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Ljusternik-Schnirelmann theory on C^1-manifolds

by

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ABSTRACT. — Let M be a complete Finsler manifold of class C^1. It is shown that if M contains a compact subset of category k (in M), then each function $f \in C^1(M, \mathbb{R})$ which is bounded below and satisfies the Palais-Smale condition must necessarily have k critical points. This should be compared with the known result that f has at least cat(M) critical points provided M is of class C^2. An application is given to an eigenvalue problem for a quasilinear differential equation involving the p-Laplacian $-\operatorname{div}(|\nabla u|^{p-2} \nabla u)$, $1 < p < \infty$.

Key words : Finsler manifold, critical point, Ljusternik-Schnirelmann theory, Ekeland's variational principle, category, genus, eigenvalue problem, p-Laplacian.

RÉSUMÉ. — Soit M une variété de Finsler complète de classe C^1. On démontre que si M contient un sous-ensemble compacte de catégorie k (dans M), alors toute fonction $f \in C^1(M, \mathbb{R})$ qui est bornée inférieurement et satisfait à la condition de Palais-Smale doit nécessairement avoir k point critique. Ce résultat est à rapprocher du théorème connu selon lequel f a au moins cat(M) point critique lorsque M est de classe C^2. On donne une application à un problème de valeur propre pour une équation

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différentielle quasi-linéaire faisant intervenir le p-Laplacien $-\text{div}(|\nabla u|^{p-2}\nabla u)$, $1<p<\infty$.

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1. INTRODUCTION

Let M be a compact C^2-manifold without boundary and $f: M \to \mathbb{R}$ a continuously differentiable function. A classical result by Ljusternik and Schnirelmann [14], *cf.* also ([8], [21]), asserts that if M is of category k [denoted $\text{cat}(M)=k$], then f has at least k distinct critical points (all definitions will be given in the next section). This result has been generalized by Palais ([16], [17]) who proved the following

1.1. THEOREM. — *Let* M *be a* C^2 *Finsler manifold (without boundary) and* $f \in C^1(M, \mathbb{R})$ *a function which is bounded below and such that for each* $c \in \mathbb{R}$ *the set* $f_c = \{x \in M : f(x) \leqq c\}$ *is complete in the Finsler metric for* M. *If* f *satisfies the Palais-Smale condition and if* $\text{cat}(M)=k$, *then* f *has at least* k *distinct critical points.*

The key ingredient in the proof of Theorem 1.1 is a deformation lemma which in its simplest form says that if c is not a critical value of f and if $\varepsilon>0$ is small enough, then there exists a mapping $\eta : [0, 1] \times M \to M$ satisfying $\eta(0, x)=x$, $f(\eta(t, x)) \leqq f(x)$ for all t and x, and $\eta(1, f_{c+\varepsilon}) \subset f_{c-\varepsilon}$ (i. e., η deforms $f_{c+\varepsilon}$ to $f_{c-\varepsilon}$). The deformation is constructed by letting $\eta(t, x)$ move along the integral lines of a pseudogradient vector field for f as t varies from 0 to 1. As is well known from the theory of ordinary differential equations, integral lines may not exist unless the vector field is locally Lipschitz continuous. To carry out the above construction it seems therefore necessary to assume that M is at least of class C^{2-} (a mapping is of class C^{2-} if it is differentiable and the derivative is locally Lipschitz continuous).

In this paper we will be concerned with a generalization of Theorem 1.1 to C^1-manifolds. Ideally, one would like to show that the conclusion remains valid if M is a C^1 Finsler manifold. Our result is slightly weaker, yet it seems to be sufficient for most of practical purposes. It asserts that if M is C^1 and contains a compact set of category k (in M), other assumptions being as in Theorem 1.1, then the conclusion still holds. Note in particular that if M contains compact sets of arbitrarily large category, then f has infinitely many critical points.

The proof of Theorem 1.1 is carried out as follows. Set

$$c_j = \inf_{\mathrm{cat}_M(A) \geqq j} \sup_{x \in A} f(x),$$

where $1 \leqq j \leqq k$ and $\mathrm{cat}_M(A)$ denotes the category of A in M. Assume for simplicity that all c_j are finite and distinct. Then, if c_j is not a critical value, one finds an A with $\mathrm{cat}_M(A) \geqq j$ and $f(x) \leqq c + \varepsilon \, \forall x \in A$. By the deformation lemma, if $B = \eta(1, A)$, then $\mathrm{cat}_M(B) \geqq j$ and $f(x) \leqq c - \varepsilon \, \forall x \in B$, a contradiction. So all c_j are critical and f has at least k critical points. As we pointed out earlier, this argument is not readily applicable if M is only of class C^1. Our proof is therefore quite different. We define

$$c_j = \inf_{A \in \Lambda_j} \sup_{x \in A} f(x),$$

where $\Lambda_j = \{ A \subset M : \mathrm{cat}_M(A) \geqq j \text{ and } A \text{ is compact} \}$. On Λ_j we introduce the Hausdorff metric *dist* and set $\Pi(A) = \sup_{x \in A} f(x)$. Again, assume for simplicity that all c_j are distinct. By Ekeland's variational principle (see the next section), there exists an $A \in \Lambda_j$ such that

$$\Pi(B) - \Pi(A) \geqq -\varepsilon \, dist(A, B), \qquad \forall B \in \Lambda_j.$$

If c_j is not a critical value, then, by slightly deforming A, we find $B \in \Lambda_j$ with $dist(A, B) \leqq s$ and $\Pi(B) - \Pi(A) < -\varepsilon s$ for all small $s > 0$. So $-\varepsilon s \leqq \Pi(B) - \Pi(A) < -\varepsilon s$, a contradiction. The idea of using Ekeland's principle to show the existence of critical points other than local minima may be found in [2] (Section 5.5), and an argument similar to the above one (using Ekeland's principle on the space of subsets) – in [20].

Suppose now that X is a Banach space and $f, g \in C^1(X, \mathbb{R})$ are two even functions. Consider the eigenvalue problem

$$\textit{Find } (x, \lambda) \in X \times \mathbb{R} \quad \textit{such that } f'(x) = \lambda g'(x) \textit{ and } g(x) = b. \tag{1}$$

Problems of this type have been studied by several authors. See e. g. [1], [3], [5], [10], [11], [19], [23]. If $b \neq g(0)$ is a regular value of g, then $M = g^{-1}(b)$ is a C^1-manifold, $0 \notin M$ and there is a one-to-one correspondence between solutions of (1) and critical points of $f|_M$. Assuming in addition that $f|_M$ is bounded below and $g \in C^2(X, \mathbb{R})$, we may pass to the quotient space $\tilde{M} = M/\sim$, where $\sim$ is the equivalence relation identifying

x with $-x$, and use Theorem 1.1 in order to obtain a lower bound for the number of solutions of (1). The assumption that $g \in C^2(X, \mathbb{R})$ [or even $g \in C^{2-}(X, \mathbb{R})$] turns out to be too restrictive for some applications, *cf.* Browder [5]. One possible approach to (1) when $g \in C^1(X, \mathbb{R})$ is by using the Galerkin approximations (*see* e. g. [5]). However, in order to carry out the limiting procedure it seems necessary to put some restrictions on f and g which are not needed in the case of $g \in C^2(X, \mathbb{R})$. A different approach has been taken by Amann [1] who has shown that the deformation lemma remains valid whenever $g \in C^1(X, \mathbb{R})$ and M is bounded and homeomorphic to the unit sphere by the radial projection mapping. From this he has derived results on (1) which generalize those in [5]. As a corollary to our generalization of Theorem 1.1 we shall show that it is neither necessary to assume that $g \in C^{2-}(X, \mathbb{R})$ nor that M is bounded and homeomorphic to the unit sphere.

