_rf$, $f_r=f-P_rf$, and $k=[a_1]+1$. Then by Theorem 2.4 (i) and Proposition 3.2,

$$\|\partial_t^kP_tf^r\|_\infty=\|\partial_t^kP_{t+r}f\|_\infty\leqslant C|f|_a^*(t+r)^{a-a_1}(t+r)^{a_1-k}$$
$$\leqslant C|f|_a^*r^{a-a_1}t^{a_1-k}.$$

Thus $|f^r|_{a_1}^*\leqslant C|f|_a^*r^{a-a_1}$. Also, since $r\leqslant 1$,

$$\|f^r\|_\infty\leqslant\|p_r\|_1\|f\|_\infty\leqslant r^{a-a_1}\|f\|_\infty,$$

so $\|f^r\|_{(a_1)}^*\leqslant C\|f\|_{(a)}^*r^{a-a_1}$. On the other hand,

$$\partial_t^kP_tf_r=\partial_t^kP_tf-\partial_t^kP_{t+r}f=-\int_t^{t+r}\partial_s^{k+1}P_sf\,ds,$$

so

$$\|\partial_t^kP_tf_r\|_\infty\leqslant\int_t^{t+r}\|\partial_s^{k+1}P_sf\|_\infty ds\leqslant C_1|f|_a^*\int_t^{t+r}s^{a-k-1}ds$$
$$\leqslant C_2|f|_a^*[t^{a-k}-(t+r)^{a-k}]\leqslant C_2|f|_a^*t^{a-k},$$

and also

$$\|\partial_t^kP_tf_r\|_\infty\leqslant r\sup_{t\leqslant s\leqslant t+r}\|\partial_s^{k+1}P_sf\|_\infty\leqslant C_1|f|_a^*rt^{a-k-1}.$$

Now apply the inequality $\min(a,b)\leqslant a^\theta b^{1-\theta}$ $(a,b>0,\ 0\leqslant\theta\leqslant 1)$ to the right hand sides of these estimates, with $\theta=a-a_0$, obtaining

$$\|\partial_t^kP_tf_r\|_\infty\leqslant C|f|_a^*r^{a-a_0}t^{a_0-k}.$$

Thus $|f_r|_{a_0}^*\leqslant C|f|_a^*r^{a-a_0}$. Also, we have

$$\|f_r\|_\infty=\Big\|\int_0^r\partial_tP_tf\,dt\Big\|_\infty\leqslant\int_0^r\|\partial_tP_tf\|_\infty dt.$$

If $\alpha \leqslant 1$, then $\alpha-\alpha_0 < 1$ and $f \in \Gamma_{\alpha-\alpha_0}$, so

$$\|f_r\|_\infty \leqslant |f|^*_{\alpha-\alpha_0} \int_0^r t^{\alpha-\alpha_0-1}dt \leqslant C|f|^*_{\alpha-\alpha_0} r^{\alpha-\alpha_0} \leqslant C\|f\|^*_{(\alpha)} r^{\alpha-\alpha_0}.$$

If $\alpha > 1$, then by Proposition 3.8,

$$\|f_r\|_\infty \leqslant C\|f\|^*_{(\alpha)} r \leqslant C\|f\|^*_{(\alpha)} r^{\alpha-\alpha_0}$$

since $\alpha-\alpha_0 \leqslant 1$, $r \leqslant 1$. In any event, we have $\|f_r\|^*_{(\alpha_0)} \leqslant C\|f\|^*_{(\alpha)} r^{\alpha-\alpha_0}$.

We now settle the general case by induction: Suppose the theorem is true when $\alpha-\alpha_0 \leqslant j-1$, and suppose $j-1 < \alpha-\alpha_0 \leqslant j$. If $f \in \Gamma^*_\alpha$ and $r > 0$, we can find $g_r \in \Gamma^*_{\alpha-1}$, $g^r \in \Gamma^*_{\alpha_1}$ with $|g_r|_{\alpha-1} \leqslant C_1|f|^*_\alpha r$, $|g^r|^*_\alpha \leqslant C_1|f|^*_\alpha r^{\alpha-\alpha_1}$, and $f = g_r+g^r$. But since $(\alpha-1)-\alpha_0 \leqslant j-1$, we can apply the inductive hypothesis to $g_r$ to find $h_r \in \Gamma_{\alpha_0}$, $h^r \in \Gamma_{\alpha_1}$ such that $g_r = h_r+h^r$,

$$|h_r|^*_{\alpha_0} \leqslant C_2|g_r|^*_{\alpha-1} r^{\alpha-1-\alpha_0} \leqslant C_1C_2|f|^*_\alpha r^{\alpha-\alpha_0},$$
$$|h^r|^*_{\alpha_1} \leqslant C_2|g_r|^*_{\alpha-1} r^{\alpha-1-\alpha_1} \leqslant C_1C_2|f|^*_\alpha r^{\alpha-\alpha_1}.$$

Thus we have merely to take $f_r = h_r$, $f^r = h^r+g^r$. This argument works just as well with $|\ |^*_\beta$ replaced by $\|\ \|^*_{(\beta)}$ ($\beta = \alpha_0, \alpha-1, \alpha, \alpha_1$), so the proof is complete.

