Lipschitz stability for the electrical impedance tomography problem: the complex case.

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Outline

Part I

The forward problem

Formulation of the problem

Given a bounded connected open set $\Omega \subset \mathbb{R}^n$ for $n \geq 2$, given γ a bounded complex valued function (admittivity function) satisfying

$$\Re\gamma \geq \lambda^{-1} > 0$$

and given $f \in H^{1/2}(\partial\Omega)$ let us consider the weak solution $u \in H^1(\Omega)$ of the problem

$$\begin{cases} \mathrm{div}\,(\gamma\nabla u) &= 0 \quad \text{in} \quad \Omega \\ u &= f \quad \text{on} \quad \partial\Omega. \end{cases}$$

Consider the Dirichlet to Neumann map $\Lambda_\gamma : H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)$ given by

$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}_{|\partial\Omega},$$

where ν is the exterior unit normal vector to $\partial\Omega$.

Part II

The inverse problem

Formulation of the problem

EIT Inverse Problem:

Determine γ from the knowledge of the Dirichlet-to Neumann map Λ_γ

Known results

Uniqueness

- $n \geq 3$ If γ is sufficiently smooth the results obtained by Sylvester and Uhlmann (1987) for the conductivity case can be extended to the complex case.
- $n = 2$ Francini (2000), Bukhgeim (2008).
- $n \geq 3, \gamma \in L^{\infty}(\Omega)$ completely open
- $n = 2$ Astala and Paivarinta for real conductivities $\gamma \in L^{\infty}(\Omega)$

Known results

Stability

- $n \geq 3$ Alessandrini (1988) has proved for the conductivity case that if
$$\|\gamma\|_{W^{2,\infty}(\Omega)} \leq E$$
then
$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq \omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|)$$
where $\omega(t) = C|\log t|^{-\eta}$.
- $n = 2$ Barcelo, Faraco and Ruiz (2007) have proved for the conductivity case that if
$$\|\gamma\|_{C^\alpha(\overline{\Omega})} \leq E$$
then logarithmic continuous dependence holds.
- Clop, Faraco, Ruiz (2009) $\gamma \in W^{\alpha,p}$, $\alpha > 0$ and $1 < p < \infty$ then
$$\|\gamma_1 - \gamma_2\|_{L^2(\Omega)} \leq \omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|)$$
where $\omega(t) = C|\log t|^{-\eta}$.

A-priori assumptions

Mandache (2001) has proved that logarithmic stability is the best possible stability also using as a-priori assumption

$$\|\gamma\|_{C^k(\overline{\Omega})} \leq E, \quad \forall k = 0, 1, 2....$$

Strategy:

Look for a-priori assumptions on γ

that are

- physically relevant
- give rise to a better type of stability

A-priori assumptions

Alessandrini and Vessella (2003) assume that

γ piecewise constant

$$\gamma(x) = \sum_{j=1}^{N} \gamma_j 1_{D_j}(x) \quad \text{a.e. in } \Omega,$$

where $D_1, ...D_N$ are known disjoint Lipschitz domains and $\gamma_1, ...\gamma_N$ are unknown real numbers.
Assuming ellipticity and $C^{1,\alpha}$ regularity at the interfaces joining contiguous domains D_j and at $\partial\Omega$ they prove Lipschitz continuous dependence of γ from Λ_γ

Part III

The main result

Main assumptions

(H1) $\Omega \subset \mathbb{R}^n$ is a bounded domain such that

$$|\Omega| \leq A,$$

and

$$\partial\Omega \text{ is of Lipschitz class with constants } r_0, L.$$

(H2) The complex conductivity γ satisfies

$$\Re\gamma \geq \lambda^{-1}, \quad |\gamma| \leq \lambda \qquad \text{a.e. in } \Omega$$

for some $\lambda \geq 1$, and is of the form

$$\gamma(x) = \sum_{j=1}^{N} \gamma_j 1_{D_j}(x),$$

where γ_j are for $j = 1, \ldots, N$ unknown complex numbers and D_j are known open sets in $\mathbb{R}^n$ which satisfy the following conditions