The paper is organized as follows. In Section 2 we collect some definitions and facts which will be useful later. In Section 3 we state and prove the main theorem. Some of its consequences and extensions are given in Section 4. In Section 5 we present an application to the boundary value problem

$$A u + f(u) = \lambda g(u) \quad \text{in } \Omega, \qquad u\big|_{\partial\Omega} = 0,$$

where $\Omega \subset \mathscr{R}^N$ is bounded and A is the p-Laplacian, $1 < p < \infty$ (in particular, $A = -\Delta$ if $p = 2$).

I would like to thank Ivar Ekeland for bringing to my attention the problem of generalizing the Ljusternik-Schnirelmann theory to C^1-manifolds.

2. PRELIMINARIES

Let M be a C^1 Banach manifold (without boundary). Denote the tangent bundle of M by T(M) and the tangent space of M at x by $T_x(M)$. Let $\| \ \|: T(M) \to [0, +\infty)$ be a continuous function such that

(i) For each $x \in M$, the restriction of $\| \ \|$ to $T_x(M)$, denoted by $\| \ \|_x$ (or sometimes simply by $\| \ \|$), is an admissible norm on $T_x(M)$;

(ii) For each $x_0 \in M$ and $k>1$ there is a trivializing neighbourhood U of x_0 such that

$$\frac{1}{k}\| \quad \|_x \leqq \| \quad \|_{x_0} \leqq k \| \quad \|_x, \qquad \forall x \in U.$$

The function $\| \quad \|$ is called *a Finsler structure* for T(M). A regular manifold together with a fixed Finsler structure for T(M) is called *a Finsler manifold.* Every paracompact C^1 Banach manifold admits a Finsler structure [16] (Theorem 2.11). For a C^1-path $\sigma:[a, b] \to M$ define *the length* of σ by

$$l(\sigma)=\int_b^a \|\sigma'(t)\| \, dt.$$

If x, y are two points in the same connected component of M, let *the distance* $\rho(x, y)$ be defined as the infimum of $l(\sigma)$ over all σ joining x and y. Then ρ is a metric for each component of M (called *the Finsler metric*), and it is consistent with the topology of M [17] (Section 2).

Let M be a Finsler manifold and $f \in C^1(M, \mathbb{R})$. Denote the differential of f at x by $df(x)$. Then $df(x)$ is an element of the cotangent space of M at x, $T_x(M)^*$. A point $x \in M$ is said to be *a critical point* of f if $df(x)=0$. The corresponding value $c=f(x)$ will be called *a critical value.* Values other than critical are *regular.* We shall repeatedly use the following notation:

$$K=\{x \in M : df(x)=0\}, \qquad K_c=K \cap f^{-1}(c),$$
$$f_c=\{x \in M : f(x) \leqq c\}.$$

If M is a Finsler manifold, then the cotangent bundle $T(M)^*$ has a dual Finsler structure given by

$$\|w\|=\sup\{\langle w, v\rangle : v \in T_x(M), \|v\|_x=1\},$$

where $w \in T_x(M)^*$ and $\langle \ , \ \rangle$ is the duality pairing between $T_x(M)^*$ and $T_x(M)$. It follows that the mapping $x \mapsto \|df(x)\|$ is well defined and continuous for $f \in C^1(M, \mathbb{R})$. A function $f \in C^1(M, \mathbb{R})$ is said to satisfy *the Palais-Smale condition at the level c*, $c \in \mathbb{R}$, [$(PS)_c$ in short] if each sequence $(x_n) \subset M$ such that $f(x_n) \to c$ and $\|df(x_n)\| \to 0$ has a convergent subsequence. This is a local version of the following compactness condition due to Palais and Smale: If $f(x_n)$ is bounded and $\|df(x_n)\| \to 0$, then a subsequence of (x_n) converges.

Let $x_0 \in M - K$. There exists a vector $V_0 \in T_{x_0}(M)$ such that $\|V_0\| = 1$ and $\langle df(x_0), V_0 \rangle > \frac{2}{3}\|df(x_0)\|$. Set $v_0 = \frac{3}{2}\|df(x_0)\| V_0$. Then

$$\|v_0\| < 2\|df(x_0)\| \quad \text{and} \quad \langle df(x_0), v_0 \rangle > \|df(x_0)\|^2.$$

Such v_0 is called *pseudogradient vector* ([16], [17]). It is easily seen that $\|v_0\| > \|df(x_0)\|$ and $\langle df(x_0), v_0 \rangle > \frac{1}{4}\|v_0\|^2$. Let $\varphi : U \to T_{x_0}(M)$ be a chart at x_0. Denote

$$g = f \circ \varphi^{-1} : \varphi(U) \subset T_{x_0}(M) \to \mathbb{R}.$$

Then $df(x_0)$ is locally represented by $g'(\varphi(x_0))$, where g' is the Fréchet derivative of g. Therefore $\langle g'(\varphi(x_0)), v_0 \rangle > \frac{1}{4}\|v_0\|^2$. Since g' is continuous,

$$\|v_0\| > \|df(x_0)\| \quad \text{and} \quad \langle g'(y), v_0 \rangle > \frac{1}{4}\|v_0\|^2, \quad \forall y \in \varphi(U) \tag{2}$$

provided U is small enough. We have proved

2.1. PROPOSITION. — *For each* $x_0 \in M - K$ *there exist a chart* $\varphi : U \to T_{x_0}(M)$ *at* x_0 *and a vector* $v_0 \in T_{x_0}(M)$ *such that* (2) *is satisfied* (*with* $g = f \circ \varphi^{-1}$).

In what follows we shall need the notions of Ljusternik-Schnirelmann category and genus. Let M be a topological space. A set $A \subset M$ is said to be *of category k in* M [denoted $\mathrm{cat}_M(A) = k$] if it can be covered by k but not $k-1$ closed sets which are contractible to a point in M. If such k does not exist, $\mathrm{cat}_M(A) = +\infty$. Let X be a real Banach space and Σ the collection of all symmetric subsets of $X - \{0\}$ which are closed in X (A is symmetric if $A = -A$). A nonempty set $A \in \Sigma$ is said to be *of genus k* [denoted $\gamma(A) = k$] if k is the smallest integer with the property that there exists an odd continuous mapping from A to $\mathbb{R}^k - \{0\}$. If there is no such k, $\gamma(A) = +\infty$, and if $A = \varnothing$, $\gamma(A) = 0$. Below we summarize pertinent properties of category and genus.