Remark. An examination of the proof shows that if $f \in \Gamma^*_\alpha$ and $j-1 < \alpha-\alpha_0 \leqslant j$, the $f_r$ we have constructed (for $r \leqslant 1$) is $(I-P_r)^j f$, and the $f^r$ we have constructed is not just in $\Gamma^*_{\alpha_1}$ but in $C^\infty$.

As a simple corollary of Theorem 3.9, we obtain the following interpolation theorem.

(3.10) THEOREM. *Let $G$ and $H$ be stratified groups, $0 < \alpha_0 < \alpha_1$, and $0 < \beta_0 < \beta_1$. Suppose $T$ is a bounded linear transformation from $\Gamma^*_{\alpha_0}(G)$ to $\Gamma^*_{\beta_0}(H)$ whose restriction to $\Gamma^*_{\alpha_1}(G)$ is bounded from $\Gamma^*_{\alpha_1}(G)$ to $\Gamma^*_{\beta_1}(H)$.*

*Then if $\alpha = \theta\alpha_1+(1-\theta)\alpha_0$, $\beta = \theta\beta_1+(1-\theta)\beta_0$ $(0 < \theta < 1)$, the restriction of $T$ to $\Gamma^*_\alpha(G)$ is bounded from $\Gamma^*_\alpha(G)$ to $\Gamma^*_\beta(H)$.*

Proof. If $f \in \Gamma^*_\alpha(G)$, for each $r > 0$ write $f = f_r+f^r$, where

$$\|f_r\|^*_{(\alpha_0)} \leqslant C\|f\|^*_{(\alpha)} r^{\alpha-\alpha_0} = C\|f\|^*_{(\alpha)} r^{\theta(\alpha_1-\alpha_0)},$$
$$\|f^r\|^*_{(\alpha_1)} \leqslant C\|f\|^*_{(\alpha)} r^{\alpha-\alpha_1} = C\|f\|^*_{(\alpha)} r^{(\theta-1)(\alpha_1-\alpha_0)}.$$

Given $s > 0$, take $r = s^{(\beta_1-\beta_0)/(\alpha_1-\alpha_0)}$ and set $(Tf)_s = T(f_r)$, $(Tf)^s = T(f^r)$. Then $Tf = (Tf)_s+(Tf)^s$, and

$$\|(Tf)_s\|^*_{(\beta_0)} \leqslant A\|f_r\|^*_{(\alpha_0)} \leqslant AC\|f\|^*_{(\alpha)} s^{\theta(\beta_1-\beta_0)} = AC\|f\|^*_{(\alpha)} s^{\beta-\beta_0},$$
$$\|(Tf)^s\|^*_{(\beta_1)} \leqslant A\|f^r\|^*_{(\alpha_1)} \leqslant AC\|f\|^*_{(\alpha)} s^{(\theta-1)(\beta_1-\beta_0)} = AC\|f\|^*_{(\alpha)} s^{\beta-\beta_1}.$$

Therefore $Tf \in \Gamma^*_\beta(H)$ and $\|Tf\|^*_{(\beta)} \leqslant C'\|f\|^*_{(\alpha)}$.



4. Our aim now is to prove the following theorem:

(4.1) THEOREM. *If $\alpha > 0$, $\Gamma_\alpha = \Gamma^*_\alpha$ and the norms $\|\ \|_{(\alpha)}$ and $\|\ \|^*_{(\alpha)}$ are equivalent.*

The proof is lengthy and will be accomplished in several steps. We begin with some lemmas.

(4.2) LEMMA. *There is a constant $C > 0$ such that for all $f \in \mathscr{C}^1$,*

$$\sup_{x,y} |f(xy)-f(x)|/|y| \leqslant C\sum_1^n \|X_jf\|_\infty.$$

Proof. See [3], Proposition 5.4.

(4.3) LEMMA. *If $0 < \alpha < 2$, there is a constant $C > 0$ such that for all $f \in \Gamma_\alpha$,*

$$\sup_{x,y} |f(xy)+f(xy^{-1})-2f(x)|/|y|^\alpha \leqslant C|f|_\alpha.$$

Proof. See [3], Proposition 5.5, for the case where $f$ has compact support. The argument given there to remove this restriction is defective, and we take this opportunity to provide a valid proof. We need only consider $\alpha > 1$, as the estimate is obvious for $\alpha \leqslant 1$. For brevity we shall write $\Delta^2_y f(x) = f(xy)+f(xy^{-1})-2f(x)$.