Main assumptions

(H3) $D_i \cap D_j = \emptyset$, $i \neq j$ such that $\cup_{j=1}^{N} \overline{D}_j = \overline{\Omega}$, ∂D_j, $j = 1, \ldots, N$ of Lipschitz class with constants r_0, L.
There exists one region D_1 such that $\partial D_1 \cap \partial\Omega$ contains an open flat portion Σ_1. For every $j \in \{2, \ldots, N\}$ there exists a walkway from D_1 to D_j, i.e. there exist $j_1, \ldots, j_M \in \{1, \ldots, N\}$ such that

$$D_{j_1} = D_1, \quad D_{j_M} = D_j$$

and, for every $k = 1, \ldots, M$

$$\partial D_{j_{k-1}} \cap \partial D_{j_k}$$

contains a flat portion Σ_k such that

$$\Sigma_k \subset \Omega, \quad \forall k = 2, \ldots, M.$$

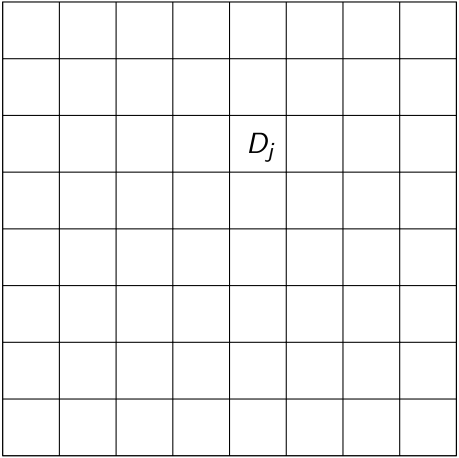
Main assumptions

There exists $P_k \in \Sigma_k$ and a rigid transformation of coordinates such that $P_k = 0$ and

$$
\begin{aligned}
\Sigma_k \cap B_{r_0/3} &= \{x \in B_{r_0/3} : x_n = 0\}, \\
D_{j_k} \cap B_{r_0/3} &= \{x \in B_{r_0/3} : x_n > 0\}, \\
D_{j_{k-1}} \cap B_{r_0/3} &= \{x \in B_{r_0/3} : x_n < 0\}.
\end{aligned}
$$

Main assumptions

A possible configuration



$$\overline{\Omega} = \cup_{j=1}^{N} \overline{D}_j$$

Main result

Theorem

Let Ω *satisfy assumption* **(H1)**. *Let* $\gamma^{(k)}$, $k = 1, 2$ *be two complex piecewise constant functions of the form*

$$\gamma^{(k)}(x) = \sum_{j=1}^{N} \gamma_j^{(k)} 1_{D_j}(x),$$

where $\gamma^{(k)}$ *satisfy for* $k = 1, 2$ *assumption* **(H2)** *and* $D_j, j = 1, \dots, N$ *satisfy assumption* **(H3)**. *Then there exists a positive constant* $C = C(r_0, L, A, n, N, \lambda)$ *such that*

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Omega)} \leq C\|\Lambda_1 - \Lambda_2\|_{\mathfrak{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))}, \qquad (1)$$

where $\Lambda_k = \Lambda_{\gamma^{(k)}}$ *for* $k = 1, 2$.

[B., Francini, 2010]

Part IV

Proof of the main result

Main ideas of the proof

The proof is constructive and is based on:

- Quantitative estimates of unique continuation of harmonic functions
- Construction of singular solutions and study of its asymptotic behaviour near the discontinuity interfaces.