2.2. PROPOSITION. — *Let* M *be a topological space and* A, $B \subset M$. *Then*

(*a*) $\mathrm{cat}_M(A) = 0$ *if and only if* $A = \varnothing$.

(*b*) $\mathrm{cat}_M(A) = 1$ *if and only if* $\bar{A}$ *is contractible to a point in* M.

(*c*) *If* $A \subset B$, *then* $\mathrm{cat}_M(A) \leq \mathrm{cat}_M(B)$.

(*d*) $\mathrm{cat}_M(A \cup B) \leq \mathrm{cat}_M(A) + \mathrm{cat}_M(B)$.

(*e*) *If* $\mathrm{cat}_M(B) < \infty$, *then* $\mathrm{cat}_M(A-B) \geqq \mathrm{cat}_M(A) - \mathrm{cat}_M(B)$.

(*f*) *If* A *is closed in* M *and* $\alpha: [0, t_0] \times A \to M$ *is a deformation of* A (*i. e.*, $\alpha(0, x) = x \, \forall x \in A$), *then* $\mathrm{cat}_M(A) \leqq \mathrm{cat}_M(\alpha(t_0, A))$.

(*g*) *If* M *is a Finsler manifold and* $A \subset M$, *then there is a neighbourhood* U *of* A *such that* $\mathrm{cat}_M(\bar{U}) = \mathrm{cat}_M(A)$.

(*h*) *If* M *is a connected Finsler manifold and* A *is a closed subset of* M, *then* $\mathrm{cat}_M(A) \leqq \dim(A) + 1$, *where* dim *denotes the covering dimension.*

Properties (*a*)–(*d*) follow directly from the definition, (*e*) follows from (*c*) and (*d*) because $A \subset (A-B) \cup B$, (*f*) is Theorem 6.2(3) in [16] and (vii) on p. 191 in [17], and (*g*), (*h*) follow from Theorems 6.3, 6.4 in [16] upon observing that each Finsler manifold is necessarily an absolute neighbourhood retract (ANR) [15] (Theorem 5).

2.3. Proposition. – *Let* $A, B \in \Sigma$. *Then*

(*a*) *If there exists an odd continuous mapping* $f: A \to B$, *then* $\gamma(A) \leqq \gamma(B)$.

(*b*) *If* $A \subset B$, *then* $\gamma(A) \leqq \gamma(B)$.

(*c*) $\gamma(A \cup B) \leqq \gamma(A) + \gamma(B)$.

(*d*) *If* $\gamma(B) < \infty$, $\gamma(\overline{A-B}) \geqq \gamma(A) - \gamma(B)$.

(*e*) *If* A *is compact, then* $\gamma(A) < \infty$ *and there exists a neighbourhood* N *of* A, $N \in \Sigma$, *such that* $\gamma(N) = \gamma(A)$.

(*f*) *If* N *is a symmetric and bounded neighbourhood of the origin in* $\mathbb{R}^k$ *and if* A *is homeomorphic to the boundary of* N *by an odd homeomorphism, then* $\gamma(A) = k$.

(*g*) *If* X_0 *is a subspace of* X *of codimension* k *and if* $\gamma(A) > k$, *then* $A \cap X_0 \neq \varnothing$.

Properties (*a*)–(*f*) may be found e. g. in [8], [19], [21] and (*g*) in [19].

In the proof of the main theorem we shall employ the following variational principle due to Ekeland [2] (Corollary 5.3.2), [9].

2.4. Proposition. – *Let* (Z, d) *be a complete metric space and* $\Pi: Z \to (-\infty, +\infty]$ *a proper* (*i. e.*, $\Pi \not\equiv +\infty$) *lower semicontinuous function which is bounded below. If* $\varepsilon > 0$ *and* $x \in Z$ *satisfy*

$$\Pi(x) \leqq \inf_{z \in Z} \Pi(z) + \varepsilon^2,$$

then there exists a $y \in Z$ *such that*

$$\Pi(y) \leqq \Pi(x), \qquad d(x, y) \leqq \varepsilon$$

and

$$\Pi(z) - \Pi(y) \geqq -\varepsilon d(y, z), \qquad \forall z \in Z.$$

3. THE MAIN THEOREM

3.1. THEOREM. – *Suppose that* M *is a* C^1 *Finsler manifold and* $f \in C^1(M, \mathbb{R})$ *is bounded below and such that* f_c *is complete in the metric* ρ *for each* $c \in \mathbb{R}$. *Define*

$$c_j = \inf_{A \in \Lambda_j} \sup_{x \in A} f(x),$$

where $\Lambda_j = \{ A \subset M : \text{cat}_M(A) \geqq j$ *and* A *is compact* $\}$. *If* $\Lambda_k \neq \varnothing$ *for some* $k \geqq 1$ *and if* f *satisfies* $(PS)_c$ *for all* $c = c_j$, $j = 1, \ldots, k$, *then* f *has at least* k *distinct critical points.*

Proof. – Assume that M is connected. This causes no loss of generality because if $M = \bigcup_i M_i$, where M_i are the connected components of M, then it follows from the definition of category that $\text{cat}_M(A) = \sum_i \text{cat}_{M_i}(A \cap M_i)$. Since $\Lambda_{j+1} \subset \Lambda_j$ for $j = 1, \ldots, k-1$ and the sets $A \in \Lambda_j$ are compact,

$$-\infty < c_1 \leqq c_2 \leqq \ldots \leqq c_k < +\infty.$$

Given j, suppose $c_j = \ldots = c_{j+p} \equiv c$ for some $p \geqq 0$. It suffices to show that

$$\text{cat}_M(K_c) \geqq p + 1. \tag{3}$$

Indeed, it follows from (3) that $\text{cat}_M(K_{c_j}) \geqq 1$, so $K_{c_j} \neq \varnothing$. This gives the correct number of critical points if all c_j are distinct. If they are not, $p > 0$ for some j. Therefore $\dim(K_{c_j}) \geqq \text{cat}_M(K_{c_j}) - 1 \geqq 1$ according to (*h*) of Proposition 2.2, so K_{c_j} is an infinite set.

Let $b > c_k$ be a real number. Define

$$\Gamma_j = \{ A \subset f_b : \text{cat}_M(A) \geqq j \text{ and A is compact} \}.$$

It is easy to see that

$$c_j = \inf_{A \in \Gamma_j} \sup_{x \in A} f(x).$$

Let $\mathcal{S}$ be the collection of all nonempty closed and bounded subsets of f_b. In $\mathcal{S}$ we introduce the Hausdorff metric *dist* [12] (§ 15. VII) given by

$$dist(\mathrm{A}, \mathrm{B}) = \max\{\sup_{a \in \mathrm{A}} \rho(a, \mathrm{B}), \sup_{b \in \mathrm{B}} \rho(b, \mathrm{A})\},$$

where $\rho(a, \mathrm{B}) = \inf_{b \in \mathrm{B}} \rho(a, b)$. Since f_b is complete, so is the space $(\mathcal{S}, dist)$ [12] (§ 29. IV).