Suppose $f \in \Gamma_\alpha$, $1 < \alpha < 2$. If $f$ is constant, the estimate is trivial; otherwise, $|f|_\alpha \neq 0$, and we set $R = (\|f\|_\infty/|f|_\alpha)^{1/\alpha}$. It will suffice to show that

$$\sup\{|\Delta^2_y f(x)|/|y|^\alpha:\ x \in G,\ |y| \leqslant R\} \leqslant C|f|_\alpha,$$

since for $|y| > R$ we have

$$|\Delta^2_y f(x)|/|y|^\alpha \leqslant |\Delta^2_y f(x)|\,|f|_\alpha/\|f\|_\infty \leqslant 4|f|_\alpha.$$

Choose $\varphi \in C^\infty_0$ such that $\|\varphi\|_\infty = 1$ and $\varphi(x) = 1$ for $|x| \leqslant 1$, and for $\varepsilon > 0$, set $\varphi_\varepsilon(x) = \varphi(\varepsilon x)$. Then $\varphi_\varepsilon f \in \Gamma_\alpha$, and from Leibniz's rule and Lemma 4.2 we see that

$$|\varphi_\varepsilon f|_\alpha \leqslant \Big(\|\varphi_\varepsilon\|_\infty + \sum_1^n \|X_j\varphi_\varepsilon\|_\infty\Big)|f|_\alpha + \Big(\|f\|_\infty + \sum_1^n \|X_jf\|_\infty\Big)|\varphi_\varepsilon|_\alpha.$$

But $\|X_j\varphi_\varepsilon\|_\infty = \varepsilon\|X_j\varphi\|_\infty$, and

$$|\varphi_\varepsilon|_\alpha = |\varphi_\varepsilon|_{\alpha-1} + \sum_1^n |X_j\varphi_\varepsilon|^\cdot_{\alpha-1} = \varepsilon^{\alpha-1}|\varphi|_{\alpha-1} + \varepsilon^\alpha \sum_1^n |X_j\varphi|_{\alpha-1}.$$

Since $\|\varphi\|_\infty = 1$, it follows that for some $A > 0$, depending only on $\varphi$,

$$|\varphi_\varepsilon f|_\alpha \leqslant (1+A\varepsilon)|f|_\alpha + A\varepsilon^{\alpha-1}\Big(\|f\|_\infty + \sum_1^n \|X_jf\|_\infty\Big).$$

Now, from the estimate (1.2) and the fact that $\varphi_\varepsilon(x) = 1$ for $|x| \leqslant 1/\varepsilon$, it follows that if $\varepsilon$ is sufficiently small, $\Delta_y^2 f(x) = \Delta_y^2(\varphi_\varepsilon f)(x)$ for $|x| \leqslant 1/2\varepsilon$ and $|y| \leqslant R$. But since $\varphi_\varepsilon f$ has compact support,

$$\begin{aligned}\sup\{|\Delta_y^2 f(x)|/|y|^\alpha\colon\ |x| \leqslant 1/2\varepsilon,\ |y| \leqslant R\} &\leqslant \sup\{|\Delta_y^2(\varphi_\varepsilon f)(x)|/|y|^\alpha\colon\ x, y \in G\}\\ &\leqslant C|\varphi_\varepsilon f|_\alpha\\ &\leqslant C\Big[(1+A\varepsilon)|f|_\alpha + A\varepsilon^{\alpha-1}\Big(\|f\|_\infty + \sum_1^n \|X_j f\|_\infty\Big)\Big].\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain the desired result.

(4.4) PROPOSITION. *If $\alpha$ is not an even integer, $\Gamma_\alpha \subset \Gamma_\alpha^*$, and $\| \ \|_{(\alpha)}$ dominates $\| \ \|_{(\alpha)}^*$.*

Proof. First suppose that $0 < \alpha < 2$ and $f \in \Gamma_\alpha$. By Theorem 2.4 (c,f),

$$\partial_t^2 P_t f(x) = \tfrac{1}{2}\int [f(xy)+f(xy^{-1})-2f(x)]\,\partial_t^2 p_t(y)\,dy.$$

Hence, by Lemma 4.3 and Theorem 2.4 (a),

$$\begin{aligned}\|\partial_t^2 P_t f\|_\infty &\leqslant C_1|f|_\alpha \int |y|^\alpha(|y|+t)^{-2-Q}\,dy\\ &\leqslant C_1|f|_\alpha\Big[\int_{|y|\leqslant t}|y|^\alpha t^{-2-Q}\,dy + \int_{|y|>t}|y|^{\alpha-2-Q}\,dy\Big]\\ &\leqslant C_2|f|_\alpha t^{\alpha-2}.\end{aligned}$$

Thus $f \in \Gamma_\alpha$ and $|f|_\alpha^* \leqslant C_2|f|_\alpha$, hence $\|f\|_{(\alpha)}^* \leqslant C\|f\|_{(\alpha)}$.

For the general case, suppose $2k < \alpha < 2k+2$ and proceed by induction on $k$. If $f \in \Gamma_\alpha$, then $\mathscr{J} f \in \Gamma_{\alpha-2} \subset \Gamma_{\alpha-2}^*$, so

$$\|\partial_t^{2k} P_t \mathscr{J} f\|_\infty \leqslant C|\mathscr{J} f|_{\alpha-2} t^{\alpha-2-2k} \leqslant C|f|_\alpha t^{\alpha-(2k+2)}.$$

But because of the differential equation $\mathscr{J} - \partial_t^2 = \mathscr{L} = 0$ governing the Poisson semigroup,

$$\partial_t^{2k} P_t \mathscr{J} f = \partial_t^{2k+2} P_t f.$$

Thus $f \in \Gamma_\alpha^*$ and $|f|_\alpha^* \leqslant C|f|_\alpha$, hence $\|f\|_{(\alpha)}^* \leqslant C\|f\|_{(\alpha)}$.

