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Littlewood-Richardson Coefficients and Kazhdan-Lusztig Polynomials

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Abstract.

We show that the Littlewood-Richardson coefficients are values at 1 of certain parabolic Kazhdan-Lusztig polynomials for affine symmetric groups. These q-analogues of Littlewood-Richardson multiplicities coincide with those previously introduced in [21] in terms of ribbon tableaux.

§1. Introduction

Let $\lambda = (\lambda_1 \geq \ldots \geq \lambda_r \geq 0)$ and $\mu = (\mu_1 \geq \ldots \geq \mu_r \geq 0)$ denote two partitions of length $\leq r$, identified in the usual way with dominant integral weights of the complex Lie algebra $\mathfrak{gl}_r$. It was shown by Lusztig [28] that the multiplicity $K_{\lambda,\mu}$ of the weight μ in the finite-dimensional irreducible representation $W(\lambda)$ of $\mathfrak{gl}_r$ with highest weight λ is the value at 1 of a certain Kazhdan-Lusztig polynomial P_{n_μ,n_λ} for the affine symmetric group $\widehat{\mathfrak{S}}_r$. (For the definition of $\widehat{\mathfrak{S}}_r$ and n_λ, see below Section 2.1). Moreover, Lusztig proved [27] that the polynomial $P_{n_\mu,n_\lambda}(q)$ is equal to the Kostka-Foulkes polynomial $K_{\lambda,\mu}(q)$ defined as the coefficient of the Schur function s_λ on the basis of Hall-Littlewood function $P_\mu(q)$ [33]. A combinatorial expression of $K_{\lambda,\mu}(q)$ had previously been given by Lascoux and Schützenberger in terms of semi-standard Young tableaux [35, 33].

It is well known that $K_{\lambda,\mu}$ is also equal to the multiplicity of $W(\lambda)$ as an irreducible component of the tensor product

$$W(\mu_1) \otimes \cdots \otimes W(\mu_r)$$

of symmetric powers of the vector representation of $\mathfrak{gl}_r$. Let now $\nu^{(1)}$, $\ldots$, $\nu^{(s)}$ be arbitrary dominant weights and let $c^{\lambda}_{\nu^{(1)},\ldots,\nu^{(s)}}$ denote the

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multiplicity of $W(\lambda)$ in

$$W(\nu^{(1)}) \otimes \cdots \otimes W(\nu^{(s)}) .$$

A q-analogue $c^{\lambda}_{\nu^{(1)},\ldots,\nu^{(s)}}(q)$ of this multiplicity has been introduced in [21] by means of certain generalizations of semi-standard Young tableaux called ribbon tableaux, and it has been proved that when the partitions $\nu^{(j)}$ have only one part μ_j

$$c^{\lambda}_{\mu_1,\ldots,\mu_r}(q) = K_{\lambda,\mu}(q).$$

The purpose of this paper is to establish that for all $\nu^{(1)}, \ldots, \nu^{(s)}, \lambda$ the $c^{\lambda}_{\nu^{(1)},\ldots,\nu^{(s)}}(q)$ are Kazhdan-Lusztig polynomials for the group $\widehat{\mathfrak{S}}_r$.

Let us outline how this result is obtained. As mentioned by Lusztig in [28], the expression of the weight multiplicity $K_{\lambda,\mu}$ as a value at 1 of a Kazhdan-Lusztig polynomial might be deduced from the conjecture of [26] for the characters of irreducible representations of GL_r over an algebraically closed field of characteristic $n \geq r$ together with the Steinberg tensor product theorem. In [30, 31] a similar conjecture was formulated for the characters of irreducible representations of $U_q(\mathfrak{gl}_r)$ when q^2 is a primitive nth root of 1. A remarkable feature of the quantum conjecture is that the restriction $n \geq r$ is no longer necessary. This conjecture is now proved due to work of Kazhdan-Lusztig and Kashiwara-Tanisaki. On the other hand Lusztig has derived in [30] an analogue of the Steinberg tensor product theorem for the quantum case. From these two facts, it is easy to deduce that the Littlewood-Richardson multiplicities are value at 1 of Kazhdan-Lusztig polynomials (see below, Section 3).

However this would not provide the link with the q-analogues defined by means of ribbon tableaux. We shall therefore follow a different approach and rely on the construction given in [22] of a canonical basis in the level 1 Fock space representation of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$. This canonical basis satisfies a formal q-analogue of Steinberg's tensor product theorem which may be formulated in terms of the combinatorics of ribbon tableaux. On the other hand, Varagnolo and Vasserot [42] have recently verified a conjecture of [22]. They proved that the coefficients of the expansion of this canonical basis on the standard basis of q-wedge products coincide with the Kazhdan-Lusztig polynomials occuring in Lusztig's conjecture. Using these two results, we are able to express the $c^{\lambda}_{\nu^{(1)},\ldots,\nu^{(s)}}(q)$ as Kazhdan-Lusztig polynomials.

More precisely, they belong to a family of parabolic analogues of Kazhdan-Lusztig polynomials introduced by Deodhar [4, 5]. There are two types of such polynomials associated with the Hecke algebra modules obtained by inducing respectively the characters $T_i \mapsto -q$ and $T_i \mapsto q^{-1}$ of a parabolic subalgebra. The $c^{\lambda}_{\nu^{(1)},\dots,\nu^{(s)}}(q)$ turn out to belong to the family denoted by $\widetilde{P}^J_{x,y}$ in [5] and by $n_{x,y}$ in [39], which is less well understood. In particular $\widetilde{P}^J_{x,y}$ may be 0 even if $x < y$ in the Bruhat ordering. Also, since the $\widetilde{P}^J_{x,y}$ are equal to alternating sums of ordinary Kazhdan-Lusztig polynomials, it is not a priori clear whether these polynomials have non-negative coefficients. However, according to experts, it seems probable that they admit a geometrical interpretation in terms of Schubert varieties of finite codimension in an affine flag manifold [1]. This would settle the positivity conjecture VI.3 of [21]. Note that in the case of two factors the polynomials $c^{\lambda}_{\nu^{(1)},\nu^{(2)}}(q)$ are known to have non-negative coefficients because of their combinatorial interpretation in terms of Yamanouchi domino tableaux given in [2].

That the non-vanishing of the polynomials $\widetilde{P}^J_{x,y}$ is a difficult problem should not be too surprising. Indeed, our result shows that this contains as a special case the non-vanishing of the Littlewood-Richardson coefficients. There has been some important recent progress by Klyachko on this classical subject [19] using toric vector bundles on the projective plane (see the reviews of Zelevinsky [44] and Fulton [8]). Maybe some new understanding will arise from the connection with affine Schubert varieties.

A few comments concerning the growing literature on q-analogues of Littlewood-Richardson coefficients are in order. In [36] Shimozono and Weyman have studied the Poincaré polynomials of isotypic components of some virtual graded GL_r-modules supported in the closure of a nilpotent conjugacy class. These are q-analogues of Littlewood-Richardson multiplicities $c^{\lambda}_{\nu^{(1)},\dots,\nu^{(s)}}$ satisfying a q-Kostant formula and a Morris-like recurrence. In the case where all partitions $\nu^{(j)}$ are rectangular (*i.e.* the corresponding weights are multiples of a single fundamental weight) and are arranged in non-increasing order of width, these polynomials have non-negative coefficients. (This is not true in general, but see [36], Conjecture 4.) In this case, a combinatorial interpretation in terms of semi-standard Young tableaux was given by Shimozono [37, 38], which shows that they coincide with the generalized Kostka-Foulkes polynomials studied by Schilling and Warnaar [34] in relation with exactly solvable lattice models and Rogers-Ramanujan type identities. A different combinatorial interpretation using rigged configurations has been conjectured by Kirillov and Shimozono [18] and recently verified [17].

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[1] Added 09/1999. This has eventually been proved by Kashiwara and Tanisaki (preprint math.RT/9908153).

It is believed that for rectangular shapes in non-increasing order these Poincaré polynomials are equal to the corresponding $c^{\lambda}_{\nu^{(1)},\dots,\nu^{(s)}}(q)$ but the reason for that is still unclear.

Let us describe more precisely the contents of this paper. The results rely mainly on four sources, namely the parabolic analogue of Kazhdan-Lusztig polynomials developed by Deodhar in [4, 5], our joint paper with Lascoux on ribbon tableaux and generalizations of Kostka-Foulkes polynomials [21], our previous note [22], and the paper of Varagnolo and Vasserot [42]. Since [22] contains no proofs, and since only a small part of [21] and [42] is needed to obtain our results, we thought it would be appropriate to provide a self-contained exposition of this material. Thus the style of the paper is openly expository and we hope it can be read without a previous knowledge of these four sources. However for what concerns parabolic Kazhdan-Lusztig polynomials, we decided to omit the proofs because they can be found in the optimum exposition by Soergel of Kazhdan-Lusztig theory from scratch [39].

So in Section 2 we explain all the necessary background on (extended) affine symmetric groups $\widehat{\mathfrak{S}}_r$ and their Hecke algebras $\widehat{H}_r$. In particular we introduce the two presentations (Coxeter-type and Bernstein-type) and give the relations between them. Following [42] we construct a representation of $\widehat{H}_r$ on the weight lattice $\mathcal{P}_r$ of $\mathfrak{gl}_r$ and introduce its two Kazhdan-Lusztig bases. The coefficients of these bases on the basis of weights are the parabolic Kazhdan-Lusztig polynomials (for various parabolic subgroups).

In Section 3, we recall the Lusztig conjecture for quantum $\mathfrak{gl}_r$ at an nth root of 1, the tensor product theorem, and using a formula of Littlewood we deduce from this that the Littlewood-Richardson coefficients are value at 1 of parabolic Kazhdan-Lusztig polynomials (Theorem 3.3).

In Section 4 we recall following [21] the definitions of ribbon tableaux and their spin, we introduce the q-analogues $c^{\lambda}_{\nu^{(1)},\dots,\nu^{(s)}}(q)$, and we state our main result (Theorem 4.1).

In Section 5 we explain the construction of [42] and consider a quotient $\mathcal{F}_r$ of $\mathcal{P}_r$ whose bases are naturally labelled by dominant integral $\mathfrak{gl}_r$-weights. This space can be identified in a natural way with the (finitized) q-deformed Fock space of Kashiwara, Miwa and Stern [15] considered in [22]. Projecting on $\mathcal{F}_r$ the Kazhdan-Lusztig involution of $\mathcal{P}_r$ one gets the involution defined in [22] in terms of q-wedge products. This implies that the canonical bases of [22] have coefficients given by some parabolic Kazhdan-Lusztig polynomials (Theorem 5.12).

In Section 6 we study the action of the center $Z(\widehat{H}_r)$ of $\widehat{H}_r$ on $\mathcal{F}_r$ and show that it can be expressed via the combinatorics of ribbon tableaux.

We then prove that the vectors $G^-_{\lambda+\rho}$ of the canonical basis indexed by non-restricted weights λ are obtained from the restricted ones by acting with an element of $Z(\widehat{H}_r)$. This should be regarded as an analogue in this setting of the Steinberg-Lusztig tensor product theorem. Then we give the proof of Theorem 4.1.

In Section 7 we review the construction of Kashiwara, Miwa and Stern of the Fock space $\mathbf{F}_\infty$ obtained by taking the limit $r \to \infty$ in $\mathcal{F}_r$. It affords a level 1 integrable representation of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$. We investigate the behaviour of the canonical bases of $\mathbf{F}_\infty$ introduced in [22] with respect to the semi-linear involution induced by the conjugation of partitions, and derive from this a symmetry of the polynomials $c^\lambda_{\nu^{(1)},\dots,\nu^{(s)}}(q)$ (Theorem 7.13) and an inversion formula for parabolic Kazhdan-Lusztig polynomials (Corollary 7.15). This formula, together with a result of Du, Parshall and Scott [7], provides an alternative proof of Soergel's character formula for tilting modules in type A (Remark 7.16).

Finally Section 8 provides some numerical tables of q-Littlewood-Richardson multiplicities and Kazhdan-Lusztig polynomials, which may serve as examples of the results discussed in the text.

§2. Affine symmetric groups and their Hecke algebras

2.1. Affine symmetric groups

Let $\widetilde{\mathfrak{S}}_r$ denote the Coxeter group of type $\widetilde{A}_{r-1}$. For $r = 2$, this is the group generated by s_0, s_1 subject to the relations $s_0^2 = s_1^2 = 1$. For $r > 2$, $\widetilde{\mathfrak{S}}_r$ is generated by $s_0, s_1, \dots, s_{r-1}$ subject to

(1) $$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},$$

(2) $$s_i s_j = s_j s_i, \qquad (i - j \neq \pm 1),$$

(3) $$s_i^2 = 1,$$

where the subscripts are understood modulo r. The subgroup generated by $s_1, \dots, s_{r-1}$ is isomorphic to the symmetric group $\mathfrak{S}_r$. The group $\widetilde{\mathfrak{S}}_r$ has a concrete realization as an affine reflection group. Let $(\epsilon_1, \dots, \epsilon_r)$ denote the standard basis of $\mathbf{R}^r$, and define a scalar product by putting $(\epsilon_i, \epsilon_j) = \delta_{ij}$. Set $\alpha_i = \epsilon_i - \epsilon_{i+1}$ $(1 \le i \le r-1)$ and $\alpha_0 = \epsilon_r - \epsilon_1$. Let $\mathfrak{h}_r$ denote a Cartan subalgebra of $\mathfrak{gl}_r$. We identify $\mathbf{R}^r$ with (the real part of) $\mathfrak{h}_r^*$ in the usual way, so that $P = P_r := \bigoplus_{i=1}^r \mathbf{Z}\epsilon_i$ becomes the weight lattice, $Q = Q_r := \bigoplus_{i=1}^{r-1} \mathbf{Z}\alpha_i$ the root lattice, α_i $(1 \le i \le r-1)$ the simple roots, $-\alpha_0$ the highest root, *etc.* For $\alpha \in \mathbf{R}^r$ and $m \in \mathbf{Z}$, denote by $S_{\alpha,m}$ the affine reflection defined by

$$S_{\alpha,m}(\lambda) = \lambda - 2\frac{(\lambda,\alpha)+m}{(\alpha,\alpha)}\,\alpha.$$

Then for any $m \in \mathbf{Z}^*$, the assignment

$$s_0 \mapsto S_{\alpha_0,m}\,, \quad s_i \mapsto S_{\alpha_i,0} \quad (1 \le i \le r-1)$$

defines a faithful representation π_m of $\widetilde{\mathfrak{S}}_r$ as a discrete subgroup of the group of affine transformations of $\mathbf{R}^r$. In coordinates, we have

$$\begin{aligned}
\pi_m(s_i)(\lambda) &= (\lambda_1,\ldots,\lambda_{i+1},\lambda_i,\ldots,\lambda_r), \qquad (1 \le i \le r-1),\\
\pi_m(s_0)(\lambda) &= (\lambda_r + m, \lambda_2, \ldots, \lambda_{r-1}, \lambda_1 - m).
\end{aligned}$$

Note that for $s \in \mathfrak{S}_r$, $\pi_m(s)$ does not depend on m. We shall therefore simplify the notation and write $s\lambda$ in place of $\pi_m(s)(\lambda)$.

This realization shows that $\widetilde{\mathfrak{S}}_r$ contains a large commutative subgroup, namely the image under π_m^{-1} of the group of translations by the vectors of the lattice mQ. Write $\mathrm{Tr}(\lambda)$ for the translation by $\lambda \in \mathbf{R}^r$, and let t_i denote the element of $\widetilde{\mathfrak{S}}_r$ corresponding to $\mathrm{Tr}(m\alpha_i)$ under π_m. Then one can check that

$$\begin{aligned}
t_1 &= (s_0 s_{r-1} s_{r-2} \cdots s_3 s_2)(s_3 s_4 \cdots s_{r-1} s_0 s_1),\\
t_2 &= (s_1 s_0 s_{r-1} \cdots s_4 s_3)(s_4 s_5 \cdots s_0 s_1 s_2),\\
&\ \vdots\\
t_{r-1} &= (s_{r-2} s_{r-3} s_{r-4} \cdots s_1 s_0)(s_1 s_2 \cdots s_{r-3} s_{r-2} s_{r-1}),\\
t_0 &= (s_{r-1} s_{r-2} s_{r-3} \cdots s_2 s_1)(s_1 s_2 \cdots s_{r-2} s_{r-1} s_0).
\end{aligned}$$

It will be convenient to enlarge $\widetilde{\mathfrak{S}}_r$ by adding the translations by vectors of the lattice mP. Abstractly, this extended affine symmetric group that we shall denote by $\widehat{\mathfrak{S}}_r$ may be defined as the group generated by $s_0, s_1, \ldots, s_{r-1}, \tau$ subject to relations (1), (2), (3) together with

$$(4) \qquad \tau s_i = s_{i+1}\tau,$$

where again subscripts are understood modulo r. It is clear that each $w \in \widehat{\mathfrak{S}}_r$ can be written in a unique way as

$$(5) \qquad w = \tau^k \sigma, \qquad (k \in \mathbf{Z},\ \sigma \in \widetilde{\mathfrak{S}}_r).$$

An alternative useful presentation is as follows. The group $\widehat{\mathfrak{S}}_r$ is generated by the elements $s_1, \dots, s_{r-1}, y_1, \dots, y_r$ subject to relations (1), (2), (3) with all indices between 1 and $r-1$ together with

$$y_i y_j = y_j y_i, \tag{6}$$
$$s_i y_j = y_j s_i \text{ for } j \neq i, i+1, \tag{7}$$
$$s_i y_i s_i = y_{i+1}. \tag{8}$$

The homomorphism π_m can then be extended to $\widehat{\mathfrak{S}}_r$ by setting

$$\pi_m(y_i) := \mathrm{Tr}(m\epsilon_i), \qquad \pi_m(\tau) := S_{\alpha_1,0} S_{\alpha_2,0} \cdots S_{\alpha_{r-1},0} \mathrm{Tr}(m\epsilon_r),$$

or in coordinates

$$\pi_m(y_i)(\lambda) = (\lambda_1, \dots, \lambda_i + m, \dots, \lambda_r), \qquad (1 \le i \le r),$$
$$\pi_m(\tau)(\lambda) = (\lambda_r + m, \lambda_1, \dots, \lambda_{r-2}, \lambda_{r-1}).$$

The following equations relate the two above presentations of $\widehat{\mathfrak{S}}_r$:

$$y_i = s_{i-1}s_{i-2}\cdots s_1 s_0 s_{r-1} s_{r-2} \cdots s_{i+1}\tau, \qquad (1 \le i \le r) \tag{9}$$
$$\tau = s_1 s_2 \cdots s_{r-1} y_r, \tag{10}$$
$$s_0 = s_{r-1}s_{r-2}\cdots s_2 s_1 s_2 \cdots s_{r-1} y_1^{-1} y_r. \tag{11}$$

(In (9) the subscripts are understood modulo r.)

Note that $\widehat{\mathfrak{S}}_r$ is not a Coxeter group. However, one can still define a Bruhat order and a length function. Let $w = \tau^k\sigma, w' = \tau^m\sigma'$ with $k, m \in \mathbf{Z}$, $\sigma, \sigma' \in \widetilde{\mathfrak{S}}_r$. We say that $w < w'$ if and only if $k = m$ and $\sigma < \sigma'$, and we put $\ell(w) := \ell(\sigma)$. Define

$$A_{r,m} := \left| \begin{array}{ll} \{\lambda \in \mathbf{R}^r \mid m > \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_r \ge 0\} & \text{if } m > 0, \\ \{\lambda \in \mathbf{R}^r \mid m < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_r \le 0\} & \text{if } m < 0, \end{array} \right.$$

and $\mathcal{A}_{r,m} := A_{r,m} \cap P$ (see Figure 1). Then $A_{r,m}$ is a fundamental domain for the action of $\widehat{\mathfrak{S}}_r$ on $\mathbf{R}^r$ via π_m, that is, each orbit intersects it in a unique point. Let $\lambda \in P$, and let ν be the intersection of $\mathcal{A}_{r,m}$ with the orbit of λ. Then there is a unique $w(\lambda, m) \in \widehat{\mathfrak{S}}_r$ of minimal length such that $\pi_m(w(\lambda, m))(\nu) = \lambda$. Let $\mathfrak{S}_{\nu,m}$ be the parabolic subgroup consisting of the w such that $\pi_m(w)(\nu) = \nu$. (Since $|\nu_1 - \nu_r| < m$, $\mathfrak{S}_{\nu,m} \subset \mathfrak{S}_r$.) Then $w(\lambda, m)$ is the minimal length representative of the coset

$$w(\lambda, m)\mathfrak{S}_{\nu,m} = \{w \in \widehat{\mathfrak{S}}_r \mid \pi_m(w)(\nu) = \lambda\}.$$

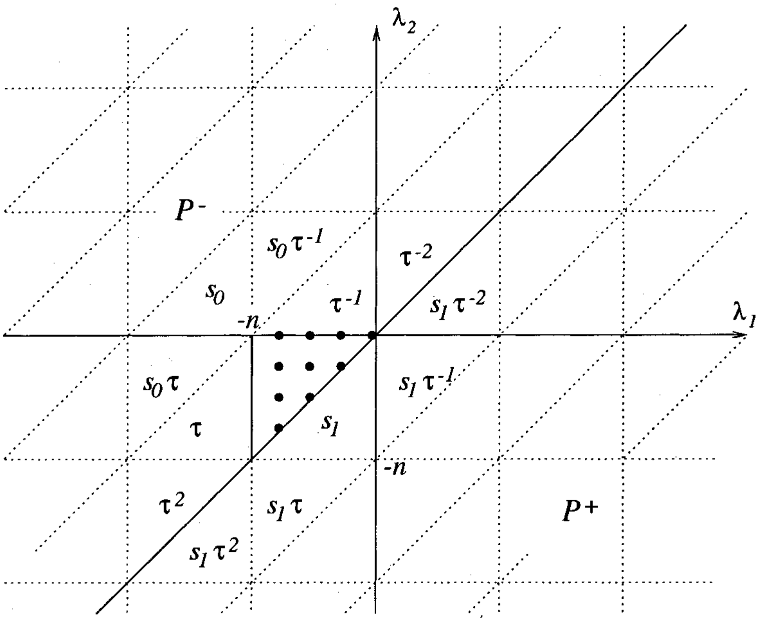

Fig. 1. The action of $\widehat{\mathfrak{S}}_2$ on P_2 via π_{-n}

In this way, we can associate to the data (λ, m) a certain element $w(\lambda, m)$ of $\widehat{\mathfrak{S}}_r$. This will allow us to pass from the indexation by weights of the Littlewood-Richardson coefficients to the indexation by elements of $\widehat{\mathfrak{S}}_r$ of the Kazhdan-Lusztig polynomials.

Example 2.1. Take $r = 3$ and $\lambda = (5, 3, 0)$. Then

$$w(\lambda, 3) = \tau^2 s_1, \quad w(\lambda, 2) = \tau^3 s_0 s_1 s_2, \quad w(\lambda, -2) = \tau^{-5} s_1 s_0 s_2 s_1 s_2 s_0.$$

◇

For $\lambda = (\lambda_1, \dots, \lambda_r) \in P$ set $\lambda_0 := \lambda_r + m$, $\lambda_{r+1} := \lambda_1 - m$ and define the descent function

$$\mathrm{desc}(\lambda, i, m) := \begin{cases} 1 & \text{if } \lambda_i > \lambda_{i+1}, \\ 0 & \text{if } \lambda_i = \lambda_{i+1}, \\ -1 & \text{if } \lambda_i < \lambda_{i+1}, \end{cases} \qquad (0 \le i \le r).$$

Note that

$$\mathrm{desc}(\lambda, 0, m) = \mathrm{desc}(\lambda, r, m)\,, \quad \mathrm{desc}(\lambda, i, m) = \mathrm{desc}(\pi_m(\tau)(\lambda), i+1, m)\,.$$

Geometrically, $\mathrm{desc}(\lambda, i, m) = 0$ means that λ lies on the reflecting hyperplane $\mathcal{H}_m$ of $\pi_m(s_i)$, *i.e.* $\pi_m(s_i)(\lambda) = \lambda$, and $\mathrm{desc}(\lambda, i, m) = \mathrm{sgn}\,(m)$ means that λ belongs to the 1/2-space defined by $\mathcal{H}_m$ which contains the fundamental domain $\mathcal{A}_{r,m}$, *i.e.* $s_i w(\lambda, m) > w(\lambda, m)$.

