LIPSCHITZ STABILITY OF SOLUTIONS TO PARAMETRIC OPTIMAL CONTROL PROBLEMS FOR PARABOLIC EQUATIONS [1]

Kazimierz Malanowski
Systems Research Institute
Polish Academy of Sciences
ul. Newelska 6, 01-447 Warszawa, Poland
and
Fredi Tröltzsch
Fakultät für Mathematik
Technische Universität Chemnitz-Zwickau
D-09107 Chemnitz, Germany

Dedicated to the 65. birthday of Prof. Lothar von Wolfersdorf

Abstract A class of parametric optimal control problems for semilinear parabolic equations is considered. Using recent regularity results for solutions of such equations, sufficient conditions are derived under which the solutions to optimal control problems are locally Lipschitz continuous functions of the parameter in the L^∞-norm. It is shown that these conditions are also necessary, provided that the dependence of data on the parameter is sufficiently strong.

1 Introduction

The presence of inequality type constraints in optimization problems introduces a nonsmoothness even if all data are smooth. That is the reason why the classical implicit function theorem can not be used in stability analysis of solutions to such problems. Instead of that, the main tool in such analysis is Robinson's implicit function theorem for so called *generalized equations* (see, [18] and [7] for extentions). This theorem allows to reduce the stability analysis for the original *nonlinear* optimization problems to such analysis for linear-quadratic accessory problems.

This approach was used by Robinson in [18] to derive sufficient conditions of local Lipschitz continuity for solutions to parametric mathematical programs in finite dimensions. Later on these results were extended to cone constrained optimization problems in abstract Hilbert or Banach spaces (see, e.g., [1, 15, 20]), including applications to optimal control [1, 8, 15].

The main difficulty in applications to optimal control problems is connected with the presence of the so called *two-norm discrepancy* (see [16]). Namely, the original nonlinear problems are well defined and differentiable in a *stronger*

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topology of L^∞-type, whereas the accessory problems are coercive in a *weaker* topology of L^2-type. Hence the natural topology in which the solutions to accessory problems are stable is L^2, while to apply Robinson's theorem we need L^∞-stability.

In case of control-constrained problems for ODEs, stability in L^2 can be strenghtened to L^∞ using Pontryagin's maximum principle. In that step, the crucial point is that the solutions of state and adjoint equations are *uniformly bounded* functions of time. The situation is much more delicate for PDEs, where weak solution are not necessarily bounded.

Some L^2-stability results were obtained for convex distributed control problems in papers on sensitivity analysis (see, e.g.,[14, 21]). Moreover, estimates of this type were derived for numerical approximations of convex distributed control problems by discretization methods. Here, the perturbation parameter is the underlying mesh size. We refer to [23] and to the references therein. Quite a few papers have been devoted to Hölder estimates in spaces of type L^2 or L^∞. We should mention, for instance, [19] and [22] where such estimates for inverse problems with respect to data perturbations were obtained. Auxiliary Hölder stability results were derived for the convergence analysis of SQP methods [2], [11], [12].

Recently, an important step was done in [24], where new regularity results for parabolic equations, due to Casas [5] and Raymond and Zidani [17], were used to get L^∞-stability for linear-quadratic optimal control problems.

In the present paper, the results of [24] together with Robinson's theorem are used to derive L^∞ stability of solutions to a class of parametric optimal control problems for semilinear parabolic equations.

It is important to evaluate how far sufficient conditions are from necessary ones. Using the approach proposed in [9], it is shown that the obtained *sufficient* stability conditions are also *necessary*, provided that the dependence of the data on the parameter is sufficiently strong. Thus, we derived a *characterization* of the Lipschitz stability property.

The organization of the paper is the following. In Section 2 we recall some needed regularity results for parabolic equations and formulate the class of optimal control problems to be studied. In Section 3 the application of Robinson's implicit function theorem in stability analysis is recalled. In Section 4 the results of [24] are used to get conditions of Lipschitz stability for the accessory problems. Sufficient conditions of local Lipschitz continuity of solutions to the original nonlinear problems are derived in Section 5, while the necessity of these conditions is discussed in Section 6.

2 Preliminaries

Let H be the Banach space of parameters endowed with the norm $\|\cdot\|_H$ and $G \subset H$ a bounded open set of feasible parameters. For any $h \in G$ consider the following semilinear parabolic initial-boundary value problem

$$\begin{aligned} y_t(x,t) + A\, y(x,t) + a(x,t,y(x,t),u(x,t),h) &= 0 \text{ in } Q, \\ \partial_\nu y(x,t) + b(x,t,y,h) &= 0 \text{ in } \Sigma, \\ y(x,0) - \chi(x) &= 0 \text{ in } \Omega. \end{aligned} \tag{2.1}$$

Here, A is the elliptic differential operator

$$A\, y := -\sum_{i,j=1}^{N} D_j\,(a_{ij} D_i\, y)$$

with sufficiently smooth coefficients $a_{ij} = a_{ij}(x)$ satisfying the condition of symmetry $a_{ij} = a_{ji}$. This equation is considered in $Q = \Omega \times (0,T)$, where $\Omega \subset I\!R^N$ $(N \geq 2)$ is a bounded domain with boundary $\partial\Omega = \Gamma$, $\Sigma = \Gamma \times (0,T)$ and $T > 0$ is a fixed time. By ∂_ν the co-normal derivative of y at Γ is denoted, where ν is the outward normal to Γ. Thus we have

$$\partial_\nu y := \sum_{i,j=1} a_{ij}\, \nu_i\, D_j\, y.$$

By $\langle \cdot, \cdot \rangle$ we shall denote the inner product in $I\!R^N$. The function u stands for a *distributed* control, while χ is a fixed initial state function. Following Casas [5] and Raymond and Zidani [17] we assume the following properties of the data:

(A1) Γ is of class $C^{2,\alpha}$ for some $\alpha \in (0,1]$. A is uniformly elliptic (see, e.g., the definition given in [5]). Its coefficients a_{ij} belong to $C^{1,\alpha}(\bar{\Omega})$.

(A2) The distributed nonlinearity $a = a(x,t,y,u,h)$ is a real-valued function defined on $\overline{Q} \times I\!R^2 \times H$ and satisfies the following Carathéodory type condition:

- **(i)** For all $(y,u,h) \in I\!R^2 \times H$, $a(\cdot,\cdot,y,u,h)$ and its first- and second order derivatives $a_y, a_u, a_{yy}, a_{yu}, a_{uu}$ (all depending on $(\cdot,\cdot,y,u,h)$) are Lebesgue measurable on $\overline{Q}$.
- **(ii)** For almost all $(x,t) \in Q$, $a(x,t,\cdot,\cdot,\cdot)$ is twice continuously differentiable with respect to $(y,u) \in I\!R^2$ on $I\!R^2 \times G$.

Throughout the paper, the control u and the perturbation h are uniformly bounded by a certain constant K.

(A3) The function a fulfils the *assumptions of boundedness*

- **(i)**

$$|a(x,t,0,u,h)| \leq a_K(x,t) \quad \forall (x,t) \in Q, |u| \leq K, h \in G, \tag{2.2}$$

where $a_K \in L^q(Q)$ and $q > \frac{N}{2} + 1$. There is a number $c_0 \in I\!R$, and a non-decreasing function $\eta : I\!R_+ \to I\!R_+$ such that

$$c_0 \leq a_y(x,t,y,u,h) \leq \eta(|y|) \tag{2.3}$$

for a.e. $(x,t) \in Q$, all $y \in I\!R$, all $|u| \le K$, $h \in G$.

(ii) The *Lipschitz condition*

$$\begin{array}{rl} & |a(x,t,y_1,u_1,h_1) - a(x,t,y_2,u_2,h_2)| \\ + & |D\,a(x,t,y_1,u_1,h_1) - D\,a(x,t,y_2,u_2,h_2)| \\ + & |D^2\,a(x,t,y_1,u_1,h_1) - D^2\,a(x,t,y_2,u_2,h_2)| \\ & \le L_K\,(|y_1 - y_2| + |u_1 - u_2| + \|h_1 - h_2\|_H) \end{array} \tag{2.4}$$

is fulfilled for almost all $(x,t) \in Q$, all $|y_i| \le K$, $|u_i| \le K$, $i = 1,2$, and all $h \in G$.

Here, D and D^2 stand for gradient and Hessian matrix with respect to the variables (y,u), while $|\cdot|$ is used to denote Euclidean norms of real numbers, 2-vectors and 2×2-matrices.

(A4) The boundary nonlinearity $b = b(x,t,y,h)$ is a real-valued function defined on $\Sigma \times I\!R \times H$. It is assumed to satisfy a Carathéodory type condition and boundedness assumptions analogously to (A2), (A3). These conditions are obtained substituting Σ by Q and deleting u in (A2), (A3).

A *weak solution* of (2.1) is understood as a function $y \in L^2(0,T;H^1(\Omega)) \cap C(\overline{Q})$ such that

$$\begin{array}{l} \int\limits_Q (-\,y \cdot p_t + \langle \nabla_x y, \nabla_x p \rangle) dxdt + \int\limits_Q a(x,t,y,u,h)\ p\ dxdt + \\ + \int\limits_\Sigma b(x,t,y,h)\ p\ dSdt - \int\limits_\Omega \chi(x) p(x,0) dx = 0 \end{array} \tag{2.5}$$

holds for all $p \in W_2^{1,1}(Q)$ satisfying $p(x,T) = 0$. The following theorem is a conclusion of a more general result proved in [5] or [17].