(4.5) PROPOSITION. *If $\alpha$ is not an integer, $\Gamma_\alpha^* \subset \Gamma_\alpha$, and $\| \ \|_{(\alpha)}^*$ dominates $\| \ \|_{(\alpha)}$.*

Proof. Let $\alpha = k+\beta$, where $k$ is an integer and $0 < \beta < 1$, and suppose $f \in \Gamma_\alpha^*$. By Proposition 3.6, $f \in \mathscr{C}^k$ and

$$\sum_{|I|\leqslant k} \|X_I f\|_\infty \leqslant C\|f\|_{(\alpha)}^*.$$

It remains to show that for $|I| \leqslant k$, $X_I f \in \Gamma_\beta$ and $|X_I f|_\beta \leqslant C|f|_\alpha^*$. Fix $I$; replacing $\alpha$ by $\alpha-|I|$, we may assume that $|I| = k$. The proof of Proposition 3.6 shows that

$$\|X_I f - X_I P_t f\|_\infty \leqslant \int_0^t \|\partial_s X_I P_s f\|_\infty ds \leqslant C_1|f|_\alpha^* t^\beta.$$

Also, by Lemmas 4.2 and 3.4,

$$|X_I P_t f(xy) - X_I P_t f(x)| \leqslant C_2|y| \sum_1^n \|X_j X_I P_t f\|_\infty \leqslant C_3|y|\,|f|_\alpha^* t^{\beta-1},$$

Hence, for all $x, y \in G$ and $t > 0$,

$$\begin{aligned}&|X_I f(xy) - X_I f(x)|\\ &\leqslant |X_I f(xy) - X_I P_t f(xy)| + |X_I P_t f(xy) - X_I P_t f(x)| + |X_I P_t f(x) - X_I f(x)|\\ &\leqslant |f|_\alpha^*(2C_1 t^\beta + C_3|y|\,t^{\beta-1}).\end{aligned}$$

Taking $t = |y|$, we are done.

(4.6) PROPOSITION. *If $k$ is a positive integer, $\Gamma_k^* \subset \Gamma_k$, and $\| \ \|_{(k)}^*$ dominates $\| \ \|_{(k)}$.*

Proof. Suppose $f \in \Gamma_k^*$. By Proposition 3.6, $f \in \mathscr{C}^{k-1}$ and

$$\sum_{|I|\leqslant k-1} \|X_I f\|_\infty \leqslant C\|f\|_{(k)}^*.$$

As in the proof of Proposition 4.5, we must show that $X_I f \in \Gamma_1$ and $|X_I f|_1 \leqslant C\|f\|_{(k)}^*$ for $|I| \leqslant k-1$, and it suffices to consider $|I| = k-1$. By Theorem 3.9 and Proposition 4.5, for each $r > 0$ we can write $f = f_r + f^r$, where $f_r \in \Gamma_{k-(1/2)}$, $f^r \in \Gamma_{k+(1/2)}$, $\|f_r\|_{(k-(1/2))} \leqslant C\|f\|_{(k)}^* r^{1/2}$, $\|f^r\|_{(k+(1/2))} \leqslant C\|f\|_{(k)}^* r^{-1/2}$. Thus by Lemma 4.3,

$$|X_I f_r(xy) + X_I f_r(xy^{-1}) - 2X_I f_r(x)| \leqslant A|X_I f_r|_{1/2}|y|^{1/2} \leqslant AC\|f\|_{(k)}^* r^{1/2}|y|^{1/2},$$

$$|X_I f^r(xy) + X_I f^r(xy^{-1}) - 2X_I f^r(x)| \leqslant A|X_I f^r|_{3/2}|y|^{3/2} \leqslant AC\|f\|_{(k)}^* r^{-1/2}|y|^{3/2}.$$

Thus for all $x, y \in G$ and $r > 0$,

$$|X_I f(xy) + X_I f(xy^{-1}) - 2X_I f(x)| \leqslant AC\|f\|_{(k)}^*(r^{1/2}|y|^{1/2} + r^{-1/2}|y|^{3/2}).$$

Taking $r = |y|$, we are done.

We have now established that $\Gamma_\alpha^* \subset \Gamma_\alpha$ for all $\alpha > 0$, and that $\Gamma_\alpha \subset \Gamma_\alpha^*$ whenever $\alpha$ is not an even integer. The most delicate part of the argument now comes in showing that $\Gamma_2 \subset \Gamma_2^*$. For this we shall need to invoke the theory of the Poisson semigroup.

