List Update with Locality of Reference

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Abstract. It is known that in practice, request sequences for the list update problem exhibit a certain degree of locality of reference. Motivated by this observation we apply the locality of reference model for the paging problem due to Albers et al. [STOC 2002/JCSS 2005] in conjunction with bijective analysis [SODA 2007] to list update. Using this framework, we prove that Move-to-Front (MTF) is the unique optimal algorithm for list update. This addresses the open question of defining an appropriate model for capturing locality of reference in the context of list update [Hester and Hirschberg ACM Comp. Surv. 1985]. Our results hold both for the standard cost function of Sleator and Tarjan [CACM 1985] and the improved cost function proposed independently by Martínez and Roura [TCS 2000] and Munro [ESA 2000]. This result resolves an open problem of Martínez and Roura, namely proposing a measure which can successfully separate MTF from all other list-update algorithms.

1 Introduction

List update is a fundamental problem in the context of on-line computation. Consider an unsorted list of l items. The input to the algorithm is a sequence of n requests that must be served in an on-line manner. Let $\mathcal{A}$ be an arbitrary on-line list update algorithm. To serve a request to an item x, $\mathcal{A}$ linearly searches the list until it finds x. If x is the i^{th} item in the list, $\mathcal{A}$ incurs a cost i to access x. Immediately after this access, $\mathcal{A}$ can move x to any position closer to the front of the list at no extra cost. This is called a *free exchange*. Also $\mathcal{A}$ can exchange any two consecutive items at a cost of 1. These are called *paid exchanges*. An efficient algorithm can thus use free and paid exchanges to minimize the overall cost of serving a sequence. This is called the *standard cost model* [5].

The competitive ratio, first introduced formally by Sleator and Tarjan [22], has served as a practical measure for the study and classification of on-line algorithms in general and list-update algorithms in particular. An algorithm is said to be α-competitive (assuming a cost-minimization problem) if the cost of serving any specific request sequence never exceeds α times the optimal cost (up to some additive constant) of an *off-line* algorithm which knows the entire request sequence. List update algorithms were among the first algorithms studied using competitive analysis. Three well-known deterministic on-line algorithms are

Move-To-Front (MTF), *Transpose*, and *Frequency-Count* (FC). MTF moves the requested item to the front of the list whereas Transpose exchanges the requested item with the item that immediately precedes it. FC maintains an access count for each item ensuring that the list always contains items in non-increasing order of frequency count. Sleator and Tarjan showed that MTF is 2-competitive, while Transpose and FC do not have constant competitive ratios [22]. Since then, several other deterministic and randomized on-line algorithms have been studied using competitive analysis. (See [16, 1, 4, 14] for some representative results.)

Notwithstanding its wide applicability, competitive analysis has some drawbacks. For certain problems, it gives unrealistically pessimistic performance ratios and fails to distinguish between algorithms that have vastly differing performance in practice. Such anomalies have led to the introduction of many alternatives to competitive analysis of on-line algorithms (see [13] for a comprehensive survey). While list update algorithms with better competitive ratio tend to have better performance in practice the validity of the cost model has been debated. More precisely, Martínez and Roura [18] and Munro [19], independently addressed the drawbacks of the standard cost model. Let $(a_1, a_2, \ldots, a_l)$ be the list currently maintained by an algorithm $\mathcal{A}$. Martínez and Roura argued that in a realistic setting a complete rearrangement of all items in the list which precede item a_i would in practice require time proportional to i, while this has cost proportional to i^2 in the standard cost model. Munro provided the example of accessing the last item of the list and then reversing the entire list. The real cost of this operation in an array or a linear link list should be $O(l)$, while it costs about $l^2/2$ in the standard cost model. As a consequence, their main objection to the standard model is that it prevents on-line algorithms from using their true power. They instead proposed a new model in which the cost of accessing the i^{th} item of the list plus the cost of reorganizing the first i items is linear in i. We will refer to this model as the *modified cost model*. Surprisingly, it turns out that the off-line optimum benefits substantially more from this realistic adjustment than the on-line algorithms do. Indeed, under this model, every on-line algorithm has amortized cost of $\Theta(l)$ per access for some arbitrary long sequences, while an optimal off-line algorithm incurs a cost of $\Theta(\log l)$ on every sequence and hence all on-line list update algorithm have a constant competitive ratio of $\Omega(l/\log l)$. One may be tempted to argue that this is proof that the new model makes the off-line optimum too powerful and hence this power should be removed, however this is not correct as in real life on-line algorithms can rearrange items at the cost indicated. Observe that the ineffectiveness of this power for improving the worst case competitive ratio does not preclude the possibility that under certain realistic input distributions (or other similar assumptions on the input) this power might be of use. Martínez and Roura observed this and posed the question: "an important open question is whether there exist alternative ways to define competitiveness such that MTF and other good online algorithms for the list update problem would be competitive, even for the [modified] cost model".