THEOREM **2.1** *Suppose that (A1)-(A4) are satisfied,* $\chi \in C(\overline{\Omega})$, $u \in L^\infty(Q)$. *Then the system (2.1) has a unique weak solution*

$$y \in L^2(0,T;H^1(\Omega)) \cap C(\overline{Q}).$$

$\diamond$

Let us introduce the following spaces

$$\begin{array}{l} W(0,T) = \{y \in L^2(0,T;H^1(\Omega)) \mid y_t \in L^2(0,T;(H^1(\Omega))')\}, \\ W^s := \{y \in W(0,T) | y_t + A\,y \in L^s(Q),\ \partial_\nu y \in L^s(\Sigma),\ y(0) \in C(\overline{\Omega})\} \\ Z^s = W^s \times L^s(Q). \end{array} \tag{2.6}$$

The space Z^s is used for elements $\zeta = (y,u)$, while adjoint states p belong to W^s. In W^s, we shall use the norm

$$\|y\|_{W^s} = \|y_t + A\,y\|_{L^s(Q)} + \|\partial_\nu y\|_{L^s(\Sigma)} + \|y(0)\|_{C(\overline{\Omega})}.$$

For $s > \max\{N/2+1, N+1\}$, this space is continuously embedded into $C(\overline{Q})$. This follows from [5] and [17]. By the definition of the norm in W^s, the operators $y_t + A\,y$ and $\partial_\nu y$ are continuous from W^s to $L^s(Q)$ and $L^s(\Sigma)$, respectively. This fact will be used in the definition of the generalized equation at the end of this section. The normal trace is defined, for instance, as in Casas [5].

For each $h \in G$ consider the following optimal control problem

(P_h) Find $\zeta_h := (y_h, u_h) \in Z^\infty$ such that

$$\mathcal{J}_h(\zeta_h) = \min_{\zeta} \mathcal{J}_h(\zeta)$$

subject to (2.1) and to the pointwise control constraints

$$r^a(x,t) \le u(x,t) \le r^b(x,t) \qquad \text{a.e on } Q, \tag{2.7}$$

where

$$\mathcal{J}_h(\zeta) := \mathcal{J}_h(y,u) := \int_Q \psi(x,t,y,u,h)\,dxdt. \tag{2.8}$$

We assume:

(A5) The real-valued function ψ satisfies the assumptions (A2), (A3) imposed on a, except the growth condition (2.3).

(A6) The functions r^a and r^b are of class $L^\infty(Q)$ and a constant $d > 0$ exists such that

$$r^b(x,y) - r^a(x,t) \ge d \quad \text{a.e. on } Q. \tag{2.9}$$

Let us introduce the Hamiltonian $\mathcal{H} = \mathcal{H}(x,t,y,u,p,h) : I\!R^{N+4} \times G \to I\!R$,

$$\mathcal{H} = \psi(x,t,y,u,h) - p \cdot a(x,t,y,u,h) \tag{2.10}$$

and the Lagrangian $\mathcal{L} : W^\infty \times L^\infty(Q) \times W(0,T) \times G \to I\!R$

$$\begin{aligned}\mathcal{L}(y,u,p,h) \;&:= \textstyle\int_Q \mathcal{H}(y,u,p,h)\,dxdt - \int_\Sigma p \cdot b(y,h)\,dSdt - \\ &\quad - \textstyle\int_\Omega p(0)(y(0) - \chi(x))dx - \int_Q (y_t + A\,y)\,p\,dxdt.\end{aligned} \tag{2.11}$$

Assume:

(A7) For a fixed reference value $h_0 \in G$ of the parameter there exists a solution $\zeta_0 = (y_0, u_0) := (y_{h_0}, u_{h_0}) \in Z^\infty$ of (P_{h_0}) and an associated adjoint state $p_0 := p_{h_0} \in Y^\infty$ such that the following first order necessary optimality conditions hold:

$$D_y\mathcal{L}(y_0, u_0, p_0,, h_0)\; y = 0 \quad \text{for all } y \in W^\infty, \tag{2.12}$$

$$D_u\mathcal{L}(y_0, u_0, p_0, h_0)(u - u_0) = \\ = \int_Q D_u\mathcal{H}(y_0, u_0, p_0, h_0)(u - u_0)\,dxdt \ge 0 \quad \text{for all } u \in \mathcal{U}, \tag{2.13}$$

where

$$\mathcal{U} := \{u \in L^\infty(Q) \mid r^a(x,t) \le u(x,t) \le r^b(x,t)\} \tag{2.14}$$

is the set of feasible controls.

In the sequel, to simplify notation, the subscript 0 will be used to denote that a given function is evaluated at the reference solution, e.g., $\mathcal{H}_0 := \mathcal{H}(x,t,y_0,u_0,p_0,h_0)$.

Condition (2.12) yields the adjoint equation

$$\begin{aligned} -(p_0)_t(x,t) + A\,p_0(x,t) &= D_y\mathcal{H}(x,t,y_0,u_0,p_0,h_0) && \text{in } Q,\\ \partial_\nu p_0(x,t) + D_y b(x,t,y_0,h_0)\,p_0(x,t) &= 0 && \text{in } \Sigma,\\ p_0(x,T) &= 0 && \text{in } \Omega. \end{aligned} \tag{2.15}$$

Without loss of generality we can assume

$$\chi = 0. \tag{2.16}$$

Define the spaces:

$$\begin{aligned} W_0^s &:= \{y \in W^s \mid y(0) = 0\}, \quad W_T^s := \{p \in W^s \mid p(T) = 0\},\\ X &:= W_T^\infty \times L^\infty(Q) \times W_0^\infty,\\ \Delta &:= L^\infty(Q) \times L^\infty(\Sigma) \times L^\infty(Q) \times L^\infty(Q) \times L^\infty(\Sigma). \end{aligned} \tag{2.17}$$

Introduce the normal cone operator $\mathcal{N}$ of the feasible set $\mathcal{U}$:

$$\mathcal{N}(u) = \begin{cases} \lambda \in \{L^\infty(Q) \mid \int_Q \lambda(v-u)\,dxdt \le 0\ \forall v \in \mathcal{U}\} & \text{if } u \in \mathcal{U},\\ \emptyset & \text{if } u \notin \mathcal{U}. \end{cases} \tag{2.18}$$

Using (2.18), the optimality system consisting of (2.12), (2.13) as well as of (2.1) and (2.7) can be expressed in the form of the following *generalized equation*

$$0 \in \mathcal{F}(\xi_0, h_0) + \mathcal{T}(\xi_0), \tag{2.19}$$

where $\xi = (y,u,p)$, while $\mathcal{F} : X \times G \to \Delta$ and $\mathcal{T} : X \to 2^\Delta$, are, respectively, a function and a set valued mapping with closed graph, given by

$$\mathcal{F}(\xi,h) = \begin{bmatrix} -p_t + A\,p - D_y\mathcal{H}(y,u,p,h) & \text{in } Q\\ \partial_\nu p + D_y b(y,h)\,p & \text{in } \Sigma\\ D_u\mathcal{H}(y,u,p,h) & \text{in } Q\\ y_t + A\,y + a(y,u,h) & \text{in } Q\\ \partial_\nu y + b(y,h) & \text{in } \Sigma \end{bmatrix}, \tag{2.20}$$

$$\mathcal{T} = [\{0\},\{0\},\mathcal{N}(u),\{0\},\{0\}]^\top. \tag{2.21}$$

3 Application of an abstract implicit function theorem

The problem that we are interested in can be formulated as follows:

> **Find conditions under which there exists a neighborhood $G_0 \subset H$ of h_0 such that for each $h \in G_0$ there is a locally unique solution $\xi_h = (y_h, u_h, p_h)$ of the generalized equation**
>
> $$0 \in \mathcal{F}(\xi, h) + \mathcal{T}(\xi), \tag{3.1}$$
>
> **where (y_h, u_h) is a local solution of $(\mathbf{P}_h)$, and ξ_h is a Lipschitz continuous function of h.**

To solve this problem, we are going to apply to (3.1), in a standard way, an abstract implicit function theorem for generalized equations [18, 7]. Note that, by our assumptions, $\mathcal{F}$ is Fréchet differentiable. Along with (3.1), let us introduce the following generalized equation, obtained from (3.1) by linearization and by perturbation of $\mathcal{F}$:

$$\delta \in \mathcal{F}(\xi_0, h_0) + D_\xi \mathcal{F}(\xi_0, h_0)(\xi - \xi_0) + \mathcal{T}(\xi), \tag{3.2}$$

where $\delta \in \Delta$.

We will denote by $\mathcal{B}_\rho^X(x_0) := \{x \in X \mid \|x - x_0\|_X \le \rho\}$ the closed ball of radius ρ around x_0 in a Banach space X.