The infinitesimal generator of the Poisson semigroup $\{P_t\}$ is $-\mathscr{J}^{1/2}$, the negative of the square root of the sub-Laplacian, defined on the domain

$$D = \{f \in \mathscr{C}\colon\ \lim_{t\to 0} t^{-1}(P_t f - f) \text{ exists in the uniform norm}\}.$$

It follows easily from Proposition 3.8 that $\Gamma_\alpha^* \subset D$ if $\alpha > 1$. Since $\Gamma_1 = \Gamma_1^*$

$\subset \Gamma^*_{1/2} = \Gamma_{1/2}$, we see that $\Gamma_2 \subset \Gamma_{3/2} = \Gamma^*_{3/2} \subset D$. We wish to study more closely the functions $\mathscr{J}^{1/2}f$, $f \in \Gamma_2$.

If $Y$ is a left-invariant differential operator on $G$, $\tilde{Y}$ will denote the right-invariant differential operator which agrees with $Y$ at 0. Thus for any $f \in \mathscr{D}'$, $Yf = f * Y\delta$ and $\tilde{Y}f = Y\delta * f$. Also, we recall from [3] that a *kernel of type* $\lambda$ ($\lambda > 0$) is a $C^\infty$ function on $G - \{0\}$ which is homogeneous of degree $\lambda - Q$. Such functions, being locally integrable on $G$, define distributions. We shall not repeat here the definition of a "kernel of type zero", which is more complicated, but simply remark that if $K$ is a kernel of type 1, $\tilde{X}_jK$ and $X_jK$ are kernels of type zero for $1 \leqslant j \leqslant n$.

If $f \in C_0^\infty$, it follows from Section 3 of [3] that

$$\mathscr{J}^{1/2}f = \mathscr{J}^{-1/2}\mathscr{J}f = (f * \mathscr{J}\delta) * R_1 = f * (\mathscr{J}\delta * R_1) = f * \tilde{\mathscr{J}}R_1,$$

where $R_1$, a kernel of type 1, is the convolution kernel of $\mathscr{J}^{-1/2}$, and $\tilde{\mathscr{J}}R_1$ is thus well defined as a distribution. (The use of the associative law is justified since everything except $R_1$ has compact support.) Let us fix $\varphi \in C_0^\infty$ such that $\varphi(x) = 1$ for $|x| \leqslant 1$ and $\varphi(x) = 0$ for $|x| \geqslant 2$. Then, if we set $G_0 = \tilde{\mathscr{J}}(\varphi R_1)$ and $G_\infty = \tilde{\mathscr{J}}((1-\varphi)R_1)$, we have

(4.7) $$\mathscr{J}^{1/2}f = f * G_0 + f * G_\infty \quad (f \in C_0^\infty).$$

(4.8) LEMMA. *If $f \in \mathscr{C}$, then $f * G_\infty$ is well-defined and is in $\Gamma_\alpha$ for all $\alpha > 0$. Moreover, $\|f * G_\infty\|_{(\alpha)} \leqslant C_\alpha \|f\|_\infty$.*

Proof. $G_\infty$ is $C^\infty$, and it agrees outside the support of $\varphi$ with the function $\tilde{\mathscr{J}}R_1$, which is homogeneous of degree $-Q-1$. Hence, for any multi-index $I$,

$$|X_I G_\infty(x)| = O(|x|^{-Q-1-|I|}) \quad \text{as} \quad x \to \infty.$$

In particular, $X_I G_\infty \in L^1$ for all $I$, so

$$X_I(f * G_\infty) = f * X_I G_\infty \in \mathscr{C}$$

for all $I$, and

$$\|X_I(f * G_\infty)\|_\infty \leqslant \|X_I G_\infty\|_1 \|f\|_\infty.$$

The assertion then follows from Lemma 4.2.

Next, if $f \in \mathscr{D}'$, $f * G_0$ is well-defined as a distribution since $G_0 \in \mathscr{E}'$, and we have

(4.9) $$f * G_0 = -\sum_1^n f * (X_j\delta * X_j\delta * (\varphi R_1)) = -\sum_1^n (f * X_j\delta) * (X_j\delta * (\varphi R_1))$$
$$= -\sum_1^n X_jf * \tilde{X}_j(\varphi R_1).$$

(4.10) LEMMA. *The mapping $g \to g * \tilde{X}_j(\varphi R_1)$ is a bounded operator on $\Gamma_1$ ($j = 1, \ldots, n$).*



This lemma will be proved in the next section. Assuming it for the moment, we establish the following proposition, which completes the proof of Theorem 4.1.

(4.11) PROPOSITION. *If $k$ is a positive integer, $\Gamma_{2k} \subset \Gamma^*_{2k}$ and $\|\ \|_{(2k)}$ dominates $\|\ \|^*_{(2k)}$.*