As well, a common objection to competitive analysis is that it relies on an optimal off-line algorithm, OPT, as a baseline for comparing on-line algorithms.

While this may be convenient, it is rather indirect: one could argue that in comparing two online algorithms $\mathcal{A}$ and $\mathcal{B}$ all the information we should need is the cost incurred by the algorithms on each request sequence. For example, for some problems OPT is too powerful, causing all on-line algorithms to seem equally bad. Certain alternative measures allow direct comparison of on-line algorithms, for example the *Max-Max Ratio* [9], *Relative Worst Order Ratio* [11], Bijective Analysis and Average Analysis [6]. These measures have been applied mostly to the paging problem as well as some other on-line problems. We are not aware of any result in the literature that applies the above measures to on-line list update algorithms.

Another issue in the analysis of on-line algorithms is that "real-life" sequences usually exhibit *locality of reference*. Informally, this property suggests that the currently requested item is likely to be requested again in the near future. For the paging problem, several models for capturing locality of reference have been proposed [23, 2, 8]. Input sequences of list update algorithms in practice show locality of reference [15, 21, 10] and on-line list update algorithms try to take advantage of this property [15, 20]. Hester and Hirschberg [15] posed the question of providing a satisfactory formal definition of locality of reference for the list update problem as an open problem. However, to the best of our knowledge, locality of reference for list update algorithms has not been formally studied. In addition, it has been commonly assumed, based on intuition and experimental evidence, that MTF is the best algorithm on sequences with high locality of reference, e.g., Hester and Hirschberg [15] claim: "move-to-front performs best when the list has a high degree of locality" (see also [3], page 327).

To this end, we introduce a natural measure of locality of reference. Perhaps not surprisingly, this measure seems to parallel MTF's behaviour as the latter has been tailored to benefit from locality of reference. This should not be construed as a drawback of the measure, but rather as evidence of the fact that the design of the MTF algorithm *optimally* incorporates the presence of locality of reference into its choices. Our theoretical proof of the optimality of MTF in this context is then perhaps not surprising, yet this fact had eluded proof until now.

Our Results. We begin by showing that all on-line list update algorithms are equivalent according to Bijective Analysis under the modified cost model. We then extend a model for locality of reference, proposed by Albers et al. [2] in the context of the paging problem to the list update problem. The validity of the extended model is supported by experimental results obtained on the Calgary Corpus, which is frequently used as a standard benchmark for evaluating the performance of compression algorithms (and by extension list update algorithms, e.g. [7]). Thus, we resolve the open problem posed by Hester and Hirschberg [15]. Our main result proves that under both the standard and the modified cost functions MTF is never outperformed in our model, while it always outperforms *any other on-line list update algorithm* in at least one instance. As mentioned earlier, Martínez and Roura [18] posed the open problem of finding an alternative measure that shows the superiority of MTF in the modified cost model and suggested that this can be done by adding some restrictions over the sequences

of requests. Our analysis technique allows us to resolve this problem as well. As noted above the model used for this proof builds upon the work of Albers et al. and Angelopoulos et al. for paging with locality of reference. As such, the results in this paper also provide evidence of the applicability of these models to problems other than paging.

2 Bijective Analysis

In this section, we first provide the formal definitions of Bijective Analysis and Average Analysis and then we show equivalence of all list update algorithms under the modified model according to these measures. We choose to employ these measures since they reflect certain desirable characteristics for comparing online algorithms: they allow for direct comparison of two algorithms without appealing to the concept of the "optimal" cost (see [6] for a more detailed discussion), and they do not evaluate the performance of the algorithm on a single "worst-case" request, but instead use the cost that the algorithm incurs on each and all request sequences. These two measures have already been successfully applied in the context of the paging problem [6].