Our sufficiency analysis is based on the following Robinson's abstract implicit function theorem (see, Theorem 2.1 and Corollary 2.2 in [18]):

THEOREM **3.1** *If*

(j) *there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that, for each $\delta \in \mathcal{B}_{\rho_1}^\Delta(0)$ there is a unique in $\mathcal{B}_{\rho_2}^X(\xi_0)$ solution to (3.2), which is Lipschitz continuous in δ,*

then

(jj) *there exist $\sigma_1 > 0$ and $\sigma_2 > 0$ such that, for each $h \in \mathcal{B}_{\sigma_1}^H(h_0)$ there is a unique in $\mathcal{B}_{\sigma_2}^X(\xi_0)$ solution to (3.1), which is Lipschitz continuous in h.*

◇

Verifying necessity of the derived sufficient conditions of Lipschitz continuity, we will consider a special situation, where the dependence upon the parameter in (3.1) is *strong* in the following sense:

$$\begin{aligned} &H = H^0 \times \Delta, \text{ where } H^0 \text{ is an arbitrary Banach space and} \\ &\mathcal{F}(\xi, h) = \mathcal{F}^0(\xi, h^0) + h^1, \text{ with } h^0 \in H^0 \text{ and } h^1 \in \Delta. \end{aligned} \tag{3.3}$$

The next theorem follows from Theorem 3 in [10]:

THEOREM **3.2** *If (3.3) holds, then* **(jj)** *implies* **(j)**. $\diamond$

Theorem 3.1 allows to deduce existence, local uniqueness and Lipschitz continuity of solutions to (3.1) from the same properties of the solutions to the *linear* generalized equation (3.2). In general, these last properties are much easier to verify than the original ones.

Let

$$\delta = (\Delta g, \Delta d) \in \Delta \tag{3.4}$$

be a vector of perturbations, where

$$\Delta g = (\Delta g_Q, \Delta g_u, \Delta g_\Sigma) \in (L^\infty(Q))^2 \times L^\infty(\Sigma),$$
$$\Delta d = (\Delta d_Q, \Delta d_\Sigma) \in L^\infty(Q) \times L^\infty(\Sigma).$$

Recall that the subscript 0 will be used to denote that a given function is evaluated at the reference solution. In view of (2.20) and (2.21), the generalized equation (3.2) takes on the form

$$(\mathrm{LO}_\delta) \qquad \begin{aligned} -q_t + A\,q + a_y^0\,q &= g_Q^0 + \Delta g_Q + D^2_{yy}\mathcal{H}_0\,z + D^2_{yu}\mathcal{H}_0 v, \\ \partial_\nu q + b_y^0\,q &= g_\Sigma^0 + \Delta g_\Sigma - p_0 \cdot D^2_{yy} b^0 z, \end{aligned} \tag{3.5}$$

$$D^2_{uy}\mathcal{H}_0\,z + D^2_{uu}\mathcal{H}_0 v\; - a_u^0\,q - g_u^0 - \Delta g_u \in \mathcal{N}(u), \tag{3.6}$$

$$\begin{aligned} z_t + A\,z + a_y^0\,z &= d_Q^0 + \Delta d_Q - a_u^0\,v, \\ \partial_\nu z + b_y^0\,z &= d_\Sigma^0 + \Delta d_\Sigma, \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} a_y^0 &= D_y\,a(y_0,u_0,h_0), \quad a_u^0 = D_u\,a(y_0,u_0,h_0), \quad b_y^0 = D_y\,b(y_0,u_0,h_0), \\ g_Q^0 &= D_y\,\psi(y_0,u_0,h_0) - D^2_{yy}\mathcal{H}_0 y_0 - D^2_{yu}\mathcal{H}_0 u_0, \\ g_\Sigma^0 &= p_0 \cdot D^2_{yy} b(y_0,h_0) y_0, \\ g_u^0 &= -D_u\,\psi(y_0,u_0,h_0) + D^2_{uy}\mathcal{H}_0\,y_0 + D^2_{uu}\mathcal{H}_0 u_0, \\ d_Q^0 &= -a(y_0,u_0,h_0) + D_y\,a(y_0,u_0,h_0)\,y_0 + D_u\,a(y_0,u_0,h_0)\,u_0, \\ d_\Sigma^0 &= -b(y_0,h_0) + D_y\,b(y_0,h_0)\,y_0. \end{aligned} \tag{3.8}$$

An inspection shows that (LO_δ) constitutes an optimality system for the following linear-quadratic accessory problem

$$\begin{aligned} &(QP_\delta) \qquad \text{Find } \zeta_\delta := (z_\delta, v_\delta) \in Z^\infty \text{ that minimizes} \\ &\mathcal{I}_\delta(\zeta) = \tfrac{1}{2}(\zeta, D^2_{\zeta\zeta}\mathcal{L}_0\zeta) + \textstyle\int_Q (g_Q^0 + \Delta g_Q) z\,dxdt + \int_Q (g_u^0 + \Delta g_u) v\,dxdt + \\ &+ \textstyle\int_\Sigma (g_\Sigma^0 + \Delta g_\Sigma)\,z\,dSdt \end{aligned} \tag{3.9}$$

subject to

$$\begin{aligned} z_t + A\,z + a_y^0\,z &= d_Q^0 + \Delta d_Q - a_u^0\,v && \text{in } Q \\ \partial_\nu z + b_y^0\,z &= d_\Sigma^0 + \Delta d_\Sigma && \text{in } \Sigma \\ z(0) &= 0 && \text{in } \Omega, \end{aligned} \tag{3.10}$$

and

$$r^a \le v \le r^b \quad \text{in } Q, \tag{3.11}$$

where the quadratic form in the cost functional $\mathcal{I}_\delta(\zeta)$ is defined on $Z^2 \times Z^2$ by

$$
\begin{aligned}
(\zeta_1, D^2_{\zeta\zeta}\mathcal{L}_0\zeta_2) &= \int_Q [z_1, v_1] \begin{bmatrix} D^2_{yy}\mathcal{H}_0 & D^2_{yu}\mathcal{H}_0 \\ D^2_{uy}\mathcal{H}_0 & D^2_{uu}\mathcal{H}_0 \end{bmatrix} \begin{bmatrix} z_2 \\ v_2 \end{bmatrix} dxdt + \\
&+ \int_\Sigma z_1 p_0 \cdot D^2_{yy} b^0 z_2 \; dSdt.
\end{aligned} \tag{3.12}
$$

Certainly, the reference solution (y_0, u_0) together with the associated adjoint state p_0 constitutes a solution of (LO_0) i.e., a stationary point for (QP_0).

4 Lipschitz stability for accessory problems

In this section, conditions are derived under which the solutions to (LO_δ), i.e., stationary points of (QP_δ), are locally Lipschitz continuous functions of the parameter δ.

For any $\alpha \geq 0$ let us introduce the sets

$$
I^\alpha = \{(x,t) \in Q \mid D_u\mathcal{H}_0(x,t) > \alpha\}, \quad J^\alpha = \{(x,t) \in Q \mid - D_u\mathcal{H}_0(x,t) > \alpha\}. \tag{4.1}
$$

Moreover, define the mapping $\mathcal{C}_0^\alpha : W_0^2 \times L^2(Q) \to U_\alpha^2 := L^2(Q) \times L^2(\Sigma) \times L^2(I^\alpha \cup J^\alpha)$

$$
\mathcal{C}_0^\alpha \zeta := \begin{pmatrix} z_t + A\,z + a_y^0\,z + a_u^0\,v & \text{in } Q \\ \partial_\nu z + b_y^0\,z & \text{in } \Sigma \\ v & \text{in } I^\alpha \cup J^\alpha \end{pmatrix}. \tag{4.2}
$$

Assume

(AC) (*Coercivity*) There exist $\alpha > 0$ and $\gamma > 0$ such that

$$
(\zeta, D^2_{\zeta\zeta}\mathcal{L}_0\zeta) \geq \gamma\, ||\zeta||_2^2 \quad \text{for all } \zeta \in \ker\, \mathcal{C}_0^\alpha, \tag{4.3}
$$

where $||\zeta||_2^2 := \gamma(||z||^2_{W(0,T)} + ||v||^2_{L^2(Q)})$.

Define the following modification $(\widetilde{\mathrm{QP}}_\delta)$ of problem (QP_δ), where the inequality constraints are modified.

$(\widetilde{QP}_\delta)$ Find $\widetilde{\zeta}_\delta := (\widetilde{z}_\delta, \widetilde{v}_\delta) \in Z^\infty$ which minimizes
$\mathcal{I}_\delta(\zeta)$ subject to (3.10) and to
$\tilde{r}^a(x,t) \leq v(x,t) \leq \tilde{r}^b(x,t)$ a.e. on Q (4.4)
(4.5)

where

$$
\tilde{r}^a = \begin{cases} r^a & \text{on } Q \setminus J^\alpha \\ r^b & \text{on } J^\alpha \end{cases}, \quad \tilde{r}^b = \begin{cases} r^b & \text{on } Q \setminus I^\alpha \\ r^a & \text{on } I^\alpha \end{cases}.
$$

This choice of $\tilde{r}^a$, $\tilde{r}^b$ yields $\tilde{r}^a = \tilde{r}^b = u_0$ on $I^\alpha \cup J^\alpha$.