Proof. Consider first the case $k = 1$. If $f \in \Gamma_2$, then $X_jf \in \Gamma_1$, so (4.9) and Lemmas 4.8 and 4.10 imply that the mapping

$$f \to f * G_0 + f * G_\infty$$

is bounded from $\Gamma_2$ to $\Gamma_1$. We know, moreover, that $\Gamma_2 \subset D$. The arguments used by Hunt [4] to characterize infinitesimal generators of probability semigroups on $G$ can then easily be extended to show that the formula (4.7) remains valid for $f \in \Gamma_2$. (See, in particular, Sections 4, 6, and 7 of [4]. The measure on the complement of the origin which Hunt calls $G$ is, in our case, $\tilde{\mathscr{J}}R_1(x)\,dx$.) In short, if $f \in \Gamma_2$ then $\mathscr{J}^{1/2}f \in \Gamma_1 = \Gamma_1^*$, so

$$\|\partial_t^3 P_t f\|_\infty = \|\partial_t^2 P_t(-\mathscr{J}^{1/2}f)\|_\infty \leqslant |\mathscr{J}^{1/2}f|_1^* t^{-1} \leqslant C_1 \|\mathscr{J}^{1/2}f\|_{(1)} t^{-1} \leqslant C_2 \|f\|_{(2)} t^{-1}.$$

Thus $f \in \Gamma_2^*$ and $\|f\|^*_{(2)} \leqslant C\|f\|_{(2)}$.

The assertion is therefore proved for $k = 1$, and the general case follows by induction on $k$ as in the proof of Proposition 4.4.

**5.** It remains to prove Lemma 4.10. The compactly supported distributions $K_j = \tilde{X}_j(\varphi R_1)$ have the following properties:

(a) $K_j$ is $C^\infty$ away from 0 and is supported in $\{x\colon |x| \leqslant 2\}$.

(b) $K_j$ agrees with a kernel of type zero (namely $\tilde{X}_jR_1$) on $\{x\colon |x| \leqslant 1\}$.

(c) As a linear functional on $C^\infty$, $K_j$ annihilates constant functions, for

$$\langle K_j, C\rangle = -\langle \varphi R_1, \tilde{X}_jC\rangle = -\langle \varphi R_1, 0\rangle = 0.$$

A compactly supported distribution having the properties (a), (b), and (c) will be called a *truncated singular kernel*. We shall prove the following generalization of Lemma 4.10 (the generalization is necessary for the proof):

(5.1) PROPOSITION. *If $K$ is a truncated singular kernel, the mapping $f \to f * K$ is a bounded operator on $\Gamma_\alpha$, $0 < \alpha < 2$.*

The proof will be accomplished by a series of lemmas.

(5.2) LEMMA. *If $K$ is a truncated singular kernel, the mapping $f \to f * K$ is a bounded operator on $\Gamma_\alpha$, $0 < \alpha < 1$.*

Proof. Korányi–Vági [5] have shown that convolution with a kernel of type zero preserves $\Gamma_\alpha \cap L^p$ ($0 < \alpha < 1$, $1 < p < \infty$), and their argument shows equally well that convolution with $K$ preserves $\Gamma_\alpha$ and that

$|f*K|_\alpha \leqslant C|f|_\alpha$ $(0<\alpha<1)$. Also,

$$|f*K(x)| = \Big|\text{P.V.}\int f(xy^{-1})K(y)\,dy\Big| = \Big|\int [f(xy^{-1})-f(x)]K(y)\,dy\Big| \leqslant C_1|f|_\alpha \int_{|y|\leqslant 2} |y|^\alpha |y|^{-Q}\,dy = C_2|f|_\alpha,$$

so that $\|f*K\|_\infty \leqslant C_2|f|_\alpha$.

Next, if $y\in G$, define the operator $\Delta_y$ on functions on $G$ by $\Delta_y f(x) = f(xy)-f(x)$.

(5.3) Lemma. *Let $F$ be a kernel of type 1. There exist constants $\varepsilon>0$, $C>0$ such that whenever* $\max(|y|, |z|, |w|)\leqslant \varepsilon|x|$,

$$|\Delta_y\Delta_z F(x)| \leqslant C|y|\,|z|\,|x|^{-Q-1},$$
$$|\Delta_y\Delta_z\Delta_w F(x)| \leqslant C|y|\,|z|\,|w|\,|x|^{-Q-2}.$$

Proof. If $x, y, z, w$ are replaced by $rx, ry, rz, rw$ $(r>0)$, both sides of these inequalities are multiplied by $r^{1-Q}$, so it suffices to prove them for $|x|=1$ and $\max(|y|, |z|, |w|)\leqslant\varepsilon$. Here $\varepsilon$ is to be taken small enough so that when $x, y, z, w$ are thus restricted, the products $xwzy$, $xwz$, $xwy$, $xzy$, $xy$, $xz$, $xw$ are bounded away from 0 (which is possible by (1.2)). In this case, since $F$ is $C^\infty$ away from 0, it follows from Taylor's theorem and (1.3) that

$$|\Delta_y\Delta_z F(x)| \leqslant C_1\|y\|\,\|z\| \leqslant C_2|y|\,|z| = C_2|y|\,|z|\,|x|^{-Q-1},$$
$$|\Delta_y\Delta_z\Delta_w F(x)| \leqslant C_3\|y\|\,\|z\|\,\|w\| \leqslant C_4|y|\,|z|\,|w| = C_4|y|\,|z|\,|w|\,|x|^{-Q-2}.$$