For the sake of simplicity, in this paper we only consider the *static* list update problem. This means that we only have accesses to list items and do not have any insert or delete operations. In particular, we have a set $S = \{a_1, a_2, \ldots, a_l\}$ of l items initially organized as a list $\mathcal{L}_0 = (a_1, a_2, \ldots, a_l)$. The results in this paper can easily be extended to the dynamic version of the problem. For an on-line algorithm $\mathcal{A}$ and a sequence σ, we denote by $\mathcal{A}(\sigma)$ the cost that $\mathcal{A}$ incurs to serve σ. We denote by $\mathcal{I}_n$ the set of all request sequences of length n, and by $\mathcal{I}_{k+1}(\sigma)$ where $|\sigma| = k$, the set of sequences in $\mathcal{I}_{k+1}$ which have σ as their prefix.

Informally, Bijective Analysis aims to pair input sequences for two algorithms $\mathcal{A}$ and $\mathcal{B}$ using a bijection in such a way that the cost of $\mathcal{A}$ on input σ is no more than the cost of $\mathcal{B}$ on the image of σ, for all request sequences σ of the same length. In this case, intuitively, $\mathcal{A}$ is no worse than $\mathcal{B}$. On the other hand, Average Analysis compares the average cost of the two algorithms over all request sequences of the same length.

Definition 1. *[6] We say that an on-line algorithm $\mathcal{A}$ is* no worse *than an on-line algorithm $\mathcal{B}$ according to Bijective Analysis if there exists an integer $n_0 \geq 1$ so that for each $n \geq n_0$, there is a bijection $b : \mathcal{I}_n \leftrightarrow \mathcal{I}_n$ satisfying $\mathcal{A}(\sigma) \leq \mathcal{B}(b(\sigma))$ for each $\sigma \in \mathcal{I}_n$. We denote this by $\mathcal{A} \preceq_b \mathcal{B}$. Otherwise we denote the situation by $\mathcal{A} \not\preceq_b \mathcal{B}$. Similarly, we say that $\mathcal{A}$ and $\mathcal{B}$ are* the same *according to Bijective Analysis if $\mathcal{A} \preceq_b \mathcal{B}$ and $B \preceq_b A$. This is denoted by $\mathcal{A} \equiv_b \mathcal{B}$. Lastly we say $\mathcal{A}$ is* better *than $\mathcal{B}$ according to Bijective Analysis if $\mathcal{A} \preceq_b \mathcal{B}$ and $\mathcal{B} \not\preceq_b \mathcal{A}$. We denote this by $\mathcal{A} \prec_b \mathcal{B}$.*

Definition 2. *[6] We say that an on-line algorithm $\mathcal{A}$ is* no worse *than an on-line algorithm $\mathcal{B}$ according to Average Analysis if there exists an integer $n_0 \geq 1$ so that for each $n \geq n_0$, $\sum_{I \in \mathcal{I}_n} \mathcal{A}(I) \leq \sum_{I \in \mathcal{I}_n} \mathcal{B}(I)$. We denote this by $\mathcal{A} \preceq_a \mathcal{B}$. Otherwise we denote the situation by $\mathcal{A} \not\preceq_a \mathcal{B}$. $\mathcal{A} \equiv_a \mathcal{B}$, and $\mathcal{A} \prec_a \mathcal{B}$ are defined as for Bijective Analysis.*

Observation 1 *[6] If $\mathcal{A} \not\preceq_a \mathcal{B}$, then $\mathcal{A} \not\preceq_b \mathcal{B}$. In addition, if $\mathcal{A} \preceq_b \mathcal{B}$, then $\mathcal{A} \preceq_a \mathcal{B}$ and similar statements hold for $\mathcal{A} \equiv_b \mathcal{B}$ and $\mathcal{A} \prec_b \mathcal{B}$.*

Suitability of the Measure Note that rather than considering a worst case sequence, these measures take into account all sequences of the same length. To be precise, bijective analysis compares the performance of two algorithms over pairs of different inputs of the same size. A natural question is if this is a reasonable comparison. To answer this, it is necessary to briefly review standard worst case analysis. Worst case analysis of an algorithm A considers the running time of A over all possible inputs of a given size n and selects as representative for this set the maximum or worst case time observed in that class. Let $I_{A,n}$ denote this worst case input of size n for A. Now when the worst case performance of A is compared to that of algorithm B, worst case analysis compares the timing of A on $I_{A,n}$ with that of B on $I_{B,n}$. Observe that in general $I_{A,n} \neq I_{B,n}$ and hence bijective analysis is no different than worst case analysis in terms of pairing different inputs of the same size. The main difference is that bijective analysis studies the performance of both algorithms across the entire spectrum on inputs of size n as opposed to the worst case. This is similar to average case analysis which also measures performance across all inputs of a given size.