Problem $(\widetilde{\mathrm{QP}}_\delta)$ coincides with the quadratic problem considered in [24], and by Theorem 4.6 in that paper we get:

PROPOSITION **4.1** *Let (A1)-(A7) and* **(AC)** *hold, then for any $\delta \in \Delta$, problem $(\widetilde{\mathrm{QP}}_\delta)$ has a unique solution $\widetilde{\zeta}_\delta = (\widetilde{z}_\delta, \widetilde{v}_\delta,) \in Z^\infty$ and a unique associated adjoint state $\widetilde{q}_\delta \in W^\infty$. Moreover, there exists a constant $c > 0$ such that*

$$\|\widetilde{z}_{\delta'} - \widetilde{z}_{\delta''}\|_{C(\bar{Q})}, \|\widetilde{v}_{\delta'} - \widetilde{v}_{\delta''}\|_{L^\infty(Q)}, \|\widetilde{q}_{\delta'} - \widetilde{q}_{\delta''}\|_{C(\bar{Q})} \leq c\,\|\delta' - \delta''\|_\Delta. \tag{4.6}$$

◇

REMARK **4.2** Assumption (2.9) is not needed to get (4.6). It will be used in Section 6 in deriving necessary conditions of Lipschitz continuity. ◇

We are going to show that for δ sufficiently small $(\widetilde{z}_\delta, \widetilde{v}_\delta, \widetilde{q}_\delta)$ is a stationary point of (QP_δ), i.e., it satisfies (LO_δ). Since the state equation (3.5) and the adjoint equation (3.7) are satisfied, it is enough to show that the variational inequality (3.6) holds. Note that, for $\delta = 0$ we have $(\widetilde{z}_0, \widetilde{v}_0) = (y_0, u_0)$, $\widetilde{q}_0 = p_0$ and the linearized generalized equation (3.2) reduces to the original nonlinear one (2.19). In particular, it follows from (2.20) and (3.6) that

$$\begin{aligned}&D^2_{uy}\mathcal{H}_0(x,t)\widetilde{z}_0(x,t) + D^2_{uu}\mathcal{H}_0(x,t)\widetilde{v}_0(x,t) + a^0_u(x,t)\widetilde{q}_0(x,t) - g^0_u(x,t) = \\ &= D_u\mathcal{H}_0(x,t).\end{aligned}$$

Hence, by (4.1) we have

$$\begin{aligned}&D^2_{uy}\mathcal{H}_0(x,t)\widetilde{z}_0(x,t) + D^2_{uu}\mathcal{H}_0(x,t)\widetilde{v}_0(x,t) + \\ &+ a^0_u(x,t)\widetilde{q}_0(x,t) - g^0_u(x,t)\begin{cases} > \alpha & \text{for } (x,t) \in I^\alpha, \\ < -\alpha & \text{for } (x,t) \in J^\alpha,\end{cases}\end{aligned}$$

and in view of (4.6), for any $\delta \in \mathcal{B}^\Delta_\rho(0)$, with $\rho > 0$ sufficiently small, we obtain

$$\begin{aligned}&D^2_{uy}\mathcal{H}_0(x,t)\widetilde{z}_\delta(x,t) + D^2_{uu}\mathcal{H}_0(x,t)\widetilde{v}_\delta(x,t) + a^0_u(x,t)\widetilde{q}_\delta(x,t) - \\ &- g^0_u(x,t) - \Delta g_u(x,t)\begin{cases} > \frac{\alpha}{2} & \text{for } (x,t) \in I^\alpha, \\ < -\frac{\alpha}{2} & \text{for } (x,t) \in J^\alpha.\end{cases}\end{aligned} \tag{4.7}$$

It can be easily seen that by (4.7) condition (3.6) is satisfied, i.e.,

$$(\widetilde{z}_\delta, \widetilde{v}_\delta, \widetilde{q}_\delta) = (z_\delta, v_\delta, q_\delta) \tag{4.8}$$

is a stationary point for (QP_δ).

Let us denote $\varsigma = \frac{\alpha}{2}$, then each stationary point $\xi_\delta := (z_\delta, v_\delta, q_\delta) \in \mathcal{B}^X_\varsigma(\xi_0)$ of (QP_δ) is a stationary point of $(\widetilde{QP}_\delta)$. Hence, by the uniqueness of stationary points of $(\widetilde{QP}_\delta)$, we arrive at:

THEOREM **4.3** *If conditions (A1)-(A7) and* **(AC)** *hold, then there exist constants* $\rho > 0$ *and* $\varsigma > 0$ *such that for each* $\delta \in \mathcal{B}^{\Delta}_{\rho}(0)$ *there is a unique in* $\mathcal{B}^{X}_{\varsigma}(\xi_0)$ *stationary point*

$$(z_\delta, v_\delta, q_\delta) \in Z^\infty \times Y^\infty$$

of (QP_δ)*. Moreover, there exists a constant* $c > 0$ *such that*

$$\|z_{\delta'} - z_{\delta''}\|_{C(\bar{Q})}, \|v_{\delta'} - v_{\delta''}\|_{L^\infty(Q)}, \|q_{\delta'} - q_{\delta''}\|_{C(\bar{Q})} \le c\,\|\delta' - \delta''\|_\Delta, \tag{4.9}$$

for all $\delta', \delta'' \in \mathcal{B}^{\Delta}_{\rho}(0)$. ◇

Note that in view of (4.1) condition **(AC)** constitutes *sufficient optimality condition for* (QP_δ). Hence we obtain:

COROLLARY **4.4** *If* (AC) *holds, then for* $\delta \in \mathcal{B}^{\Delta}_{\rho}(0)$, (z_δ, v_δ) *in Theorem 4.3 is a locally unique solution of* (QP_δ) *and* q_δ *is the associated adjoint state.*

◇

5 Lipschitz stability for nonlinear problems: sufficiency

In this section, sufficient conditions of Lipschitz stability of the solutions to the original nonlinear problem (P_h) are derived. The proof will be based on Theorem 3.1 as well as on the results of Section 4. Applying Theorem 3.1 to the generalized equation (2.19) and using Theorem 4.3 we obtain:

THEOREM **5.1** *If (A1)-(A7) and* **(AC)** *hold, then there exist constants* $\rho_1 > 0$, $\rho_2 > 0$ *and* $\ell > 0$ *with the following property: for each* $h \in \mathcal{B}^{H}_{\rho_1}(h_0)$ *there exists a unique in* $\mathcal{B}^{Z^\infty \times W^\infty}_{\rho_2}(\xi_0)$ *stationary point* $\xi_h = (y_h, u_h, p_h)$ *of* (P_h) *and*

$$\|y_{h'} - y_{h''}\|_{C(\bar{Q})}, \|u_{h'} - u_{h''}\|_{L^\infty(Q)}, \|p_{h'} - p_{h''}\|_{C(\bar{Q})} \le \ell\,\|h' - h''\|_H, \tag{5.1}$$

for all $h', h'' \in \mathcal{B}^{H}_{\rho_1}(h_0)$. ◇

Now we are going to show that (y_h, u_h) is a local solution of (P_h). As in case of the reference point, the subscript h will denote that the relevant function is evaluated at ξ_h. In particular

$$\begin{aligned}(\zeta_1, D^2_{\zeta\zeta}\mathcal{L}_h\zeta_2) &= \int_Q [y_1, u_1] \begin{bmatrix} D^2_{yy}\mathcal{H}_h & D^2_{yu}\mathcal{H}_h \\ D^2_{uy}\mathcal{H}_h & D^2_{uu}\mathcal{H}_h \end{bmatrix} \begin{bmatrix} y_2 \\ u_2 \end{bmatrix} dxdt + \\ &\quad + \int_\Sigma z_1\; p_h \cdot D^2_{yy} b^h z_2 \; dSdt. \end{aligned} \tag{5.2}$$

$$\mathcal{C}^\alpha_h : W^2_0 \times L^2(Q) \to U^2_\alpha,$$

$$\mathcal{C}^\alpha_h \zeta := \begin{pmatrix} y_t + A\,y + a^h_y\,y + a^h_u\,u & \text{in } Q \\ \partial_\nu y + b^h_y\,y & \text{in } \Sigma \\ u & \text{in } I^\alpha \cup J^\alpha \end{pmatrix}. \tag{5.3}$$

LEMMA **5.2** *If* **(AC)** *holds, then there exist constants* $\varepsilon > 0$ *and* $\rho > 0$ *such that*

$$(\zeta, D^2_{\zeta\zeta}\mathcal{L}_h\zeta) \geq \frac{\gamma}{2}||\zeta||_2^2 \quad \text{holds for all } \zeta \in \ker\, \mathcal{C}_h^\alpha, \tag{5.4}$$

provided that $||y_0 - y_h||_{C(\bar{Q})} + ||u_0 - u_h||_{L^\infty(Q)} \leq \varepsilon$ *and* $h \in \mathcal{B}_\rho^H(h_0)$. ◇