Lemma 2.1. *Let $\lambda \in P$, $i \in \{0, \dots, r-1\}$ and $m \in \mathbf{Z}^*$. Let $\nu = w(\lambda, m)^{-1}(\lambda)$ be the point of $\mathcal{A}_{r,m}$ congruent to λ under π_m. One has the three following alternatives:*

$$\begin{array}{llll}
\text{(i)} & \mathrm{desc}(\lambda,i,m) = \mathrm{sgn}\,(m) & \Longleftrightarrow & s_i w(\lambda,m) = w(s_i\lambda,m) > w(\lambda,m), \\
\text{(ii)} & \mathrm{desc}(\lambda,i,m) = 0 & \Longleftrightarrow & s_i w(\lambda,m) = w(\lambda,m)s_j \\
 & & & \textit{for some } s_j \in \mathfrak{S}_{\nu,m}, \\
\text{(iii)} & \mathrm{desc}(\lambda,i,m) = -\mathrm{sgn}\,(m) & \Longleftrightarrow & s_i w(\lambda,m) = w(s_i\lambda,m) < w(\lambda,m).
\end{array}$$

Proof — This is a reformulation of Lemma 2.1 (iii) of [4]. Indeed, $\mathrm{desc}(\lambda, i, m) = -\mathrm{sgn}\,(m)$ if and only if $s_i w(\lambda, m) < w(\lambda, m)$ and in this case $s_i w(\lambda, m) = w(s_i\lambda, m)$ by [4]. Also, $\mathrm{desc}(\lambda, i, m) = 0$ if and only if $s_i w(\lambda, m)\nu = w(\lambda, m)\nu$ which shows that $s_i w(\lambda, m)$ belongs to the same coset as $w(\lambda, m)$ and is not minimal in this coset. In this case, by [4], there exists $s_j \in \mathfrak{S}_{\nu,m}$ such that $s_i w(\lambda, m) = w(\lambda, m)s_j$. Finally, $\mathrm{desc}(\lambda, i, m) = \mathrm{sgn}\,(m)$ if and only if $s_i w(\lambda, m) > w(\lambda, m)$ and $s_i w(\lambda, m)\nu \neq w(\lambda, m)\nu$. In that case, again by [4], $s_i w(\lambda, m)$ is minimal in its coset and thus equal to $w(s_i\lambda, m)$. □

If ν is regular, that is, $\mathfrak{S}_{\nu,m} = \{1\}$, then case (ii) does not occur and we obtain the following criterion

$$\text{(12)} \qquad s_i w > w \iff \mathrm{desc}(w\nu, i, m) = \mathrm{sgn}\,(m), \qquad (w \in \widehat{\mathfrak{S}}_r).$$

In particular, taking $m = r$ and $\nu = \rho := (r-1, r-2, \dots, 1, 0)$ we get that

$$\text{(13)} \qquad s_i w > w \iff \mathrm{desc}(w\rho, i, r) = 1, \qquad (w \in \widehat{\mathfrak{S}}_r).$$

For $\lambda \in P$, set $y^\lambda := y_1^{\lambda_1} \cdots y_r^{\lambda_r}$. Every $w \in \widehat{\mathfrak{S}}_r$ has a unique decomposition of the form $w = y^\lambda s$, where $\lambda \in P$ and $s \in \mathfrak{S}_r$. Therefore each coset $w\mathfrak{S}_r$ contains a unique element y^λ. It follows from (8) that for $s \in \mathfrak{S}_r$, $sy^\lambda = y^{s\lambda}s$. This implies that each double coset $\mathfrak{S}_r w \mathfrak{S}_r$ in $\widehat{\mathfrak{S}}_r$ contains a unique element y^λ with $\lambda \in P^+ := \{\mu \in P \mid \mu_1 \geq \mu_2 \geq \dots \geq \mu_r\}$, the set of dominant weights. For $\lambda \in P^+$, we denote by n_λ the element of maximal length in $\mathfrak{S}_r y^\lambda \mathfrak{S}_r$.

Lemma 2.2. *Let $\lambda \in P^+$, $\mu \in P^- := -P^+$ and $s \in \mathfrak{S}_r$. We have*

$$\ell(sy^\lambda) = \ell(s) + \ell(y^\lambda), \qquad \ell(y^\mu s) = \ell(y^\mu) + \ell(s).$$

In particular $n_\lambda = w_0 y^\lambda$, *where* w_0 *denotes the longest element of* $\mathfrak{S}_r$.

Proof — If $\lambda \in P^+$ then $\alpha := y^\lambda \rho$ satisfies $\alpha_1 > \alpha_2 > \cdots > \alpha_r$. Let $s = s_{i_1} \cdots s_{i_k}$ be a reduced decomposition of s. By repeated applications of (13) we see that $\ell(sy^\lambda) = \ell(y^\lambda) + k$, which proves the first statement. The case of μ is similar. Finally, $w_0 y^\lambda$ belongs to the double coset of y^λ and for $s \in \mathfrak{S}_r$ $(s \neq 1)$, $\ell(sw_0y^\lambda) = \ell(sw_0)+\ell(y^\lambda) < \ell(w_0)+\ell(y^\lambda)$ so that sw_0y^λ is not maximal. The argument is similar for right multiplication by s, since $w_0 y^\lambda = y^{w_0\lambda} w_0$, and $w_0\lambda \in P^-$. □

Example 2.2. Take $r = 3$. Then,

$$\begin{aligned} n_{(2,1,0)} &= s_2 s_1 s_2 y_1^2 y_2 = s_2 s_1 s_2 s_0 s_2 s_1 s_2 \tau^3, \\ n_{(1,1,1)} &= s_2 s_1 s_2 y_1 y_2 y_3 = s_2 s_1 s_2 \tau^3. \end{aligned}$$

◇

In fact, Lemma 2.2 easily follows from a general formula of Iwahori and Matsumoto ([11], Prop. 1.23) which in our case reads

$$\ell(sy^\lambda) = \sum_{\substack{i<j \\ s(i)<s(j)}} |\lambda_i - \lambda_j| + \sum_{\substack{i<j \\ s(i)>s(j)}} |\lambda_i - \lambda_j + 1|, \tag{14}$$

where $s \in \mathfrak{S}_r$ and $\lambda \in P$. In particular, if $\lambda \in P^+$ then $\ell(y^\lambda) = \sum_{i=1}^r (r+1-2i)\lambda_i$, which shows that

$$\ell(y^\lambda) + \ell(y^\mu) = \ell(y^{\lambda+\mu}), \qquad (\lambda, \mu \in P^+). \tag{15}$$

Lemma 2.3. *Let* $\lambda \in P^+$ *and set* $\lambda^* := w_0(-\lambda)$. *Then, for all* $n \geq r$ *one has*

$$w(n\lambda + \rho, -n) = n_{\lambda^*}\, \tau^{-r+1}.$$

Proof — Since $n \geq r$, the weight

$$\nu := \pi_{-n}(\tau^{r-1} w_0)(\rho) = (1-n, 2-n, \ldots, r-1-n, 0)$$

belongs to $\mathcal{A}_{r,-n}$ and we have

$$n\lambda + \rho = \pi_{-n}(y^{-\lambda})(\rho) = \pi_{-n}(y^{-\lambda} w_0 \tau^{-r+1})(\nu).$$

The stabilizer of ν in $\pi_{-n}(\widehat{\mathfrak{S}}_r)$ is trivial, that is, ν is a regular weight. Therefore we get

$$w(n\lambda + \rho, -n) = y^{-\lambda} w_0 \tau^{-r+1} = w_0 y^{w_0(-\lambda)} \tau^{-r+1} = n_{\lambda^*}\, \tau^{-r+1}.$$

□

2.2. Affine Hecke algebras

The Hecke algebra $\widehat{H}_r := H(\widehat{\mathfrak{S}}_r)$ is the algebra over $\mathbf{Z}[q, q^{-1}]$ with basis T_w ($w \in \widehat{\mathfrak{S}}_r$) and multiplication defined by

$$T_w T_{w'} = T_{ww'} \text{ if } \ell(ww') = \ell(w) + \ell(w'), \tag{16}$$

$$(T_{s_i} - q^{-1})(T_{s_i} + q) = 0. \tag{17}$$

There is a canonical involution $x \mapsto \overline{x}$ of $\widehat{H}_r$ defined as the unique ring homomorphism such that $\overline{q} = q^{-1}$ and $\overline{T_w} = (T_{w^{-1}})^{-1}$.

To simplify notation, we put $T_i := T_{s_i}$ and we write τ instead of T_τ. Then we have the two following presentations of $\widehat{H}_r$ corresponding to the two above presentations of $\widehat{\mathfrak{S}}_r$ (see [27, 29]). First, $\widehat{H}_r$ is the algebra generated by T_i ($0 \le i \le r-1$) and an invertible element τ subject to the relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \tag{18}$$

$$T_i T_j = T_j T_i, \qquad (i - j \neq \pm 1), \tag{19}$$

$$(T_i - q^{-1})(T_i + q) = 0, \tag{20}$$

$$\tau T_i = T_{i+1} \tau. \tag{21}$$

Alternatively, $\widehat{H}_r$ is the algebra generated by T_i ($1 \le i \le r-1$) and invertible elements Y_i ($1 \le i \le r$) subject to the relations (18), (19), (20) with subscripts between 1 and $r-1$ together with

$$Y_i Y_j = Y_j Y_i, \tag{22}$$

$$T_i Y_j = Y_j T_i \text{ for } j \neq i, i+1, \tag{23}$$

$$T_i Y_i T_i = Y_{i+1}. \tag{24}$$

The following equations relate the two above presentations of $\widehat{H}_r$:

$$Y_i = T_{i-1} T_{i-2} \cdots T_1 T_0^{-1} T_{r-1}^{-1} T_{r-2}^{-1} \cdots T_{i+1}^{-1} \tau, \qquad (1 \le i \le r) \tag{25}$$

$$\tau = T_1^{-1} T_2^{-1} \cdots T_{r-1}^{-1} Y_r, \tag{26}$$

$$T_0 = T_{r-1}^{-1} T_{r-2}^{-1} \cdots T_2^{-1} T_1^{-1} T_2^{-1} \cdots T_{r-1}^{-1} Y_1^{-1} Y_r. \tag{27}$$

(In (18)(19)(21)(25) the subscripts are understood modulo r.)

Note that for $\lambda \in P$, we have two natural elements in $\widehat{H}_r$ corresponding to the translation by λ, namely, $Y^\lambda := Y_1^{\lambda_1} \cdots Y_r^{\lambda_r}$ and $T_\lambda := T_{y^\lambda}$. They do not coincide in general. (For example if $r = 3$, $T_{y_2} = T_1 T_0 \tau$ and $Y_2 = T_1 T_0^{-1} \tau$.) In fact the T_λ do not commute in general. However it follows from (15) that $T_\lambda T_\mu = T_{\lambda+\mu} = T_\mu T_\lambda$ if $\lambda, \mu \in P^+$. Let $\lambda \in P$

be written as $\lambda = \lambda' - \lambda''$ with $\lambda', \lambda'' \in P^+$. Bernstein has introduced an element $\widehat{T}_\lambda \in \widehat{H}_r$ by

$$\widehat{T}_\lambda := T_{\lambda'} T_{\lambda''}^{-1}.$$

This element is well-defined, *i.e.* it does not depend on the choice of λ' and λ'', and $\widehat{T}_\lambda \widehat{T}_\mu = \widehat{T}_{\lambda+\mu} = \widehat{T}_\mu \widehat{T}_\lambda$ for all $\lambda, \mu \in P$. With this notation one can check that

$$Y^\lambda = \overline{\widehat{T}_\lambda} = T_{-\lambda'}^{-1} T_{-\lambda''} = T_{-\lambda''} T_{-\lambda'}^{-1}. \tag{28}$$

In particular, if $\lambda \in P^-$ then

$$Y^\lambda = T_\lambda. \tag{29}$$

2.3. Action of $\widehat{H}_r$ on the weight lattice

Let $\mathcal{P} = \mathcal{P}_r := \mathbf{Z}[q, q^{-1}] \otimes_{\mathbf{Z}} P$. We shall use the descent function to q-deform the representation π_m of $\widehat{\mathfrak{S}}_r$ on P into a representation Π_m of $\widehat{H}_r$ on $\mathcal{P}$. Indeed, it follows from Lemma 2.1 that $\widehat{H}_r$ acts on $\mathcal{P}$ by $\Pi_m(\tau)(\lambda) := \pi_m(\tau)(\lambda)$ and for $0 \le i \le r-1$,

$$\Pi_m(T_i)(\lambda) := \begin{cases} \pi_m(s_i)(\lambda) & \text{if } \operatorname{desc}(\lambda, i, m) = \operatorname{sgn}(m), \\ q^{-1}\lambda & \text{if } \operatorname{desc}(\lambda, i, m) = 0, \\ \pi_m(s_i)(\lambda) + (q^{-1} - q)\lambda & \text{if } \operatorname{desc}(\lambda, i, m) = -\operatorname{sgn}(m). \end{cases}$$

Warning From now on in order to simplify the notation we shall often omit the dependence on m and write for example $T_i\lambda$ in place of $\Pi_m(T_i)(\lambda)$, or $s_i\lambda$ in place of $\pi_m(s_i)(\lambda)$. We hope that this will not create confusion.

In terms of the Kazhdan-Lusztig elements $C_i' := T_i + q$ and $C_i := T_i - q^{-1}$ we have

$$C_i'\lambda = \begin{cases} s_i\lambda + q\lambda & \text{if } \operatorname{desc}(\lambda, i, m) = \operatorname{sgn}(m), \\ (q + q^{-1})\lambda & \text{if } \operatorname{desc}(\lambda, i, m) = 0, \\ s_i\lambda + q^{-1}\lambda & \text{if } \operatorname{desc}(\lambda, i, m) = -\operatorname{sgn}(m), \end{cases} \qquad (0 \le i \le r-1),$$

$$C_i\lambda = \begin{cases} s_i\lambda - q^{-1}\lambda & \text{if } \operatorname{desc}(\lambda, i, m) = \operatorname{sgn}(m), \\ 0 & \text{if } \operatorname{desc}(\lambda, i, m) = 0, \\ s_i\lambda - q\lambda & \text{if } \operatorname{desc}(\lambda, i, m) = -\operatorname{sgn}(m), \end{cases} \qquad (0 \le i \le r-1).$$

These formulas show that the $\widehat{H}_r$-module $\mathcal{P}$ decomposes as

$$\mathcal{P} = \bigoplus_{\nu \in \mathcal{A}_{r,m}} \widehat{H}_r \nu.$$

Moreover, each summand of the right-hand side is isomorphic to an induced module. Indeed, let $\widehat{H}_{\nu,m}$ be the subalgebra of $\widehat{H}_r$ generated by the T_i such that $s_i\nu = \nu$, and let $\mathbf{1}_{q^{-1}}$ denote the 1-dimensional $\widehat{H}_{\nu,m}$-module in which T_i acts by multiplication by q^{-1}. Then

$$\widehat{H}_r\nu \simeq \widehat{H}_r \otimes_{\widehat{H}_{\nu,m}} \mathbf{1}_{q^{-1}},$$

the isomorphism being given by

$$(30) \qquad \lambda \in \widehat{\mathfrak{S}}_r\nu \mapsto T_{w(\lambda,m)} \otimes 1.$$

In particular $\lambda = T_{w(\lambda,m)}\nu$.

2.4. Kazhdan-Lusztig polynomials

The module $\mathcal{P}_\nu := \widehat{H}_r\nu$ is a parabolic module of the type considered by Deodhar in [4, 5]. (Note that if ν is a regular weight, then $\mathcal{P}_\nu$ is just the regular representation of $\widehat{H}_r$.) Therefore $\mathcal{P}_\nu$ has two Kazhdan-Lusztig bases constructed as follows (see [39]). Define a semi-linear involution on $\mathcal{P}_\nu$ by

$$\overline{q} := q^{-1}, \qquad \overline{x\nu} := \overline{x}\nu \quad (x \in \widehat{H}_r),$$

and two lattices

$$L_\nu^+ := \bigoplus_{\lambda\in\widehat{\mathfrak{S}}_r\nu} \mathbf{Z}[q]\lambda, \qquad L_\nu^- := \bigoplus_{\lambda\in\widehat{\mathfrak{S}}_r\nu} \mathbf{Z}[q^{-1}]\lambda.$$

Then there are two bases C_λ^+, C_λ^- $(\lambda \in \widehat{\mathfrak{S}}_r\nu)$ characterized by

$$\overline{C_\lambda^+} = C_\lambda^+, \qquad \overline{C_\lambda^-} = C_\lambda^-,$$

and

$$C_\lambda^+ \equiv \lambda \bmod qL_\nu^+, \qquad C_\lambda^- \equiv \lambda \bmod q^{-1}L_\nu^-.$$

When ν is regular these bases coincide with the Kazhdan-Lusztig bases C'_w and C_w respectively under the isomorphism (30).

These bases can be computed recursively as follows [39]. First, by definition, $C_\nu^+ = C_\nu^- = \nu$, and more generally $C_{\tau^k\nu}^+ = C_{\tau^k\nu}^- = \tau^k\nu$ $(k \in \mathbf{Z})$. Let $\lambda \in \widehat{\mathfrak{S}}_r\nu$ and suppose that C_μ^+ (*resp.* C_μ^-) has already been calculated for all $\mu < \lambda$, that is, such that $w(\mu,m) < w(\lambda,m)$. Then compute $v_\lambda^+ = C'_i C_\mu^+$ (*resp.* $v_\lambda^- = C_i C_\mu^-$) where μ and i satisfy $s_i(\mu) = \lambda$

and $\mathrm{desc}(\mu, i, m) = \mathrm{sgn}\,(m)$. Then v_λ^+ (*resp.* v_λ^-) is invariant under the bar-involution and belongs to L_ν^+ (*resp.* L_ν^-). Write

$$v_\lambda^+ \equiv \lambda + \sum_\alpha a_\alpha \alpha \bmod qL_\nu^+, \qquad (\textit{resp. } v_\lambda^- \equiv \lambda + \sum_\beta b_\beta \beta \bmod q^{-1}L_\nu^-),$$

where $a_\alpha,\ b_\beta \in \mathbf{Z}$. The weights α, β occuring in the right-hand side are certainly $< \lambda$ and we obtain

$$C_\lambda^+ = v_\lambda^+ - \sum_\alpha a_\alpha C_\alpha^+ \qquad (\textit{resp. } C_\lambda^- = v_\lambda^- - \sum_\beta b_\beta C_\beta^-).$$

Example 2.3. Let us take $r = 3$, $m = -2$ and compute $C^-_{(0,6,1)}$. We have $w((0,6,1),-2) = s_2 s_0 s_1 s_2 s_0 \tau^{-4}$ and

$$\nu := w((0,6,1),-2)^{-1}(0,6,1) = (-1,0,0).$$

Clearly,

$$C^-_{(2,2,3)} = C^-_{\tau^{-4}(-1,0,0)} = (2,2,3).$$

Then we compute successively ($t = q^{-1}$)

$$\begin{aligned}
v^-_{(1,2,4)} &= C^-_{(1,2,4)} = (1,2,4) - t(2,2,3),\\
v^-_{(1,4,2)} &= C^-_{(1,4,2)} = (1,4,2) - t(1,2,4) - t(2,3,2) + t^2(2,2,3),\\
v^-_{(4,1,2)} &= C^-_{(4,1,2)} = (4,1,2) - t(1,4,2) - t(2,1,4) + t^2(1,2,4)\\
&\qquad -t(3,2,2) + t^2(2,3,2),\\
v^-_{(0,1,6)} &= C^-_{(0,1,6)} = (0,1,6) - t(4,1,2) - t(0,4,3) + t^2(1,4,2)\\
&\qquad +t^2(2,2,3) - t(1,2,4) - t(0,2,5)\\
&\qquad +t^2(3,2,2) + t^2(0,3,4) - t^3(2,3,2),\\
v^-_{(0,6,1)} &= (0,6,1) - t(0,1,6) - t(4,2,1) + t^2(4,1,2) - t(0,3,4)\\
&\qquad +(0,4,3) + 2t^2(1,2,4) - 2t(1,4,2)\\
&\qquad +2t^2(2,3,2) - 2t^3(2,2,3) - t(0,5,2)\\
&\qquad +t^2(0,2,5) + t^2(0,4,3) - t^3(0,3,4).
\end{aligned}$$

We see that $v^-_{(0,6,1)} \equiv (0,6,1) + (0,4,3) \bmod tL_\nu^-$. Thus subtracting the previously calculated element

$$C^-_{(0,4,3)} = (0,4,3) - t(0,3,4) - t(1,4,2) + t^2(1,2,4) + t^2(2,3,2) - t^3(2,2,3)$$

we get

$$C^-_{(0,6,1)} = (0,6,1) - t(0,1,6) - t(4,2,1) + t^2(4,1,2) + t^2(1,2,4) \\ -t(1,4,2) + t^2(2,3,2) - t^3(2,2,3) - t(0,5,2) \\ +t^2(0,2,5) + t^2(0,4,3) - t^3(0,3,4).$$

◇

Put

$$C'_w = \sum_{x\in\widehat{\mathfrak{S}}_r} P_{x,w}(q)\, T_x.$$

Then

$$C_w = \sum_{x\in\widehat{\mathfrak{S}}_r} P_{x,w}(-q^{-1})\, T_x.$$

The $P_{x,w}$ are the Kazhdan-Lusztig polynomials (up to a factor $q^{\ell(w)-\ell(x)}$ and the change of variable $q \mapsto q^{-2}$). Similarly for $\lambda \in \widehat{\mathfrak{S}}_r\nu$ write

$$C^+_\lambda = \sum_{\mu\in\widehat{\mathfrak{S}}_r\nu} P^+_{\mu,\lambda}(q)\,\mu, \qquad C^-_\lambda = \sum_{\mu\in\widehat{\mathfrak{S}}_r\nu} P^-_{\mu,\lambda}(-q^{-1})\,\mu.$$

Then $P^+_{\mu,\lambda}$ and $P^-_{\mu,\lambda}$ are respectively equal to Deodhar's polynomials $P^J_{w(\mu,m),w(\lambda,m)}$ and $\widetilde{P}^J_{w(\mu,m),w(\lambda,m)}$ (again up to a factor $q^{\ell(w)-\ell(x)}$ and the change of variable $q \mapsto q^{-2}$), where J is the set of indices i of the Coxeter generators $s_i \in \mathfrak{S}_{\nu,m}$. Their expression in terms of ordinary Kazhdan-Lusztig polynomials is given by

Theorem 2.4 (Deodhar [4, 5]). *Let $w_{0,\nu}$ be the longest element of $\mathfrak{S}_{\nu,m}$. Then*

$$P^+_{\mu,\lambda} = P_{w(\mu,m)w_{0,\nu}\,,\,w(\lambda,m)w_{0,\nu}}, \qquad P^-_{\mu,\lambda} = \sum_{z\in\mathfrak{S}_{\nu,m}} (-q)^{\ell(z)} P_{w(\mu,m)z\,,\,w(\lambda,m)}.$$

We shall also need the following simple observation (see [39], Remark 3.2.4). Suppose that $\mathrm{desc}(\lambda,i,m) = \mathrm{desc}(\mu,i,m) = -\mathrm{sgn}\,(m)$. Then

$$P^+_{s_i\mu,\lambda} = qP^+_{\mu,\lambda}, \qquad P^-_{s_i\mu,\lambda} = qP^-_{\mu,\lambda}. \tag{31}$$

This follows from the fact that if $\mathrm{desc}(\lambda,i,m) = -\mathrm{sgn}\,(m)$ then

$$C'_iC^+_\lambda = (q+q^{-1})\,C^+_\lambda, \qquad C_iC^-_\lambda = -(q+q^{-1})\,C^-_\lambda.$$

§3. Littlewood-Richardson coefficients and Kazhdan-Lusztig polynomials

3.1. The Lusztig conjecture

Let $U_q(\mathfrak{gl}_r)$ be the quantum enveloping algebra of $\mathfrak{gl}_r$. This is a $\mathbf{Q}(q)$-algebra with generators $E_i,\ F_i,\ q^{\pm\epsilon_j}$ $(1 \le i \le r-1,\ 1 \le j \le r)$. The relations are standard [14] and will be omitted. To avoid confusion when q is specialized to a complex number, we shall write $K_j^{\pm}$ in place of $q^{\pm\epsilon_j}$. Let $U_{q,\mathbf{Z}}(\mathfrak{gl}_r)$ denote the $\mathbf{Z}[q,q^{-1}]$-subalgebra generated by the elements

$$E_i^{(k)} := \frac{E_i^k}{[k]!},\quad F_i^{(k)} := \frac{F_i^k}{[k]!},\quad K_j^{\pm},\qquad (k \in \mathbf{N}),$$

where $[k]! := [k][k-1]\cdots[2][1]$ and $[k] := (q^k - q^{-k})/(q-q^{-1})$. Let $\zeta \in \mathbf{C}$ be such that ζ^2 is a primitive nth root of 1. One defines $U_\zeta(\mathfrak{gl}_r) := U_{q,\mathbf{Z}}(\mathfrak{gl}_r) \otimes_{\mathbf{Z}[q,q^{-1}]} \mathbf{C}$ where $\mathbf{Z}[q,q^{-1}]$ acts on $\mathbf{C}$ by $q \mapsto \zeta$ [30, 31].