The following theorem proves that under the modified cost model all list update algorithms are equivalent. This result parallels the equivalence of all lazy paging algorithms under Bijective Analysis as shown in [6].

Theorem 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two arbitrary on-line list update algorithms. Under the modified cost model, we have $\mathcal{A} \equiv_b \mathcal{B}$.*

Proof. We prove that for every $n \geq 1$ there is a bijection $b^n : \mathcal{I}_n \leftrightarrow \mathcal{I}_n$ so that $\mathcal{A}(\sigma) \leq \mathcal{B}(b^n(\sigma))$ for each $\sigma \in I_n$. We show this by induction on n, the length of sequences. Since $\mathcal{A}$ and $\mathcal{B}$ start with the same initial list, they incur the same cost on each sequence of length 1. Therefore the statement trivially holds for $n = 1$. Assume that it is true for $n = k$. Thus there is a bijection $b^k : \mathcal{I}_k \leftrightarrow \mathcal{I}_k$ so that $\mathcal{A}(\sigma) \leq \mathcal{B}(b^k(\sigma))$ for each $\sigma \in I_k$. Let σ be an arbitrary sequence of length k and $\sigma' = b^k(\sigma)$. We map $\mathcal{I}_{k+1}(\sigma)$ to $\mathcal{I}_{k+1}(\sigma')$ as follows. Let $\mathcal{L}(\mathcal{A}, \sigma) = (a_1, a_2, \ldots, a_l)$ be the list maintained by $\mathcal{A}$ after serving σ and $\mathcal{L}(\mathcal{B}, \sigma') = (b_1, b_2, \ldots, b_l)$ be the list maintained by $\mathcal{B}$ after serving σ'. Consider an arbitrary sequence $\sigma_1 \in \mathcal{I}_{k+1}(\sigma)$ and let its last request be to item a_i. We map σ_1 to the sequence $\sigma_2 \in \mathcal{I}_{k+1}(\sigma')$ that has b_i as its last request. Since $\mathcal{A}(\sigma) \leq \mathcal{B}(\sigma')$ and $\mathcal{A}$'s cost on the last request of σ_1 is the same as $\mathcal{B}$'s cost on the last request of σ_2, we have $\mathcal{A}(\sigma_1) \leq \mathcal{B}(\sigma_2)$. Therefore we get the desired mapping from $\mathcal{I}_{k+1}(\sigma)$ to $\mathcal{I}_{k+1}(\sigma')$. We obtain a bijection $b^{k+1} : \mathcal{I}_{k+1} \leftrightarrow \mathcal{I}_{k+1}$ by considering the above mapping for each sequence $\sigma \in \mathcal{I}_k$. Thus our induction statement is true and we have $\mathcal{A} \preceq_b \mathcal{B}$. Using a similar argument, we can show $\mathcal{B} \preceq_b \mathcal{A}$. Therefore we have $\mathcal{A} \equiv_b \mathcal{B}$.

We will call a list update algorithm *economical* if it does not use paid exchanges. Since an economical list update algorithm does not incur any cost for reorganizing the list we can prove the following statement using an argument analogous to the proof of Theorem 1.

Corollary 1. *All economical on-line list update algorithms are equivalent according to Bijective Analysis under the standard cost model.*

These results show that so long as we consider all possible request sequences, all on-line list update algorithms are equivalent in a strong sense. However, as stated earlier, in practice request sequences tend to exhibit locality of reference. Therefore, the algorithm can focus on input sequences with this property. In the next section we show that we can use such an assumption to prove the superiority of MTF.

3 List Update with Locality of Reference

As stated in the Introduction, several models have been proposed for paging with locality of reference [23, 2, 8]. In this paper, we consider the model of Albers et al. [2], in which a request sequence has high locality of reference if the number of distinct requests in a window of size n is small. In Section 4 we will present experimental evidence which supports the validity of this model for the list update problem. Consider a function that represents the maximum number of distinct items in a window of size n, on a given request sequence. For the paging problem, extensive experiments with real data show that this function can be bounded by a concave function for most practical request sequences [2]. Let f be an increasing concave function. We say that a request sequence is *consistent* with f if the number of distinct requests in any window of size n is at most $f(n)$, for any $n \in \mathcal{N}$. In order to model locality, we restrict the request sequences to those consistent with a concave function f. Let $\mathcal{I}^f$ denote the set of such sequences. We can easily modify the definitions of Bijective Analysis and Average Analysis (Definition 1 and Definition 2) by replacing $\mathcal{I}$ with $\mathcal{I}^f$ throughout. We denote the corresponding relations by $\mathcal{A} \preceq_b^f \mathcal{B}$, $\mathcal{A} \preceq_a^f \mathcal{B}$, etc. Observe that the performance of list update algorithms are now evaluated within the subset of request sequences of a given length that are consistent with f, which we denote as $\mathcal{I}_n^f$, where n is the length of the requence sequences.