Proof Let $\zeta = (y, u) \in \ker\, \mathcal{C}_h^\alpha$ be given and define $\widetilde{\zeta} = (\widetilde{y}, u)$, where $\widetilde{y} \in W_0^2$ is the solution to

$$\begin{aligned} \widetilde{y}_t + A\widetilde{y} + a_y^0\, \widetilde{y} + a_u^0\, u &= 0 \\ \partial_\nu \widetilde{y} + b_y^0\, \widetilde{y} &= 0 \\ \widetilde{y}(0) &= 0. \end{aligned}$$

Notice that $\widetilde{\zeta} \in \ker\, \mathcal{C}_0^\alpha$, hence $(\widetilde{\zeta}, D^2_{\zeta\zeta}\mathcal{L}_0\widetilde{\zeta}) \geq \gamma\, ||\widetilde{\zeta}||_2^2$. Thanks to Theorem 5.1 and the Lipschitz properties of a, b, there is a $L > 0$ such that

$$\max\{||a_y^0 - a_y^h||_{L^\infty(Q)}, ||a_u^0 - a_u^h||_{L^\infty(Q)}, ||b_y^0 - b_y^h||_{L^\infty(\Sigma)}\} \leq L\,(\varepsilon + \rho). \tag{5.5}$$

and

$$|(\zeta, (D^2_{\zeta\zeta}\mathcal{L}_h - D^2_{\zeta\zeta}\mathcal{L}_0)\zeta)| \leq c\, L\,(\varepsilon + \rho)\, ||\zeta||_2^2. \tag{5.6}$$

Moreover, $w = y - \widetilde{y}$ solves

$$\begin{aligned} w_t + Aw + a_y^0\, w &= (a_y^0 - a_y^h)\, y + (a_u^0 - a_u^h)\, u, \\ \partial_\nu w + b_y^0\, w &= (b_y^0 - b_y^h)\, y, \\ w(0) &= 0. \end{aligned}$$

Now the L^2-theory of parabolic equations yields the existence of a constant $C > 0$, independent of y, u, h, such that

$$\begin{aligned} ||w||_{W(0,T)} = ||y - \widetilde{y}||_{W(0,T)} &\leq C\,(\varepsilon + \rho)\,(||y||_{L^2(Q)} + ||u||_{L^2(Q)}) \\ &\leq c\,(\varepsilon + \rho)\, ||\zeta||_2. \end{aligned}$$

Therefore, for any $\zeta \in \ker\, \mathcal{C}_h^\alpha$ there exists $\widetilde{\zeta} \in \ker\, \mathcal{C}_0^\alpha$ such that

$$||\zeta - \widetilde{\zeta}||_2 \leq c\,(\varepsilon + \rho)\, ||\zeta||_2. \tag{5.7}$$

Condition (5.4) follows easily from (5.6), (5.7) by a standard argument. □

LEMMA **5.3** *For* $\rho_1 > 0$ *sufficiently small, there exist constants* $\widetilde{\rho} > 0$ *and* $\widetilde{\gamma} > 0$ *such that for all* $h \in \mathcal{B}_{\rho_1}^H(h_0)$ *we have*

$$\begin{aligned} &\mathcal{J}_h(\zeta) \geq \mathcal{J}_h(\zeta_h) + \widetilde{\gamma}||\zeta - \zeta_h||_2^2 \\ &\text{for all feasible } \zeta \text{ such that } ||\zeta - \zeta_h||_{Z^\infty} \leq \widetilde{\rho}, \end{aligned} \tag{5.8}$$

i.e., in view of Theorem 5.1, ζ_h *is a, locally isolated in* Z^∞*, local solution of* (P_h). ◇

Proof This result can be shown using the same arguments as in the proof of the abstract Theorem 1 in [8]. We refer also to the detailed discussion of an analogous result in the case of elliptic boundary control in [6]. The statement of our Theorem can be derived in exactly the same way. Therefore, we omit the corresponding lengthy and tedious estimates. □

6 Lipschitz stability for nonlinear problems: necessity

In this section, we are going to show that **(AC)** is not only a sufficient but also a *necessary* condition of local Lipschitz continuity of solutions to (P_h), provided that the dependence of data upon the parameter h is *sufficiently strong*, in the sense that (3.3) holds.

Let us start with some preliminary results. Introduce the mapping

$$\begin{aligned} &\mathcal{S}_h : L^2(Q) \to L^2(Q) \times L^2(\Sigma), \\ &\mathcal{S}_h v = (\mathcal{S}_h^Q, \mathcal{S}_h^\Sigma) = (z, z|_\Sigma), \end{aligned} \tag{6.1}$$

given by the solution of the following boundary value problem

$$\begin{aligned} &z_t + Az + a_y^h z + a_u^h v = 0 && \text{in } Q, \\ &\partial_\nu z + b_y^h z = 0 && \text{in } \Sigma, \\ &z(0) = 0 && \text{in } \Omega. \end{aligned} \tag{6.2}$$

The mapping $\mathcal{S}_h : L^2(Q) \to L^2(Q) \times L^2(\Sigma)$ is compact. This property is obtained by the following arguments: $\mathcal{S}_h^Q$ is continuous from $L^2(Q)$ to $W(0,T)$. Therefore, the linear mappings $v \mapsto z$ and $v \mapsto z_t$ are bounded from $L^2(Q)$ to $L^2(0,T;H^1(\Omega))$ and $L^2(0,T;H^1(\Omega)')$, respectively. We have the inclusions $H^1(\Omega) := B_0 \subset L^2(\Omega) \subset B_1 := H^1(\Omega)'$, where the embedding $B_0 \subset L^2(\Omega)$ is compact. A well known result by Aubin [3] yields that $\mathcal{S}_h^Q : L^2(Q) \to L^2(Q)$ is compact. The embedding $B_0 \subset B := H^{3/4}(\Omega)$ is compact as well, and $B_0 \subset B \subset B_1$. Applying Aubin's result again, we find that the mapping $v \mapsto z$ is compact from $L^2(Q)$ to $L^2(0,T;H^{3/4}(\Omega))$. The trace operator $z \mapsto z|_\Sigma$ is continuous from $L^2(0,T;H^{3/4}(\Omega))$ to $L^2(0,T;H^{1/2}(\Gamma))$. This implies the compactness of $v \mapsto z|_\Sigma$ from $L^2(Q)$ to $L^2(0,T;H^{1/2}(\Gamma)) \subset L^2(\Sigma)$, so that $\mathcal{S}_h^\Sigma$ is compact, too.

On the other hand by recent results of Raymond and Zidani (see, Theorem 3.1 in [17]) we have

$$\mathcal{S}_h \text{ is bounded from } L^r(Q) \text{ into } L^\infty(Q) \times L^\infty(\Sigma), \text{ for } r > \frac{N}{2} + 1. \tag{6.3}$$

It follows from (5.3) and (6.2) that $\zeta = (z, v) \in \ker \mathcal{C}_h^\alpha$ if and only if

$$\zeta = (\mathcal{S}_h^Q v, v), \quad \text{with } v \in V^2, \tag{6.4}$$

where

$$V^p := \{v \in L^p(Q) \mid v(x,t) = 0 \text{ a.e. on } I^\alpha \cup J^\alpha\}, \quad p \in [1, \infty]. \tag{6.5}$$

By (2.11) and (6.4), for any $\zeta = (z, v) \in \ker \mathcal{C}_h^\alpha$ we have

$$(\zeta, D^2_{\zeta\zeta}\mathcal{L}(\zeta_h, p_h, h)\zeta) = (v, (\mathcal{K}_h + D^2_{uu}\mathcal{H}(\zeta_h, p_h, h))\, v)_{V^2}, \tag{6.6}$$

where $D^2_{uu}\mathcal{H}(\zeta_h, p_h, h) : L^2(Q) \to L^2(Q)$ is the linear mapping given by

$$(D^2_{uu}\mathcal{H}(\zeta_h, p_h, h)v)(x,t) = D^2_{uu}\mathcal{H}(x,t,\zeta_h(x,t), p_h(x,t), h)u(x,t)$$

and

$$\begin{aligned} \mathcal{K}_h \quad &: V^2 \to V^2, \\ (\mathcal{K}_h u)(x,t) \quad &= [(\mathcal{S}^Q_h)^*(D^2_{yy}\mathcal{H}(\zeta_h, p_h, h)\cdot \mathcal{S}^Q_h\, u) + 2\, D^2_{yu}\mathcal{H}(\zeta_h, p_h, h)\cdot(\mathcal{S}^Q_h\, u) + \\ &+ (\mathcal{S}^\Sigma_h)^*(D^2_{yy} b(z_h, h)\cdot \mathcal{S}^\Sigma_h\, u)](x,t) \quad \text{a.e. on } Q \setminus (I^\alpha \cup J^\alpha), \end{aligned} \tag{6.7}$$

Note that **(AC)** is *equivalent* to the condition that the quadratic form (6.6) is coercive at $h = h_0$.