Let $\lambda \in P_r^+$. There is a unique finite-dimensional $U_q(\mathfrak{gl}_r)$-module (of type 1) $W_q(\lambda)$ with highest weight λ. Its character is the same as in the classical case and is given by Weyl's character formula

$$\operatorname{ch} W_q(\lambda) = s_\lambda(e^{\epsilon_1},\dots,e^{\epsilon_r}), \tag{32}$$

where s_λ denotes the Schur function (see [33]). Fix a highest weight vector $u_\lambda \in W_q(\lambda)$ and denote by $W_{q,\mathbf{Z}}(\lambda)$ the $U_{q,\mathbf{Z}}(\mathfrak{gl}_r)$-submodule of $W_q(\lambda)$ generated by acting on u_λ. Finally, put

$$W_\zeta(\lambda) := W_{q,\mathbf{Z}}(\lambda) \otimes_{\mathbf{Z}[q,q^{-1}]} \mathbf{C}.$$

This is a $U_\zeta(\mathfrak{gl}_r)$-module called a Weyl module [30]. By definition $\operatorname{ch} W_\zeta(\lambda) = \operatorname{ch} W_q(\lambda)$.

There is a unique simple quotient of $W_\zeta(\lambda)$ denoted by $L(\lambda)$. Its character is given in terms of the characters of the Weyl modules by the Lusztig conjecture. Put $m = -n$ (this assumption will be in force for the whole Section 3) and consider the action of $\widehat{\mathfrak{S}}_r$ on P via π_m. For $\lambda \in P^+$ write $\nu := w(\lambda+\rho,m)^{-1}(\lambda+\rho)$. Then

Theorem 3.1 (Kazhdan-Lusztig, Kashiwara-Tanisaki).

$$\operatorname{ch} L(\lambda) = \sum_w (-1)^{\ell(w(\lambda+\rho,m))-\ell(w)}\, P_{w,w(\lambda+\rho,m)}(1)\ \operatorname{ch} W_\zeta(w(\nu)-\rho),$$

where the sum runs over the $w \in \widehat{\mathfrak{S}}_r$ such that $w < w(\lambda+\rho,m)$ and $w(\nu) - \rho \in P^+$.

Note that if λ is a singular weight the coefficient of a given Weyl module $W_\zeta(\mu)$ in the right-hand side of Theorem 3.1 is an alternating sum of $P_{w,w(\lambda,m)}(1)$ over the stabilizer $\mathfrak{S}_{\nu,m}$. In fact, using the notation of Section 2.4 one can rewrite Theorem 3.1 as

$$\operatorname{ch} L(\lambda) = \sum_{\mu} P^-_{\mu+\rho,\lambda+\rho}(-1) \operatorname{ch} W_\zeta(\mu), \tag{33}$$

where the sum is over the $\mu \in P^+$ such that $\mu + \rho \in \widehat{\mathfrak{S}}_r \nu$.

Example 3.1. Take $r = 3$, $n = 2$ and $\lambda = (4,0,0)$. Then $\lambda + \rho = (6,1,0)$ and for $m = -n = -2$, one has

$$\begin{aligned} C^-_{(6,1,0)} &= (6,1,0) - q^{-1}(6,0,1) - q^{-1}(1,6,0) + q^{-2}(0,6,1) \\ &+q^{-2}(1,0,6) - q^{-3}(0,1,6) - q^{-1}(5,2,0) + q^{-2}(5,0,2) \\ &+q^{-2}(2,5,0) - q^{-3}(0,5,2) - q^{-3}(2,0,5) + q^{-4}(0,2,5) \\ &+q^{-2}(4,3,0) - q^{-3}(4,0,3) - q^{-3}(3,4,0) + q^{-4}(0,4,3) \\ &+q^{-4}(3,0,4) - q^{-5}(0,3,4). \end{aligned}$$

It follows that the character of $L(4,0,0)$ for $\zeta^2 = -1$ is given by

$$\operatorname{ch} L(4,0,0) = \operatorname{ch} W_\zeta(4,0,0) - \operatorname{ch} W_\zeta(3,1,0) + \operatorname{ch} W_\zeta(2,2,0).$$

◇

3.2. The tensor product theorem

Let Fr denote the Frobenius map from $U_\zeta(\mathfrak{gl}_r)$ to the (classical) enveloping algebra $U(\mathfrak{gl}_r)$ [30, 3]. This is the algebra homomorphism defined by $\mathrm{Fr}(K_j) = 1$ and

$$\mathrm{Fr}(E_i^{(k)}) = \begin{cases} E_i^{(k/n)} & \text{if } n \text{ divides } k, \\ 0 & \text{otherwise,} \end{cases} \quad \mathrm{Fr}(F_i^{(k)}) = \begin{cases} F_i^{(k/n)} & \text{if } n \text{ divides } k, \\ 0 & \text{otherwise.} \end{cases}$$

(Here we slightly abuse notation and denote by the same symbols the Chevalley generators of $U_\zeta(\mathfrak{gl}_r)$ and those of $U(\mathfrak{gl}_r)$.) Given a $U(\mathfrak{gl}_r)$-module M, one can thus define a $U_\zeta(\mathfrak{gl}_r)$-module M^{Fr} by composing the action of $U(\mathfrak{gl}_r)$ with Fr. If M is a finite-dimensional module with character the symmetric Laurent polynomial $\operatorname{ch} M = \varphi(e^{\epsilon_1}, \ldots, e^{\epsilon_r})$, then

$$\operatorname{ch} M^{\mathrm{Fr}} = p_n(\varphi)(e^{\epsilon_1}, \ldots, e^{\epsilon_r}) := \varphi(e^{n\epsilon_1}, \ldots, e^{n\epsilon_r}),$$

the so-called plethysm of φ with the power sum p_n (see [33]). In particular, the character of the pullback $W(\lambda)^{\mathrm{Fr}}$ of the classical Weyl module $W(\lambda)$ is the plethysm $p_n(s_\lambda)$.

Theorem 3.2 (Lusztig [30]). *Let $\lambda \in P^+$. Write $\lambda = \lambda^{(0)} + n\lambda^{(1)}$, where $\lambda^{(0)}$ is n-restricted, that is,*

$$0 \leq \lambda_i^{(0)} - \lambda_{i+1}^{(0)} < n \qquad (1 \leq i \leq r-1).$$

The simple $U_\zeta(\mathfrak{gl}_r)$-module $L(\lambda)$ is isomorphic to the tensor product

$$L(\lambda) \simeq L(\lambda^{(0)}) \otimes W(\lambda^{(1)})^{\mathrm{Fr}}.$$

Consider now the particular case when λ is a partition whose parts are all divisible by n. Then, writing $n\lambda$ in place of λ, we deduce from Theorem 3.2 and Eq. (33) that $p_n(s_\lambda) = \mathrm{ch}\, L(n\lambda)$ is given by

$$(34) \quad p_n(s_\lambda) = \sum_\mu P^-_{\mu+\rho, n\lambda+\rho}(-1) \mathrm{ch}\, W_\zeta(\mu) = \sum_\mu P^-_{\mu+\rho, n\lambda+\rho}(-1) s_\mu,$$

where the sum is over the $\mu \in P^+$ such that $\mu + \rho \in \widehat{\mathfrak{S}}_r(n\lambda + \rho) = \widehat{\mathfrak{S}}_r\rho$.

3.3. Expression of the Littlewood-Richardson coefficients

Let $\lambda \in \mathbb{P}_r^+ = \{\lambda \in P \,|\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0\}$, the set of partitions of length $l(\lambda) \leq r$. It is a well-known result of Littlewood [25] that the coefficients in the expansion of $p_n(s_\lambda)$ on the basis of Schur functions are Littlewood-Richardson multiplicities. More precisely, if $\mu \in \mathbb{P}_r^+$ is such that $\mu + \rho \in \widehat{\mathfrak{S}}_r\rho$ then there is a unique expression

$$\mu + \rho = \gamma + n\alpha, \qquad (\gamma = s\rho,\ s \in \mathfrak{S}_r,\ \alpha \in \mathbf{N}^r)$$

such that $i < j$ and $\gamma_i \equiv \gamma_j\ (n)$ implies $\gamma_i > \gamma_j$. Then for $k \in \{0, 1, \ldots, n-1\}$ the subsequence of α consisting of the α_i such that $\gamma_i \equiv k - r$ is a partition $\mu^{(k)}$ (possibly empty), and one has [25]

$$(35) \qquad \langle p_n(s_\lambda)\,;\, s_\mu\rangle = (-1)^{\ell(s)} \langle s_\lambda\,;\, s_{\mu^{(0)}} \cdots s_{\mu^{(n-1)}}\rangle$$

where $\langle\cdot\,;\,\cdot\rangle$ denotes the standard scalar product of the algebra of symmetric functions for which the s_λ form an orthonormal basis. The n-tuple of partitions $(\mu^{(0)}, \ldots, \mu^{(n-1)})$ is called the n-quotient of μ and $(-1)^{\ell(s)}$ the n-sign of μ, denoted $\varepsilon_n(\mu)$. Conversely, provided that r is large enough, given an arbitrary n-tuple of partitions $(\mu^{(0)}, \ldots, \mu^{(n-1)})$ there exists a unique $\mu \in \mathbb{P}_r^+$ such that $\mu + \rho \in \widehat{\mathfrak{S}}_r\rho$ and μ has $(\mu^{(0)}, \ldots, \mu^{(n-1)})$ as n-quotient (see [33, 13]).

Example 3.2. Let $r = 8$, $n = 3$, and $\mu = (6, 6, 4, 4, 4, 3, 2, 1)$. Then

$$\mu + \rho = (13, 12, 9, 8, 7, 5, 3, 1) = (7, 6, 3, 5, 4, 2, 0, 1) + 3\,(2, 2, 2, 1, 1, 1, 1, 0).$$

Thus the 3-quotient of μ is

$$(\mu^{(0)}, \mu^{(1)}, \mu^{(2)}) = ((1,1),(2,2,1),(2,1)).$$

◇

Let us define the Littlewood-Richardson coefficient

$$\begin{aligned} c^{\lambda}_{\mu^{(0)},\dots,\mu^{(n-1)}} &:= \langle s_{\mu^{(0)}} \cdots s_{\mu^{(n-1)}} \; ; \; s_\lambda \rangle \\ &= [W(\mu^{(0)}) \otimes \cdots \otimes W(\mu^{(n-1)}) \; : \; W(\lambda)] . \end{aligned}$$

Combining (34) and (35), we have obtained

Theorem 3.3. *Let $\lambda, \mu^{(0)}, \dots, \mu^{(n-1)}$ be partitions and denote by μ the partition with n-quotient $(\mu^{(0)}, \dots, \mu^{(n-1)})$. Take $r \geq l(\mu)$, the number of parts of μ. Then,*

$$c^{\lambda}_{\mu^{(0)},\dots,\mu^{(n-1)}} = P^{-}_{\mu+\rho, n\lambda+\rho}(1)$$

where the right-hand side is a Kazhdan-Lusztig polynomial of parabolic type for $\widehat{\mathfrak{S}}_r$ with $m = -n$. In other words, setting

$$\nu = w(n\lambda + \rho, -n)^{-1}(n\lambda + \rho),$$

one has in terms of the (ordinary) Kazhdan-Lusztig polynomials for $\widehat{\mathfrak{S}}_r$

$$c^{\lambda}_{\mu^{(0)},\dots,\mu^{(n-1)}} = \sum_{z \in \mathfrak{S}_{\nu,-n}} (-1)^{\ell(z)} P_{w(\mu+\rho,-n)z\,,\,w(n\lambda+\rho,-n)}(1).$$

If $l(\lambda) > r$ the polynomial $P^{-}_{\mu+\rho,n\lambda+\rho}$ is not defined, but in this case $l(\lambda) > l(\mu)$ and it is easy to see that $c^{\lambda}_{\mu^{(0)},\dots,\mu^{(n-1)}} = 0$.

Note that if $w = \tau^k \sigma$, $w' = \tau^m \sigma'$ with $k, m \in \mathbf{Z}$, $\sigma, \sigma' \in \widetilde{\mathfrak{S}}_r$, then $P_{w,w'}$ is nonzero only if $k = m$ and then $P_{w,w'} = P_{\sigma,\sigma'}$. Thus the Kazhdan-Lusztig polynomials above are in fact polynomials for $\widetilde{\mathfrak{S}}_r$.

Example 3.3. Take $r = 3$ and $n = -m = 2$. The dominant weights occuring in the expansion of $C^{-}_{(6,3,0)}$ are

$$(6,3,0),\ (6,2,1),\ (5,4,0),\ (4,3,2),$$

with respective coefficients

$$1,\ -q^{-1},\ -q^{-1},\ q^{-2}.$$

This gives the following expressions for some Littlewood-Richardson coefficients (which are all equal to 1):

$$c^{(2,1)}_{(1),(2)} = P^-_{(6,3,0),(6,3,0)}(1), \qquad c^{(2,1)}_{\emptyset,(2,1)} = P^-_{(6,2,1),(6,3,0)}(1),$$

$$c^{(2,1)}_{(2),(1)} = P^-_{(5,4,0),(6,3,0)}(1), \qquad c^{(2,1)}_{(1),(1,1)} = P^-_{(4,3,2),(6,3,0)}(1).$$

In terms of ordinary Kazhdan-Lusztig polynomials for $\widetilde{\mathfrak{S}}_3$ we can write for example

$$c^{(2,1)}_{(1),(1,1)} = P_{s_2s_0s_2\,,\,s_2s_0s_2s_1s_2s_0s_2}(1) - P_{s_2s_0s_2s_1\,,\,s_2s_0s_2s_1s_2s_0s_2}(1) = 2-1.$$

◇

Example 3.4. Let us express the coefficient $c^{(3,2,1)}_{(2,1),(2,1)} = 2$ in terms of Kazhdan-Lusztig polynomials. We take $r=4$, $\lambda=(3,2,1)$ and $\mu=(4,4,2,2)$ so that μ has 2-quotient $((2,1);(2,1))$. It follows that

$$c^{(3,2,1)}_{(2,1),(2,1)} = P^-_{(7,6,3,2),(9,6,3,0)}(1).$$

This Kazhdan-Lusztig polynomial corresponds to the following elements of $\widetilde{\mathfrak{S}}_4$:

$$w((9,6,3,0),-2) = s_1s_2s_1s_3s_2s_1s_0s_1s_3s_2s_1s_3s_0s_1s_3s_2s_0\tau^{-10},$$
$$w((7,6,3,2),-2) = s_1s_2s_1s_3s_2s_1s_0s_1s_3s_2s_0\tau^{-10}.$$

◇

Observe that if $n \geq r$, then the n-quotient of the partition $n\mu = (n\mu_1, \ldots, n\mu_r)$ is just $((\mu_1), \ldots, (\mu_r), \emptyset, \ldots, \emptyset)$ up to reordering, and therefore Theorem 3.3 gives

$$P^-_{n\mu+\rho,n\lambda+\rho}(1) = c^\lambda_{(\mu_1),\ldots,(\mu_r)} = K_{\lambda,\mu},$$

the Kostka number. On the other hand, taking into account Lemma 2.3 and the fact that the weight $n\lambda+\rho$ is regular, one also has

$$P^-_{n\mu+\rho,n\lambda+\rho}(1) = P_{n_{\mu^*}\,\tau^{-r+1},n_{\lambda^*}\,\tau^{-r+1}}(1) = P_{n_{\mu^*},n_{\lambda^*}}(1).$$

Hence

$$P_{n_\mu,n_\lambda}(1) = K_{\lambda^*,\mu^*} = K_{\lambda,\mu}$$

since the weight multiplicities of the contragredient representation $W(\lambda^*)$ are equal to those of $W(\lambda)$, and we recover the expression of [28] for the weight multiplicities.

Thus we see that the modular Lusztig conjecture with its restriction $n \geq r$ is enough to express the weight multiplicities in terms of Kazhdan-Lusztig polynomials, but for what concerns the general tensor product multiplicities we need the case $n < r$ and the quantum Lusztig conjecture.

§4. Littlewood-Richardson coefficients and ribbon tableaux

4.1. Ribbon tableaux

Let us start from the well-known formula

$$h_\mu = \sum_\lambda |\mathrm{Tab}\,(\lambda,\mu)|\, s_\lambda, \tag{36}$$

where $h_\mu := h_{\mu_1} \dots h_{\mu_r}$ is a product of complete homogeneous symmetric functions and $\mathrm{Tab}\,(\lambda,\mu)$ denotes the set of semi-standard Young tableaux of shape λ and weight μ [33]. Let $n \in \mathbf{N}^*$. Semi-standard n-ribbon tableaux are combinatorial objects which replace ordinary Young tableaux when one substitutes the plethysm $p_n(h_\mu)$ in place of h_μ in (36). More precisely, denoting by $\mathrm{Tab}_n(\lambda,\mu)$ the set of n-ribbon tableaux of shape λ and weight μ (to be defined below), one has

$$p_n(h_\mu) = \sum_\lambda \varepsilon_n(\lambda)\, |\mathrm{Tab}_n(\lambda,\mu)|\, s_\lambda, \tag{37}$$

where $\varepsilon_n(\lambda)$ is the n-sign of λ.

A ribbon tableau of weight $\mu = (1, 1, \dots, 1)$ is called standard. Standard ribbon tableaux were introduced by Stanton and White [41] in relation with generalizations of the Robinson-Schensted correspondence for the complex reflection groups $G(n,1,r) = (\mathbf{Z}/n\mathbf{Z}) \wr \mathfrak{S}_r$. In particular, the case $n = 2$ (domino tableaux) is related to Weyl groups of type B, C, D, and therefore to the geometry of flag manifolds for classical groups [23] and to the classification of the primitive ideals of classical enveloping algebras [1, 9]. Semi-standard domino tableaux were introduced in [2] for calculating the multiplicities of the symmetric and alternating square of an irreducible representation of $\mathfrak{gl}_r$ (see also [16, 24]). In an attempt to extend the results of [2] to higher degree plethysms, semi-standard n-ribbon tableaux were defined in [21] and several conjectures were formulated. We shall give a brief review of [21] refering to the paper for more detail.

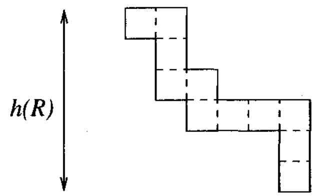

Fig. 2. An 11-ribbon of height $h(R) = 6$

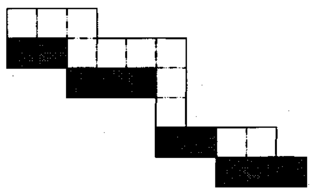

Fig. 3. A skew diagram θ with its subdiagram $\theta\!\downarrow$ shaded

A ribbon is a connected skew Young diagram of width 1, *i.e.* which does not contain any 2×2 square (see Figure 2). The rightmost and bottommost cell is called the origin of the ribbon. An n-ribbon is a ribbon made of n square cells. Let θ be a skew Young diagram, and let $\theta\!\downarrow$ be the horizontal strip made of the bottom cells of the columns of θ (see Figure 3). We say that θ is a horizontal n-ribbon strip of weight m if it can be tiled by m n-ribbons the origins of which lie in $\theta\!\downarrow$. One can check that if such a tiling exists, it is unique (see below Lemma 6.3 and Figure 7). Now, an n-ribbon tableau T of shape λ/ν and weight $\mu = (\mu_1, \ldots, \mu_r)$ is defined as a chain of partitions

$$\nu = \alpha^0 \subset \alpha^1 \subset \cdots \subset \alpha^r = \lambda$$

such that α^i/α^{i-1} is a horizontal n-ribbon strip of weight μ_i. Graphically, T may be described by numbering each n-ribbon of α^i/α^{i-1} with the number i (see Figure 4). We denote by $\mathrm{Tab}_n(\lambda/\nu, \mu)$ the set of n-ribbon tableaux of shape λ/ν and weight μ. Define the spin of a ribbon R as $\mathrm{spin}(R) := h(R) - 1$ where $h(R)$ is the height of R, and the spin of a ribbon tableau T as the sum of the spins of its ribbons. Then the sign $(-1)^{\mathrm{spin}(T)}$ depends only on the shape λ/ν of T and is equal to the n-sign $\varepsilon_n(\lambda)$ when ν is empty. We denote it in general by $\varepsilon_n(\lambda/\nu)$.

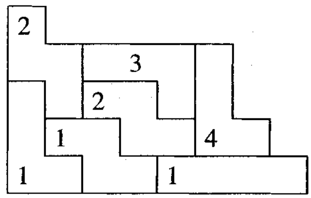

Fig. 4. A 4-ribbon tableau of shape $(8,7,6,6,1)$, weight $(3,2,1,1)$ and spin 9

4.2. A q-analogue of the Littlewood-Richardson coefficients

Using a classical formula for multiplying a monomial symmetric function by a Schur function one can easily derive Eq. (37). Note that since $h_{s\mu} = h_\mu$ $(s \in \mathfrak{S}_r)$, (37) implies that

$$|\mathrm{Tab}_n(\lambda, s\mu)| = |\mathrm{Tab}_n(\lambda,\mu)|, \qquad (s \in \mathfrak{S}_r). \tag{38}$$

Let φ_n denote the adjoint of the endomorphism $f \mapsto p_n(f)$ of the space of symmetric functions with respect to $\langle \cdot\, ; \cdot \rangle$. Recall from Section 3.3 the definition of the n-quotient $(\lambda^{(0)}, \cdots, \lambda^{(n-1)})$ of a partition λ of length r such that $\lambda + \rho \in \widehat{\mathfrak{S}}_r \rho$ (for the action of $\widehat{\mathfrak{S}}_r$ on weights via π_n). Then (35) is equivalent to

$$\varphi_n(s_\lambda) = \varepsilon_n(\lambda)\, s_{\lambda^{(0)}} \cdots s_{\lambda^{(n-1)}}, \tag{39}$$

where we put $\varepsilon_n(\lambda) = 0$ if $\lambda + \rho \notin \widehat{\mathfrak{S}}_r \rho$. By (37) we have

$$|\mathrm{Tab}_n(\lambda,\mu)| = \varepsilon_n(\lambda)\, \langle p_n(h_\mu)\, ; s_\lambda \rangle = \varepsilon_n(\lambda)\, \langle h_\mu\, ; \varphi_n(s_\lambda) \rangle.$$

Recalling that the basis dual to $\{h_\mu\}$ is the basis $\{m_\mu\}$ of monomial symmetric functions, we thus have

$$s_{\lambda^{(0)}} \cdots s_{\lambda^{(n-1)}} = \sum_{\mu \in P^+} |\mathrm{Tab}_n(\lambda,\mu)|\, m_\mu. \tag{40}$$

Hence, putting $x^T := x_1^{\alpha_1} \cdots x_r^{\alpha_r}$ for a ribbon tableau T of weight $\alpha = (\alpha_1, \dots, \alpha_r)$, we get using (38)

(41)

$$s_{\lambda^{(0)}} \cdots s_{\lambda^{(n-1)}} = \sum_{\mu \in P^+} \left(\sum_{\beta \in \mathfrak{S}_r \mu} \left(\sum_{T \in \mathrm{Tab}_n(\lambda,\beta)} x^T \right) \right) = \sum_{T \in \mathrm{Tab}_n(\lambda,\cdot)} x^T$$

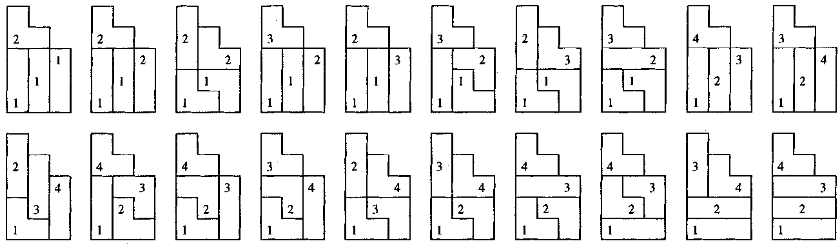

Fig. 5. The 3-ribbon tableaux of shape $(3,3,3,2,1)$ and dominant weight

where we denote by $\mathrm{Tab}_n(\lambda,\cdot)$ the set of n-ribbon tableaux of shape λ (and arbitrary weight).