(5.4) Lemma. *Let $F$ be a kernel of type 1 and $K$ a truncated singular kernel. Then $F*K$ is $C^\infty$ away from 0, and for $i, j = 1, \ldots, n$,*

$$|X_j(F*K)(x)| = O(|x|^{-1-Q}) \quad \text{as} \quad x\to\infty,$$
$$|X_iX_j(F*K)(x)| = O(|x|^{-2-Q}) \quad \text{as} \quad x\to\infty.$$

Proof. $F*K$ is well-defined as a distribution since $F\in\mathcal{D}'$, $K\in\mathcal{E}'$, and since $F$ and $K$ are $C^\infty$ away from 0 it follows easily that $F*K$ is. Let $\varepsilon$ be as in Lemma 5.3, and assume that $|x|\geqslant 2/\varepsilon$. As $K$ annihilates constants, for any $z\in G$,

$$\Delta_z(F*K)(x) = \Delta_z\int [F(xy^{-1})-F(x)]K(y)\,dy = \int \Delta_z\Delta_{y^{-1}}F(x)K(y)\,dy.$$

Since $K(y)=0$ for $|y|\geqslant 2$, Lemma 5.3 implies that for $|z|\leqslant 2$,

$$|\Delta_z(F*K)(x)| \leqslant C\int_{|y|\leqslant 2} |z|\,|y|\,|x|^{-1-Q}|y|^{-Q}\,dy \leqslant C'|z|\,|x|^{-1-Q}.$$

Take $z=\exp(tX_j)$; then $|z|$ is proportional to $t$, so dividing both sides of this inequality by $t$ and letting $t\to 0$,

$$|X_j(F*K)(x)| \leqslant C''|x|^{-1-Q} \quad (|x|\geqslant 2/\varepsilon).$$



The second estimate follows similarly: if $|z|\leqslant 2$, $|w|\leqslant 2$,

$$|\Delta_w\Delta_z(F*K)(x)| = \Big|\int \Delta_w\Delta_z\Delta_{y^{-1}}F(x)K(y)\,dy\Big| \leqslant C\int_{|y|\leqslant 2} |w|\,|z|\,|y|\,|x|^{-Q-2}|y|^{-Q}\,dy = C'|w|\,|z|\,|x|^{-Q-2}.$$

Take $w=\exp(sX_i)$, $z=\exp(tX_j)$, divide both sides by $st$ and let $s\to 0$, $t\to 0$, obtaining

$$|X_iX_j(F*K)(x)| \leqslant C''|x|^{-Q-2} \quad (|x|\geqslant 2/\varepsilon).$$

(5.5) Lemma. *There exist kernels $F_1, F_2, \ldots, F_n$ of type 1 such that for all $f\in\mathcal{E}'$, $f = \sum_1^n X_if*F_i$.*

Proof. See [3], Lemma 4.12.

(5.6) Lemma. *If $K$ is a truncated singular kernel, the mapping $f\to f*K$ is a bounded operator on $\Gamma_\alpha$, $1<\alpha<2$.*

Proof. Let $1<\alpha<2$, and suppose $f\in\Gamma_\alpha$ has compact support. We claim that then $f*K\in\Gamma_\alpha$ and there is a constant $C>0$, independent of $f$, such that $\|f*K\|_{(\alpha)}\leqslant C\|f\|_{(\alpha)}$. To begin with, by Lemma 5.2 we know that $f*K\in\Gamma_{\alpha-1}$ and $\|f*K\|_{(\alpha-1)}\leqslant C\|f\|_{(\alpha-1)}$, so we must show that $X_j(F*K)\in\Gamma_{\alpha-1}$ for $j=1,\ldots,n$ and

$$\sum_1^n \|X_j(F*K)\|_{(\alpha-1)} \leqslant C\sum_1^n \|X_jf\|_{(\alpha-1)}. \tag{5.7}$$

Write $f=\sum_1^n X_if*F_i$ as in Lemma 5.5. Then

$$X_j(f*K) = X_j\Big(\sum_1^n X_if*F_i*K\Big) = \sum_1^n X_if*X_j(F_i*K). \tag{5.8}$$

Now $K$ agrees with a kernel $K_0$ of type zero on the set $\{x\colon |x|<1\}$, so

$$F_i*K = F_i*K_0+F_i*(K-K_0).$$

By Proposition 1.13 of [3], $F_i*K_0$ is a kernel of type 1. Also, the integrals defining $X_I(F_i*(K-K_0))$, for any $I$, are absolutely and uniformly convergent, since $F_i\in L^{r-\varepsilon}+L^{r+\varepsilon}$ $(r=Q/(Q-1))$, and $X_I(K-K_0)\in L^p$ $(1<p<\infty, |I|\geqslant 0)$. Hence $F_i*(K-K_0)$ is a $C^\infty$ function. Choose $\varphi\in C_0^\infty$ with $\varphi(x)=1$ for $|x|\leqslant 1$ and $\varphi(x)=0$ for $|x|\geqslant 2$. Then

$$\begin{aligned} X_j(F_i*K) &= X_j\big(\varphi(F_i*K)\big)+X_j\big((1-\varphi)(F_i*K)\big) \\ &= X_j\big(\varphi(F_i*K_0)\big)+X_j\big(\varphi(F_i*(K-K_0))\big)+X_j\big((1-\varphi)(F_i*K)\big) \\ &= H_1+H_2+H_3. \end{aligned}$$