In our further analysis we will require that the abstract condition (3.3) holds. In view of (2.20), condition (3.3) is satisfied if the following *strong dependence* condition holds

(SD)
$$\begin{aligned} &H = H^0 \times L^\infty(Q) \times L^\infty(\Sigma) \times L^\infty(Q) \times \\ &\times L^\infty(Q) \times L^\infty(\Sigma) \end{aligned} \tag{6.8}$$

and

$$\begin{aligned} &\mathcal{J}_h(\zeta) = \\ &= \int_Q (\psi(x,t,y,u,h^0) + y(x,t)h^1(x,t) + u(x,t)h^3(x,t))\; dxdt + \\ &+ \int_\Sigma y(x,t)h^2(x,t)\; dSdt, \end{aligned} \tag{6.9}$$

$$\begin{aligned} &y_t + Ay + a(y,u,h^0) + h^4 = 0, \\ &\partial_\nu y + b(y,h^0) + h^5 = 0, \end{aligned} \tag{6.10}$$

where

$h^0 \in H^0,\ h^1 \in L^\infty(Q),\ h^2 \in L^\infty(\Sigma),\ h^3 \in L^\infty(Q),$
$h^4 \in L^\infty(Q),\ h^5 \in L^\infty(\Sigma).$

We assume that **(jj)** in Theorem 3.1 holds, where the solution $\xi_h = (y_h, u_h, p_h)$ to (3.1) corresponds to a local solution (y_h, u_h) of (P_h) and the associated adjoint state p_h. In other words, we assume the L^∞-Lipschitz stability of local solutions of (P_h) and the associated adjoint states with respect to the parameter. We would like to show that this Lipschitz stability implies that **(AC)** holds, provided that **(SD)** is satisfied. The idea of the proof is very similar to that in [9]. It uses Theorem 3.2 and it is based on a construction of a small perturbation of the reference value h_0 of the parameter, such that, in a neighborhood of the perturbed value $\widehat{h}$ of h_0, the constraints in the problems (P_h) can be treated as being of *equality* type.

We proceed in a similar way as in [9]. In view of (6.8)- (6.10), the first order optimality conditions for (P_h) can be written in the form:

$$
\begin{aligned}
&-p_t(x,t)+Ap(x,t)-D_y\mathcal{H}(x,t,y(x,t),u(x,t),p(x,y),h^0)-\\
&-h^1(x,t)=0 \qquad \text{in } Q,\\
&\partial_\nu p(x,t)+D_y b(x,t,y(x,t),h^0)-h^2(x,t)=0 \qquad \text{in } \Sigma,\\
&p(x,T)=0 \qquad \text{in } \Omega.
\end{aligned} \tag{6.11}
$$

$$
\begin{aligned}
&(D_u\mathcal{H}(x,t,y(x,t),u(x,t),p(x,t),h^0)+h^3(x,t))(v-u(x,t))\geq 0\\
&\text{for all } v\in[r^a(x,t),r^b(x,t)] \text{ and a.a. } (x,t)\in Q.
\end{aligned} \tag{6.12}
$$

Let $h_0=(h_0^0,h_0^1,h_0^2,h_0^3,h_0^4,h_0^5)$ be the reference value of the parameter. Define the following set:

$$
K=\{(x,t)\in Q \mid v_0(x,t)\leq \frac{1}{2}(r^a(x,t)+r^b(x,t))\}.
$$

Let us choose any $\alpha<\frac{\sigma_1}{2}$ and $\epsilon<\min\{\sigma_2,d\}$, where σ_1,σ_2 and d are given in **(jj)** of Theorem 3.1, and in (A6), respectively. Introduce the following variations Δu and Δh of the reference control u_0 and parameter h_0, respectively:

$$
\Delta u(x,t)=\begin{cases} 0 & \text{on } I^\alpha\cup J^\alpha,\\ \epsilon & \text{on } K\setminus I^\alpha,\\ -\epsilon & \text{on } [Q\setminus K]\setminus J^\alpha,\end{cases} \tag{6.13}
$$

$$
\begin{aligned}
&\Delta h^1=D_y\mathcal{H}(y_0,u_0,p_0,h_0^0)-D_y\mathcal{H}(y_0,u_0+\Delta u,p_0,h_0^0),\\
&\Delta h^3=\begin{cases} -D_u\mathcal{H}(y_0,u_0+\Delta u,p_0,h_0^0)-h_0^3 & \text{on } Q\setminus(I^\alpha\cup J^\alpha),\\ 0 & \text{on } I^\alpha\cup J^\alpha,\end{cases}\\
&\Delta h^4=a(y_0,u_0,h_0^0)-a(y_0,u_0+\Delta u,h_0^0),\\
&\Delta h^0=0,\quad \Delta h^2=0,\quad \Delta h^5=0.
\end{aligned} \tag{6.14}
$$

Note that Δh^3 is chosen in such a way that (6.12) is satisfied at $u_{\widehat{h}}=u_0+\Delta u$. On $(I^\alpha\cup J^\alpha)$ it is satisfied for $\Delta h^3(t)=0$, since $\Delta u(t)=0$ on that set. On $Q\setminus(I^\alpha\cup J^\alpha)$ we put $D_u\mathcal{H}(y_0,u_0+\Delta u,p_0,h_0^0)+\widehat{h}^3=D_u\mathcal{H}(y_0,u_0+\Delta u,p_0,h_0^0)+h_0^3+\Delta h^3=0$, i.e., $\Delta h_0^3=-D_u\mathcal{H}(y_0,u_0+\Delta u,p_0,h_0^0)-h_0^3$.

Let us denote $\widehat{h}=h_0+\Delta h$. A simple calculation shows that

$$
\xi_{\widehat{h}}:=(y_{\widehat{h}},u_{\widehat{h}},p_{\widehat{h}}):=(y_0,u_0+\Delta u,p_0) \tag{6.15}
$$

is a solution of the optimality system (6.10)-(6.12), i.e., of the generalized equation (2.19) with h_0 substituted by $\widehat{h}$.

Note that, in view of (6.13), the control constraints for $u_{\widehat{h}}$ are active on the set $I^\alpha\cup J^\alpha$ and they are *nonactive with the margin* $\epsilon>0$ on the complement of this set:

$$
u_{\widehat{h}}(x,t)\begin{cases} =r^a(x,t) & \text{on } I^\alpha,\\ =r^b(x,t) & \text{on } J^\alpha,\\ \in[r^a(x,t)+\epsilon,r^b(x,t)-\epsilon] & \text{on } Q\setminus(I^\alpha\cup J^\alpha).\end{cases} \tag{6.16}
$$

Moreover, in view of (4.1), (6.13) and (6.14)

$$\begin{aligned}&D_u\mathcal{H}(x,t,y_{\widehat{h}}(x,t),u_{\widehat{h}}(x,t),p_{\widehat{h}}(x,t),\widehat{h}^0)+\widehat{h}^3=\\&=D_u\mathcal{H}(x,t,y_0(x,t),u_0(x,t)+\Delta u,p_0(x,t),{h_0}^0)+{h_0}^3\\&=D_u\mathcal{H}(x,t,y_0(x,t),u_0(x,t),p_0(x,t),{h_0}^0)+{h_0}^3\begin{cases}>\alpha & \text{on } I^\alpha,\\<-\alpha & \text{on } J^\alpha.\end{cases}\end{aligned}\tag{6.17}$$

Lemma **6.1** *For $\alpha>0$ and $\epsilon>0$ sufficiently small*

$$(\zeta,D^2_{\zeta\zeta}\mathcal{L}(\zeta_{\widehat{h}},p_{\widehat{h}},\widehat{h})\zeta)\geq 0 \ \ \textit{for all } \zeta\in\ker\,\mathcal{C}^\alpha_{\widehat{h}}.\tag{6.18}$$

$\diamond$

Proof Since

$$\begin{aligned}&|-D_u\mathcal{H}(y_0,u_0+\Delta u,p_0,h_0^0)-h_0^3|\leq\\&\leq|D_u\mathcal{H}(y_0,u_0,p_0,h_0^0)-D_u\mathcal{H}(y_0,u_0+\Delta u,p_0,h_0^0)|+\\&+|-D_u\mathcal{H}(y_0,u_0,p_0,h_0^0)-h_0^3|,\end{aligned}$$

then, in view of (6.14),

$$\begin{aligned}&|\Delta h^3(x,t)|\leq|D_u\mathcal{H}(y_0,u_0,p_0,h_0^0)-D_u\mathcal{H}(y_0,u_0+\Delta u,p_0,h_0^0)|\\&\text{on } Q\setminus(I^\alpha\cup J^\alpha).\end{aligned}$$

Hence, it follows from (6.13) and (6.14) that, shrinking $\epsilon>0$ if necessary, we get $\widehat{h}\in\mathcal{B}^H_{\sigma_1}(h_0)$, i.e., $(y_{\widehat{h}},u_{\widehat{h}})$ is a locally unique solution of $(\mathrm{P}_{\widehat{h}})$ and $p_{\widehat{h}}$ is the associated adjoint state.