Note that the inductive argument used to prove that all on-line list update algorithms are equivalent according to Bijective Analysis (Theorem 1) does not necessarily carry through under concave analysis because the bijection of the proof may map a sequence in $\mathcal{I}^f$ to one not in $\mathcal{I}^f$.

Definition 3. *Let $\mathcal{A}$ and $\mathcal{B}$ be list update algorithms, and f be a concave function. $\mathcal{A}$ is said to* (m, f)-dominate *$\mathcal{B}$ for some integer m, if we have*

$$\sum_{\sigma \in \mathcal{I}_m^f} \mathcal{A}(\sigma) \leq \sum_{\sigma \in \mathcal{I}_m^f} \mathcal{B}(\sigma).$$

$\mathcal{A}$ is said to dominate *$\mathcal{B}$ if there exists an integer $m_0 \geq 1$ so that for each $m \geq m_0$ and every concave function f, $\mathcal{A}$ (m, f)-dominates $\mathcal{B}$.*

Observation 2 *$\mathcal{A} \preceq_a^f \mathcal{B}$ if and only if there exists an integer $m_0 \geq 1$ so that $\mathcal{A}$ (m, f)-dominates $\mathcal{B}$ for each $m \geq m_0$.*

Lemma 1. *For every on-line list update algorithm $\mathcal{A}$, MTF dominates $\mathcal{A}$.*

Proof. Let f be an arbitrary concave function and m be a positive integer. For any $1 \leq i \leq m$, let $\mathcal{F}_{i,m}(\mathcal{A})$ be the total cost $\mathcal{A}$ incurs on the i^{th} request of all sequences in $\mathcal{I}_m^f$. We will first show that $\mathcal{F}_{i,m}(MTF) \leq \mathcal{F}_{i,m}(\mathcal{A})$ for any $1 \leq i \leq m$. For $i = 1$, we have $\mathcal{F}_{1,m}(MTF) = \mathcal{F}_{1,m}(\mathcal{A})$, as all algorithms start with the same list. Now suppose that $i > 1$. Let σ be an arbitrary sequence of length $i-1$, T_σ denote the set of all sequences in $\mathcal{I}_m^f$ that have σ as their prefix, and $\mathcal{F}_{i,m}(\mathcal{A} \,|\, \sigma)$ be the total cost $\mathcal{A}$ incurs on the i^{th} request of all sequences in T_σ. Denote by $\mathcal{L}(MTF, \sigma) = (a_1, a_2, \ldots, a_l)$ and $\mathcal{L}(\mathcal{A}, \sigma) = (b_1, b_2, \ldots, b_l)$ the lists maintained by MTF and $\mathcal{A}$ after serving σ, respectively. Suppose that c_j (resp., d_j) sequences in T_σ have a_j (resp., b_j) as their i^{th} request, for $1 \leq j \leq l$. Note that $\sum_{1 \leq j \leq l} c_j = \sum_{1 \leq j \leq l} d_j = |T_\sigma|$ and $(d_1, d_2, \ldots, d_l)$ is a permutation of $(c_1, c_2, \ldots, c_l)$.

We first show that $c_{j+1} \leq c_j$ for $1 \leq j < l$. Let C_j and C_{j+1} denote the set of sequences in T_σ that have a_j and a_{j+1} as their i^{th} request. We provide a one-to-one mapping from C_{j+1} to C_j which proves that $|C_{j+1}| \leq |C_j|$. We map every sequence τ in C_{j+1} to a sequence τ' in C_j by replacing every a_j with a_{j+1} and every a_{j+1} by a_j, starting from position i. Since a_j occurs before a_{j+1} in MTF's list after serving σ, we know that the last request to a_j occurs after the last request to a_{j+1} in σ. Therefore if τ is consistent with f, so is τ'. Thus every sequence in C_{j+1} is mapped to a unique sequence in C_j and we have $c_{j+1} = |C_{j+1}| \leq |C_j| = c_j$.