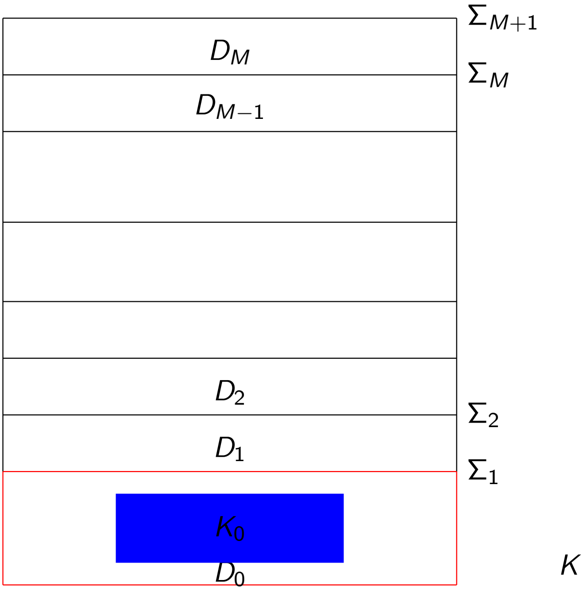
Estimates of unique continuation

We extend Ω to $\Omega_0 = \Omega \cup D_0$ by adding an open cylinder D_0 whose basis is the flat portion Σ_1 of $\partial\Omega \cap \overline{D}_1$ and with height greater than r_0. Let $K_0 = \{x \in D_0 : \mathrm{dist}(x, \partial D_0) \geq r_0/2\}$.
Extend γ on Ω_0 setting it equal to 1 in D_0. Let us consider the chain of domains connecting D_0 to D_M, $M \leq N$ where

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(D_M)} = \|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Omega)}.$$

and consider the *walkway* $K \subset \cup_{j=0}^{M} \overline{D}_j$ with Lipschitz boundary such that $\overline{K} \cap \partial D_j = \Sigma_j \cup \Sigma_{j+1}$ for $j = 1, \ldots, M$ and $\mathrm{dist}(K, \partial(\cup_{j=0}^{M} \overline{D}_j) \setminus \{\Sigma_{M+1} \cup D_0\}) > r_0/16$.

The set K



Estimates of unique continuation

Theorem

Let $v \in H^1(K)$ *be a solution to*

$$div(\gamma \nabla v) = 0 \quad in \quad K,$$

such that

$$\|v\|_{L^\infty(K_0)} \le \epsilon_0,$$

$$|v(x)| \le (\epsilon_0 + E_0) dist(x, \Sigma_{M+1})^{1-\frac{n}{2}} \quad for\ every \quad x \in K.$$

Then

$$|v(\tilde{x})| \le C \left(\frac{\epsilon_0}{E_0 + \epsilon_0} \right)^{\tau^{(M+1)N_1} \delta_1^{M+1} \tau_r} (E_0 + \epsilon_0) r^{\left(1-\frac{n}{2}\right)(1-\tau_r)},$$

where $\tilde{x} = P_{M+1} - 2r\nu(P_{M+1})$, $r \in (0, r_0)$ *and* $\tau, \tau_r, \delta_1 \in (0,1)$, $N_1 = \frac{A}{r_0^n}$ *and* C *depends on* r_0, L, A, n, λ, N.

Singular solutions

Difficulty:

Lack of a result of existence of the Green's function in Ω for $n \geq 3$

- $n \geq 3$ Hofmann and Kim (2007) existence of Green's function for weak solutions satisfying interior Hoelder continuity estimates .
- $n = 2$ Dong and Kim (2009) existence of Green's function

Singular solutions

Theorem

Let γ satisfy assumptions **(H1)-(H3)** *in Ω_0. For $y \in K$ there exists a unique function $G(\cdot, y)$, continuous in $\Omega \setminus \{y\}$*

$$\int_{\Omega_0} \gamma \nabla G(\cdot, y) \cdot \nabla \phi = \phi(y), \quad \forall \phi \in C_0^\infty(\Omega_0),$$

$$\|G(\cdot, y)\|_{H^1(\Omega \setminus B_r(y))} \leq C r^{1-n/2}, \quad \forall r < d_y/2,$$

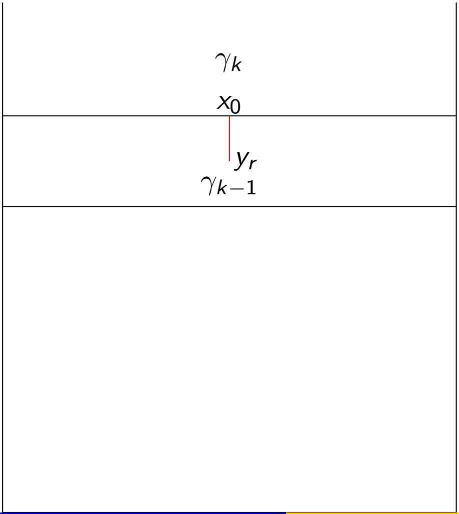
$$G(x, y) = G(y, x) \quad \textit{for every } x, y \in K.$$