Note that, in view of (6.16), the constraints (2.7) in $(\mathrm{P}_{\widehat{h}})$ can be *locally* treated as equality type constraints:

$$u_{\widehat{h}}(x,t)\begin{cases}=r^a(x,t) & \text{on } I^\alpha,\\=r^b(x,t) & \text{on } J^\alpha,\\\text{free} & \text{on } Q\setminus(I^\alpha\cup J^\alpha),\end{cases}\tag{6.19}$$

in the sense that, for any Δu, such that

$$|\Delta u(x,t)|\begin{cases}=0 & \text{on } I^\alpha\cup J^\alpha,\\\leq\epsilon & \text{on } Q\setminus(I^\alpha\cup J^\alpha),\end{cases}\tag{6.20}$$

the control function $u=u_{\widehat{h}}+\Delta u$ is feasible for $(\mathrm{P}_{\widehat{h}})$. In particular, (2.13) together with (6.20) implies

$$D_u\mathcal{H}(x,t,y_{\widehat{h}}(x,t),u_{\widehat{h}}(x,t),p_{\widehat{h}}(x,t),\widehat{h}^0)+\widehat{h}^3=0 \quad \text{on } Q\setminus(I^\alpha\cup J^\alpha).\tag{6.21}$$

Let $\zeta = (y,u)$ be feasible for $(\mathrm{P}_{\widehat{h}})$, where $u = u_{\widehat{h}} + \Delta u$, and Δu is any increment satisfying (6.20). Using standard perturbation results for parabolic equations and the notation (6.1), (6.2) we find that

$$\begin{aligned} & y = y_{\widehat{h}} + \Delta y + o(\Delta y), \\ & \text{where } \Delta y = \mathcal{S}_{\widehat{h}} \Delta u \ \text{ and } \ \tfrac{\|o(\Delta y)\|_{L^2}}{\|\Delta y\|_{L^2}} \to 0 \text{ as } \|\Delta y\|_{C(\bar{Q})} \to 0. \end{aligned} \tag{6.22}$$

By (6.4) and (6.22)

$$\Delta \zeta = (\Delta y, \Delta u) \in \ker \mathcal{C}^{\alpha}_{\widehat{h}}. \tag{6.23}$$

Using the second order Taylor expansion at $\xi_{\widehat{h}}$ and taking advantage of necessary optimality conditions (2.12) and (6.21) as well as of (6.22) we get

$$\begin{aligned} 0 \ & \leq \mathcal{J}_{\widehat{h}}(\zeta) - \mathcal{J}_{\widehat{h}}(\zeta_{\widehat{h}}) = \mathcal{L}(\zeta_{\widehat{h}} + \Delta\zeta, p_{\widehat{h}}, \widehat{h}) - \mathcal{L}(\zeta_{\widehat{h}}, p_{\widehat{h}}, \widehat{h}) = \\ & = (\Delta \zeta, D^2_{\zeta\zeta} \mathcal{L}(\zeta_{\widehat{h}}, p_{\widehat{h}}, \widehat{h}) \Delta\zeta) + r(\Delta\zeta), \end{aligned} \tag{6.24}$$

where $\frac{|r(\Delta\zeta)|}{\|\Delta\zeta\|_{Z^2}} \to 0$ as $\|\Delta\zeta\|_{Z^\infty} \to 0$.

Passing to the limit in (6.24) and using (6.23) we obtain

$$(\Delta\zeta, D^2_{\zeta\zeta} \mathcal{L}(\zeta_{\widehat{h}}, p_{\widehat{h}}, \widehat{h}) \Delta\zeta) \geq 0 \quad \text{for all } \Delta\zeta \in \{\zeta \in \ker \mathcal{C}^{\alpha}_{\widehat{h}} \mid \Delta u \in V^\infty\}.$$

By density of the embedding $V^\infty \subset V^2$ we arrive at (6.18). □

LEMMA **6.2** *If* **(j)** *holds with a Lipschitz constant* $\ell > 0$, *then*

$$\|(\mathcal{K}_{\widehat{h}} + D^2_{uu}\mathcal{H}(\zeta_{\widehat{h}}, p_{\widehat{h}}, \widehat{h}))v\|_{V^\infty} \geq \ell^{-1} \|v\|_{V^\infty}. \tag{6.25}$$

◇

Proof Let us introduce the generalized equation $(\widehat{\mathrm{LO}}_\delta)$ analogous to (LO_δ), which is the linearization of the optimality system (2.19) evaluated at $(\xi_{\widehat{h}}, \widehat{h})$ rather than at (ξ_0, h_0). For $\delta = 0$, $(\widehat{\mathrm{LO}}_0)$ has a locally unique solution

$$(\widehat{\zeta}_0, \widehat{q}_0) = (\xi_{\widehat{h}}, p_{\widehat{h}}). \tag{6.26}$$

Moreover, it follows from Theorem 3.2 that there exists $\rho_3 > 0$, such that, for all $\delta \in \mathcal{B}^{\Delta}_{\rho_3}(0)$ there exists a locally unique solution $(\widehat{\zeta}_\delta, \widehat{q}_\delta) := (\widehat{z}_\delta, \widehat{v}_\delta, \widehat{q}_\delta)$ of $(\widehat{\mathrm{LO}}_\delta)$, which is Lipschitz continuous with modulus ℓ.

Note that by (6.16) and (6.26)

$$\widehat{v}_0(x,t) \begin{cases} = r^a(x,t) & \text{on } I^\alpha, \\ = r^b(x,t) & \text{on } J^\alpha, \\ \in [r^a(x,t) + \epsilon, r^b(x,t) - \epsilon] & \text{on } Q \setminus (I^\alpha \cup J^\alpha). \end{cases} \tag{6.27}$$

On the other hand by (3.6), (3.8), (6.17) and (6.26)

$$D^2_{uy}\mathcal{H}_{\widehat{h}}\,\widehat{z}_0 + D^2_{uu}\mathcal{H}_0\widehat{v}_0 - a^{\widehat{h}}_u\,\widehat{q}_0 - g^{\widehat{h}}_u \begin{cases} > \alpha & \text{on } I^\alpha, \\ < -\alpha & \text{on } J^\alpha. \end{cases} \tag{6.28}$$

In view of the Lipschitz continuity of $(\widehat{z}_\delta, \widehat{v}_\delta, \widehat{q}_\delta)$ around $(\widehat{z}_0, \widehat{v}_0, \widehat{q}_0)$, we can shrink $\rho_3 > 0$ so that

$$\begin{array}{l} \widehat{v}_\delta(x,t) \in [r^a(x,t) + \frac{\epsilon}{2}, r^b(x,t) - \frac{\epsilon}{2}] \text{ on } Q \setminus (I^\alpha \cup J^\alpha) \\ \text{and} \\ D^2_{uy}\mathcal{H}_{\widehat{h}}\,\widehat{z}_0 + D^2_{uu}\mathcal{H}_0\widehat{v}_0 - a^{\widehat{h}}_u\,\widehat{q}_0 - g^{\widehat{h}}_u - \Delta g_u \begin{cases} > \frac{\alpha}{2} & \text{on } I^\alpha, \\ < -\frac{\alpha}{2} & \text{on } J^\alpha. \end{cases} \end{array} \tag{6.29}$$

for all $\delta = (\Delta g_Q, \Delta g_\Sigma, \Delta g_u, \Delta d_Q, \Delta d_\Sigma) \in \mathcal{B}^\Delta_{\rho_3}(0)$.

In the same way as in (6.19) and in (6.21), relations (6.29) imply that, for all $\delta \in \mathcal{B}^\Delta_{\rho_3}(0)$ we have

$$\widehat{v}_\delta(x,t) \begin{cases} = r^a(x,t) & \text{on } I^\alpha, \\ = r^b(x,t) & \text{on } J^\alpha, \end{cases} \tag{6.30}$$

$$D^2_{uy}\mathcal{H}_{\widehat{h}}\,\widehat{z}_0 + D^2_{uu}\mathcal{H}_0\widehat{v}_0 - a^{\widehat{h}}_u\,\widehat{q}_0 - g^{\widehat{h}}_u - \Delta g_u = 0 \text{ on } Q \setminus (I^\alpha \cup J^\alpha) \tag{6.31}$$

Let us use (3.7) and (3.5) to find $\widehat{z}_\delta$ and $\widehat{q}_\delta$ as functions of $\widehat{v}_\delta$ and substitute to (6.31). Taking advantage of definitions (6.1) and (6.7), after straightforward but tedious calculations we obtain

$$\begin{array}{l} (\mathcal{K}_{\widehat{h}} + D^2_{uu}\mathcal{H}(\zeta_{\widehat{h}}, p_{\widehat{h}}, \widehat{h}))\widehat{v}_\delta = s(\Delta g_Q, \Delta g_\Sigma, \Delta d_Q, \Delta d_\Sigma) + \Delta g_u, \\ \text{in } V^\infty, \end{array} \tag{6.32}$$

where $s(\cdot,\cdot,\cdot,\cdot)$ is an affine function. By **(j)**, equation (6.32) has a unique solution for any $\delta = (\Delta g_Q, \Delta g_\Sigma, \Delta g_u, \Delta d_Q, \Delta d_\Sigma) \in \mathcal{B}^\Delta_{\rho_3}(0)$. Putting $\delta = (0, 0, \Delta g_u, 0, 0)$ we obtain from (6.32)

$$(\mathcal{K}_{\widehat{h}} + D^2_{uu}\mathcal{H}(\zeta_{\widehat{h}}, p_{\widehat{h}}, \widehat{h}))(\widehat{v}_\delta - \widehat{v}_0) = \Delta g_u.$$

Since by **(j)**, the unique solution $(\widehat{v}_\delta - \widehat{v}_0)$ to this equation is a Lipschitz continuous function of Δg_u with modulus ℓ, we arrive at (6.25). □