Now we can introduce a q-analogue of (41) via the spin of ribbon tableaux and set

$$G(\lambda^{(0)},\ldots,\lambda^{(n-1)};q,x) := \sum_{T\in\mathrm{Tab}_n(\lambda,\cdot)} q^{\mathrm{spin}(T)}x^T. \tag{42}$$

It was proved in [21] that this function is symmetric with respect to the variables x_i. (This is not clear a priori, and the proof will be recalled below (see Remark 6.5).) Thus, expanding on the basis of Schur functions we get

$$G(\lambda^{(0)},\ldots,\lambda^{(n-1)};q,x) = \sum_{\nu} c^{\nu}_{\lambda^{(0)},\ldots,\lambda^{(n-1)}}(q)\, s_\nu(x), \tag{43}$$

where the $c^{\nu}_{\lambda^{(0)},\ldots,\lambda^{(n-1)}}(q)\in\mathbf{Z}[q]$ are some q-analogues of the Littlewood-Richardson coefficients. The symmetric function (43) is the function $\widetilde{G}^{(n)}_{\lambda}(x;q)$ of [21] up to the change of variable $q\mapsto q^{-2}$ and rescaling by an appropriate power of q.

Example 4.1. The partition having as 3-quotient $=((1),(1,1),(1))$ is $\mu=(3,3,3,2,1)$. Thus the symmetric function $G((1),(1,1),(1);q)$ is calculated by enumerating the 3-ribbon tableaux of shape μ and dominant weight, and counting their spin (see Figure 5). One obtains

$$\begin{aligned}G((1),(1,1),(1);q) &= q^7 m_{(3,1)} + (q^7+q^5)m_{(2,2)}\\ &\quad +(2q^7+2q^5+q^3)m_{(2,1,1)}\\ &\quad +(3q^7+5q^5+3q^3+q)m_{(1,1,1,1)}\\ &= q^7 s_{(3,1)} + q^5 s_{(2,2)} + (q^5+q^3)s_{(2,1,1)}\\ &\quad +q s_{(1,1,1,1)}.\end{aligned}$$

◇

We can now state our main result, which is the q-analog of Theorem 3.3.

Theorem 4.1. *With the notation of Theorem* 3.3

$$\begin{aligned} c^{\lambda}_{\mu^{(0)},\dots,\mu^{(n-1)}}(q) &= P^{-}_{\mu+\rho,n\lambda+\rho}(q) \\ &= \sum_{z\in\mathfrak{S}_{\nu,-n}} (-q)^{\ell(z)} P_{w(\mu+\rho,-n)z\,,\,w(n\lambda+\rho,-n)}(q). \end{aligned}$$

The next two sections will be devoted to the proof of Theorem 4.1. This proof does not rely on the Lusztig conjecture and thus will give an independent proof of Theorem 3.3.

§5. Canonical bases and Kazhdan-Lusztig polynomials

5.1. Another basis of $\mathcal{P}$

The basis of $\mathcal{P}$ consisting of the weights λ is adapted to the Coxeter-type presentation of $\widehat{H}_r$ in terms of the generators $T_0,\dots,T_{r-1},\tau$. There is another natural basis adapted to the Bernstein presentation in terms of $T_1,\dots,T_{r-1},Y_1,\dots,Y_r$, which is defined as follows. Fix $m\in\mathbf{Z}^*$ and consider the action of $\widehat{H}_r$ via Π_m. Every $\lambda\in P$ has a unique expression as $\lambda=m\beta+\gamma$ ($\beta,\gamma\in P$, $\gamma\in\mathfrak{S}_r\mathcal{A}_{r,m}$). We define $V_\lambda:=Y^\beta\gamma$. In other words, the basis $\{V_\lambda\}$ is characterized by

$$V_\gamma=\gamma \qquad (\gamma\in\mathfrak{S}_r\mathcal{A}_{r,m}), \tag{44}$$

$$Y^\beta V_\lambda=V_{\lambda+m\beta} \qquad (\lambda,\beta\in P). \tag{45}$$

Example 5.1. Take $r=2$ and $m=-2$. Then

$$\begin{aligned} &V_{(-1,-2)}=Y_2(-1,0)=T_1\tau(-1,0)=(-1,-2),\\ &V_{(-2,-1)}=Y_1(0,-1)=T_0^{-1}\tau(0,-1)=(-2,-1),\\ &V_{(2,-1)}=Y_1^{-1}(0,-1)=\tau^{-1}T_0(0,-1)=(2,-1),\\ &V_{(-1,2)}=Y_2^{-1}(-1,0)=\tau^{-1}T_1^{-1}(-1,0)=(-1,2)+(q-q^{-1})(0,1). \end{aligned}$$

Take $r=3$ and $m=-3$. Then

$$\begin{aligned} V_{(-2,-1,3)} &= Y_3^{-1}(-2,-1,0)=\tau^{-1}T_1^{-1}T_2^{-1}(-2,-1,0)\\ &= (-2,-1,3)+(q-q^{-1})(0,-1,1)+(q-q^{-1})(-2,0,2)\\ &\quad +(q-q^{-1})^2(-1,0,1). \end{aligned}$$

◇

Remark 5.1. Let $n = |m|$. The basis $\{V_\lambda\}$ can be naturally identified with the basis of monomial tensors of a certain $U_q(\widehat{\mathfrak{sl}}_n)$-module (see Section 7.1). ◇

As illustrated by Example 5.1, in some cases the vectors V_λ and λ coincide. This is made more precise in the following

Proposition 5.2. *If $\lambda = m\beta + \gamma$ as above with $\beta \in P^-$ then $V_\lambda = \lambda$. In particular, if $m < 0$ and $\lambda \in P^+$, or $m > 0$ and $\lambda \in P^-$, then $V_\lambda = \lambda$.*

Proof — Put $s = w(\gamma, m)$ and $\nu = s^{-1}\gamma$. Then by (29) and Lemma 2.2

$$V_\lambda = Y^\beta \gamma = T_\beta T_s \nu = T_{y^\beta s}\, \nu.$$

For $\sigma \neq 1$ in $\mathfrak{S}_{\nu,m} \subset \mathfrak{S}_r$ one has $\ell(s\sigma) > \ell(s)$ (because s is minimal in its coset $s\mathfrak{S}_{\nu,m}$) and $s\sigma \in \mathfrak{S}_r$. Hence by Lemma 2.2

$$\ell(y^\beta s\sigma) = \ell(y^\beta) + \ell(s\sigma) > \ell(y^\beta) + \ell(s) = \ell(y^\beta s).$$

Therefore $y^\beta s$ is also minimal in its coset, that is $w(\lambda, m) = y^\beta s$, and

$$V_\lambda = T_{y^\beta s}\, \nu = T_{w(\lambda,m)}\, \nu = \lambda.$$

□

The next proposition gives a key relation between the bar involution and the basis V_λ. It will result from the following

Lemma 5.3. *Let $\beta \in P$ and $s \in \mathfrak{S}_r$. Then*

$$\overline{(Y^\beta T_s)} = T_{w_0}^{-1} Y^{w_0\beta} T_{w_0 s}.$$

Proof — Recall that $\ell(w_0 s) + \ell(s) = \ell(w_0)$, hence $T_{w_0 s} T_{s^{-1}} = T_{w_0}$ and $\overline{T_s} = T_{w_0}^{-1} T_{w_0 s}$. Write $\beta = \beta' - \beta''$ with $\beta', \beta'' \in P^+$. By (28) we have $\overline{Y^\beta} = T_{\beta'} T_{\beta''}^{-1}$. Hence, $\overline{(Y^\beta T_s)} = T_{\beta'} T_{\beta''}^{-1} T_{w_0}^{-1} T_{w_0 s}$. Now, using Lemma 2.2 we see that

$$T_{\beta'} T_{\beta''}^{-1} T_{w_0}^{-1} = T_{\beta'} T_{w_0}^{-1} T_{w_0\beta''}^{-1} = T_{w_0}^{-1} T_{w_0\beta'} T_{w_0\beta''}^{-1}$$

because $\beta', \beta'' \in P^+$. Now, $w_0\beta = (-w_0\beta'') - (-w_0\beta')$, with $-w_0\beta''$, $-w_0\beta' \in P^+$. Hence, using again (28), $\overline{(Y^\beta T_s)} = T_{w_0}^{-1} Y^{w_0\beta} T_{w_0 s}$. □

Proposition 5.4. *Let $\lambda \in P$ and let $\nu \in \mathcal{A}_{r,m}$ be the point congruent to λ. Then*

$$\overline{V_\lambda} = q^{-\ell(w_{0,\nu})} T_{w_0}^{-1} V_{w_0\lambda},$$

where $w_{0,\nu}$ is the longest element in the stabilizer $\mathfrak{S}_{\nu,m}$.

Proof — By Lemma 5.3, $\overline{V_\lambda} = \overline{(Y^\beta T_s)}\,\nu = T_{w_0}^{-1} Y^{w_0\beta} T_{w_0 s}\,\nu$. The minimal length of an element $\sigma \in \mathfrak{S}_r$ such that $\sigma\,\nu = (w_0 s)\nu$ is $\ell(w_0 s) - \ell(w_{0,\nu})$. Hence $T_{w_0 s}\,\nu = q^{-\ell(w_{0,\nu})}(w_0 s)\nu$, and this proves the proposition. □

Example 5.2. Take $m = -2$ and $\lambda = (2,0)$. Then,

$$\begin{aligned} \overline{V_{(2,0)}} &= \overline{Y_1^{-1}}(0,0) = \tau^{-1}T_0^{-1}(0,0) = (2,0) + (q-q^{-1})(0,2), \\ T_1^{-1}V_{(0,2)} &= T_1^{-1}Y_2^{-1}(0,0) = T_1^{-1}\tau^{-1}T_1^{-1}(0,0) = q(2,0) + (q^2-1)(0,2). \end{aligned}$$

◇

5.2. Action of $\widehat{H}_r$ on the basis V_λ

The next lemma allows to compute the action of $\widehat{H}_r$ on $\{V_\lambda\}$.

Lemma 5.5. *Let $i \in \{1,\ldots,r-1\}$ and $k \in \mathbf{Z}$. There holds*

$$T_i Y_i^k = Y_{i+1}^k T_i + (q-q^{-1}) Y_{i+1} \frac{Y_i^k - Y_{i+1}^k}{Y_i - Y_{i+1}}.$$

In other words,

$$T_i Y_i^k = \begin{cases} Y_{i+1}^k T_i + (q-q^{-1}) \sum_{j=1}^{k} Y_i^{k-j} Y_{i+1}^j, & (k \geq 0), \\ Y_{i+1}^k T_i + (q^{-1}-q) \sum_{j=1}^{-k} Y_i^{-j} Y_{i+1}^{j+k}, & (k < 0). \end{cases}$$

Proof — It follows from (24) (20) by a straightforward computation. □

Let $\lambda \in P$ and $1 \leq i \leq r-1$. Write $\lambda = m\beta + \gamma$ with $\beta, \gamma \in P$ and $\gamma \in \mathfrak{S}_r \mathcal{A}_{r,m}$. Then $V_\lambda = (\prod_{j \neq i,i+1} Y_j^{\beta_j})(Y_i Y_{i+1})^{\beta_{i+1}} Y_i^{\beta_i - \beta_{i+1}} V_\gamma$. Since T_i commutes with Y_j ($j \neq i, i+1$) and $Y_i Y_{i+1}$, we have

$$T_i V_\lambda = \left(\prod_{j \neq i,i+1} Y_j^{\beta_j} \right) (Y_i Y_{i+1})^{\beta_{i+1}} T_i Y_i^{\beta_i - \beta_{i+1}} V_\gamma.$$

Thus to compute T_iV_λ we can use the commutation relation of Lemma 5.5 with $k = \beta_i - \beta_{i+1}$ together with the fact that since $V_\gamma = \gamma$, we have

$$T_iV_\gamma = \begin{cases} V_{s_i\gamma} & \text{if } \operatorname{desc}(\gamma, i, m) = \operatorname{sgn}(m), \\ q^{-1}V_{s_i\gamma} & \text{if } \operatorname{desc}(\gamma, i, m) = 0, \\ V_{s_i\gamma} + (q^{-1} - q)V_\gamma & \text{if } \operatorname{desc}(\gamma, i, m) = -\operatorname{sgn}(m). \end{cases}$$

5.3. Projection on the positive Weyl chamber

From now on we fix $n \geq 2$ and we assume that $\widehat{H}_r$ acts on $\mathcal{P}_r$ via Π_{-n}. Introduce the $\mathbf{Z}[q, q^{-1}]$-submodule

$$\mathcal{J}_r := \sum_{i=1}^{r-1} \operatorname{im} C'_i \subset \mathcal{P}_r,$$

and define $\mathcal{F}_r := \mathcal{P}_r/\mathcal{J}_r$. The image of $\lambda \in P$ in $\mathcal{F}_r$ under the natural projection

$$\operatorname{pr} : \mathcal{P}_r \longrightarrow \mathcal{F}_r$$

will be denoted by $[\lambda] = [\lambda_1, \ldots, \lambda_r]$. For $v \in \mathcal{P}_r$ we have by definition

$$\operatorname{pr}(C'_iv) = 0 = \operatorname{pr}(T_iv) + q \operatorname{pr}(v).$$

Hence taking $v = \lambda \in P$, we obtain that if $\lambda_i < \lambda_{i+1}$ then $[\lambda] = -q^{-1}[s_i\lambda]$, and if $\lambda_i = \lambda_{i+1}$ then $[\lambda] = 0$. This implies that a spanning set of $\mathcal{F}_r$ is given by the $[\lambda]$ such that $\lambda_1 > \lambda_2 > \ldots > \lambda_r$. We put $P^{++} := \{\lambda \in P \mid \lambda_1 > \lambda_2 > \ldots > \lambda_r\}$.

Lemma 5.6. *$\{[\lambda] \mid \lambda \in P^{++}\}$ is a basis of $\mathcal{F}_r$.*

Proof — Suppose that $\sum_{\lambda \in P^{++}} a_\lambda[\lambda] = 0$. Then $\sum_{\lambda \in P^{++}} a_\lambda\lambda \in \mathcal{J}_r$. Recall that

$$C_{w_0} = \sum_{s \in \mathfrak{S}_r} (-q)^{\ell(s)-\ell(w_0)}T_s = \overline{C_{w_0}} = \sum_{s \in \mathfrak{S}_r} (-q)^{-\ell(s)+\ell(w_0)}T_s^{-1}$$

satisfies $C_{w_0}C'_i = 0$ $(1 \leq i \leq r-1)$. Hence $\mathcal{J}_r \subset \ker C_{w_0}$. Thus

$$C_{w_0}\Big(\sum_{\lambda \in P^{++}} a_\lambda\lambda\Big) = \sum_{\lambda \in P^{++},\ s \in \mathfrak{S}_r} a_\lambda(-q)^{-\ell(s)+\ell(w_0)}s\lambda = 0,$$

which implies that $a_\lambda = 0$ for all $\lambda \in P^{++}$. □

Note that for $v \in \mathcal{P}_r$, $\overline{C'_iv} = C'_i\overline{v}$. Hence $\overline{\mathcal{J}_r} \subset \mathcal{J}_r$ and one can define a semi-linear involution on $\mathcal{F}_r$ by

$$\overline{\operatorname{pr}(v)} := \operatorname{pr}(\overline{v}) \qquad (v \in \mathcal{P}_r). \tag{46}$$

Let us define

$$|\lambda\rangle := q^{-\ell(w_0)}\,\mathrm{pr}\,(V_\lambda). \tag{47}$$

Then, by Proposition 5.2, for $\lambda \in P^{++}$ we have $|\lambda\rangle = q^{-\ell(w_0)}\,[\lambda]$, so that $\{|\lambda\rangle \mid \lambda \in P^{++}\}$ is also a basis of $\mathcal{F}_r$. The next proposition shows that it is also useful to work with the vectors $|\lambda\rangle$ associated with arbitrary weights $\lambda \in P$, which can be thought of as some q-wedge products (see below Section 7.2).

Proposition 5.7. *For $\lambda \in P$, we have*

$$\overline{|\lambda\rangle} = (-1)^{\ell(w_0)} q^{\ell(w_0)-\ell(w_{0,\nu})}\,|w_0\lambda\rangle.$$

Proof — By Proposition 5.4 we have $\overline{V_\lambda} = q^{-\ell(w_{0,\nu})}T_{w_0}^{-1}V_{w_0\lambda}$. But for all $v \in \mathcal{P}_r$,

$$\mathrm{pr}\,(T_{w_0}^{-1}v) = (-q)^{-\ell(w_0)}\,\mathrm{pr}\,(v).$$

Thus,

$$\begin{aligned}\overline{|\lambda\rangle} = q^{\ell(w_0)}\,\mathrm{pr}\,(\overline{V_\lambda}) &= (-1)^{\ell(w_0)} q^{-\ell(w_{0,\nu})}\mathrm{pr}\,(V_{w_0\lambda}) \\ &= (-1)^{\ell(w_0)} q^{\ell(w_0)-\ell(w_{0,\nu})}\,|w_0\lambda\rangle.\end{aligned}$$

□

Remark 5.8. It is easy to check that the exponent $\ell(w_0)-\ell(w_{0,\nu})$ of q is equal to the number of pairs (i,j) with $1 \le i < j \le r$ such that $\lambda_i - \lambda_j$ is not divisible by n. ◇

The next proposition gives a set of straightening rules to express an element $|\mu\rangle$ with $\mu \notin P^{++}$ on the basis $\{|\lambda\rangle \mid \lambda \in P^{++}\}$.

Proposition 5.9. *Let $\mu \in P$ be such that $\mu_i < \mu_{i+1}$. Write $\mu_{i+1} = \mu_i + kn + j$ with $k \ge 0$ and $0 \le j < n$. Then*

$$|\mu\rangle = -|s_i\mu\rangle \quad \textit{if } j = 0, \tag{48}$$

$$|\mu\rangle = -q^{-1}|s_i\mu\rangle \quad \textit{if } k = 0, \tag{49}$$

$$|\mu\rangle = -q^{-1}|s_i\mu\rangle - |y_i^{-k}y_{i+1}^{k}\mu\rangle - q^{-1}|y_i^{k}y_{i+1}^{-k}s_i\mu\rangle \quad \textit{otherwise.} \tag{50}$$

Proof — To simplify the notation, let us write $l = \mu_i$ and $m = \mu_{i+1}$. Since the relations only involve components i and $i+1$ we shall also use the shorthand notations (k,l) and $|k,l\rangle$ in place of $V_{(\mu_1,\ldots,\mu_{i-1},k,l,\mu_{i+2},\ldots,\mu_r)} \in \mathcal{P}_r$ and $|(\mu_1,\ldots,\mu_{i-1},k,l,\mu_{i+2},\ldots,\mu_r)\rangle \in \mathcal{F}_r$.

Suppose $j = 0$. It follows from Section 5.2 that $T_i(l,l) = q^{-1}(l,l)$. Hence $(l,l) \in \operatorname{im} C'_i$. Since $(Y_i^{-k}+Y_{i+1}^{-k})C'_i = C'_i(Y_i^{-k}+Y_{i+1}^{-k})$ we also have $(Y_i^{-k}+Y_{i+1}^{-k})(l,l) = (m,l)+(l,m) \in \operatorname{im} C'_i$, and thus $|l,m\rangle + |m,l\rangle = 0$.

Suppose $k = 0$. Then $T_i(l,m) = (m,l)$ by Section 5.2, and $C'_i(l,m) = (m,l) + q(l,m) \in \operatorname{im} C'_i$, which gives $|l,m\rangle = -q^{-1}|m,l\rangle$.

Finally suppose that $j,k > 0$. By the previous case $(m, l+kn) + q(l+kn, m) \in \operatorname{im} C'_i$. Applying $Y_i^k + Y_{i+1}^k$ we get that $(m,l) + (m-kn, l+kn) + q(l,m) + q(l+kn, m-kn) \in \operatorname{im} C'_i$, which gives the third claim. □

Example 5.3. Take $r = 2$ and $n = 2$. Then

$$|1,4\rangle = -q^{-1}\,|4,1\rangle - |3,2\rangle - q^{-1}\,|2,3\rangle,$$

by Eq. (50), and $|2,3\rangle = -q^{-1}\,|3,2\rangle$ by Eq. (49). Thus

$$|1,4\rangle = -q^{-1}\,|4,1\rangle + (q^{-2}-1)\,|3,2\rangle.$$

Hence, by Proposition 5.7, $\overline{|4,1\rangle} = |4,1\rangle + (q-q^{-1})\,|3,2\rangle$. ◇

For $\mu \in P^{++}$ write $\overline{|\mu\rangle} = \sum_{\lambda\in P^{++}} a_{\lambda\mu}(q)\,|\lambda\rangle$. Using Proposition 5.7 and Proposition 5.9, we easily see that the coefficients $a_{\lambda\mu}(q)$ satisfy the following properties

Corollary 5.10. (i) *The coefficients $a_{\lambda\mu}(q)$ are invariant under translation of λ and μ by $\epsilon_1 + \cdots + \epsilon_r$. Hence it is enough to describe the $a_{\lambda\mu}(q)$ for which $\lambda-\rho$ and $\mu-\rho$ have non-negative components, i.e. $\lambda-\rho$ and $\mu-\rho$ are partitions.*

(ii) *If $a_{\lambda\mu}(q) \neq 0$ then $\lambda \in \widetilde{\mathfrak{S}}_r\mu$. In particular, if $\lambda-\rho$ and $\mu-\rho$ are partitions, they are partitions of the same integer k.*

(iii) *The matrix $\mathbf{A}_k$ with entries the $a_{\lambda\mu}(q)$ for which $\lambda-\rho$ and $\mu-\rho$ are partitions of k is lower unitriangular if the columns and rows are indexed in decreasing lexicographic order.*

Example 5.4. For $n = 2$ and $r = 3$, the matrices $\mathbf{A}_k$ for $k = 2,3,4$ are

(4,1,0)	(3,2,0)
1	0
$q-q^{-1}$	1

(5,1,0)	(4,2,0)	(3,2,1)
1	0	0
0	1	0
$q-q^{-1}$	0	1

(6,1,0)	(5,2,0)	(4,3,0)	(4,2,1)
1	0	0	0
$q-q^{-1}$	1	0	0
$q^{-2}-1$	$q-q^{-1}$	1	0
0	q^2-1	$q-q^{-1}$	1

◇

5.4. Canonical bases of $\mathcal{F}_r$

Let $\mathcal{L}^+$ (resp. $\mathcal{L}^-$) be the $\mathbf{Z}[q]$ (*resp.* $\mathbf{Z}[q^{-1}]$)-lattice in $\mathcal{F}_r$ with basis $\{|\lambda\rangle \mid \lambda \in P^{++}\}$. The fact that the matrix of the bar involution is unitriangular on the basis $\{|\lambda\rangle \mid \lambda \in P^{++}\}$ implies by a classical argument (see [32], 7.10 and [6]) that

Theorem 5.11. *There exist bases $\{G^+_\lambda \mid \lambda \in P^{++}\}$, $\{G^-_\lambda \mid \lambda \in P^{++}\}$ of $\mathcal{F}_r$ characterized by:*

(i) $\overline{G^+_\lambda} = G^+_\lambda$, $\overline{G^-_\lambda} = G^-_\lambda$,

(ii) $G^+_\lambda \equiv |\lambda\rangle \bmod q\mathcal{L}^+$, $G^-_\lambda \equiv |\lambda\rangle \bmod q^{-1}\mathcal{L}^-$.

These bases were introduced in [22] (in the limit $r \to \infty$, *cf.* Section 7), using Proposition 5.7 as the definition of the bar involution on $\mathcal{F}$. Set

$$G^+_\mu = \sum_\lambda c_{\lambda,\mu}(q)\,|\lambda\rangle\,, \qquad G^-_\lambda = \sum_\mu l_{\lambda,\mu}(-q^{-1})\,|\mu\rangle\,.$$

Let $\mathbf{C}_k$ and $\mathbf{L}_k$ denote respectively the matrices with entries the coefficients $c_{\lambda\mu}(q)$ and $l_{\lambda\mu}(q)$ for which $\lambda-\rho$ and $\mu-\rho$ are partitions of k.