Therefore $(c_1, c_2, \ldots, c_l)$ is a permutation of $(d_1, d_2, \ldots, d_l)$ in non-increasing order, and thus $\mathcal{F}_{i,m}(MTF \,|\, \sigma) = \sum_{1 \leq j \leq l} j \times c_j \leq \sum_{1 \leq j \leq l} j \times d_j = \mathcal{F}_{i,m}(\mathcal{A} \,|\, \sigma)$. Now since

$$\mathcal{F}_{i,m}(MTF) = \sum_{\sigma \in \mathcal{I}_{i-1}} \mathcal{F}_{i,m}(MTF \,|\, \sigma) \text{ and } \mathcal{F}_{i,m}(\mathcal{A}) = \sum_{\sigma \in \mathcal{I}_{i-1}} \mathcal{F}_{i,m}(\mathcal{A} \,|\, \sigma),$$

we get $\mathcal{F}_{i,m}(MTF) \leq F_{i,m}(\mathcal{A})$. We have

$$\sum_{\sigma \in \mathcal{I}_m^f} MTF(\sigma) = \sum_{1 \leq i \leq m} \mathcal{F}_{i,m}(MTF) \leq \sum_{1 \leq i \leq m} \mathcal{F}_{i,m}(\mathcal{A}) = \sum_{\sigma \in \mathcal{I}_m^f} \mathcal{A}(\sigma).$$

Thus MTF (m, f)-dominates $\mathcal{A}$ for every concave function f, and every integer $m \geq 1$. Hence MTF dominates $\mathcal{A}$.

Corollary 2. *For any concave function f and any on-line list update algorithm $\mathcal{A}$,*

$$MTF \preceq_a^f \mathcal{A}.$$

Therefore MTF is an optimal algorithm according to Average Analysis, when we classify the input sequences by locality of reference. A natural question is whether MTF is a unique optimum or not, i.e., is there an on-line list update algorithm $\mathcal{A}$ that dominates MTF?

Lemma 2. *No on-line list update algorithm (other than MTF itself) dominates MTF.*

Proof. Assume by way of contradiction that an on-line list update algorithm $\mathcal{A}$ dominates MTF and that $\mathcal{A}$ is different from MTF. According to the definition, there exists an integer $m_0 \geq 1$ so that for each $m \geq m_0$ and every concave function f, $\mathcal{A}$ (m, f)-dominates MTF, i.e.,

$$\sum_{\sigma \in \mathcal{I}_m^f} \mathcal{A}(\sigma) \leq \sum_{\sigma \in \mathcal{I}_m^f} MTF(\sigma).$$

Following the proof of Lemma 1, this holds only if $\mathcal{F}_{i,m}(\mathcal{A} \,|\, \sigma) = \mathcal{F}_{i,m}(MTF \,|\, \sigma)$ for every $m \geq m_0$, $2 \leq i \leq m$, and every sequence σ of length $i-1$. Let $\sigma \in \mathcal{I}_{i-1}^f$ be a sequence so that $\mathcal{L}(\mathcal{A}, \sigma)$ is different from $\mathcal{L}(MTF, \sigma)$, k be the largest index so that $y = a_k \neq b_k = x$ (for a_k and b_k defined as in Lemma 1, and p be the smallest index so that $\sigma[p..i-1]$ contains at most $k-1$ distinct items. Select the concave function f so that $\lfloor f(i-p) \rfloor = \lfloor f(i-p+1) \rfloor = k-1$. Since $y \in \sigma[p..i-1]$ and $x \notin \sigma[p..i-1]$, we have $c_k = 0 < d_k$ (the sequence of length $m > i$ obtained by repeating y in any position starting from i^{th} position is consistent with f). Therefore

$$\mathcal{F}_{i,m}(MTF \,|\, \sigma) = \sum_{1 \leq j \leq l} j \times c_j < \sum_{1 \leq j \leq l} j \times d_j = \mathcal{F}_{i,m}(\mathcal{A} \,|\, \sigma),$$

which is a contradiction.

Theorem 2. *Let $\mathcal{A}$ be an on-line list update algorithm other than MTF. Then $MTF \preceq_b^f \mathcal{A}$ and there exists at least one concave function f so that*

$$\mathcal{A} \not\preceq_a^f MTF, \qquad \text{which implies} \qquad \mathcal{A} \not\preceq_b^f MTF.$$

We can prove separation with respect to Bijective Analysis between MTF and specific algorithms, e.g., Transpose, for a much larger family of concave functions.