$H_1$ is a truncated singular kernel, so by Lemma 5.2,

$$X_if*H_1\in\Gamma_{\alpha-1}, \quad \|X_if*H_1\|_{(\alpha-1)} \leqslant C_1\|X_if\|_{(\alpha-1)}.$$

$H_3$ vanishes for $|x| \leqslant 1$ and equals $X_j(F_i * K)(x)$ for $|x| \geqslant 2$. Thus by Lemma 5.4, $H_3 \in C^\infty$, $H_3 \in L^1$, and $X_k H_3 \in L^1$ for $k = 1, \ldots, n$. The same is true of $H_2$, in fact $H_2 \in C_0^\infty$. Therefore

$$\|X_i f * (H_2 + H_3)\|_\infty \leqslant \|H_2 + H_3\|_1 \|X_i f\|_\infty,$$

and, by Lemma 4.2, $X_i f * (H_2 + H_3) \in \Gamma_{\alpha-1}$ and

$$|X_i f * (H_2 + H_3)|_{\alpha-1} \leqslant C_2 \Big\{ \|X_i f * (H_2 + H_3)\|_\infty + \sum_1^n \|X_k (X_i f * (H_2 + H_3))\|_\infty \Big\}$$
$$\leqslant C_2 \Big\{ \|H_2 + H_3\|_1 + \sum_1^n \|X_k (H_2 + H_3)\|_1 \Big\} \|X_i f\|_\infty .$$

Combining all these estimates, we see that

$$\|X_i f * X_j (F_i * K)\|_{(\alpha-1)} \leqslant C \|X_i f\|_{(\alpha-1)}.$$

In view of (5.8), we have proved the desired estimate (5.7).

To complete the proof of the lemma, we need to remove the restriction that $f$ have compact support. If $f \in \Gamma_\alpha$ is arbitrary, we still have $f * K \in \Gamma_{\alpha-1}$ and $\|f * K\|_{(\alpha-1)} \leqslant C \|f\|_{(\alpha-1)}$ by Lemma 5.2. To handle derivatives we proceed as in the proof of Lemma 4.3; choose $\varphi \in C_0^\infty$ with $\|\varphi\|_\infty = 1$ and $\varphi(x) = 1$ for $|x| \leqslant 1$, and set $\varphi_\varepsilon(x) = \varphi(\varepsilon x)$, $f_\varepsilon = \varphi_\varepsilon f$. Then $\|\varphi_\varepsilon\|_{(\alpha)} \to \|\varphi\|_\infty = 1$ as $\varepsilon \to 0$, so

$$\overline{\lim_{\varepsilon \to 0}} \|f_\varepsilon\|_{(\alpha)} \leqslant \overline{\lim_{\varepsilon \to 0}} \|\varphi_\varepsilon\|_{(\alpha)} \|f\|_{(\alpha)} = \|f\|_{(\alpha)}.$$

Moreover, by the preceding results,

$$\|f_\varepsilon * K\|_{(\alpha)} \leqslant C \|f_\varepsilon\|_{(\alpha)}.$$

But since $K$ is supported in $\{x \colon |x| \leqslant 2\}$, from (1.2) it follows that for $\varepsilon$ sufficiently small, if $|x| \leqslant 1/2\varepsilon$ and $|y| \leqslant 2^{1/(\alpha-1)}$,

$$f * K(x) = f_\varepsilon * K(x), \quad X_j(f * K)(x) = X_j(f_\varepsilon * K)(x),$$
$$\Delta_y X_j(f * K)(x) = \Delta_y X_j(f_\varepsilon * K)(x).$$

Hence

$$\|X_j(f * K)\|_\infty \leqslant \overline{\lim} \|X_j(f_\varepsilon * K)\|_\infty \leqslant \overline{\lim}\, C \|f_\varepsilon\|_{(\alpha)} \leqslant C \|f\|_{(\alpha)},$$

and

$$\sup \{ |\Delta_y X_j(f * K)(x)| / |y|^{\alpha-1} \colon x \in G,\ |y| \leqslant 2^{1/(\alpha-1)} \}$$
$$\leqslant \overline{\lim} |X_j(f_\varepsilon * K)|_{\alpha-1} \leqslant \overline{\lim}\, C \|f_\varepsilon\|_{(\alpha)} \leqslant C \|f\|_{(\alpha)}.$$



Also

$$\sup \{ |\Delta_y X_j(f * K)(x)| / |y|^{\alpha-1} \colon x \in G,\ |y| > 2^{1/(\alpha-1)} \}$$
$$\leqslant \|X_j(f * K)\|_\infty \leqslant C \|f\|_{(\alpha)}.$$

Thus $|X_j(f * K)|_{\alpha-1} \leqslant C \|f\|_{(\alpha)}$, and we are done.

Proposition 5.1 is now an immediate consequence of Lemmas 5.2 and 5.6, Propositions 4.4, 4.5, and 4.6, and Theorem 3.10.

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