In order to continue the proof we shall need two lemmas.

3.2. LEMMA. — *$(\Gamma_j, dist)$ is a complete metric space.*

Proof. — It suffices to show that Γ_j is closed in $\mathcal{S}$. Let (A_n) be a sequence in Γ_j and let $\mathrm{A}_n \to \mathrm{A}$. It is easy to see that A is compact. Let U be a neighbourhood of A in M such that $\mathrm{cat}_\mathrm{M}(\mathrm{U}) = \mathrm{cat}_\mathrm{M}(\mathrm{A})$ [*cf.* Proposition 2.2 (*g*)]. Since $\mathrm{A}_n \to \mathrm{A}$, $\mathrm{A}_n \subset \mathrm{U}$ for almost all n. Hence $\mathrm{cat}_\mathrm{M}(\mathrm{A}) = \mathrm{cat}_\mathrm{M}(\mathrm{U}) \geqq \mathrm{cat}_\mathrm{M}(\mathrm{A}_n) \geqq j$, so $\mathrm{A} \in \Gamma_j$. □

3.3. LEMMA. — *The function $\Pi : \Gamma_j \to \mathbb{R}$ defined by*

$$\Pi(\mathrm{A}) = \sup_{x \in \mathrm{A}} f(x)$$

is lower semicontinuous.

Proof. — Let $\mathrm{A}_n \to \mathrm{A}$. For each $x \in \mathrm{A}$ there is a sequence (x_n) such that $x_n \to x$ and $x_n \in \mathrm{A}_n$. Therefore

$$f(x) = \lim f(x_n) \leqq \liminf \Pi(\mathrm{A}_n).$$

Since x was chosen arbitrarily, $\Pi(\mathrm{A}) \leqq \liminf \Pi(\mathrm{A}_n)$. □

Proof of Theorem 3.1 *continued.* — Recall that we want to show that $\mathrm{cat}_\mathrm{M}(\mathrm{K}_c) \geqq p+1$ [*cf.* (3)]. Suppose $\mathrm{cat}_\mathrm{M}(\mathrm{K}_c) \leqq p$. Denote

$$\mathrm{N}_\delta(\mathrm{K}_c) = \{x \in \mathrm{M} : \rho(x, \mathrm{K}_c) \leqq \delta\}.$$

Since f satisfies $(\mathrm{PS})_c$, K_c is compact. It is therefore possible [via Proposition 2.2 (*g*)] to choose $\delta > 0$ so that $\mathrm{cat}_\mathrm{M}(\mathrm{N}_{2\delta}(\mathrm{K}_c)) = \mathrm{cat}_\mathrm{M}(\mathrm{K}_c) \leqq p$. Let $k \in \left(1, \frac{3}{2}\right)$ be a fixed number. Using $(\mathrm{PS})_c$, we may find an arbitrarily small $\varepsilon > 0$ with the property that

$$\|df(x)\| \geqq 6\varepsilon, \qquad \forall x \in f^{-1}([c-\varepsilon, c+\varepsilon]) - \mathrm{N}_\delta(\mathrm{K}_c). \tag{4}$$

Suppose $\varepsilon < \delta < 1$. Choose an $\mathrm{A}_1 \in \Gamma_{j+p}$ such that $\Pi(\mathrm{A}_1) \leqq c + \varepsilon^2$. Let $\mathrm{A}_2 = \overline{\mathrm{A}_1 - \mathrm{N}_{2\delta}(\mathrm{K}_c)}$. Then $\Pi(\mathrm{A}_2) \leqq c + \varepsilon^2$ and, by Proposition 2.2, $\mathrm{cat}_\mathrm{M}(\mathrm{A}_2) \geqq \mathrm{cat}_\mathrm{M}(\mathrm{A}_1) - \mathrm{cat}_\mathrm{M}(\mathrm{N}_\delta(\mathrm{K}_c)) \geqq j+p-p = j$. So $\mathrm{A}_2 \in \Gamma_j$. By Proposition 2.4 and Lemmas 3.2 and 3.3, there is an $A \in \Gamma_j$ such that

$$\Pi(A) \leqq c + \varepsilon^2, \qquad dist(A, A_2) \leqq \varepsilon$$

and

$$\Pi(B) - \Pi(A) \geqq -\varepsilon\, dist(A, B), \qquad \forall B \in \Gamma_j. \tag{5}$$

Since $\varepsilon < \delta$ and $dist\,(A, A_2) \leqq \varepsilon$,

$$A \cap N_\delta(K_c) = \varnothing. \tag{6}$$

Our goal now is to obtain a contradiction by constructing a $B \in \Gamma_j$ which will fail to satisfy (5). Denote

$$S = A \cap \left\{ x \in M : f(x) \geqq c - \frac{1}{2}\varepsilon \right\}.$$

Then $S \neq \varnothing$ because $A \in \Gamma_j$. Given $x_i \in S$, choose a chart $\varphi_i : U_i \to T_{x_i}(M)$ at x_i such that

$$\frac{1}{k} \| \quad \|_x \leqq \| \quad \|_{x_i} \leqq k \| \quad \|_x, \qquad \forall x \in U_i. \tag{7}$$

It follows from (6) and Proposition 2.1 that if U_i is sufficiently small, then

$$U_i \subset f^{-1}([c-\varepsilon, c+\varepsilon]) - N_\delta(K_c) \tag{8}$$

and there exists a vector v_i satisfying (2) $\forall y \in \varphi(U_i)$. Let $V_i \subset U_i$ be an open neighbourhood of x_i such that

$$\rho(x, M - U_i) \geqq \delta_i \qquad \text{and} \qquad \rho_i(\varphi_i(x), T_{x_i}(M) - \varphi_i(U_i)) \geqq \delta_i, \qquad \forall x \in V_i,$$

where $\delta_i > 0$ and

$$\rho_i(\varphi_i(x), T_{x_i}(M) - \varphi_i(U_i)) = \inf\{ \| \varphi_i(x) - z \|_{x_i} : z \in T_{x_i}(M) - \varphi_i(U_i) \}.$$

Proceeding in this way for each $x_i \in S$ we obtain an open covering (V_i) of S. Since S is compact, there exists a finite subcovering $V_1, \ldots, V_m$, to which we may subordinate a continuous partition of unity $\xi_1, \ldots, \xi_m$. Let $\chi : M \to [0, 1]$ be a continuous function such that $\chi \equiv 1$ on S and $\chi \equiv 0$ on $M - \bigcup_{i=1}^{m} V_i$, and let $\psi_i = \chi \xi_i$. The sets $U_1, \ldots, U_m$ cover S, and by construction,

$$\sum_{i=1}^{m} \psi_i(x) \equiv 1 \quad on\ S, \qquad \psi_1 = \ldots = \psi_m \equiv 0 \quad \text{on } M - \bigcup_{i=1}^{m} V_i$$

and

$$\rho(x, M-U_i) \geqq \delta_0, \qquad \rho_i(\varphi_i(x), T_{x_i}(M)-\varphi_i(U_i)) \geqq \delta_0 \qquad \forall x \in \operatorname{supp} \psi_i, \tag{9}$$

where $\delta_0 = \min\{\delta_1, \ldots, \delta_m\}$.