Theorem 3. *For all concave functions f such that $f(l) < l$ (l is the size of list),*

$$Transpose \not\preceq_b^f MTF.$$

Proof. Let $\mathcal{L}_0 = (a_1, a_2, \ldots, a_l)$ be the initial list. Assume by way of contradiction that $Transpose \preceq_b^f MTF$. Therefore there is an integer $n_0 \geq 1$ so that for each $n \geq n_0$, there is a bijection $b : \mathcal{I}_n^f \leftrightarrow \mathcal{I}_n^f$ satisfying $Transpose(\sigma) \leq MTF(b(\sigma))$ for each $\sigma \in \mathcal{I}_n^f$. Now consider a sequence σ of length $m \geq n_0$ obtained by considering the prefix of the infinite sequence $a_l a_{l-1} a_l a_{l-1} \ldots$. Transpose incurs a cost of l on each request and we have $Transpose(\sigma) = m \times l$. Note that σ is consistent with f, because it has two distinct items.[1] Thus $\sigma \in \mathcal{I}_m^f$ and from the assumption there should exist some sequence $\sigma' \in \mathcal{I}_m^f$ so that $m \times l = Transpose(\sigma) \leq MTF(\sigma')$. Therefore MTF should incur a

[1] We can assume that $f(2) = 2$ because otherwise we are restricted to sequences that contain only one item.

cost of l on each request of σ'. Hence σ' should be a prefix of the sequence $a_l a_{l-1} a_{l-2} \ldots a_1 a_l a_{l-1} a_{l-2} \ldots a_1 \ldots$. Now any window of size l in σ' has l distinct items. Since we started with $f(l) < l$, σ' is not consistent with f and this contradicts the assumption that $\sigma' \in \mathcal{I}_m^f$.

4 Experimental Results and Analysis

In this section we test the validity of the locality of reference assumption as described in Section 3 against experimental data. For our experiments, we considered the fourteen files of the Calgary Compression Corpus [24] which are frequently used as a standard benchmark for file compression. Recall that list update algorithms can be used in a very direct way in file compression. For each file, we computed the maximum number of characters in windows of all possible sizes, up to the size of the whole file. Figures 1 and 2 show the resulting graphs. Note that since we observed that the maximum number of distinct items does not change much as we increase the size of window to values more than 3500, we only show the results for windows of size up to 3500.

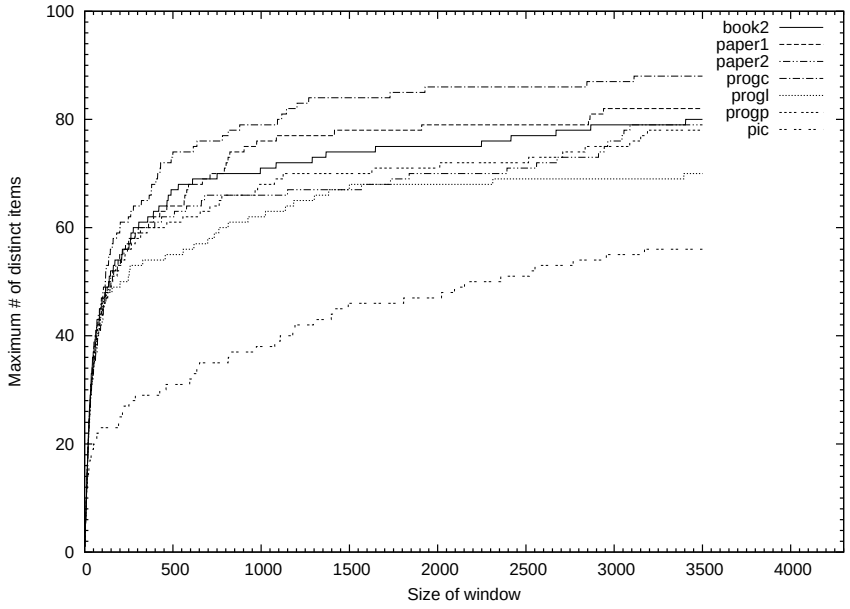

Fig. 1. Maximum number of distinct characters in windows of size up to 3500 for the files in Calgary Compression Corpus.