Example 5.5. For $r=3$ and $n=2$ we have

$\mathbf{C}_4 =$

(6,1,0)	(5,2,0)	(4,3,0)	(4,2,1)
1	0	0	0
q	1	0	0
0	q	1	0
q	q^2	q	1

$\mathbf{L}_4 =$

(6,1,0)	(5,2,0)	(4,3,0)	(4,2,1)
1	q	q^2	0
0	1	q	0
0	0	1	q
0	0	0	1

◇

Clearly, if $c_{\lambda,\mu}$ or $l_{\lambda,\mu} \neq 0$, then λ and μ lie on the same orbit under $\widehat{\mathfrak{S}}_r$. Let ν be the point of $\mathcal{A}_{r,-n}$ on this orbit. Write $\widehat{w}_\lambda := w(w_0\lambda, -n)w_{0,\nu}$ and similarly $\widehat{w}_\mu := w(w_0\mu, -n)w_{0,\nu}$. The main result of this section is

Theorem 5.12 (Varagnolo, Vasserot [42]). *With the above notation, we have*

$$l_{\lambda,\mu} = P^-_{\mu,\lambda}, \tag{51}$$

a parabolic Kazhdan-Lusztig polynomial for the action of $\widehat{\mathfrak{S}}_r$ *on* P_r *via* π_{-n}, *and*

$$c_{\lambda,\mu} = \sum_{s\in\mathfrak{S}_r} (-q)^{\ell(s)} P_{s\widehat{w}_\lambda\,,\,\widehat{w}_\mu}. \tag{52}$$

Remark 5.13. (i) In view of Theorem 2.4, it follows from Eq. (52) that $c_{\lambda,\mu}$ is also a parabolic Kazhdan-Lusztig polynomial of negative type with respect to the parabolic subgroup $\mathfrak{S}_r$ of $\widehat{\mathfrak{S}}_r$, (but for the right $\widehat{H}_r$-module $\mathbf{1}_{q^{-1}} \otimes_{H_r} \widehat{H}_r$). This agrees with the expression obtained by Goodman and Wenzl when $\mu - \rho$ is a n-regular partition [10].
(ii) Let $\overline{\mathcal{F}}_r$ denote the specialization of $\mathcal{F}_r$ at $q = 1$. Define a $\mathbf{Z}$-linear map ι from the Grothendieck group of finite-dimensional representations of $U_\zeta(\mathfrak{gl}_r)$ to $\overline{\mathcal{F}}_r$ by

$$\iota[W(\lambda)] = |\lambda + \rho\rangle \qquad (\lambda \in P_r^+).$$

Then comparing Theorem 5.12 and the Lusztig conjecture (33) we see that $\iota[L(\lambda)] = G^-_{\lambda+\rho}$.

Proof — Consider the element $D_\lambda := \mathrm{pr}\,(C^-_\lambda) \in \mathcal{F}_r$. Then $\overline{D_\lambda} = D_\lambda$ by (46). Since $\lambda \in P^{++}$, $\mathrm{desc}(\lambda, i, -n) = 1$ for all $i = 1, \ldots, r-1$. Therefore using (31) we see that

$$D_\lambda = [r]! \sum_{\mu\in P^{++}} P^-_{\mu,\lambda}(-q^{-1})\,|\mu\rangle.$$

Hence $(1/[r]!)D_\lambda$ is bar invariant and congruent to $|\lambda\rangle$ modulo $q^{-1}\mathcal{L}^-$. Thus $D_\lambda = [r]!\,G^-_\lambda$ and (51) is proved.

Next put $E_\mu := \mathrm{pr}\,(C^+_{w_0\mu}) \in \mathcal{F}_r$. Then $\overline{E_\mu} = E_\mu$. We have

$$E_\mu = \mathrm{pr}\left(\sum_{\alpha\in\widehat{\mathfrak{S}}_r\nu} P^+_{\alpha,w_0\mu}\,\alpha\right) = \sum_{\lambda\in P^{++}} \left(\sum_{s\in\mathfrak{S}_r} (-q)^{-\ell(s)} P^+_{s\lambda,w_0\mu}\right) q^{\ell(w_0)}\,|\lambda\rangle.$$

This shows that $E_\mu \equiv (-1)^{\ell(w_0)}\,|\mu\rangle \bmod q\mathcal{L}^+$. Hence, $E_\mu = (-1)^{\ell(w_0)}\,G^+_\mu$. It follows that

$$\begin{aligned} c_{\lambda,\mu} &= \sum_{s\in\mathfrak{S}_r} (-q)^{\ell(w_0)-\ell(s)} P^+_{s\lambda,w_0\mu} \\ &= \sum_{\sigma\in\mathfrak{S}_r} (-q)^{\ell(\sigma)} P_{w(\sigma w_0\lambda,-n)w_{0,\nu}\,,\,w(w_0\mu,-n)w_{0,\nu}} \end{aligned}$$

by Theorem 2.4. Finally, since $w_0\lambda \in P^{--}$ we have $w(\sigma w_0\lambda,-n) = \sigma w(w_0\lambda,-n)$ for all $\sigma\in\mathfrak{S}_r$, and we get (52). □

§6. A q-analogue of the tensor product theorem

6.1. Action of $Z(\widehat{H}_r)$ on $\mathcal{F}_r$

By a result of Bernstein (see [28], Th. 8.1), the center $Z(\widehat{H}_r)$ of $\widehat{H}_r$ is the algebra of symmetric Laurent polynomials in the elements Y_i. Clearly, $Z(\widehat{H}_r)$ leaves invariant the submodule $\mathcal{J}_r$. It follows that $Z(\widehat{H}_r)$ acts on $\mathcal{F}_r = \mathcal{P}_r/\mathcal{J}_r$. This action can be computed via (45) and (47). In particular $B_k = \sum_{i=1}^r Y_i^k$ acts by

$$B_k\,|\lambda\rangle = \sum_{j=1}^{r} |\lambda - nk\epsilon_j\rangle, \qquad (k\in\mathbf{Z}^*). \tag{53}$$

Note that the right-hand side of (53) may involve terms $|\mu\rangle$ with $\mu\notin P^+$ which have to be expressed on the basis $\{|\lambda\rangle \mid \lambda\in P^{++}\}$ by repeated applications of Proposition 5.9.

Example 6.1. Take $r=4$ and $n=2$. We have

$$B_{-2}\,|3,2,1,0\rangle = |7,2,1,0\rangle + |3,6,1,0\rangle + |3,2,5,0\rangle + |3,2,1,4\rangle.$$

By Proposition 5.9,

$$\begin{aligned} &|3,6,1,0\rangle = -q^{-1}\,|6,3,1,0\rangle + (q^{-2}-1)\,|5,4,1,0\rangle, \\ &|3,2,5,0\rangle = -q^{-1}\,|3,5,2,0\rangle + (q^{-2}-1)\,|3,4,3,0\rangle = q^{-1}\,|5,3,2,0\rangle, \\ &|3,2,1,4\rangle = -q^{-1}\,|3,2,4,1\rangle + (q^{-2}-1)\,|3,2,3,2\rangle = -q^{-2}\,|4,3,2,1\rangle, \end{aligned}$$

which yields

$$\begin{aligned} B_{-2}\,|3,2,1,0\rangle &= |7,2,1,0\rangle - q^{-1}|6,3,1,0\rangle \\ &\quad + (q^{-2}-1)|5,4,1,0\rangle + q^{-1}|5,3,2,0\rangle - q^{-2}|4,3,2,1\rangle. \end{aligned}$$

◇

The compatibility of the bar involution with this action is given by the next

Proposition 6.1. *For $u \in \mathcal{F}_r$ and $z \in Z(\widehat{H}_r)$ one has*

$$\overline{zu} = z\overline{u}.$$

Proof — Since z is a symmetric Laurent polynomial in the Y_i, we see using Lemma 5.3 that $\overline{z} = T_{w_0}^{-1} z T_{w_0} = z$. □

6.2. Action of $Z(\widehat{H}_r)$ and ribbon tableaux

We shall now show that the straightening relations can be avoided provided that one uses appropriate linear bases of $Z(\widehat{H}_r)$. For $d \in [1,r] := \{1,2,\ldots,r\}$ and $m \in \mathbf{N}^*$ define

$$\widetilde{\mathcal{U}}_d := \sum_{1\le i_1<i_2<\ldots<i_d\le r} Y_{i_1} Y_{i_2} \cdots Y_{i_d}, \tag{54}$$

$$\widetilde{\mathcal{V}}_d := \sum_{1\le i_1<i_2<\ldots<i_d\le r} Y_{i_1}^{-1} Y_{i_2}^{-1} \cdots Y_{i_d}^{-1}, \tag{55}$$

$$\mathcal{U}_m := \sum_{1\le i_1\le i_2\le\ldots\le i_m\le r} Y_{i_1} Y_{i_2} \cdots Y_{i_m}, \tag{56}$$

$$\mathcal{V}_m := \sum_{1\le i_1\le i_2\le\ldots\le i_m\le r} Y_{i_1}^{-1} Y_{i_2}^{-1} \cdots Y_{i_m}^{-1}. \tag{57}$$

For $\alpha \in [1,r]^s$ set $\widetilde{\mathcal{U}}_\alpha := \widetilde{\mathcal{U}}_{\alpha_1} \cdots \widetilde{\mathcal{U}}_{\alpha_s}$, $\widetilde{\mathcal{V}}_\alpha := \widetilde{\mathcal{V}}_{\alpha_1} \cdots \widetilde{\mathcal{V}}_{\alpha_s}$, and for $\beta \in \mathbf{N}^{*s}$ set $\mathcal{U}_\beta := \mathcal{U}_{\beta_1} \cdots \mathcal{U}_{\beta_s}$, $\mathcal{V}_\beta := \mathcal{V}_{\beta_1} \cdots \mathcal{V}_{\beta_s}$. In other words, using the notation of [33] for symmetric functions,

$$\widetilde{\mathcal{U}}_\alpha = e_\alpha(Y_1,\ldots,Y_r), \qquad \widetilde{\mathcal{V}}_\alpha = e_\alpha(Y_1^{-1},\ldots,Y_r^{-1}),$$
$$\mathcal{U}_\beta = h_\beta(Y_1,\ldots,Y_r), \qquad \mathcal{V}_\beta = h_\beta(Y_1^{-1},\ldots,Y_r^{-1}).$$

The following formulas were obtained in [21]. They will allow us to relate ribbon tableaux to Kazhdan-Lusztig polynomials. Put

$$L^{(n)}_{\lambda/\nu\,,\,\mu}(q) := \sum_{T\in \mathrm{Tab}_n(\lambda/\nu\,,\,\mu)} q^{\mathrm{spin}(T)}. \tag{58}$$

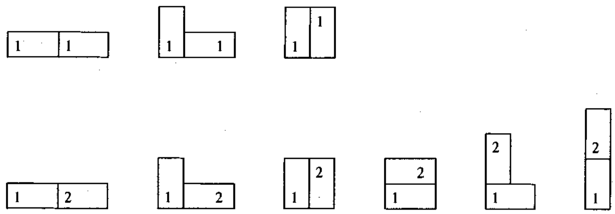

Fig. 6. The domino tableaux of weight (2) and (1,1)

Theorem 6.2. *Let $\nu \in \mathbb{P}_r^+$ and $\alpha \in [1,r]^s$. Set $k = |\alpha| := \alpha_1 + \cdots + \alpha_s$. We have*

$$\widetilde{\mathcal{U}}_\alpha \, |\nu+\rho\rangle = (-q)^{-(n-1)k} \sum_{\mu \in \mathbb{P}_r^+} L^{(n)}_{\nu'/\mu',\alpha}(-q) \; |\mu+\rho\rangle, \tag{59}$$

$$\widetilde{\mathcal{V}}_\alpha \, |\nu+\rho\rangle = (-q)^{-(n-1)k} \sum_{\lambda \in \mathbb{P}_r^+} L^{(n)}_{\lambda'/\nu',\alpha}(-q) \; |\lambda+\rho\rangle, \tag{60}$$

where for $\lambda \in \mathbb{P}_r^+$, λ' denotes the conjugate partition.

Note that in (59) (60) λ', μ', ν' may be partitions of length $s > r$.

Example 6.2. Let us redo the calculation of Example 6.1 using domino tableaux. Clearly, $B_{-2} = \widetilde{\mathcal{V}}_{(1,1)} - 2\widetilde{\mathcal{V}}_{(2)}$. Now, applying Theorem 6.2 we have

$$\begin{aligned} \widetilde{\mathcal{V}}_{(2)}\,|\rho\rangle &= q^{-2}\,|(1,1,1,1)+\rho\rangle - q^{-1}\,|(2,1,1)+\rho\rangle + |(2,2)+\rho\rangle, \\ \widetilde{\mathcal{V}}_{(1,1)}\,|\rho\rangle &= q^{-2}\,|(1,1,1,1)+\rho\rangle - q^{-1}\,|(2,1,1)+\rho\rangle \\ &\quad +(1+q^{-2})\,|(2,2)+\rho\rangle - q^{-1}|(3,1)+\rho\rangle + |(4)+\rho\rangle \end{aligned}$$

as illustrated by Figure 6, and we recover the result of Example 6.1. ◇

The proof of Theorem 6.2 is based on the following simple combinatorial lemma.

Lemma 6.3. *Let λ, $\nu \in \mathbb{P}_r^+$ and $k \in [1,r]$. Put $\beta = \epsilon_1 + \cdots + \epsilon_k$. The skew Young diagram λ'/ν' is a horizontal n-ribbon strip of weight k if and only if there exist $s, \sigma \in \mathfrak{S}_r$ such that $\nu + \rho + s(n\beta) = \sigma(\lambda+\rho)$. If this is the case,*

$$\ell(\sigma) = (n-1)k - \mathrm{spin}(\lambda'/\nu').$$

Proof — The proof is elementary and is left to the reader. □

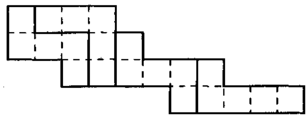

Fig. 7. A horizontal 5-ribbon strip of weight 4 and spin 7

Example 6.3. Take $r = 11$, $\lambda = (4,4,4,4,3,2,2,2,1,1,1)$ and $\nu = (2,2,1,1,1,1)$. Then $\lambda'/\nu' = (11,8,5,4)/(6,2)$ is a horizontal 5-ribbon strip of weight 4. Indeed

$$(12,11,14,13,7,6,9,3,2,1,5) = \nu + \rho + (0,0,5,5,0,0,5,0,0,0,5)$$

is a permutation of $\lambda + \rho$. This permutation has length 9, thus the spin of λ'/ν' is equal to $4.4 - 9 = 7$, as can be checked on Figure 7. ◇

Proof of Theorem 6.2— Since $\widetilde{\mathcal{V}}_\alpha := \widetilde{\mathcal{V}}_{\alpha_1} \cdots \widetilde{\mathcal{V}}_{\alpha_s}$, it is enough to prove the theorem in the case $\alpha = (k)$. Let $\beta = \epsilon_1 + \cdots + \epsilon_k$. Observe that we can reformulate (55) as $\widetilde{\mathcal{V}}_k = \sum_{\zeta \in \mathfrak{S}_r \beta} Y^{-\zeta}$. Hence we have

$$\widetilde{\mathcal{V}}_k \, |\nu + \rho\rangle = \sum_{\gamma \in \mathfrak{S}_r n\beta} |\nu + \rho + \gamma\rangle .$$

If $\xi := \nu + \rho + \gamma \notin P^{++}$ we have to use the straightening relations of Proposition 5.9 to express $|\xi\rangle$ on the basis $\{|\lambda\rangle \mid \lambda \in P^{++}\}$. But if $\xi_i < \xi_{i+1}$ then clearly we must have $\xi_i < \xi_{i+1} < \xi_i + n$, and we need only the simple relation (49). It follows that $|\xi\rangle = (-q)^{-\ell(\sigma)} |\lambda + \rho\rangle$, where $\lambda + \rho$ is the decreasing reordering of ξ and σ is the permutation mapping ξ into $\lambda + \rho$. By Lemma 6.3, $\ell(\sigma) = (n-1)k - \mathrm{spin}(\lambda'/\nu')$ and we are done. The proof for $\widetilde{\mathcal{U}}_k$ is similar. □

We now deduce from Theorem 6.2 similar formulas for the operators $\mathcal{U}_\beta$ and $\mathcal{V}_\beta$.

Theorem 6.4. *Let $\nu \in \mathbb{P}_r^+$ and $\beta \in \mathbf{N}^{*s}$. We have*

$$\mathcal{U}_\beta \, |\nu + \rho\rangle = \sum_{\mu \in \mathbb{P}_r^+} L^{(n)}_{\nu/\mu,\,\beta}(-q^{-1}) \; |\mu + \rho\rangle , \tag{61}$$

$$\mathcal{V}_\beta \, |\nu + \rho\rangle = \sum_{\lambda \in \mathbb{P}_r^+} L^{(n)}_{\lambda/\nu,\,\beta}(-q^{-1}) \; |\lambda + \rho\rangle . \tag{62}$$

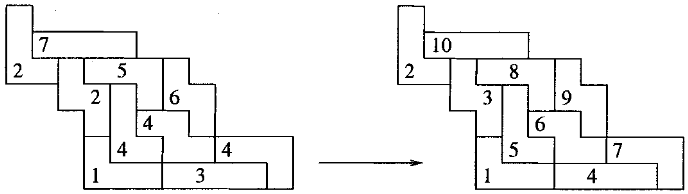

Fig. 8. Standardization $T \to \mathcal{T}$ of a ribbon tableau

Proof — Again, it is enough to prove this for $\beta = (k)$. Recall that a composition of $k \in \mathbf{N}$ is an ordered partition of k, that is, a sequence $\alpha = (\alpha_1, \dots, \alpha_s)$ of positive integers such that $\sum_i \alpha_i = k$. We denote this by $\alpha \models k$ and we call s the length $l(\alpha)$ of α. There is a classical formula for expressing the symmetric function h_k in terms of the e_α, namely

$$h_k = \sum_{\alpha \models k} (-1)^{k-l(\alpha)} e_\alpha .$$

Thus by Theorem 6.2, we have

$$\mathcal{V}_k \,|\nu + \rho\rangle = (-q)^{-(n-1)k} \sum_\lambda \left(\sum_{\alpha \models k} (-1)^{k-l(\alpha)} L^{(n)}_{\lambda'/\nu',\alpha}(-q) \right) |\lambda + \rho\rangle .$$

Recall that for a ribbon tableau T, $(-1)^{\mathrm{spin}(T)} = \varepsilon_n(\lambda/\nu)$ depends only on the shape λ/ν of T. It is clear that $\varepsilon_n(\lambda'/\nu') = (-1)^{(n-1)k} \varepsilon_n(\lambda/\nu)$, hence we are reduced to prove that

$$q^{-(n-1)k} \sum_{\alpha \models k} (-1)^{k-l(\alpha)} L^{(n)}_{\lambda'/\nu',\alpha}(q) = L^{(n)}_{\lambda/\nu,k}(q^{-1}) .$$

To do this, we associate with each ribbon tableau T of weight α a standard ribbon tableau $\mathcal{T}$ of weight $(1, \dots, 1)$ as follows. Consider two ribbons R and R' of T, numbered i and i' respectively. We say that $R < R'$ if $i < i'$, or $i = i'$ and R is to the left of R'. Clearly this is a total order. Now $\mathcal{T}$ is the tableau with the same shape and inner ribbon structure as T, whose ribbons are numbered $1, 2, 3, \dots$ in the previous total order (see Figure 8).

Let us fix a skew shape λ'/ν' and consider the set $\mathcal{E}$ of all ribbon tableaux of this shape and of arbitrary weight $\alpha \models k$. For $T \in \mathcal{E}$ of weight α, write $\epsilon(T) := (-1)^{k-l(\alpha)}$. We want to prove that

(63)
$$\sum_{T\in\mathcal{E}} \epsilon(T)\, q^{\mathrm{spin}(T)} = \begin{cases} q^{(n-1)k}\, L^{(n)}_{\lambda/\nu\,,\,k}(q^{-1}) & \text{if } \lambda/\nu \text{ is a horizontal } n\text{-ribbon strip,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{T} \in \mathcal{E}$ be a standard tableau, and let $\mathcal{E}_{\mathcal{T}} \subset \mathcal{E}$ denote the subset consisting of those tableaux T whose standardization gives $\mathcal{T}$. We say that $\mathcal{T}$ is a column if for all $i = 1, \dots, k-1$ the ribbon R_{i+1} numbered $i+1$ lies above the ribbon R_i numbered i, that is, if the origin of R_{i+1} lies in a row strictly above the origin of R_i. Eq. (63) will follow from the more precise statement

$$(64) \qquad \sum_{T\in\mathcal{E}_{\mathcal{T}}} \epsilon(T)\, q^{\mathrm{spin}(T)} = \begin{cases} q^{\mathrm{spin}(\mathcal{T})} & \text{if } \mathcal{T} \text{ is a column,} \\ 0 & \text{otherwise.} \end{cases}$$

Now this is very easy. First, by definition all $T \in \mathcal{E}_{\mathcal{T}}$ have the same inner ribbon structure, hence the same spin, and we can simplify the powers of q of both sides of (64). Then we only have to observe that a tableau $T \in \mathcal{E}_{\mathcal{T}}$ is specified by the numbering of its ribbons, *i.e.* by a map $f_T : [1,k] \longrightarrow [1,k]$ satisfying

(i) $f_T(i+1) = f_T(i)$ or $f_T(i+1) = f_T(i) + 1$,
(ii) if R_{i+1} lies above R_i in $\mathcal{T}$ then $f_T(i+1) = f_T(i)+1$.

Let $a(\mathcal{T})$ be the number of i's such that R_{i+1} is not above R_i. Then clearly $|\mathcal{E}_{\mathcal{T}}| = 2^{a(\mathcal{T})}$ and more precisely the number of $T \in \mathcal{E}_{\mathcal{T}}$ such that $f_T(k) = j$ (*i.e.* $\epsilon(T) = (-1)^{k-j}$) is equal to $\binom{a(\mathcal{T})}{j}$. Hence by the binomial theorem

$$\sum_{T\in\mathcal{E}_{\mathcal{T}}} \epsilon(T) = \begin{cases} 1 & \text{if } a(\mathcal{T}) = 0, \text{ i.e. } \mathcal{T} \text{ is a column,} \\ 0 & \text{otherwise.} \end{cases}$$

To finish the proof we need only note that λ/ν is a horizontal n-ribbon strip if and only if there exists a (necessarily unique) column tableau $\mathcal{T}$ of shape λ'/ν', and in this case $\mathrm{spin}(\mathcal{T}) = (n-1)k - \mathrm{spin}(\lambda/\nu)$. □

Remark 6.5. Since the $\mathcal{V}_m$ commute, $\mathcal{V}_\beta$ is invariant under permutation of β. Hence Theorem 6.4 implies that $L^{(n)}_{\lambda/\nu\,,\,\beta}(q)$ is also invariant under permutation of β. This proves that the polynomial (42) is indeed symmetric. ◇

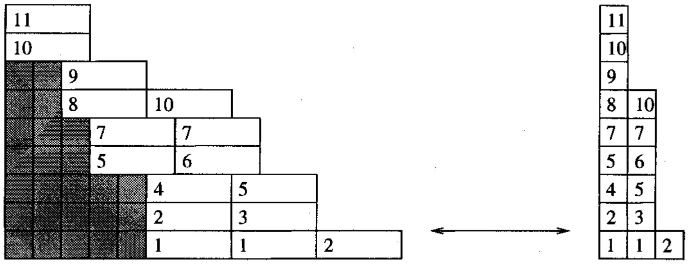

Fig. 9. Correspondence between n-ribbon tableaux of spin 0 and n-restricted inner shape, and ordinary tableaux

6.3. Action of $Z(\widehat{H}_r)$ on the canonical basis $\{G^-_\lambda\}$

For $\lambda \in \mathbb{P}^+_r$, define

$$S_\lambda := s_\lambda(Y_1^{-1}, \dots, Y_r^{-1}) \in Z(\widehat{H}_r),$$

where s_λ is the Schur function. The following theorem is a formal analogue of Theorem 3.2.