Moreover for $x \in D_k$, $x_0 \in \Sigma_k$ and $y_r = x_0 + r\nu$,

$$\nabla_x G(x, y_r) = \nabla_x \Gamma_k(x, y_r) + O(1)$$

Singular solutions

$$\operatorname{div}\Big(\Big(\gamma_{k-1}1_{\mathbb{R}^n_-}+\gamma_k 1_{\mathbb{R}^n_+}\Big)\nabla_x\Gamma_k(\cdot,y)\Big)=\delta_y$$



Σ_k

$$\nabla_x G(x,y_r)=\nabla_x\Gamma_k(x,y_r)+O(1)$$

Proof

Let $D_0, D_1, \ldots, D_M$ be the chain of domains such that

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(D_M)} = E := \|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Omega)}.$$

and let

$$\delta_k := \|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\cup_{j=0}^k D_j)}$$

for $k = 1, \cdots, M$.

Proof

Note that, for $y, z \in D_0$ we have

$$\int_\Omega (\gamma^{(1)} - \gamma^{(2)}) \nabla G_1(\cdot, y) \cdot \nabla G_2(\cdot, z) = < (\Lambda_1 - \Lambda_2) G_1(\cdot, y), \overline{G_2(\cdot, z)} >$$

so that for $y, z \in K_0$

$$\left| \int_\Omega (\gamma^{(1)} - \gamma^{(2)}) \nabla G_1(\cdot, y) \cdot \nabla G_2(\cdot, z) \right| \le C\epsilon$$

Then

$$\left| \int_{\Omega \setminus \cup_{j=1}^{k-1} D_j} (\gamma^{(1)}(x) - \gamma^{(2)}(x)) \nabla_x G_1(x, y) \cdot \nabla_x G_2(x, z)\, dx \right| \le C(\epsilon + \delta_{k-1})$$

Define, for $y, z \in K \cap \cup_{j=1}^{k-1} D_j$,

$$S_{k-1}(y, z) := \int_{\Omega \setminus \cup_{j=1}^{k-1} D_j} (\gamma^{(1)}(x) - \gamma^{(2)}(x)) \nabla_x G_1(x, y) \cdot \nabla_x G_2(x, z)\, dx$$

Proof

We have

$$|S_{k-1}(y,z)| \le C(\epsilon + \delta_{k-1}) \quad \forall y, z \in K_0,$$

$$|S_{k-1}(y,z)| \le C E \left(d(y,\Sigma_k) d(z,\Sigma_k)\right)^{1-\frac{n}{2}} \forall y, z \in K \cap \cup_{j=1}^{k-1} D_j,$$

and

$$\operatorname{div}\left(\gamma^{(1)} \nabla S_{k-1}(\cdot, z)\right) = 0, \quad \operatorname{div}\left(\gamma^{(2)} \nabla S_{k-1}(y, \cdot)\right) = 0 \quad \text{in } K \cap \cup_{j=1}^{k-1} D_j.$$

Using the Hoelder estimates of unique continuation in $K \cap W_{k-1}$ we get

$$|S_{k-1}(y_r, y_r)| \le C \left(\frac{\epsilon + \delta_{k-1}}{\epsilon + \delta_{k-1} + E}\right)^{\alpha_r} (\epsilon + \delta_{k-1} + E) r^{2-n}$$