Fix a number $t \in (0, \delta_0/(1+k^2))$ and let

$$\alpha_1(t, x) = \begin{cases} \varphi_1^{-1}\left(\varphi_1(x) - t\psi_1(x)\dfrac{v_1}{\|v_1\|}\right) & \text{if } x \in U_1 \\ x & \text{otherwise.} \end{cases}$$

According to (9), α_1 is well defined and continuous. For an arbitrary point $x \in U_1$, let $\sigma_1(s) = \alpha_1(s, x)$, $0 \leqq s \leqq t$. Since σ_1 is a path joining x to $\alpha_1(t, x)$,

$$\rho(x, \alpha_1(t, x)) \leqq \int_0^t \|\sigma_1'(s)\| ds \leqq k \int_0^t \left\| \frac{d}{ds} \varphi_1(\sigma_1(s)) \right\|_{x_1} ds = k\psi_1(x)t \tag{10}$$

according to (7) and the definition of α_1. Denote $g = f \circ \varphi_1^{-1}$ and let $x \in U_1$. Then it follows from the mean value theorem and Proposition 2.1 that for some $\theta \in (0, 1)$,

$$\begin{aligned} f \circ \alpha_1(t, x) - f(x) &= g\left(\varphi_1(x) - t\psi_1(x)\frac{v_1}{\|v_1\|}\right) - g(\varphi_1(x)) \\ &= -t\psi_1(x)\left\langle g'\left(\varphi_1(x) - \theta t\psi_1(x)\frac{v_1}{\|v_1\|}\right), \frac{v_1}{\|v_1\|}\right\rangle \leqq -\frac{1}{4}t\psi_1(x)\|v_1\| \\ &\leqq -\frac{1}{4}t\psi_1(x)\|df(x_1)\|. \end{aligned} \tag{11}$$

Therefore, employing (8) and (4),

$$f \circ \alpha_1(t, x) - f(x) \leqq -\frac{3}{2}\varepsilon t\psi_1(x). \tag{12}$$

Note that (10) and (12) are satisfied for all $x \in M$ because $\psi_1(x) = 0$ whenever $x \notin U_1$. Note also that $\alpha_1(t, A) \in \Gamma_j$ according to (*f*) of Proposition 2.2.

Let

$$\alpha_2(t, x) = \begin{cases} \varphi_2^{-1}\left(\varphi_2(\alpha_1(t, x)) - t\psi_2(x)\dfrac{v_2}{\|v_2\|}\right) & \text{if } \alpha_1(t, x) \in U_2 \\ \alpha_1(t, x) & \text{otherwise.} \end{cases}$$

We need to show that α_2 is well defined and continuous. Let $x \in \operatorname{supp} \psi_2$ and let σ be a path joining x to $\alpha_1(t, x)$. Since $\rho(x, M-U_2) \geqq \delta_0$ and

$$\rho(x, \alpha_1(t, x)) \leqq kt < \frac{\delta_0 k}{1+k^2} \leqq \frac{1}{2}\delta_0 \tag{13}$$

according to (9) and (10), $\alpha_1(t, x) \in U_2$ and $l(\sigma) \geqq \delta_0$ if σ leaves U_2. Therefore

$$\rho(x, \alpha_1(t, x)) = \inf\{l(\sigma) : \sigma \text{ joins } x \text{ to } \alpha_1(t, x) \text{ and } \sigma \subset U_2\}.$$

Furthermore, if $\sigma \subset U_2$,

$$l(\sigma) = \int_a^b \|\sigma'(s)\| \, ds \geqq \frac{1}{k}\int_a^b \left\| \frac{d}{ds}\varphi_2(\sigma(s)) \right\|_{x2}$$

$$\geqq \frac{1}{k}\left\| \int_a^b \frac{d}{ds}\varphi_2(\sigma(s))\, ds \right\|_{x2} = \frac{1}{k}\|\varphi_2(\alpha_1(t, x)) - \varphi_2(x)\|_{x2}.$$

Combining these two facts,

$$\|\varphi_2(\alpha_1(t, x)) - \varphi_2(x)\|_{x2} \leqq k\,\rho(x, \alpha_1(t, x)) \leqq k^2 t.$$

Hence by the triangle inequality,

$$\left\| \varphi_2(\alpha_1(t, x)) - t\psi_2(x)\frac{v_2}{\|v_2\|} - \varphi_2(x) \right\|_{x2} \leqq \|\varphi_2(\alpha_1(t, x)) - \varphi_2(x)\|_{x2} + \left\| t\psi_2(x)\frac{v_2}{\|v_2\|} \right\|_{x2} \leqq k^2 t + t < \delta_0.$$

So it follows from (9) that $\varphi_2(\alpha_1(t, x)) - t\psi_2(x)\frac{v_2}{\|v_2\|} \in \varphi(U_2)$ and α_2 is well defined. If $\alpha_1(t, x)$ is sufficiently close to the boundary of U_2, then $x \notin \operatorname{supp} \psi_2$ according to (9) and (13). Hence for such x, $\alpha_2(t, x) = \alpha_1(t, x)$. Therefore α_2 is continuous. Set

$$\sigma_2(s) = \varphi_2^{-1}\left(\varphi_2(\alpha_1(t, x)) - s\psi_2(x)\frac{v_2}{\|v_2\|} \right), \qquad 0 \leqq s \leqq t.$$

Then $\rho(\alpha_1(t, x), \alpha_2(t, x)) \leqq l(\sigma_2) \leqq k\psi_2(x)t$ [*cf.* the argument of (10)]. This and (10) yield

$$\rho(x, \alpha_2(t, x)) \leqq k(\psi_1(x) + \psi_2(x))t.$$

The same argument as in (11) and (12) implies that

$$f \circ \alpha_2(t, x) - f \circ \alpha_1(t, x) \leqq -\frac{3}{2}\varepsilon t \psi_2(x).$$

So by (12),

$$f\circ\alpha_2(t,x)-f(x)\leqq-\frac{3}{2}\varepsilon t(\psi_1(x)+\psi_2(x)).$$

Since $\alpha_2(t, A)$ was obtained from $\alpha_1(t, A)$ by continuous deformation, $\alpha_2(t, A)\in\Gamma_j$.