Theorem 6.6. *Let $\lambda \in P^+$. Write $\lambda = \lambda^{(0)} + n\lambda^{(1)}$, where $\lambda^{(0)}$ is n-restricted, that is,*

$$0 \le \lambda_i^{(0)} - \lambda_{i+1}^{(0)} < n \qquad (1 \le i \le r-1),$$

and $\lambda_r^{(1)} \ge 0$. Then $G^-_{\lambda+\rho} = S_{\lambda^{(1)}}\, G^-_{\lambda^{(0)}+\rho}$.

Proof — By definition of the basis G^-, we have to prove that $F_\lambda := S_{\lambda^{(1)}}\, G^-_{\lambda^{(0)}+\rho}$ satisfies

$$\overline{F_\lambda} = F_\lambda \qquad \text{and} \qquad F_\lambda \equiv |\lambda + \rho\rangle \quad \mathrm{mod}\ q^{-1}\mathcal{L}^-.$$

The first property is clear by Proposition 6.1. Indeed, S_λ is a $\mathbf{Q}$-linear combination of products of elements B_{-i}. To prove the second one we observe that by Theorem 6.4 for all $\nu \in \mathbb{P}^+_r$ and $\alpha \in \mathbf{N}^{*s}$, $\mathcal{V}_\alpha\,|\nu+\rho\rangle \in \mathcal{L}^-$. Since $G^-_{\lambda^{(0)}+\rho} \equiv |\lambda^{(0)} + \rho\rangle$ mod $q^{-1}\mathcal{L}^-$, and $S_{\lambda^{(1)}}$ is a $\mathbf{Z}$-linear combination of operators $\mathcal{V}_\alpha$ we thus have

$$F_\lambda \equiv S_{\lambda^{(1)}}\,|\lambda^{(0)} + \rho\rangle \quad \mathrm{mod}\ q^{-1}\mathcal{L}^-.$$

In fact Theorem 6.4 implies

$$\mathcal{V}_\alpha\,|\nu+\rho\rangle \equiv \sum_T |\mathrm{sh}(T) + \rho\rangle \quad \mathrm{mod}\ q^{-1}\mathcal{L}^-,$$

where the sum is over the n-ribbon tableaux of weight α, spin 0 and inner shape ν, and $\mathrm{sh}(T)$ stands for the outer shape of T. Therefore for all α

$$\mathcal{V}_\alpha\,|\lambda^{(0)}+\rho\rangle \equiv \sum_{T'} |\mathrm{sh}(T')+\rho\rangle \mod q^{-1}\mathcal{L}^-,$$

where the sum is over the n-ribbon tableaux T' of weight α with inner shape $\lambda^{(0)}$ whose ribbons are all horizontal. Now $\lambda^{(0)}$ being n-restricted, there is an obvious bijection between the set of these tableaux T' and the set $\mathrm{Tab}\,(\cdot,\alpha)$ of ordinary Young tableaux of weight α (see Figure 9). Hence, for all α

$$\mathcal{V}_\alpha\,|\lambda^{(0)}+\rho\rangle \equiv \sum_\beta |\mathrm{Tab}\,(\beta,\alpha)|\,|\lambda^{(0)}+n\beta+\rho\rangle \mod q^{-1}\mathcal{L}^-.$$

Comparing with the well-known formula $h_\alpha = \sum_\beta |\mathrm{Tab}\,(\beta,\alpha)|\, s_\beta$ which yields

$$\mathcal{V}_\alpha = \sum_\beta |\mathrm{Tab}\,(\beta,\alpha)|\, S_\beta,$$

we deduce that for all β,

$$S_\beta\,|\lambda^{(0)}+\rho\rangle \equiv |\lambda^{(0)}+n\beta+\rho\rangle \mod q^{-1}\mathcal{L}^-,$$

and putting $\beta=\lambda^{(1)}$ we are done. □

6.4. Proof of Theorem 4.1

Let us write in the ring of symmetric functions $s_\lambda = \sum_\nu \kappa_{\lambda,\nu}\, h_\nu$. Then we also have $m_\nu = \sum_\lambda \kappa_{\lambda,\nu}\, s_\lambda$. Hence

$$G(\mu^{(0)},\ldots,\mu^{(n-1)};q) := \sum_\nu L^{(n)}_{\mu,\nu}(q)\, m_\nu = \sum_\lambda \left(\sum_\nu \kappa_{\lambda,\nu}\, L^{(n)}_{\mu,\nu}(q)\right) s_\lambda,$$

which gives

$$c^\lambda_{\mu^{(0)},\ldots,\mu^{(n-1)}}(q) = \sum_\nu \kappa_{\lambda,\nu}\, L^{(n)}_{\mu,\nu}(q).$$

On the other hand, by Theorem 6.6 and Theorem 5.12 we have

$$S_\lambda\,|\rho\rangle = G^-_{n\lambda+\rho} = \sum_\mu P^-_{\mu+\rho,n\lambda+\rho}(-q^{-1})\,|\mu+\rho\rangle.$$

Finally, using Theorem 6.4 we get

$$S_\lambda\,|\rho\rangle = \sum_\nu \kappa_{\lambda,\nu}\,\mathcal{V}_\nu\,|\rho\rangle = \sum_\mu \left(\sum_\nu \kappa_{\lambda,\nu}\,L^{(n)}_{\mu,\nu}(-q^{-1})\right)|\mu+\rho\rangle$$
$$= \sum_\mu c^\lambda_{\mu^{(0)},\dots,\mu^{(n-1)}}(-q^{-1})\,|\mu+\rho\rangle,$$

and by comparing the coefficients of $|\mu+\rho\rangle$ we have

$$c^\lambda_{\mu^{(0)},\dots,\mu^{(n-1)}}(q) = P^-_{\mu+\rho,n\lambda+\rho}(q).$$

□

§7. An inversion formula for Kazhdan-Lusztig polynomials

In this section we extend the coefficients to $\mathbf{Q}(q)$ and work with

$$\mathbf{P}_r := \mathbf{Q}(q)\otimes_{\mathbf{Z}[q,q^{-1}]}\mathcal{P}_r, \quad \mathbf{F}_r := \mathbf{Q}(q)\otimes_{\mathbf{Z}[q,q^{-1}]}\mathcal{F}_r,$$

$$\widehat{\mathbf{H}}_r := \mathbf{Q}(q)\otimes_{\mathbf{Z}[q,q^{-1}]}\widehat{H}_r.$$

7.1. Action of $U_q(\widehat{\mathfrak{sl}}_n)$ on the weight lattice of $\mathfrak{gl}_r$

Let $U_q(\widehat{\mathfrak{sl}}_n)$ be the quantum enveloping algebra of the affine Lie algebra $\widehat{\mathfrak{sl}}_n$. This is a $\mathbf{Q}(q)$-algebra with generators $e_i,\ f_i,\ q^{\pm h_i}$ $(0 \le i \le n-1)$. The standard relations can be found for example in [20] and will be omitted. There is a canonical involution $x \mapsto \overline{x}$ of $U_q(\widehat{\mathfrak{sl}}_n)$ defined as the unique ring homomorphism such that $\overline{q} = q^{-1}$, $\overline{e_i} = e_i$, and $\overline{f_i} = f_i$.

Using the basis $\{V_\lambda\}$ for $m = -n$ one can define an action of $U_q(\widehat{\mathfrak{sl}}_n)$ on $\mathbf{P}_r$. First we start with the trivial case $r = 1$, where $\mathbf{P}_r$ reduces to $\mathbf{P}_1 = \bigoplus_{l\in\mathbf{Z}}\mathbf{Q}(q)V_l$. It is immediate to check that the formulas

$$f_iV_l := \delta_{l\equiv i}\,V_{l+1}, \qquad e_iV_l := \delta_{l\equiv i+1}\,V_{l-1}, \qquad q^{\pm h_i}V_l := q^{\pm(\delta_{l\equiv i}-\delta_{l\equiv i+1})}\,V_l,$$

extend to an action of $U_q(\widehat{\mathfrak{sl}}_n)$ on $\mathbf{P}_1$. Here, $a \equiv b$ means congruence modulo n and $\delta_{a\equiv b}$ is the Kronecker δ equal to 1 if $a \equiv b$ and to 0 otherwise. Then using the comultiplication

$$\begin{cases} \Delta f_i = f_i\otimes 1 + q^{h_i}\otimes f_i, \\ \Delta e_i = e_i\otimes q^{-h_i} + 1\otimes e_i, \\ \Delta q^{\pm h_i} = q^{\pm h_i}\otimes q^{\pm h_i}, \end{cases} \tag{65}$$

and identifying $\mathbf{P}_r$ with $\mathbf{P}_1^{\otimes r}$ by $V_\lambda \mapsto V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}$, we obtain the following formulas

$$(66) \qquad f_i V_\lambda := \sum_{\substack{j=1\\ \lambda_j \equiv i}}^{r} q^{\sum_{k=1}^{j-1}(\delta_{\lambda_k \equiv i} - \delta_{\lambda_k \equiv i+1})} V_{\lambda+\epsilon_j},$$

$$(67) \qquad e_i V_\lambda := \sum_{\substack{j=1\\ \lambda_j \equiv i+1}}^{r} q^{-\sum_{k=j+1}^{r}(\delta_{\lambda_k \equiv i} - \delta_{\lambda_k \equiv i+1})} V_{\lambda-\epsilon_j}.$$

Proposition 7.1. *This action of $U_q(\widehat{\mathfrak{sl}}_n)$ on $\mathbf{P}_r$ commutes with the action of $\widehat{\mathbf{H}}_r$ via Π_{-n}.*

Proof — It is clear from (66) (67) that

$$f_i Y^\mu V_\lambda = f_i V_{\lambda - n\mu} = Y_\mu f_i V_\lambda, \qquad e_i Y^\mu V_\lambda = e_i V_{\lambda-n\mu} = Y_\mu e_i V_\lambda,$$

that is, the action of $U_q(\widehat{\mathfrak{sl}}_n)$ commutes with the operators Y^μ. Hence, recalling the discussion of Section 5.2, we see that it is enough to prove that $f_i T_j V_\gamma = T_j f_i V_\gamma$ for $\gamma \in \mathfrak{S}_r \mathcal{A}_{r,-n}$ and $1 \le j \le r-1$. Moreover, since T_j only acts on components j and $j+1$ of γ, we can assume that $r = 2$. Then the claim is verified by a direct computation. For example on the one hand

$$f_0 T_1\, V_{(1-n,0)} = f_0\, V_{(0,1-n)} = V_{(1,1-n)},$$

and on the other hand

$$T_1 f_0\, V_{(1-n,0)} = q^{-1} T_1 V_{(1-n,1)} = q^{-1} T_1 Y_2^{-1} V_{(1-n,1-n)}$$

$$= q^{-1}(Y_1^{-1} T_1 + (q-q^{-1}) Y_1^{-1}) V_{(1-n,1-n)} = V_{(1,1-n)}.$$

□

Remark 7.2. This action of $U_q(\widehat{\mathfrak{sl}}_n)$ does not commute with the positive level action Π_n of $\widehat{\mathbf{H}}_r$. For example if $r = 2$ and $n = 3$

$$f_2 \Pi_3(T_1)(V_{(2,0)}) = q^{-1} V_{(0,3)}, \qquad \Pi_3(T_1)(f_2 V_{(2,0)}) = q V_{(0,3)}.$$

However, one can easily obtain an action commuting with Π_n by simply replacing the comultiplication Δ of (65) by its opposite

$$\Delta^{\mathrm{op}} f_i = f_i \otimes q^{h_i} + 1 \otimes f_i, \quad \Delta^{\mathrm{op}} e_i = e_i \otimes 1 + q^{-h_i} \otimes e_i.$$

◇

The action of $U_q(\widehat{\mathfrak{sl}}_n)$ is compatible with the bar involution of $\mathbf{P}_r$ in the following sense.

Proposition 7.3. *For $x \in U_q(\widehat{\mathfrak{sl}}_n)$ and $v \in \mathbf{P}_r$ one has $\overline{(xv)} = \overline{x}\,\overline{v}$. In other words,*

$$\overline{f_i v} = f_i \overline{v}, \qquad \overline{e_i v} = e_i \overline{v} \qquad\qquad (0 \le i \le n-1).$$

Proof — We can assume that $v = V_\lambda$. Then by (70) and Proposition 5.7 we have

$$(68) \qquad \overline{f_i V_\lambda} := q^{-\ell(w_{0,\xi})}\, T_{w_0}^{-1} \left(\sum_{\substack{j=1 \\ \lambda_j \equiv i}}^{r} q^{\sum_{k=1}^{j-1}(\delta_{\lambda_k \equiv i} - \delta_{\lambda_k \equiv i+1})}\, V_{w_0(\lambda+\epsilon_j)} \right).$$

Here, $\xi \in \mathcal{A}_{r,-n}$ is the point congruent to $\lambda + \epsilon_j$, which does not depend on j because λ_j is required to be $\equiv i$. On the other hand, since f_i commutes with $T_{w_0}^{-1}$ by Proposition 7.1,

$$f_i \overline{V_\lambda} = q^{-\ell(w_{0,\nu})}\, T_{w_0}^{-1} \left(\sum_{\substack{j=1 \\ \lambda_{r+1-j} \equiv i}}^{r} q^{\sum_{k=1}^{j-1}(\delta_{\lambda_{r+1-k} \equiv i} - \delta_{\lambda_{r+1-k} \equiv i+1})}\, V_{w_0 \lambda + \epsilon_j} \right)$$

$$(69) \qquad = q^{-\ell(w_{0,\nu})}\, T_{w_0}^{-1} \left(\sum_{\substack{j=1 \\ \lambda_j \equiv i}}^{r} q^{\sum_{k=1}^{r-j}(\delta_{\lambda_{r+1-k} \equiv i} - \delta_{\lambda_{r+1-k} \equiv i+1})}\, V_{w_0(\lambda+\epsilon_j)} \right).$$

It remains to check that the coefficients of $T_{w_0}^{-1} V_{w_0(\lambda+\epsilon_j)}$ in (68) and (69) are equal, which is equivalent to

$$\sum_{k=1}^{r} (\delta_{\lambda_k \equiv i} - \delta_{\lambda_k \equiv i+1}) - 1 = \ell(w_{0,\nu}) - \ell(w_{0,\xi}).$$

This is elementary, using for instance Remark 5.8. The formula for e_i is similar and its proof is omitted. □

7.2. Action of $U_q(\widehat{\mathfrak{sl}}_n)$ on $\mathbf{F}_r$

Since the action of $U_q(\widehat{\mathfrak{sl}}_n)$ on $\mathbf{P}_r$ commutes with the action of $\widehat{\mathbf{H}}_r$, the subspace $\mathbf{I}_r := \mathbf{Q}(q) \otimes_{\mathbf{Z}[q,q^{-1}]} \mathcal{J}_r$ is stable under $U_q(\widehat{\mathfrak{sl}}_n)$ and we obtain an induced action of $U_q(\widehat{\mathfrak{sl}}_n)$ on $\mathbf{F}_r$. As explained in Section 7.1, the vector V_λ should be regarded as a monomial tensor $V_\lambda \equiv V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}$. Hence its projection $|\lambda\rangle$ on $\mathbf{F}_r$ should be thought of as some q-wedge product $|\lambda\rangle \equiv V_{\lambda_1} \wedge_q \cdots \wedge_q V_{\lambda_r}$ with the anticommutation relations replaced by the straightening rules of Proposition 5.9. The action on $|\lambda\rangle$ of the generators of $U_q(\widehat{\mathfrak{sl}}_n)$ is obtained by projecting (66), (67):

$$f_i|\lambda\rangle := \sum_{\substack{j=1 \\ \lambda_j \equiv i}}^{r} q^{\sum_{k=1}^{j-1}(\delta_{\lambda_k \equiv i} - \delta_{\lambda_k \equiv i+1})} |\lambda + \epsilon_j\rangle, (0 \le i \le n-1), \tag{70}$$

$$e_i|\lambda\rangle := \sum_{\substack{j=1 \\ \lambda_j \equiv i+1}}^{r} q^{-\sum_{k=j+1}^{r}(\delta_{\lambda_k \equiv i} - \delta_{\lambda_k \equiv i+1})} |\lambda - \epsilon_j\rangle, (0 \le i \le n-1). \tag{71}$$

Note that if $\lambda \in P^{++}$ then $\lambda \pm \epsilon_j \in P^+$. It follows that either $|\lambda \pm \epsilon_j\rangle$ belongs to the basis $\{|\lambda\rangle \mid \lambda \in P^{++}\}$, or $|\lambda \pm \epsilon_j\rangle = 0$. Hence, Eq. (70) (71) require no straightening relation and are very simple to use in practice. The compatibility of the bar involution with this action is given by the next

Proposition 7.4. *For $u \in \mathbf{F}_r$ and $0 \le i \le n-1$ one has*

$$\overline{f_i u} = f_i \overline{u}, \qquad \overline{e_i u} = e_i \overline{u}.$$

Proof — This follows immediately from (46) and Proposition 7.3. □

7.3. The Fock space $\mathbf{F}_\infty$

For $s \ge r$ define a linear map $\varphi_{r,s} : \mathbf{F}_r \longrightarrow \mathbf{F}_s$ by

$$\varphi_{r,s}(|\lambda\rangle) := |\lambda_1, \ldots, \lambda_r, -r, -r-1, \ldots, -s+1\rangle \qquad (\lambda \in P_r^+).$$

Then clearly $\varphi_{s,t} \circ \varphi_{r,s} = \varphi_{r,t}$. Let $\mathbf{F}_\infty := \varinjlim \mathbf{F}_r$ be the direct limit of the vector spaces $\mathbf{F}_r$ with respect to the maps $\varphi_{r,s}$. Each $|\lambda\rangle$ in $\mathbf{F}_r$ has an image $\varphi_r(|\lambda\rangle) \in \mathbf{F}_\infty$, which should be thought of as some infinite q-wedge

$$\varphi_r(|\lambda\rangle) \equiv V_{\lambda_1} \wedge_q \cdots \wedge_q V_{\lambda_r} \wedge_q V_{-r} \wedge_q V_{-r-1} \wedge_q \cdots$$

Lemma 7.5. (i) *If* $\lambda_r \le -r$ *then* $\varphi_r(|\lambda\rangle) = 0$.
(ii) *If* $\lambda \in P_r^{++}$ *and* $\lambda_r > -r$ *then* $\varphi_r(|\lambda\rangle) \neq 0$.

Proof — (i) Write $\lambda_r = k \le -r$ and consider the element

$$\varphi_{r,-k+1}(|\lambda\rangle) = |\lambda_1, \dots, \lambda_r, -r, -r-1, \dots, k\rangle.$$

By applying Proposition 5.9 one checks easily that $|k, -r, -r-1, \dots, k\rangle$ straightens to 0 in F_{-k-r+2}. Therefore $\varphi_{r,-k+1}(|\lambda\rangle) = 0$, hence $\varphi_r(|\lambda\rangle) = 0$. (ii) By Lemma 5.6 if $\lambda \in P_r^{++}$ and $\lambda_r > -r$ then $\varphi_{r,s}(|\lambda\rangle) \neq 0$ for all $s > r$. Hence $\varphi_r(|\lambda\rangle) \neq 0$. □

Let $\mathbb{P}^+$ denote the set of all partitions, *i.e.* of all finite non-increasing sequences of positive integers. Put $\rho_r^* := (0, -1, \dots, -r+1)$, and for $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{P}^+$ define

$$|\alpha) := \varphi_s(|\alpha + \rho_s^*\rangle).$$

It readily follows from Lemma 5.6 and Lemma 7.5 that $\{|\alpha) \mid \alpha \in \mathbb{P}^+\}$ is a basis of $\mathbf{F}_\infty$. We define a grading on $\mathbf{F}_\infty$ by requiring that

$$\deg |\alpha) := \sum_{i=1}^{s} \alpha_i.$$

Then for all $\lambda \in P_r$, $\varphi_r(|\lambda\rangle)$ is homogeneous of degree

$$\deg \varphi_r(|\lambda\rangle) = \sum_{i=1}^{r} (\lambda_i + i - 1).$$

In particular, if $\sum_{i=1}^{r} (\lambda_i + i - 1) < 0$ then $\varphi_r(|\lambda\rangle) = 0$.

7.4. Action of $U_q(\widehat{\mathfrak{sl}}_n)$ on $\mathbf{F}_\infty$

Let $\lambda \in \mathcal{P}_r$. It follows easily from (70) that

$$f_i\, \varphi_{r,s}(|\lambda\rangle) = \varphi_{r+1,s}\, f_i\, \varphi_{r,r+1}\,(|\lambda\rangle) \tag{72}$$

for all $s > r$. Hence one can define an endomorphism f_i of $\mathbf{F}_\infty$ by

$$f_i\, \varphi_r(|\lambda\rangle) = \varphi_{r+1}\, f_i\, \varphi_{r,r+1}\,(|\lambda\rangle) \tag{73}$$

and thus get an action of $U_q^-(\widehat{\mathfrak{sl}}_n)$ on $\mathbf{F}_\infty$.

On the basis $\{|\alpha) \mid \alpha \in \mathbb{P}^+\}$ this action is expressed as follows. Let α and β be two Young diagrams such that β is obtained from α by adding a cell γ whose content is $\equiv i \bmod n$. Such a cell is called a removable

i-cell of β, or an indent i-cell of α. Let $I_i^r(\alpha,\beta)$ (*resp.* $R_i^r(\alpha,\beta)$) be the number of indent i-cells of α (*resp.* of removable i-cells of α) situated to the right of γ (γ not included). Set $N_i^r(\alpha,\beta) = I_i^r(\alpha,\beta) - R_i^r(\alpha,\beta)$. Then Eq. (70) gives

$$f_i|\alpha\rangle = \sum_\beta q^{N_i^r(\alpha,\beta)}|\beta\rangle, \tag{74}$$

where the sum is over all partitions β such that β/α is an i-cell.

Defining an action of $U_q^+(\widehat{\mathfrak{sl}}_n)$ is not as straightforward since there is no formula like (72) for e_i. For example if $n=2$,

$$\begin{aligned}
&e_1\,|2\rangle = |1\rangle,\\
&e_1\,|2,-1\rangle = q^{-1}\,|1,-1\rangle,\\
&e_1\,|2,-1,-2\rangle = |1,-1,-2\rangle + |2,-1,-3\rangle,\\
&e_1\,|2,-1,-2,-3\rangle = q^{-1}\,|1,-1,-2,-3\rangle,
\end{aligned}$$

and in general

$$\begin{aligned}
&e_1\varphi_{1,2r}\,|2\rangle = q^{-1}\varphi_{1,2r}e_1|2\rangle,\\
&e_1\varphi_{1,2r+1}\,|2\rangle = \varphi_{1,2r+1}e_1|2\rangle + |2,-1,\dots,-2r+1,-2r-1\rangle.
\end{aligned}$$

However, one can check that putting $e_i\varphi_r(|\lambda\rangle) := q^{-\delta_{i\equiv r}}\,\varphi_r(e_i|\lambda\rangle)$ one gets a well-defined action of $U_q^+(\widehat{\mathfrak{sl}}_n)$ compatible with (73) (see [15]). Its combinatorial description is given by

$$e_i|\beta\rangle = \sum_\alpha q^{-N_i^l(\alpha,\beta)}|\alpha\rangle, \tag{75}$$

where the sum is over all partitions α such that β/α is an i-cell, and $N_i^l(\alpha,\beta)$ is defined as $N_i^r(\alpha,\beta)$ but replacing right by left.

In contrast to $\mathbf{F}_r$, the representation $\mathbf{F}_\infty$ has primitive vectors, *i.e.* vectors annihilated by all e_i. In particular the vector $|0\rangle$ labelled by the unique partition of 0 is primitive. In fact $\mathbf{F}_\infty$ is a level 1 highest weight integrable representation of $U_q(\widehat{\mathfrak{sl}}_n)$, while $\mathbf{F}_r$ is a level 0 representation (without highest weight). As shown by Kashiwara, Miwa and Stern [15], the decomposition of $\mathbf{F}_\infty$ into simple $U_q(\widehat{\mathfrak{sl}}_n)$-modules is obtained by considering the limit $r\to\infty$ of the action of $Z(\widehat{\mathbf{H}}_r)$ on $\mathbf{F}_r$.