As can be seen from these graphs, the curves have an overall concave shape. We should note that for some of the input files, the function we obtained is

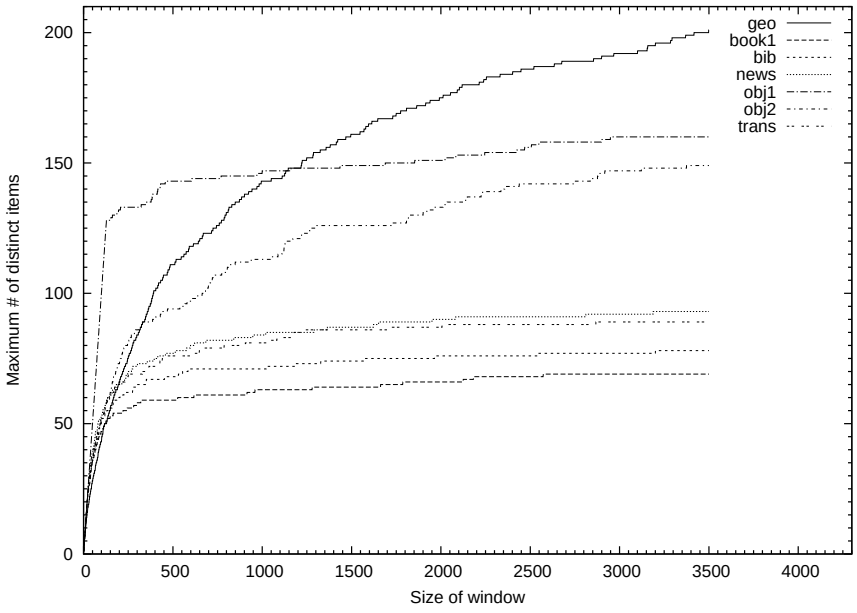

Fig. 2. Maximum number of distinct characters in windows of size up to 3500 for the files in Calgary Compression Corpus.

not concave for some intervals. However, this is not a major concern, since we can bound said function by any concave function f which is such that $f(i)$ is an upper bound on the maximum number of distinct items in windows of size i. For instance, we can take the upper convex hull of the data points. In fact, Albers et al. [2] observed that similar non-concavity (mostly localized within small intervals) was present in their experimental results concerning locality of reference in typical request sequences for the paging problem. Albers et al. put forth this argument to justify the fact that local small deviations from concavity do not impose a serious problem.

Albers and Mitzenmacher [3] compared the efficiency of MTF and Timestamp (TS) algorithms for compressing the files of the Calgary Compression Corpus. TS is a list update algorithm that is 2-competitive [1]. After accessing an item a, TS inserts a in front of the first item b that appears before a in the list and was requested at most once since the last request for a. If there is no such item b, or if this is the first access to a, TS does not reorganize the list. They compared MTF and TS in two settings: with or without Burrows-Wheeler transform (BWT). Informally, BWT transforms a string to one of its permutations that has more locality of reference, which is hence more readily compressible [12, 17]. Their results show that although TS outperforms MTF on compression without BWT, MTF usually has better performance when we use BWT. This is consistent with

our results as BWT is a transform designed with the goal of increasing the locality of reference in the representation of the string.

5 Conclusions

In this paper we addressed certain open questions concerning the well-studied list update problem. We first considered the issue of modeling locality of reference for typical request sequences for this problem. We provided experimental evidence which suggests that the concave-function model of Albers et al., originally devised for the context of paging algorithms, can satisfactorily model locality of reference within the domain of list update. We then combined this model with two recently proposed measures for comparing online algorithms, namely Bijective Analysis and Average Analysis. Our choice was based on the fact that these measures allow direct comparison of two online algorithms, by considering their relative performance on all requests sequences of the same length, rather than on some specific pathological sequences. These measures have been previously applied with success in separating several paging algorithms, a situation which has long been known but cannot be resolved by resorting solely to competitive analysis.

Using the above framework, we showed that while all list update algorithms are equivalent in the modified-cost model, when locality of reference is considered, MTF emerges as the sole optimum online algorithm for the problem. This resolves an open problem posed by Martínez and Roura. We believe that our techniques might well be applicable to other problems in which competitive analysis has failed to yield satisfactory results such as the online bin packing, but this remains the subject of future work.

The model proposed is, to our knowledge, the first that both incorporates locality of reference and achieves full separation of MTF. However locality of reference is a phenomenon which has only recently begun to be thoroughly understood. Thus we fully expect that future further refinements of the model by researchers in the field will reflect even more faithfully locality of reference as it is observed in practice.

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