Proceeding as above, we eventually define

$$\alpha_m(t,x)=\begin{cases}\varphi_m^{-1}\left(\varphi_m(\alpha_{m-1}(t,x))-t\psi_m(x)\dfrac{v_m}{\|v_m\|}\right) & \text{if } \alpha_{m-1}(t,x)\in U_m\\ \alpha_{m-1}(t,x) & \text{otherwise,}\end{cases}$$

and show that

$$\rho(x,\alpha_m(t,x))\leqq k(\psi_1(x)+\ldots+\psi_m(x))t\leqq kt, \tag{14}$$

$$f\circ\alpha_m(t,x)-f(x)\leqq-\frac{3}{2}\varepsilon t(\psi_1(x)+\ldots+\psi_m(x)) \tag{15}$$

and $\alpha_m(t, A)\in\Gamma_j$. Let $B=\alpha_m(t, A)$. By (14), $dist(A, B)\leqq kt$. Since $\Pi(B)\geqq c$ and $f\circ\alpha_m(t, x)\leqq f(x)$,

$$\sup_{x\in A} f\circ\alpha_m(t,x)=\sup_{x\in S} f\circ\alpha_m(t,x). \tag{16}$$

Recall that $k<\frac{3}{2}$ and $\psi_1(x)+\ldots+\psi_m(x)=1$ on S. Using this, (5), (15) and (16), we obtain

$$-\frac{3}{2}\varepsilon t<-\varepsilon kt\leqq-\varepsilon\, dist(A,B)\leqq\Pi(B)-\Pi(A)$$

$$=\sup_{x\in S} f\circ\alpha_m(t,x)-\sup_{x\in S} f(x)\leqq\sup_{x\in S}(f\circ\alpha_m(t,x)-f(x))\leqq-\frac{3}{2}\varepsilon t,$$

a contradiction. □

3.4. Remark. – Condition $(PS)_c$ in Theorem 3.1 may be replaced with the following weaker one: If there is a sequence (x_n) such that $f(x_n)\to c$ and $\|df(x_n)\|\to 0$, then c is a critical value and $\inf_{n\in\mathbb{N}}\rho(x_n, K_c)=0$. This is seen by verifying that the previous argument applies if (3) is modified to

(3′) *either* K_c *is not compact or* $\mathrm{cat}_M(K_c)\geqq p+1$.

A still weaker (but insufficient for our purposes) version of $(PS)_c$ has been introduced in [4]. It says that c is a critical value whenever there exists a sequence (x_n) such that $f(x_n)\to c$ and $\|df(x_n)\|\to 0$.

4. RELATED RESULTS

4.1. COROLLARY. — *Suppose that* M *is a closed symmetric* C^1*-submanifold of a real Banach space* X *and* $0 \notin M$. *Suppose also that* $f \in C^1(M, \mathbb{R})$ *is even and bounded below. Define*

$$c_j = \inf_{A \in \Gamma_j} \sup_{x \in A} f(x),$$

where $\Gamma_j = \{A \subset M : A \in \Sigma, \gamma(A) \geqq j$ *and* A *is compact*$\}$. *If* $\Gamma_k \neq \varnothing$ *for some* $k \geqq 1$ *and if* f *satisfies* $(PS)_c$ *for all* $c = c_j$, $j = 1, \ldots, k$, *then* f *has at least* k *distinct pairs of critical points.*

Proof. — Let $\tilde{M} = M/\sim$, where $\sim$ is the equivalence relation identifying x with $-x$, and let $\tilde{f} : \tilde{M} \to \mathbb{R}$ be the function induced by f. It is clear that $\tilde{M}$ and $\tilde{f}$ satisfy the hypotheses of Theorem 3.1. Furthermore, if $A \in \Gamma_k$, then, setting $\tilde{X} = (X - \{0\})/\sim$ and $\tilde{A} = A/\sim$, $\mathrm{cat}_{\tilde{X}}(\tilde{A}) = \gamma(A) \geqq k$ [18] (Theorem 3.7). Since $\tilde{M} \subset \tilde{X}$, it follows from the definition of category that $\mathrm{cat}_{\tilde{M}}(\tilde{A}) \geqq \mathrm{cat}_{\tilde{X}}(\tilde{A}) \geqq k$. So $\tilde{A} \in \Lambda_k$. Hence $\tilde{f}$ possesses at least k, and f at least k pairs of critical points. □

A different (and perhaps more natural) proof of the corollary may be obtained by modifying the argument of Theorem 3.1 (category should be replaced with genus and the mappings α_i should be odd in u).

The assumption that f is bounded below in Theorem 3.1 was used only in order to assure that $c_1 > -\infty$. It is therefore easy to see that the following stronger results are valid.

4.2. COROLLARY. — *Suppose that* M *is a* C^1 *Finsler manifold and* $f \in C^1(M, \mathbb{R})$ *is such that* f_c *is complete in the metric* ρ *for each* $c \in \mathbb{R}$. *Let* c_j *and* Λ_j *be defined as in Theorem* 3.1. *If* $\Lambda_k \neq \varnothing$ *for some* $k \geqq 1$, $c_m > -\infty$ *for some* m, $1 \leqq m \leqq k$, *and* $(PS)_c$ *is satisfied for all* $c = c_j$, $m \leqq j \leqq k$, *then* f *has at least* $k - m + 1$ *distinct critical points.*

4.3. COROLLARY. — *Suppose that* M *is a closed symmetric* C^1*-submanifold of a real Banach space* X *and* $0 \notin M$. *Suppose also that* $f \in C^1(M, \mathbb{R})$ *is an even function. Let* c_j *and* Γ_j *be defined as in Corollary* 4.1. *If* $\Gamma_k \neq \varnothing$ *for some* $k \geqq 1$, $c_m > -\infty$ *for some* m, $1 \leqq m \leqq k$, *and* $(PS)_c$ *is satisfied for all* $c = c_j$, $m \leqq j \leqq k$, *then* f *has at least* $k - m + 1$ *distinct pairs of critical points.*

A function $f : X \to \mathbb{R}$, where X is a Banach space, is said to be *Gâteaux differentiable* if for each $x \in X$ there exists a linear mapping $f'(x) \in X^*$ such that

$$\frac{d}{dt}f(x+ty)\big|_{t=0}=\langle f'(x), y\rangle, \qquad \forall y\in X. \tag{17}$$

Let us remark that there is a different—weaker—definition of Gâteaux differentiability in which it is not required that the left-hand side of (17) be linear in y (*see* e. g. [3], [8]). A function $f: M\to\mathbb{R}$, where M is a Finsler manifold, will be called *Gâteaux differentiable* if for each $x\in M$ and each chart $\varphi: U\to T_x(M)$ at x, $f\circ\varphi^{-1}$ is Gâteaux differentiable. The Gâteaux derivative df is *strong-to-weak* continuous* if for each $x\in M$, each sequence $x_n\to x$ and each chart $\varphi: U\to T_x(M)$ at x,

$$(f\circ\varphi^{-1})'(\varphi(x_n))\to(f\circ\varphi^{-1})'(\varphi(x))$$

in the weak* topology of $T_x(M)^*$.

4.4. REMARK. — Theorem 3.1 and Corollaries 4.1-4.3 remain valid if f, instead of being C^1, is continuous and Gâteaux differentiable with the derivative strong-to-weak* continuous. This follows by observing that in the proofs of Proposition 2.1 and Theorem 3.1 only the above weaker smoothness assumption has been used.