7.5. Action of $\mathbf{H}_\infty$ on $\mathbf{F}_\infty$

Let $\lambda\in P_r$. It follows from the easily checked relations

$$\begin{cases} |-s,-r,-r-1,\dots,-s\rangle = 0,\\ |-r,-r-1,\dots,-s,-r\rangle = 0,\end{cases} \quad (s\ge r\ge 0) \tag{76}$$

that the vector $\varphi_s\, B_k\, \varphi_{r,s}(|\lambda\rangle)$ is independent of s for $s > r$ large enough. Hence one can define endomorphisms B_k of $\mathbf{F}_\infty$ by

$$B_k\varphi_r(|\lambda\rangle) := \varphi_s\, B_k\, \varphi_{r,s}(|\lambda\rangle) \qquad (k \in \mathbf{Z}^*,\ s \gg 1). \tag{77}$$

By construction, these endomorphisms commute with the action of $U_q(\widehat{\mathfrak{sl}}_n)$ on $\mathbf{F}_\infty$. However they no longer generate a commutative algebra but a Heisenberg algebra. Indeed, it was proved by Kashiwara, Miwa and Stern [15] that

$$[B_k, B_l] = \begin{cases} k\,\dfrac{1-q^{-2nk}}{1-q^{-2k}} & \text{if } k=-l, \\ 0 & \text{otherwise.} \end{cases} \tag{78}$$

We shall denote this Heisenberg algebra by $\mathbf{H}_\infty$. The elements $\widetilde{\mathcal{U}}_\beta$, $\widetilde{\mathcal{V}}_\beta$, $\mathcal{U}_\beta$, $\mathcal{V}_\beta$ of $Z(\widehat{H}_r)$ also give rise to well-defined elements of $\mathbf{H}_\infty$ that we still denote by $\widetilde{\mathcal{U}}_\beta$, $\widetilde{\mathcal{V}}_\beta$, $\mathcal{U}_\beta$, $\mathcal{V}_\beta$. By Theorem 6.2 and Theorem 6.4, their action on the basis $\{|\nu\rangle,\ \nu \in \mathbb{P}^+\}$ of $\mathbf{F}_\infty$ is given by

$$\widetilde{\mathcal{U}}_\beta\,|\nu\rangle = q^{-(n-1)k} \sum_{\mu\in\mathbb{P}^+} L^{(n)}_{\nu'/\mu',\,\beta}(-q)\ |\mu\rangle, \tag{79}$$

$$\widetilde{\mathcal{V}}_\beta\,|\nu\rangle = q^{-(n-1)k} \sum_{\lambda\in\mathbb{P}^+} L^{(n)}_{\lambda'/\nu',\,\beta}(-q)\ |\lambda\rangle, \tag{80}$$

$$\mathcal{U}_\beta\,|\nu\rangle = \sum_{\mu\in\mathbb{P}^+} L^{(n)}_{\nu/\mu\,,\,\beta}(-q^{-1})\ |\mu\rangle, \tag{81}$$

$$\mathcal{V}_\beta\,|\nu\rangle = \sum_{\lambda\in\mathbb{P}^+} L^{(n)}_{\lambda/\nu\,,\,\beta}(-q^{-1})\ |\lambda\rangle, \tag{82}$$

where $k = |\beta|$. It was shown in [15] that $\mathbf{F}_\infty$ is irreducible under the commuting actions of $U_q(\widehat{\mathfrak{sl}}_n)$ and $\mathbf{H}_\infty$. It follows that $\{\mathcal{V}_\beta\,|0\rangle,\ \beta \in \mathbb{P}^+\}$ is a basis of the space of primitive vectors of $\mathbf{F}_\infty$ for $U_q(\widehat{\mathfrak{sl}}_n)$.

7.6. The bar involution of $\mathbf{F}_\infty$

Before introducing the involution we need the following lemmas.

Lemma 7.6. *Let $\mu \in P_{m+1}$ such that $\mu_i > -m$ $(i = 1,\dots,m+1)$ and $\sum_i(\mu_i + i - 1) \le m$. Then $|\mu\rangle = 0$.*

Proof — We have

$$|\mu\rangle = \sum_{\lambda\in P^{++}_{m+1}} x_\lambda\,|\lambda\rangle$$

for some coefficients x_λ. Because of the hypothesis $\mu_i > -m$ and of the form of the straightening relations, the components of the weights λ occuring in this sum must all be $> -m$. On the other hand, setting $\alpha_i = \lambda_i + i - 1$ we see that α is a partition with $|\alpha| \le m$, hence $l(\alpha) \le m$. Thus the last component of all the λ must be $= -m$, and the sum is empty. □

Lemma 7.7. *Let $\lambda \in P_r$ and let $m \ge r$. Assume that $\lambda_i > -m$ $(i = 1, \dots, r)$ and $\sum_i (\lambda_i + i - 1) \le m$. Then*

$$\begin{aligned} & |-m, \lambda_1, \dots, \lambda_r, -r, \dots, -m+1\rangle \\ &= (-1)^m q^{-a(\lambda)} \, |\lambda_1, \dots, \lambda_r, -r, \dots, -m+1, -m\rangle, \end{aligned}$$

where $a(\lambda) = \sharp\{j \le r \mid \lambda_j \not\equiv -m\} + \sharp\{-r \ge j \ge -m+1 \mid j \not\equiv -m\}$.

Proof — Consider the straightening of

$$\nu = |-m, \lambda_1, \dots, \lambda_r, -r, \dots, -m+1\rangle$$

computed by means of Proposition 5.9. At each step, if the third rule (50) has to be used, then only the first term of the right-hand side may be non-zero. Indeed the two other terms involve weights μ which satisfy the hypothesis of Lemma 7.6. Therefore the straightening of ν is simply obtained by reordering its components and multiplying by the appropriate sign and power of q. □

If λ satisfies the hypothesis of Lemma 7.7, then repeated applications of this lemma show that for $p \ge m$,

$$\begin{aligned} & (-1)^{\binom{p}{2}} q^{b(\lambda,p)} \, |-p, \dots, -r, \lambda_r, \dots, \lambda_1\rangle \\ &= (-1)^{\binom{m}{2}} q^{b(\lambda,m)} \, |-m, \dots, -r, \lambda_r, \dots, \lambda_1, -m-1, \dots, -p\rangle \end{aligned}$$

Here $b(\lambda, p)$ is the number of pairs (i, j) of components of the vector $(\lambda_1, \dots, \lambda_r, -r, \dots, -p)$ with $i \not\equiv j \mod n$. In other words, using Proposition 5.7 and Remark 5.8

$$\overline{\varphi_{r,p}(|\lambda\rangle)} = \varphi_{m,p}(\overline{\varphi_{r,m}(|\lambda\rangle)}).$$

Thus we can define a semi-linear involution on $\mathbf{F}_\infty$ by putting

$$(83) \quad \overline{\varphi_r(|\lambda\rangle)} := \varphi_m(\overline{\varphi_{r,m}(|\lambda\rangle)}) \quad (\lambda \in P_r, \ \deg \varphi_r(|\lambda\rangle) = m, \ \lambda_i > -m).$$

In particular, for $\alpha \in \mathbb{P}^+$ and $s \ge |\alpha|$, we have $\overline{|\alpha\rangle} = \varphi_s(\overline{|\alpha + \rho_s^*\rangle})$.

Proposition 7.8. *For $\alpha \in \mathbb{P}^+$, $0 \le i \le n-1$ and $k \in \mathbf{N}^*$ we have*

$$\overline{f_i\,|\alpha)} = f_i\,\overline{|\alpha)}, \quad \overline{e_i\,|\alpha)} = e_i\,\overline{|\alpha)},$$

$$\overline{B_{-k}\,|\alpha)} = B_{-k}\,\overline{|\alpha)}, \quad \overline{B_k\,|\alpha)} = q^{2(n-1)k} B_k\,\overline{|\alpha)}.$$

Proof — For f_i and B_{-k}, the proof readily follows from Proposition 7.4, Proposition 6.1 and (73) (77) (83). (Note that the condition $\lambda_i > -m$ in (83) is preserved by the action of these lowering operators.) Let us prove the statement for B_k. We argue by induction on $\deg|\alpha)$. In degree 0, the unique basis vector is $|0)$ and we have $B_k|0) = B_k\overline{|0)} = 0$, so the claim is trivially verified. Let us assume that the result is proved for all $|\alpha)$ of degree $\le m$. Since the action of the operators B_{-l} and f_i on $|0)$ generates the whole Fock space, it is enough to prove that

$$\overline{B_k\,f_i v} = q^{2(n-1)k} B_k\,\overline{f_i v}, \qquad \overline{B_k\,B_{-l} v} = q^{2(n-1)k} B_k\,\overline{B_{-l} v}$$

for all v of degree $\le m$. Now B_k and f_i commute, so

$$\overline{B_k f_i v} = \overline{f_i B_k v} = f_i \overline{B_k v} = f_i(q^{2(n-1)k} B_k \overline{v}) = q^{2(n-1)k} B_k f_i \overline{v}$$
$$= q^{2(n-1)k} B_k \overline{f_i v}.$$

If $l \neq k$ we know that B_k and B_{-l} commute and we can argue similarly. Finally if $l = k$, by (78),

$$\begin{aligned}
\overline{B_k\,B_{-k} v} &= \overline{B_{-k}\,B_k v} + k\,\frac{1-q^{2(n-1)k}}{1-q^{2k}}\overline{v} \\
&= q^{2(n-1)k} B_{-k} B_k \overline{v} + k\,\frac{1-q^{2(n-1)k}}{1-q^{2k}}\overline{v} \\
&= q^{2(n-1)k}\left(B_{-k}B_k + k\,\frac{1-q^{-2(n-1)k}}{1-q^{-2k}}\right)\overline{v} = q^{2(n-1)k} B_k B_{-k}\overline{v} \\
&= q^{2(n-1)k} B_k \overline{B_{-k} v}.
\end{aligned}$$

The proof for e_i is similar, using the commutation relation

$$[e_i, f_j] = \delta_{ij}\frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}.$$

□

Proposition 7.8 implies that for $|\beta| = k$,

$$\overline{\widetilde{\mathcal{V}}_\beta\,|\alpha)} = \widetilde{\mathcal{V}}_\beta\,\overline{|\alpha)}, \qquad \overline{\mathcal{V}_\beta\,|\alpha)} = \mathcal{V}_\beta\,\overline{|\alpha)}, \tag{84}$$

$$\overline{\widetilde{\mathcal{U}}_\beta\,|\alpha)} = q^{2(n-1)k}\,\widetilde{\mathcal{U}}_\beta\,\overline{|\alpha)}, \qquad \overline{\mathcal{U}_\beta\,|\alpha)} = q^{2(n-1)k}\,\mathcal{U}_\beta\,\overline{|\alpha)}. \tag{85}$$

7.7. The scalar product of $\mathbf{F}_\infty$

Define a scalar product on $\mathbf{F}_\infty$ by $\langle |\alpha\rangle\,,\,|\beta\rangle\rangle = \delta_{\alpha\beta}$.

Proposition 7.9. *For $u, v \in \mathbf{F}_\infty$ one has*

$$\langle f_i u\,,\,v\rangle = \langle u\,,\,q^{h_i-1}e_i v\rangle,\quad \langle e_i u\,,\,v\rangle = \langle u\,,\,q^{-h_i-1}f_i v\rangle,$$
$$\langle \widetilde{\mathcal{V}}_\alpha u\,,\,v\rangle = \langle u\,,\,\widetilde{\mathcal{U}}_\alpha v\rangle,\qquad \langle \mathcal{V}_\alpha u\,,\,v\rangle = \langle u\,,\,\mathcal{U}_\alpha v\rangle.$$

Proof — This follows immediately from (74) (75) (79) (80) (81) (82).

7.8. Symmetry of the bar involution

Define a semi-linear involution $v \mapsto v'$ on $\mathbf{F}_\infty$ by $|\alpha\rangle' := |\alpha'\rangle$, where α' is the partition conjugate to $\alpha \in \mathbb{P}^+$.

Proposition 7.10. *For $u \in \mathbf{F}_\infty$ and $\beta \in \mathbb{P}^+$ with $|\beta| = k$, there holds*

$$(e_i u)' = q^{h_{-i}-1}e_{-i}u',\qquad (f_i u)' = q^{-h_{-i}-1}f_{-i}u',$$
$$(\mathcal{V}_\beta u)' = (-q)^{(n-1)k}\,\widetilde{\mathcal{V}}_\beta u',\qquad (\mathcal{U}_\beta u)' = (-q)^{(n-1)k}\,\widetilde{\mathcal{U}}_\beta u'.$$

Proof — This also follows from (74) (75) (79) (80) (81) (82). □

Let $S_\beta = \sum_\alpha \kappa_{\beta,\alpha}\mathcal{V}_\alpha$ be the element of $\mathbf{H}_\infty$ corresponding to the Schur function s_β. The third equation above implies that

$$(S_\beta u)' = (-q)^{(n-1)k}\,S_{\beta'}u'. \tag{86}$$

Theorem 7.11. *For $u, v \in \mathbf{F}_\infty$ we have*

$$\langle \overline{u}\,,\,v\rangle = \langle u'\,,\,\overline{v'}\rangle\,.$$

Proof — The proof is by induction on the degree d of u and v. If $d = 0$ this is clear. So let us assume that the theorem is proved in degree $d < m$. The operators e_i, f_i, $\widetilde{\mathcal{U}}_k$, $\widetilde{\mathcal{V}}_k$, $\mathcal{U}_k$, $\mathcal{V}_k$ are homogeneous of respective degree -1, $+1$, $-kn$, $+kn$, $-kn$, $+kn$. Since $\mathbf{F}_\infty$ is generated by the action of the operators f_i and $\mathcal{V}_k$ on the highest weight vector $|0\rangle$, it is enough to prove that

$$\langle \overline{(f_i u)}, v\rangle = \langle (f_i u)', \overline{v'}\rangle, \tag{87}$$
$$\langle \overline{(\mathcal{V}_k w)}, v\rangle = \langle (\mathcal{V}_k w)', \overline{v'}\rangle, \tag{88}$$

for all u, v, w with $\deg u = m-1$, $\deg v = m$, $\deg w = m - kn$.

Let us prove (87). We have

$$\langle\overline{(f_i u)}, v\rangle = \langle f_i\,\overline{u}, v\rangle = \langle\overline{u}, q^{h_i-1}e_i v\rangle = \langle u', \overline{(q^{h_i-1}e_i v)'}\rangle.$$

The first equality comes from Proposition 7.8, the second from Proposition 7.9 and the third from the fact that $\deg u < m$. Now, by Proposition 7.8, 7.9 and 7.10

$$\begin{aligned}\langle u', \overline{(q^{h_i-1}e_i v)'}\rangle &= \langle u', \overline{e_{-i}v'}\rangle = \langle u', e_{-i}\overline{v'}\rangle \\ &= \langle q^{-h_{-i}-1}f_{-i}u', \overline{v'}\rangle = \langle (f_i u)', \overline{v'}\rangle,\end{aligned}$$

and (87) is proved.

The proof of (88) is similar. We have

$$\langle\overline{(\mathcal{V}_k w)}, v\rangle = \langle\mathcal{V}_k\overline{w}, v\rangle = \langle\overline{w}, \mathcal{U}_k v\rangle = \langle w', \overline{(\mathcal{U}_k v)'}\rangle.$$

The first equality comes from (84), the second from Proposition 7.9 and the third from the fact that $\deg w < m$. Then, using again Proposition 7.8, 7.9 and 7.10,

$$\begin{aligned}\langle w', \overline{(\mathcal{U}_k v)'}\rangle &= \langle w', \overline{(-q)^{(n-1)k}\widetilde{\mathcal{U}}_k(u')}\rangle = \langle w', (-q)^{(n-1)k}\widetilde{\mathcal{U}}_k(\overline{u'})\rangle \\ &= \langle (-q)^{(n-1)k}\widetilde{\mathcal{V}}_k w', \overline{v'}\rangle = \langle (\mathcal{V}_k v)', \overline{w'}\rangle,\end{aligned}$$

and (88) is proved. □

7.9. Canonical bases of $\mathbf{F}_\infty$

For $\beta \in \mathbb{P}^+$ write $\overline{|\beta)} = \sum_{\alpha\in\mathbb{P}^+} b_{\alpha,\beta}(q)\,|\alpha)$. Then, for $|\alpha| = |\beta| \leq r$ it follows from (83) that we have

$$b_{\alpha,\beta}(q) = a_{\alpha+\rho_r^*,\,\beta+\rho_r^*}(q) = a_{\alpha+\rho_r,\,\beta+\rho_r}(q),$$

where the coefficients $a_{\lambda,\mu}(q)$ ($\lambda,\ \mu \in P_r^{++}$) have been defined in Section 5.3. Hence by Corollary 5.10 the matrix

$$\mathbf{B}_k := [b_{\alpha,\beta}(q)], \qquad (\alpha, \beta \vdash k)$$

is unitriangular, and one can define canonical bases $\{\mathcal{G}_\alpha^+ \mid \alpha \in \mathbb{P}^+\}$, $\{\mathcal{G}_\alpha^- \mid \alpha \in \mathbb{P}^+\}$ of $\mathbf{F}_\infty$ characterized by:

(i) $\overline{\mathcal{G}_\alpha^+} = \mathcal{G}_\alpha^+$, $\quad\overline{\mathcal{G}_\alpha^-} = \mathcal{G}_\alpha^-$,

(ii) $\mathcal{G}_\alpha^+ \equiv |\alpha) \bmod q\mathcal{L}_\infty^+$, $\quad\mathcal{G}_\alpha^- \equiv |\alpha) \bmod q^{-1}\mathcal{L}_\infty^-$, where $\mathcal{L}_\infty^+$ (*resp.* $\mathcal{L}_\infty^-$) is the $\mathbf{Z}[q]$-submodule (*resp.* $\mathbf{Z}[q^{-1}]$-submodule) spanned by the vectors $|\alpha)$. Set

$$\mathcal{G}_\beta^+ = \sum_\alpha d_{\alpha,\beta}(q)\,|\alpha)\,, \qquad \mathcal{G}_\alpha^- = \sum_\beta e_{\alpha,\beta}(-q^{-1})\,|\beta)\,,$$

and

$$\mathbf{D}_k := [d_{\alpha,\beta}(q)], \qquad \mathbf{E}_k := [e_{\alpha,\beta}(q)], \qquad (\alpha, \beta \vdash k).$$

Then, for $r \geq k$ we have

$$d_{\alpha,\beta}(q) = c_{\alpha+\rho_r,\,\beta+\rho_r}(q), \qquad e_{\alpha,\beta}(q) = l_{\alpha+\rho_r,\,\beta+\rho_r}(q).$$

Hence by Theorem 5.12 we get

$$e_{\alpha,\beta} = P^-_{\beta+\rho_r,\alpha+\rho_r}, \tag{89}$$

a parabolic Kazhdan-Lusztig polynomial for $\widehat{\mathfrak{S}}_r$ associated with the parabolic subgroup $\mathfrak{S}_{\nu,-n}$ which stabilizes the point $\nu \in \mathcal{A}_{r,-n}$ congruent to $\alpha + \rho_r$ and $\beta + \rho_r$. Also, putting $\widehat{u}_\alpha := w(w_0(\alpha + \rho_r), -n)\, w_{0,\nu}$ and $\widehat{u}_\beta := w(w_0(\beta + \rho_r), -n)\, w_{0,\nu}$, we have

$$d_{\alpha,\beta} = \sum_{s\in\mathfrak{S}_r} (-q)^{\ell(s)} P_{s\widehat{u}_\alpha\,,\,\widehat{u}_\beta}. \tag{90}$$

Note that by Theorem 2.4 this is also a parabolic Kazhdan-Lusztig polynomial of negative type associated with the subgroup $\mathfrak{S}_r \subset \widehat{\mathfrak{S}}_r$. It is interesting to give another expression of $d_{\alpha,\beta}$ in terms of the action π_n (instead of π_{-n}). Let $\underline{P}_r = P_r/\mathbf{Z}(1,\dots,1)$ and $\lambda \mapsto \underline{\lambda}$ be the natural projection $P_r \to \underline{P}_r$. The action π_n of $\widehat{\mathfrak{S}}_r$ on P_r induces an action $\underline{\pi}_n$ of $\widetilde{\mathfrak{S}}_r$ on $\underline{P}_r$ with fundamental alcove $\underline{\mathcal{A}}_{r,n} := \{\underline{\lambda} \in \underline{P}_r \mid \lambda_1 \geq \cdots \geq \lambda_r,\ \lambda_1 - \lambda_r \leq n\}$. Let ξ be the point of $\underline{\mathcal{A}}_{r,n}$ congruent to $\underline{\alpha + \rho_r}$ and $\underline{\beta + \rho_r}$ under $\underline{\pi}_n$, and let $w_{0,\xi}$ denote the longest element of its stabilizer. Consider the projection $\underline{\cdot} : \widehat{\mathfrak{S}}_r \to \widetilde{\mathfrak{S}}_r$ defined by $\underline{\sigma\tau^k} = \sigma$ ($k \in \mathbf{Z}$, $\sigma \in \widetilde{\mathfrak{S}}_r$), and the automorphism $\sharp$ of $\widehat{\mathfrak{S}}_r$ defined by $s_i^\sharp = s_{-i}$ ($i \in \mathbf{Z}/r\mathbf{Z}$). It is easy to check that, for $\lambda \in P_r^+$, $\underline{w(w_0\lambda, -n)} = \underline{(w(\lambda, n))}^\sharp$. It follows that

$$d_{\alpha,\beta} = \sum_{s\in\mathfrak{S}_r} (-q)^{\ell(s)} P_{s\widehat{v}_\alpha\,,\,\widehat{v}_\beta}, \tag{91}$$

where $\widehat{v}_\alpha$, $\widehat{v}_\beta$ are given by $\widehat{v}_\alpha = \underline{w(\alpha + \rho_r, n)}w_{0,\xi}$, $\widehat{v}_\beta = \underline{w(\beta + \rho_r, n)}w_{0,\xi}$.

Remark 7.12. Consider the $U_q(\widehat{\mathfrak{sl}}_n)$-submodule M of $\mathbf{F}_\infty$ generated by $|0)$. This is an irreducible integrable representation with highest weight Λ_0. By Proposition 7.8, the bar involution of $\mathbf{F}_\infty$ induces the Kashiwara involution of M, and it follows that the subset $\{\mathcal{G}_\alpha^+ \mid \alpha \text{ is } n\text{-regular}\}$ is the global lower crystal basis of M (see [20]). The expression (90) and (91) of the coefficients of this basis as Kazhdan-Lusztig polynomials have been obtained independently by Vasserot, Varagnolo [42] and by Goodman, Wenzl [10] respectively. ◇

It follows from Theorem 6.6 that the basis $\mathcal{G}_\alpha^-$ satisfies the following analogue of the Steinberg-Lusztig tensor product theorem. Let $\alpha \in \mathbb{P}^+$ of length r. Write $\alpha = \alpha^{(0)} + n\alpha^{(1)}$, where $\alpha^{(0)}$ is n-restricted, that is,

$$0 \le \alpha_i^{(0)} - \alpha_{i+1}^{(0)} < n \qquad (1 \le i \le r-1).$$

Then $\mathcal{G}_\alpha^- = S_{\alpha^{(1)}}\, \mathcal{G}_{\alpha^{(0)}}^-$. Taking $\alpha^{(0)} = (0)$ and writing $n\alpha$ in place of α we obtain that

$$\mathcal{G}_{n\alpha}^- = S_\alpha\, |0). \tag{92}$$

We can now prove the following symmetry of the basis $\{\mathcal{G}_\alpha^-\}$.

Theorem 7.13. *Let λ, $\mu^{(0)}, \dots, \mu^{(n-1)}$ be partitions. Set $k = |\lambda|$. There holds*

(i) $(\mathcal{G}_{n\lambda}^-)' = (-q)^{(n-1)k}\, \mathcal{G}_{n\lambda'}^-$,

(ii) $c^{\lambda}_{\mu^{(0)},\dots,\mu^{(n-1)}}(q^{-1}) = q^{-(n-1)k}\, c^{\lambda'}_{(\mu^{(n-1)})',\dots,(\mu^{(0)})'}(q)$.

Proof — By (92) and (86) we have

$$(\mathcal{G}_{n\lambda}^-)' = (S_\lambda\, |0))' = (-q)^{(n-1)k}\, S_{\lambda'}\, |0) = (-q)^{(n-1)k}\, \mathcal{G}_{n\lambda'}^-.$$

The second equation follows now from the fact that if μ is the partition with n-quotient $(\mu^{(0)}, \dots, \mu^{(n-1)})$ then the conjugate partition μ' has n-quotient $((\mu^{(n-1)})', \dots, (\mu^{(0)})')$. □

Let $\{\mathcal{G}_\alpha^*\}$ denote the basis of $\mathbf{F}_\infty$ adjoint to $\{\mathcal{G}_\alpha^+\}$ for the above scalar product. In other words, $\langle \mathcal{G}_\alpha^*, \mathcal{G}_\beta^+ \rangle = \delta_{\alpha,\beta}$. Write

$$\mathcal{G}_\alpha^* = \sum_\beta g_{\alpha,\beta}(q)\, |\beta), \qquad \text{and} \qquad \mathbf{G}_k := [g_{\alpha,\beta}(q)], \qquad (\alpha, \beta \vdash k).$$

Since $\{|\alpha)\}$ is an orthonormal basis, we have $\mathbf{G}_k = \mathbf{D}_k^{-1}$.