LEMMA **6.3** *If (6.25) holds, then*

$$|D^2_{uu}\mathcal{H}(\zeta_{\widehat{h}}(x,t), p_{\widehat{h}}(x,t), \widehat{h})| \geq \ell^{-1} \quad \textit{for a.a. } (x,t) \in Q \setminus (I^\alpha \cup J^\alpha). \tag{6.33}$$

◇

Proof Suppose that (6.33) is violated, i.e., there exists a set $S \subset Q \setminus (I^\alpha \cup J^\alpha)$ of positive measure and a constant $\epsilon > 0$ such that

$$|D^2_{uu}\mathcal{H}(\zeta_{\widehat{h}}(x,t), p_{\widehat{h}}(x,t), \widehat{h})| \leq \ell^{-1} - \epsilon \quad \text{for a.a. } (x,t) \in S. \tag{6.34}$$

Let $R \subset S$ be any subset of positive measure. Choose

$$\bar{v}(t) = \begin{cases} 1 & \text{on } R, \\ 0 & \text{on } Q \setminus (I^\alpha \cup J^\alpha) \setminus R. \end{cases}$$

By (6.3), $\|\mathcal{S}_{\widehat{h}}\bar{v}\|_{L^\infty} \to 0$ as meas $R \to 0$. So, in view of (6.7) and (6.34), for meas R sufficiently small we get $\|(\mathcal{K}_{\widehat{h}} + D^2_{uu}\mathcal{H}(\zeta_{\widehat{h}}, p_{\widehat{h}}, \widehat{h}))\bar{v}\|_{V^\infty} \le \ell^{-1} - \epsilon/2$, while $\|\bar{v}\|_{V^\infty} = 1$. That violates (6.25) and completes the proof. □

Lemma **6.4** *If (6.25) holds, then*

$$\begin{aligned} & (v, (\mathcal{K}_{\widehat{h}} + D^2_{uu}\mathcal{H}(\zeta_{\widehat{h}}, p_{\widehat{h}}, \widehat{h}))v)_{V^2} && \ge \ell^{-1}\|v\|^2_{V^2} \quad \textit{for all } \zeta \in V^2, \\ \textit{i.e.,} \qquad & (\zeta, D^2_{\zeta\zeta}\mathcal{L}(\zeta_{\widehat{h}}, p_{\widehat{h}}, \widehat{h}))\zeta) && \ge \ell^{-1}\|\zeta\|^2_{Z^2} \quad \textit{for all } \zeta \in \ker \mathcal{C}^\alpha_{\widehat{h}}. \end{aligned} \tag{6.35}$$

◇

Proof By a well known property of the spectrum of self-adjoint operators in a Hilbert space (see, e.g., Theorem 2, p.320 in [25]) we have

$$\begin{aligned} & \min\{\mu \in I\!R \mid \mu \in \sigma\} = \\ & = \inf\{(v, (\mathcal{K}_{\widehat{h}} + D^2_{uu}\mathcal{H}(\zeta_{\widehat{h}}, p_{\widehat{h}}, \widehat{h}))v)_{V^2} \mid v \in V^2,\ \|v\|_{V^2} = 1\}, \end{aligned}$$

where σ is the spectrum of $(\mathcal{K}_{\widehat{h}} + D^2_{uu}\mathcal{H}(\zeta_{\widehat{h}}, p_{\widehat{h}}, \widehat{h}))\ :\ V^2 \to V^2$. Hence, in view of (6.18), condition (6.35) will be satisfied if the operator

$$\begin{aligned} & \mathcal{K}_{\widehat{h}} + (D^2_{uu}\mathcal{H}(\zeta_{\widehat{h}}, p_{\widehat{h}}, \widehat{h}) - \mu) \cdot I\ :\ V^2 \to V^2 \\ & \text{is invertible for any } \mu \in [0, \ell^{-1}). \end{aligned} \tag{6.36}$$

Note that by (6.33), the real function

$$(D^2_{uu}\mathcal{H}(\zeta_{\widehat{h}}, p_{\widehat{h}}, \widehat{h})(x,t) - \mu)^{-1}$$

is nonnegative, bounded and measurable on $Q \setminus (I^\alpha \cup J^\alpha)$ for any $\mu \in [0, \ell^{-1})$. Define the operators

$$\mathcal{M}^p_{\widehat{h}} := [(D^2_{uu}\mathcal{H}(\zeta_{\widehat{h}}, p_{\widehat{h}}, \widehat{h}) - \mu)^{-1} \cdot \mathcal{K}_{\widehat{h}} + I]\ :\ V^p \to V^p \quad p \in [1, \infty]. \tag{6.37}$$

By (6.25) $\mathcal{M}^\infty_{\widehat{h}}$ is invertible. It can be easily seen that (6.36) is satisfied if $\mathcal{M}^2_{\widehat{h}}$ is invertible. Note that, in view of compactness of $\mathcal{S}_{\widehat{h}}\ :\ L^2(Q) \to L^2(Q) \times L^2(\Sigma)$ and of the definition (6.7), the mapping

$$(D^2_{uu}\mathcal{H}(\zeta_{\widehat{h}}, p_{\widehat{h}}, \widehat{h}) - \mu)^{-1} \cdot \mathcal{K}_{\widehat{h}}\ :\ V^2 \to V^2$$

is compact. Therefore, $\mathcal{M}^2_{\widehat{h}}$ is a Fredholm operator. By well known properties of Fredholm operators (see, e.g., Theorem VI.6 in [4]), the range of $\mathcal{M}^2_{\widehat{h}}$ is closed in V^2. Choose any $b \in V^2$ and let $\{b_i\} \subset V^\infty$ be such that $b_i \to b$ in V^2. By invertibility of $\mathcal{M}^\infty_{\widehat{h}}$, for each b_i there exists a unique solution $v_i \in V^\infty \subset V^2$ of the equation $\mathcal{M}^\infty_{\widehat{h}} v = \mathcal{M}^2_{\widehat{h}} v = b_i$, i.e., $b_i \in$ range $\mathcal{M}^2_{\widehat{h}}$. In view of closedness of the range, we have $b \in$ range $\mathcal{M}^2_{\widehat{h}}$. Since $b \in V^2$ is arbitrary, it shows that range $\mathcal{M}^2_{\widehat{h}} = V^2$, for any $\mu \in [0, \ell^{-1})$. By the Fredholm theory, the inverse $(\mathcal{M}^2_{\widehat{h}})^{-1} : V^2 \to V^2$ exists and is bounded. That shows that (6.36) holds and completes the proof of (6.35). □

We can formulate now the principal result of this paper, i.e., a *characterization* of the Lipschitz stability property for solutions to (P_h).

THEOREM **6.5** *If (A1)-(A7) hold, then* **(AC)** *is a sufficient condition in order that*

(LC) *there exist constants $\rho_1 > 0$, $\rho_2 > 0$ and $\ell > 0$ such that for each $h \in \mathcal{B}^H_{\rho_1}(h_0)$ there exist a unique in $\mathcal{B}^{Z^\infty}_{\rho_2}(\zeta_0)$ solution $\zeta_h = (y_h, u_h)$ of (P_h) and the associated adjoint state $p_h \in W^\infty$. Moreover,*

$$\|y_{h'} - y_{h''}\|_{C(\bar{Q})}, \|u_{h'} - u_{h''}\|_{L^\infty(Q)}, \|p_{h'} - p_{h''}\|_{C(\bar{Q})} \le \ell \, \|h' - h''\|_H, \quad (6.38)$$

for all $h', h'' \in \mathcal{B}^H_{\rho_1}(h_0)$.

If, in addition, **(SD)** *holds, then* **(AC)** *is* necessary *for* **(LC)** *to be satisfied.* ◇

Proof Sufficiency follows immediately from Theorem 5.1 and Lemma 5.3. To show necessity, note that from (6.35) we have

$$\begin{aligned}&(v, (\mathcal{K}_0 + D^2_{uu}\mathcal{H}(\zeta_0, p_0, h_0))v)_{V^2} \ge \ell^{-1}\|v\|^2_{V^2} - \\ &-[(v, (\mathcal{K}_{\widehat{h}} + D^2_{uu}\mathcal{H}(\zeta_{\widehat{h}}, p_{\widehat{h}}, \widehat{h}))v)_{V^2} - (v, (\mathcal{K}_0 + D^2_{uu}\mathcal{H}(\zeta_0, p_0, h_0))v)_{V^2}].\end{aligned} \quad (6.39)$$

By (6.7) and (6.38), choosing sufficiently small α and ϵ in (6.13), we obtain

$$\begin{aligned}&|(v, (\mathcal{K}_{\widehat{h}} + D^2_{uu}\mathcal{H}(\zeta_{\widehat{h}}, p_{\widehat{h}}, \widehat{h}))v)_{V^2} - (v, (\mathcal{K}_0 + D^2_{uu}\mathcal{H}(\zeta_0, p_0, h_0))v)_{V^2}| \\ &\qquad \le \tfrac{\ell^{-1}}{2}\|v\|^2_{V^2}.\end{aligned} \quad (6.40)$$

In view of (6.6), conditions (6.39) and (6.40) show that **(AC)** holds. □

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