5. AN APPLICATION

Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$ and let f, g be two continuous real-valued functions. Fix a number $p\in(1,\infty)$ and denote

$$F(t)=\int_0^t f(s)\,ds \qquad\text{and}\qquad G(t)=\int_0^t g(s)\,ds.$$

We will be concerned with the following eigenvalue problem: *Given $b\in\mathbb{R}$, find a function u and a real number λ such that*

$$\left.\begin{aligned}&-\operatorname{div}(|\nabla u|^{p-2}\nabla u)+f(u)=\lambda g(u)\quad\text{in }\Omega,\qquad u|_{\partial\Omega}=0\\ &\text{and}\\ &\frac{1}{p}\int_\Omega|\nabla u|^p\,dx+\int_\Omega F(u)\,dx=b.\end{aligned}\right\} \tag{18}$$

Suppose that f and g satisfy the growth restriction

$$|f(t)|,\ |g(t)| \leqq a_1 + a_2 |t|^r, \tag{19}$$

where $1 \leqq r < \dfrac{Np}{N-p} - 1$ if $N > p$, $1 \leqq r < \infty$ otherwise. Let $H \equiv H_0^{1,p}(\Omega)$ be the usual Sobolev space (of real-valued functions) with the norm $\|u\| = \left(\int_\Omega |\nabla u|^p\right)^{1/p}$. Define

$$\Phi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \int_\Omega F(u)\,dx \equiv \frac{1}{p}\|u\|^p + \Phi_1(u),$$

$$\psi(u) = -\int_\Omega G(u)\,dx.$$

The following result follows from standard arguments in Sobolev spaces.

5.1. Proposition. — *Suppose that f, g satisfy* (19). *Then*

(i) $\Phi, \psi \in C^1(H, \mathbb{R})$ *and*

$$\langle \Phi'(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u.\nabla v\,dx + \int_\Omega f(u)\,v\,dx,$$

$$\langle \psi'(u), v\rangle = -\int_\Omega g(u)\,v\,dx, \qquad \forall v \in H.$$

(ii) Φ_1' and ψ' are completely continuous (i. e., they map weakly convergent sequences to strongly convergent ones).

(iii) Φ_1 and ψ are continuous with respect to weak convergence in H.

The proof for $p=2$ may be found e. g. in [19] (Appendix B). If $p \neq 2$, the argument of [19] applies upon observing that the Sobolev embedding $H \hookrightarrow L^{r+1}(\Omega)$ is compact.

5.2. Lemma. — (i) Φ' maps bounded sets to bounded sets.

(ii) If $u_n \to \bar{u}$ weakly in H and $\Phi'(u_n)$ converges strongly, then $u_n \to \bar{u}$ strongly.

Proof. — Let $A : H \to H^*$ be the mapping given by

$$\langle A u, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u.\nabla v\,dx.$$

Then $\Phi' = A + \Phi_1'$. Since $|\langle A u, v\rangle| \leqq \|u\|^{p-1}\|v\|$ and Φ_1' is completely continuous, the conclusion follows.

(ii) It is easy to verify that there is a constant $\alpha > 0$ such that

$$\langle A u - A v, u - v \rangle \geqq \alpha \| u - v \|^p. \tag{20}$$

Suppose that $u_n \to \bar{u}$ weakly and $\Phi'(u_n)$ is strongly convergent. Then also $A u_n$ is strongly convergent. It follows therefore from (20) with $u = u_n$ and $v = \bar{u}$ that $u_n \to \bar{u}$ strongly. □

A much more general version of Proposition 5.1 and Lemma 5.2 may be found in [5].

Suppose that b is a regular value of Φ. Then $M = \Phi^{-1}(b)$ is a C^1-manifold and $u \in M$ is a critical point of $\tilde{\psi} = \psi|_M$ if and only if $\psi'(u) = \mu \Phi'(u)$ for some $\mu \in \mathbb{R}$. Assume also that $\psi'(u) \neq 0$ on M. It follows that $\mu \neq 0$ for such ψ and, according to Proposition 5.1, there is a one-to-one correspondence between critical points of $\tilde{\psi}$ and weak solutions of (18) [$\lambda = \mu^{-1}$ in (18)].

5.3. Theorem. — *Suppose that $f, g \in C(\mathbb{R}, \mathbb{R})$ are two odd functions satisfying* (19). *Suppose also that $G(t) > 0$ for almost all t and there exist positive constants d_1, d_2, d_3 such that $F(t) \geqq -d_1 |t|^p - d_2$ and $G(t) \geqq -F(t) - d_3$. If $b > 0$ is a regular value of Φ, then* (18) *has infinitely many weak solutions.*

Note that under the hypotheses of Theorem 5.3 Φ need not be in $C^{2-}(H, \mathbb{R})$. In particular, if $1 < p < 2$, then Φ' cannot be locally Lipschitz continuous [*cf.* (20)]. If $p \geqq 2$ and $f \in C^1(\mathbb{R}, \mathbb{R})$, then $\Phi \in C^2(H, \mathbb{R})$ provided f' satisfies the growth restriction

$$|f'(t)| \leqq a_4 + a_5 |t|^{r-1},$$

where r is as in (19). Note also that under the present hypotheses $M = \Phi^{-1}(b)$ need neither be bounded nor radially homeomorphic to the unit sphere in H.

Proof of Theorem 5.3. — We shall keep the notation introduced earlier in this section. First we show that $\psi'(u) \neq 0$ if $u \neq 0$. Assume without loss of generality that ess sup $u > 0$. Since $G(t) > 0$ for almost all t, there exist $\beta > \alpha > 0$ such that meas $\{x \in \Omega : u(x) \geqq \beta\} > 0$, meas $\{x \in \Omega : u(x) \leqq \alpha\} > 0$ and $g(t) > 0$ on $[\alpha, \beta]$. By [7] (Theorem 1), meas $\{x \in \Omega : \alpha \leqq u(x) \leqq \beta\} > 0$. Hence $g \circ u > 0$ on a set of positive measure. So $\psi'(u) \neq 0$ and there is a one-to-one correspondence between critical points of $\tilde{\psi}$ and weak solutions of (18).