Theorem 7.14. *For $\alpha \in \mathbb{P}^+$ one has $(\mathcal{G}_\alpha^*)' = \mathcal{G}_{\alpha'}^-$.*

Proof — We have to prove that $(\mathcal{G}_\alpha^*)'$ satisfies the two defining properties of $\mathcal{G}_{\alpha'}^-$, namely

$$(\mathcal{G}_\alpha^*)' \equiv \alpha' \bmod q^{-1}\mathcal{L}_\infty^-, \qquad \overline{(\mathcal{G}_\alpha^*)'} = (\mathcal{G}_\alpha^*)'.$$

The first property is obvious. Indeed by definition $\mathcal{G}_\alpha^+ \equiv |\alpha) \bmod q\mathcal{L}_\infty^+$. Since $\mathbf{G}_k = \mathbf{D}_k^{-1}$, we deduce that $\mathcal{G}_\alpha^* \equiv |\alpha) \bmod q\mathcal{L}_\infty^+$, which implies that $(\mathcal{G}_\alpha^*)' \equiv \alpha' \bmod q^{-1}\mathcal{L}_\infty^-$. The second property is equivalent to

$$\langle \overline{(\mathcal{G}_\alpha^*)'}\,,\,(\mathcal{G}_\beta^+)' \rangle = \delta_{\alpha,\beta}, \qquad (\alpha,\ \beta \vdash k),$$

because $\{(\mathcal{G}_\alpha^*)'\}$ is the basis adjoint to $\{(\mathcal{G}_\beta^+)'\}$. Now, by Theorem 7.11,

$$\langle \overline{(\mathcal{G}_\alpha^*)'}\,,\,(\mathcal{G}_\beta^+)' \rangle = \langle \mathcal{G}_\alpha^*\,,\,\overline{\mathcal{G}_\beta^+} \rangle = \langle \mathcal{G}_\alpha^*\,,\,\mathcal{G}_\beta^+ \rangle = \delta_{\alpha,\beta}.$$

□

Corollary 7.15. *Let* $\mathbf{J}_k = [j_{\alpha,\beta}(q)]_{\alpha,\beta\vdash k} := [e_{\alpha',\beta'}(-q)]^{-1}_{\alpha,\beta\vdash k}$. *Then* $\mathbf{J}_k = \mathbf{D}_k$. *In other words, we have*

$$\sum_{\gamma\vdash k} e_{\alpha',\gamma'}(-q)\, d_{\gamma,\beta}(q) = \delta_{\alpha,\beta},$$

where $e_{\alpha',\gamma'}$ *and* $d_{\gamma,\beta}$ *are the parabolic Kazhdan-Lusztig polynomials given by* (89) (90).

Remark 7.16. (i) Let $\alpha,\ \beta$ be two partitions of k and take $r \geq k$. By Lusztig's conjecture (33), it follows from Corollary 7.15 that

$$d_{\alpha,\beta}(1) = j_{\alpha,\beta}(1) = [W(\alpha') : L(\beta')],$$

the multiplicity of the simple $U_\zeta(\mathfrak{gl}_r)$-module $L(\beta')$ in the Weyl module $W(\alpha')$, as was conjectured in [22], Conjecture 5.2.
(ii) For $\lambda \in P_r^+$, let $T(\lambda)$ denote the indecomposable tilting $U_\zeta(\mathfrak{gl}_r)$-module with highest weight λ. By Proposition 8.2 of [7] which states that

$$[W(\alpha') : L(\beta')] = [T(\beta) : W(\alpha)],$$

we see that $[T(\beta) : W(\alpha)] = d_{\alpha,\beta}(1)$. Taking into account (91) we thus get another proof of the character formula of Soergel [40] in type A. Note that we do not need to deduce the formula for singular weights from that for regular weights (see [39], Remark 7.2). In particular, we see that the formula is also valid for $n < r$, when all integral weights are singular.

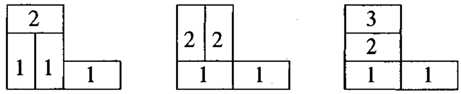

Fig. 10. The Yamanouchi domino tableaux of shape (42^2)

§8. Tables

We illustrate our results by giving some tables of q-Littlewood-Richardson coefficients and of polynomials $d_{\alpha,\beta}(q)$. These tables are q-analogues of those calculated by James in [12], which were the starting point of our investigation. They have been computed using the package FOCK written by the authors and available as a part of the environment ACE [43].

8.1. Canonical highest weight vectors of the Fock space representation of $U_q(\widehat{\mathfrak{sl}}_2)$

The following tables give the coefficients $e_{2\alpha,\beta}(-q^{-1})$ of the expansion of $\mathcal{G}^-_{2\alpha}$ on the standard basis $\{|\beta\rangle\}$ for $n = 2$ up to partitions of 10. They should be read by columns, *e.g.*

$$\mathcal{G}^-_{(4)} = |4\rangle - q^{-1}\,|3,1\rangle + q^{-2}\,|2,2\rangle.$$

These vectors form a basis of the subspace of primitive vectors of $\mathbf{F}_\infty$. Their coefficients are the q-analogues $c^\lambda_{\mu^{(0)},\mu^{(1)}}(-q^{-1})$ of the Littlewood-Richardson multiplicities for all partitions $\mu^{(0)}$, $\mu^{(1)}$ with $|\mu^{(0)}|+|\mu^{(1)}| \leq 5$. They are easily computed using the combinatorial description of [2] in terms of Yamanouchi domino tableaux. For example the row labelled (42^2) is given by the tableaux of Figure 10.

	(2)
(2)	1
(1^2)	$-q^{-1}$

	(4)	(2^2)
(4)	1	0
(31)	$-q^{-1}$	0
(2^2)	q^{-2}	1
(21^2)	0	$-q^{-1}$
(1^4)	0	q^{-2}

	(6)	(42)	(2^3)
(6)	1	0	0
(51)	$-q^{-1}$	0	0
(42)	q^{-2}	1	0
(41^2)	0	$-q^{-1}$	0
(3^2)	$-q^{-3}$	$-q^{-1}$	0
(31^3)	0	q^{-2}	0
(2^3)	0	q^{-2}	1
(2^21^2)	0	$-q^{-3}$	$-q^{-1}$
(21^4)	0	0	q^{-2}
(1^6)	0	0	$-q^{-3}$

	(8)	(62)	(4^2)	(42^2)	(2^4)
(8)	1	0	0	0	0
(71)	$-q^{-1}$	0	0	0	0
(62)	q^{-2}	1	0	0	0
(61^2)	0	$-q^{-1}$	0	0	0
(53)	$-q^{-3}$	$-q^{-1}$	0	0	0
(51^3)	0	q^{-2}	0	0	0
(4^2)	q^{-4}	q^{-2}	1	0	0
(431)	0	0	$-q^{-1}$	0	0
(42^2)	0	q^{-2}	q^{-2}	1	0
(421^2)	0	$-q^{-3}$	0	$-q^{-1}$	0
(41^4)	0	0	0	q^{-2}	0
(3^22)	0	$-q^{-3}$	0	$-q^{-1}$	0
(3^21^2)	0	q^{-4}	q^{-2}	q^{-2}	0
(32^21)	0	0	$-q^{-3}$	0	0
(31^5)	0	0	0	$-q^{-3}$	0
(2^4)	0	0	q^{-4}	q^{-2}	1
(2^31^2)	0	0	0	$-q^{-3}$	$-q^{-1}$
(2^21^4)	0	0	0	q^{-4}	q^{-2}
(21^6)	0	0	0	0	$-q^{-3}$
(1^8)	0	0	0	0	q^{-4}

	(10)	(82)	(64)	(62^2)	(4^22)	(42^3)	(2^5)
(10)	1	0	0	0	0	0	0
(91)	$-q^{-1}$	0	0	0	0	0	0
(82)	q^{-2}	1	0	0	0	0	0
(81^2)	0	$-q^{-1}$	0	0	0	0	0
(73)	$-q^{-3}$	$-q^{-1}$	0	0	0	0	0
(71^3)	0	q^{-2}	0	0	0	0	0
(64)	q^{-4}	q^{-2}	1	0	0	0	0
(631)	0	0	$-q^{-1}$	0	0	0	0
(62^2)	0	q^{-2}	q^{-2}	1	0	0	0
(621^2)	0	$-q^{-3}$	0	$-q^{-1}$	0	0	0
(61^4)	0	0	0	q^{-2}	0	0	0
(5^2)	$-q^{-5}$	$-q^{-3}$	$-q^{-1}$	0	0	0	0
(532)	0	$-q^{-3}$	0	$-q^{-1}$	0	0	0
(531^2)	0	q^{-4}	q^{-2}	q^{-2}	0	0	0
(52^21)	0	0	$-q^{-3}$	0	0	0	0
(51^5)	0	0	0	$-q^{-3}$	0	0	0
(4^22)	0	q^{-4}	q^{-2}	q^{-2}	1	0	0
(4^21^2)	0	$-q^{-5}$	$-q^{-3}$	$-q^{-3}$	$-q^{-1}$	0	0
(43^2)	0	0	$-q^{-3}$	0	$-q^{-1}$	0	0
(431^3)	0	0	0	0	q^{-2}	0	0
(42^3)	0	0	q^{-4}	q^{-2}	q^{-2}	1	0
(42^21^2)	0	0	0	$-q^{-3}$	$-q^{-3}$	$-q^{-1}$	0
(421^4)	0	0	0	q^{-4}	0	q^{-2}	0
(41^6)	0	0	0	0	0	$-q^{-3}$	0
(3^31)	0	0	q^{-4}	0	q^{-2}	0	0
(3^22^2)	0	0	$-q^{-5}$	$-q^{-3}$	$-q^{-3}$	$-q^{-1}$	0
(3^221^2)	0	0	0	q^{-4}	0	q^{-2}	0
(3^21^4)	0	0	0	$-q^{-5}$	$-q^{-3}$	$-q^{-3}$	0
(32^21^3)	0	0	0	0	q^{-4}	0	0
(31^7)	0	0	0	0	0	q^{-4}	0
(2^5)	0	0	0	0	q^{-4}	q^{-2}	1
(2^41^2)	0	0	0	0	$-q^{-5}$	$-q^{-3}$	$-q^{-1}$
(2^31^4)	0	0	0	0	0	q^{-4}	q^{-2}
(2^21^6)	0	0	0	0	0	$-q^{-5}$	$-q^{-3}$
(21^8)	0	0	0	0	0	0	q^{-4}
(1^{10})	0	0	0	0	0	0	$-q^{-5}$

8.2. Basis $\{\mathcal{G}_\beta^+\}$ of the Fock space representation of $U_q(\widehat{\mathfrak{sl}}_2)$

The following tables give the coefficients $d_{\alpha,\beta}(q)$ of the expansion of $\mathcal{G}_\beta^+$ on the standard basis $\{|\alpha\rangle\}$ for $n = 2$ up to partitions of 10. They should be read by columns, *e.g.*

$$\mathcal{G}_{(3,1)}^+ = |3,1\rangle + q\,|2,2\rangle + q^2\,|2,1,1\rangle.$$

Each square matrix corresponds to a weight space of $\mathbf{F}_\infty$. (The weight space containing $|10\rangle$ being too large, the corresponding matrix had to be displayed on two pages.) The 1-dimensional weight spaces corresponding to the partitions (1), (2, 1), (3, 2, 1), (4, 3, 2, 1) have been omitted.

(2)	1	0
(1^2)	q	1

(3)	1	0
(1^3)	q	1

(4)	1	0	0	0	0
(31)	q	1	0	0	0
(2^2)	0	q	1	0	0
(21^2)	q	q^2	q	1	0
(1^4)	q^2	0	0	q	1

(5)	1	0	0	0	0
(32)	0	1	0	0	0
(31^2)	q	q	1	0	0
(2^21)	0	q^2	q	1	0
(1^5)	q^2	0	q	0	1

(41)	1	0
(21^3)	q	1

(6)	1	0	0	0	0	0	0	0	0	0
(51)	q	1	0	0	0	0	0	0	0	0
(42)	0	q	1	0	0	0	0	0	0	0
(41^2)	q	q^2	q	1	0	0	0	0	0	0
(3^2)	0	0	q	0	1	0	0	0	0	0
(31^3)	q^2	q	q^2	q	q	1	0	0	0	0
(2^3)	0	0	q^2	q	q	0	1	0	0	0
(2^21^2)	0	q^2	q^3	q^2	q^2	q	q	1	0	0
(21^4)	q^2	q^3	0	q	0	q^2	0	q	1	0
(1^6)	q^3	0	0	q^2	0	0	0	0	q	1

(7)	1	0	0	0	0	0	0	0	0	0
(52)	0	1	0	0	0	0	0	0	0	0
(51^2)	q	q	1	0	0	0	0	0	0	0
(421)	0	q^2	q	1	0	0	0	0	0	0
(3^21)	0	0	0	q	1	0	0	0	0	0
(32^2)	0	0	q	q^2	q	1	0	0	0	0
(321^2)	0	q	q^2	q^3	q^2	q	1	0	0	0
(31^4)	q^2	q^2	q	0	0	0	q	1	0	0
(2^21^3)	0	q^3	q^2	0	0	q	q^2	q	1	0
(1^7)	q^3	0	q^2	0	0	0	0	q	0	1

(61)	1	0	0	0	0
(43)	0	1	0	0	0
(41^3)	q	q	1	0	0
(2^31)	0	q^2	q	1	0
(21^5)	q^2	0	q	0	1

(521)	1	0
(321^3)	q	1

(8)	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(71)	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(62)	0	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(61^2)	q	q^2	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(53)	0	0	q	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(51^3)	q^2	q	q^2	q	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(4^2)	0	0	0	0	q	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
(431)	0	0	q	0	q^2	0	q	1	0	0	0	0	0	0	0	0	0	0	0	0
(42^2)	0	0	q^2	q	0	0	0	q	1	0	0	0	0	0	0	0	0	0	0	0
(421^2)	0	q^2	$q+q^3$	q^2	q^2	q	q	q^2	q	1	0	0	0	0	0	0	0	0	0	0
(41^4)	q^2	q^3	q^2	q	q^3	q^2	q^2	0	0	q	1	0	0	0	0	0	0	0	0	0
(3^22)	0	0	0	0	0	0	q	q^2	q	0	0	1	0	0	0	0	0	0	0	0
(3^21^2)	0	0	q^2	0	0	0	q^2	q^3	q^2	q	0	q	1	0	0	0	0	0	0	0
(32^21)	0	0	q^3	q^2	q^2	q	$q+q^3$	q^4	$q+q^3$	q^2	0	q^2	q	1	0	0	0	0	0	0
(31^5)	q^3	q^2	q^3	q^2	0	q	0	0	0	q^2	q	0	q	0	1	0	0	0	0	0
(2^4)	0	0	0	0	q^3	q^2	q^2	0	0	0	0	0	0	q	0	1	0	0	0	0
(2^31^2)	0	0	q^3	q^2	q^4	q^3	q^3	0	q	q^2	q	0	q	q^2	0	q	1	0	0	0
(2^21^4)	0	q^3	q^4	q^3	0	q^2	0	0	q^2	q^3	q^2	0	q^2	0	q	0	q	1	0	0
(21^6)	q^3	q^4	0	q^2	0	q^3	0	0	0	0	q	0	0	0	q^2	0	0	q	1	0
(1^8)	q^4	0	0	q^3	0	0	0	0	0	0	q^2	0	0	0	0	0	0	0	q	1

(9)	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(72)	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(71^2)	q	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(621)	0	q^2	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(54)	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(531)	0	0	0	q	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(52^2)	0	0	q	q^2	0	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0
(521^2)	0	q	q^2	q^3	q	q^2	q	1	0	0	0	0	0	0	0	0	0	0	0	0
(51^4)	q^2	q^2	q	0	q^2	0	0	q	1	0	0	0	0	0	0	0	0	0	0	0
(4^21)	0	0	0	0	q^2	q	0	0	0	1	0	0	0	0	0	0	0	0	0	0
(421^3)	0	q^3	q^2	q	q^3	q^2	q	q^2	q	q	1	0	0	0	0	0	0	0	0	0
(3^3)	0	0	0	0	0	q^2	q	0	0	q	0	1	0	0	0	0	0	0	0	0
(3^21^3)	0	0	0	q^2	0	q^3	q^2	0	0	q^2	q	q	1	0	0	0	0	0	0	0
(32^3)	0	0	0	0	q^2	q^3	q^2	q	0	q^2	0	q	0	1	0	0	0	0	0	0
(32^21^2)	0	0	q^2	q^3	q^3	q^4	$q+q^3$	q^2	q	q^3	q^2	q^2	q	q	1	0	0	0	0	0
(321^4)	0	q^2	q^3	q^4	0	0	q^2	q	q^2	0	q^3	0	q^2	0	q	1	0	0	0	0
(31^6)	q^3	q^3	q^2	0	0	0	0	q^2	q	0	0	0	0	0	0	q	1	0	0	0
(2^41)	0	0	0	0	q^4	0	0	q^3	q^2	0	0	0	0	q^2	q	0	0	1	0	0
(2^21^5)	0	q^4	q^3	0	0	0	q^2	q^3	q^2	0	0	0	0	0	q	q^2	q	0	1	0
(1^9)	q^4	0	q^3	0	0	0	0	0	q^2	0	0	0	0	0	0	0	q	0	0	1

(81)	1	0	0	0	0	0	0	0	0	0
(63)	0	1	0	0	0	0	0	0	0	0
(61^3)	q	q	1	0	0	0	0	0	0	0
(432)	0	0	0	1	0	0	0	0	0	0
(431^2)	0	q	0	q	1	0	0	0	0	0
(42^21)	0	q^2	q	q^2	q	1	0	0	0	0
(41^5)	q^2	q^2	q	0	q	0	1	0	0	0
(3^221)	0	0	0	q^3	q^2	q	0	1	0	0
(2^31^3)	0	q^3	q^2	0	q^2	q	q	0	1	0
(21^7)	q^3	0	q^2	0	0	0	q	0	0	1

(721)	1	0	0	0	0
(541)	0	1	0	0	0
(521^3)	q	q	1	0	0
(32^31)	0	q^2	q	1	0
(321^5)	q^2	0	q	0	1

(10)	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(91)	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(82)	0	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(81^2)	q	q^2	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(73)	0	0	q	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
(71^3)	q^2	q	q^2	q	q	1	0	0	0	0	0	0	0	0	0	0	0	0
(64)	0	0	0	0	q	0	1	0	0	0	0	0	0	0	0	0	0	0
(631)	0	0	q	0	q^2	0	q	1	0	0	0	0	0	0	0	0	0	0
(622)	0	0	q^2	q	0	0	0	q	1	0	0	0	0	0	0	0	0	0
(621^2)	0	q^2	$q+q^3$	q^2	q^2	q	q	q^2	q	1	0	0	0	0	0	0	0	0
(61^4)	q^2	q^3	q^2	q	q^3	q^2	q^2	0	0	q	1	0	0	0	0	0	0	0
(5^2)	0	0	0	0	0	0	q	0	0	0	0	1	0	0	0	0	0	0
(532)	0	0	0	0	0	0	q	q^2	q	0	0	0	1	0	0	0	0	0
(531^2)	0	0	q^2	0	q	0	$2q^2$	q^3	q^2	q	0	q	q	1	0	0	0	0
(52^21)	0	0	q^3	q^2	q^2	q	q^3	q^4	$q+q^3$	q^2	0	0	q^2	q	1	0	0	0
(51^5)	q^3	q^2	q^3	q^2	q^2	q	q^3	0	0	q^2	q	q^2	0	q	0	1	0	0
(4^22)	0	0	0	0	0	0	q^2	0	0	0	0	q	q	0	0	0	1	0
(4^21^2)	0	0	0	0	q^2	0	q^3	0	0	0	0	q^2	q^2	q	0	0	q	1
(43^2)	0	0	0	0	0	0	0	0	q	0	0	0	q^2	0	0	0	q	0
(431^3)	0	0	q^2	0	q^3	0	q^2	q	q^2	q	0	q	q^3	q^2	0	0	q^2	q
(42^3)	0	0	0	0	q^3	q^2	q^2	0	q^2	q	0	q	q^3	q^2	q	0	q^2	q
(42^21^2)	0	0	q^3	q^2	q^4	q^3	$2q^3$	q^2	$q+q^3$	$2q^2$	q	$2q^2$	q^4	q^3	q^2	0	q^3	q^2
(421^4)	0	q^3	q^4+q^2	q^3	q^3	q^2	q^4	q^3	q^2	$q+q^3$	q^2	q^3	0	q^2	0	q	0	q
(41^6)	q^3	q^4	q^3	q^2	q^4	q^3	0	0	0	q^2	q	0	0	q^3	0	q^2	0	q^2
(3^31)	0	0	0	0	0	0	0	0	q^2	0	0	q	q^3	q^2	q	0	q^2	q
(3^22^2)	0	0	0	0	0	0	q^3	0	q^3	q^2	0	$2q^2$	q^4	q^3	q^2	0	q^3	q^2
(3^221^2)	0	0	0	0	0	0	q^4	q^3	q^4+q^2	q^3	0	$2q^3$	q^5	q^4	q^3	0	q^4	$q+q^3$
(3^21^4)	0	0	q^3	0	0	0	0	q^4	q^3	q^2	0	0	0	0	0	0	0	q^2
(32^21^3)	0	0	q^4	q^3	q^3	q^2	q^4	q^5	q^4+q^2	$2q^3$	q^2	q^3	0	q^2	q	q	0	$q+q^3$
(31^7)	q^4	q^3	q^4	q^3	0	q^2	0	0	0	q^3	q^2	0	0	0	0	q	0	0
(2^5)	0	0	0	0	0	0	q^4	0	0	q^3	q^2	q^3	0	0	0	0	0	0
(2^41^2)	0	0	0	0	q^4	q^3	q^5	0	0	q^4	q^3	q^4	0	q^3	q^2	q^2	0	q^2
(2^31^4)	0	0	q^4	q^3	q^5	q^4	0	0	q^2	q^3	q^2	0	0	q^4	q^3	q^3	0	q^3
(2^21^6)	0	q^4	q^5	q^4	0	q^3	0	0	q^3	q^4	q^3	0	0	0	0	q^2	0	0
(21^8)	q^4	q^5	0	q^3	0	q^4	0	0	0	0	q^2	0	0	0	0	q^3	0	0
(1^{10})	q^5	0	0	q^4	0	0	0	0	0	0	q^3	0	0	0	0	0	0	0

(43^2)	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(431^2)	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(42^3)	q	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(42^21^2)	q^2	q	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(421^4)	0	q^2	0	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0
(41^6)	0	0	0	0	q	1	0	0	0	0	0	0	0	0	0	0	0	0
(3^31)	q	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
(3^22^2)	q^2	0	q	0	0	0	q	1	0	0	0	0	0	0	0	0	0	0
(3^221^2)	$q+q^3$	q^2	q^2	q	0	0	q^2	q	1	0	0	0	0	0	0	0	0	0
(3^21^4)	q^2	q^3	0	q^2	q	0	0	0	q	1	0	0	0	0	0	0	0	0
(32^21^3)	q^3	q^4	q^2	$q+q^3$	q^2	0	0	q	q^2	q	1	0	0	0	0	0	0	0
(31^7)	0	0	0	0	q^2	q	0	0	0	q	0	1	0	0	0	0	0	0
(2^5)	0	0	q^2	q	0	0	0	q	0	0	0	0	1	0	0	0	0	0
(2^41^2)	0	0	q^3	q^2	0	0	0	q^2	0	0	q	0	q	1	0	0	0	0
(2^31^4)	0	0	0	q	q^2	q	0	0	0	q	q^2	0	0	q	1	0	0	0
(2^21^6)	0	0	0	q^2	q^3	q^2	0	0	0	q^2	0	q	0	0	q	1	0	0
(21^8)	0	0	0	0	0	q	0	0	0	0	0	q^2	0	0	0	q	1	0
(1^{10})	0	0	0	0	0	q^2	0	0	0	0	0	0	0	0	0	0	q	1

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