Proof

Using the asymptotic behaviour of $\nabla_x G_j$ we have

$$|\gamma_k^{(1)} - \gamma_k^{(2)}| \leq C(r^{n-2}|S_{k-1}(y_r, y_r)| + E r^{n-2}) \quad \forall r < r_0/8$$

from the Hoelder estimates for $S_{k-1}(y_r, y_r)$ we derive

$$|\gamma_k^{(1)} - \gamma_k^{(2)}| \leq C(\epsilon + \delta_{k-1} + E) \left[\left(\frac{\epsilon + \delta_{k-1}}{\epsilon + \delta_{k-1} + E} \right)^{(\tau^{kN_1}\delta_1^k \tau_r)^2} + \left(\frac{r}{r_0} \right)^{n-2} \right]$$

and we get

$$\delta_k \leq C(\epsilon + \delta_{k-1} + E)\omega \left(\frac{\epsilon + \delta_{k-1}}{\epsilon + \delta_{k-1} + E} \right). \tag{2}$$

where

$$\omega(t) = \begin{cases} |\log t|^{-(n-2)/4} & \text{for} \quad 0 < t \leq \frac{1}{e^n} \\ n^{-(n-2)/4} & \text{for} \quad t \geq \frac{1}{e^n} \end{cases}$$

Proof

Since $\delta_0 = 0$ iterating we obtain

$$\delta_k + \epsilon \leq (C+1)^k (E+\epsilon)\omega_k\left(\frac{\epsilon}{\epsilon+E}\right),$$

where ω_k is the composition of ω with itself k times.
Now we recall that $E = \delta_M$ and, hence,

$$E + \epsilon \leq (C+1)^M (\epsilon+E)\omega_M\left(\frac{\epsilon}{\epsilon+E}\right)$$

Now, either $E \leq \epsilon$ and this proves Lipschitz stability, or $E > \epsilon$ and we can write

$$E \leq 2(C+1)^M E\omega_M\left(\frac{\epsilon}{2E}\right),$$

from which

$$E \leq \frac{\epsilon}{2\omega_M^{-1}\left(\frac{1}{2(C+1)^M}\right)}$$

Part V

Conclusions

Final remarks

Remark1

We expect that Lipschitz continuous dependence of the admittivities on the data still holds replacing the flatness condition on the interfaces with $C^{1,\alpha}$ regularity.

Remark2

In our stability estimate the constant $C = C(r_0, L, A, n, N, \lambda)$ and it diverges exponentially in the number N of the subdomains of the partition of Ω.
Rondi has proved in (2006) that

$$C \geq Aexp(BN^{1/2n})$$

with A and B absolute constants. This inequality can be interpreted as a resolution limit in the determination of γ from Λ_γ . If ϵ is the error on the measured data and we can tolerate an error up to $\overline{C}\epsilon$ ($\overline{C}$ error amplification tolerance) and if all the domains D_j have size r_0^n then $r_0 = N^{-1/n}$ and we get

$$r_0 \geq \left(\frac{1}{B}\log\frac{\overline{C}}{A}\right)^{-(2n-1)/n}$$

So if we fix $\overline{C}$ no detail smaller than $(\frac{1}{B}\log\frac{\overline{C}}{A})^{-(2n-1)/n}$ can be detected.

Remark3

Observe that in our result we can replace the full Dirichlet to Neumann map with the local Dirichlet to Neumann map. More precisely, let Σ be an open portion of $\partial\Omega$ containing a flat open subset. Let $H^{1/2}_{co}(\Sigma) = \left\{\phi \in H^{1/2}(\partial\Omega) \,:\, \text{supp}\,\phi \subset \Sigma\right\}$ and define the local Dirichlet to Neumann map in the following way: for $\phi \in H^{1/2}_{co}(\Sigma)$ let

$$< \Lambda^{\Sigma}_{\gamma}\phi, \psi >= \int_{\Omega} \gamma \nabla u \nabla \overline{v} \quad \text{for } \phi, \psi \in H^{1/2}_{co}(\Sigma)$$

where u solves equation $\text{div}(\gamma\nabla u) = 0$ and $u = \phi$ on $\partial\Omega$ and $v \in H^1(\Omega)$ such that $v = \psi$ on $\partial\Omega$.

We observe that in our proof we apply the Dirichlet to Neumann map to functions whose support is contained in a neighborhood of the flat portion of $\partial\Omega$. Hence we have

$$\|\gamma^1 - \gamma^2\|_{L^{\infty}(\Omega)} \leq C\|\Lambda^{\Sigma}_1 - \Lambda^{\Sigma}_2\|_{\mathfrak{L}(H^{1/2}_{co}, H^{-1/2}_{co})}.$$