It is clear that M is symmetric. We claim that it contains compact subsets of arbitrarily large genus, i. e., $\Gamma_k \neq \varnothing$ for any $k \geqq 1$ (Γ_k was defined in Corollary 4.1). Since H is separable, there exists a biorthogonal system

$(e_m, e_n^*)_{m, n \in \mathbb{N}}$ such that $e_m \in \mathrm{H}$, $e_n^* \in \mathrm{H}^*$, the e_m's are linearly dense in H and the e_n^*'s are total for H [13] (Proposition 1.f.3). Let us remark that we could in particular choose $(e_m)_{m \in \mathbb{N}}$ to be a Schauder basis for H [which exists according to [22] (Section 4.9.4)], and then find biorthogonal functionals e_n^*. Denote

$$\mathrm{H}_m = \mathrm{span}\{e_1, \ldots, e_m\}, \qquad \mathrm{H}_m^{\perp} = \mathrm{cl}\ \mathit{span}\{e_{m+1}, e_{m+2}, \ldots\}$$

(cl is the closure). If m is large enough,

$$\inf\left\{\frac{1}{p}\|u\|^p - d_1\int_\Omega |u|^p\,dx : u \in \mathrm{H}_m^{\perp},\ \|u\| = 1\right\} > 0. \tag{21}$$

Indeed, otherwise there would exist a sequence (u_m) such that $u_m \in \mathrm{H}_m^{\perp}$, $\|u_m\| = 1$ *and*

$$\frac{1}{p}\|u_m\|^p - d_1\int_\Omega |u_m|^p\,dx \leqq \frac{1}{m}. \tag{22}$$

Since $\langle e_n^*, u_m\rangle = 0$ for all $m \geqq n$ and the e_n^*'s are total, $u_m \to 0$ weakly in H. Therefore $u_m \to 0$ strongly in $\mathrm{L}^p(\Omega)$, a contradiction to (22). It follows from (21) and the assumption on F that for some $\alpha > 0$,

$$\Phi(u) \geqq \frac{1}{p}\|u\|^p - \int_\Omega (d_1|u|^p + d_2)\,dx \geqq \alpha\|u\|^p - \int_\Omega d_2\,dx, \qquad \forall u \in \mathrm{H}_m^{\perp}.$$

In particular, the sets $\mathrm{H}_m^{\perp} \cap \mathrm{M}$ and $\mathrm{H}_m^{\perp} \cap \Phi_b$ are bounded. Let $\mathrm{E}_k = \mathrm{span}\{e_{m+1}, \ldots, e_{m+k}\}$. Then $\dim \mathrm{E}_k = k$ and $\mathrm{E}_k \cap \Phi_b$ is a bounded and symmetric neighbourhood of $0 \in \mathrm{E}_k$. Therefore $\gamma(\mathrm{E}_k \cap \mathrm{M}) = k$ according to (*f*) of Proposition 2.3. Since $\mathrm{E}_k \cap \mathrm{M}$ is compact, $\Gamma_k \neq \varnothing$.

Let $j > m$. Since $\mathrm{G}(t) > 0$ for almost all t, $\mathrm{G} \circ u > 0$ on a set of positive measure for any $u \in \mathrm{M}$ (*cf.* the argument at the beginning of the proof). Therefore $\tilde{\Psi} < 0$ on M and

$$c_j = \inf_{\mathrm{A} \in \Gamma_j} \sup_{u \in \mathrm{A}} \tilde{\Psi}(u) < 0. \tag{23}$$

Let $\mathrm{A} \in \Gamma_j$. Then $\mathrm{A} \cap \mathrm{H}_m^{\perp} \neq \varnothing$ by (*g*) of Proposition 2.3. Since $\mathrm{H}_m^{\perp} \cap \mathrm{M}$ is a bounded set, $\tilde{\Psi}|_{\mathrm{H}_m^{\perp} \cap \mathrm{M}}$ is bounded below. Hence $c_j > -\infty$ whenever $j > m$.

We shall show that $\tilde{\Psi}$ satisfies $(\mathrm{PS})_c$ for any $c < 0$. The conclusion of the theorem will then follow from Corollary 4.3. Suppose $\tilde{\Psi}(u_n) \to c < 0$ and $\|d\tilde{\Psi}(u_n)\| \to 0$. Since

$$\tilde{\Psi}(u) = -\int_\Omega \mathrm{G}(u)\,dx \leqq \int_\Omega (\mathrm{F}(u) + d_3)\,dx = b + \int_\Omega d_3\,dx - \frac{1}{p}\|u\|^p, \tag{24}$$

the sequence (u_n) is bounded. Assume after passing to a subsequence that $u_n \to \bar{u}$ weakly in H. Since $\tilde{\psi}(\bar{u}) = c < 0$ [by (iii) of Proposition 5.1], $\bar{u} \neq 0$. Let $J : H \to H^*$ be the duality mapping [*see* e.g. [6] (Section 4)]. Recall that J and J^{-1} are continuous, $\|Ju\| = \|u\|$ and $\langle Ju, u \rangle = \|u\|^2$, $\forall u \in H$. Define the projection mapping $P_u : H \to T_u(M)$ by

$$P_u v = v - \frac{\langle \Phi'(u), v \rangle}{\|\Phi'(u)\|^2} J^{-1} \Phi'(u).$$

Note that $\langle \psi'(u), v \rangle = \langle d\tilde{\psi}(u), v \rangle$ whenever $v \in T_u(M)$. Since

$$|\langle \psi'(u), P_u v \rangle| = |\langle d\tilde{\psi}(u), P_u v \rangle| \leqq \|d\tilde{\psi}(u)\| \, \|P_u v\| \leqq 2 \|d\tilde{\psi}(u)\| \, \|v\|$$

and $d\tilde{\psi}(u_n) \to 0$,

$$\sup_{\|v\| \leqq 1} \left\{ \langle \psi'(u_n), v \rangle - \frac{\langle \Phi'(u_n), v \rangle}{\|\Phi'(u_n)\|^2} \langle \psi'(u_n), J^{-1} \Phi'(u_n) \rangle \right\} \to 0.$$

Therefore

$$\psi'(u_n) - \frac{\langle \psi'(u_n), J^{-1} \Phi'(u_n) \rangle}{\|\Phi'(u_n)\|^2} \Phi'(u_n) \to 0. \tag{25}$$

Since $\psi'(u_n) \to \psi'(\bar{u}) \neq 0$ and $\Phi'(u_n)$ is bounded [by (i) of Lemma 5.2], $\Phi'(u_n)$ (or a subsequence of it) is strongly convergent. By (ii) of Lemma 5.2, $u_n \to \bar{u}$ strongly. □

5.4. REMARKS. — (i) It is well known that functionals of the type considered here do not satisfy $(PS)_0$. To see this for $\tilde{\psi}$, let $(u_n) \subset M$ be a sequence converging weakly to 0. Then $\tilde{\psi}(u_n) \to 0$ and $\psi'(u_n) \to 0$. Since $0 \notin M$, no subsequence of (u_n) converges strongly, and it follows from (ii) of Lemma 5.2 that $\Phi'(u_n) \neq 0$ for almost all n. Therefore $d\tilde{\psi}(u_n) \to 0$ (recall that $d\tilde{\psi}(u_n) \to 0$ if and only if (25) is satisfied).

(ii) The numbers c_j in (23) tend to zero as $j \to \infty$. Indeed, observe that $A \cap H_{j-1}^{\perp} \neq \varnothing$ for any $A \in \Gamma_j$ [by (*g*) of Proposition 2.3]. Furthermore,

$$\varepsilon_j = \inf \{ \tilde{\psi}(u) : u \in H_j^{\perp} \cap M \} \to 0 \quad \text{as } j \to \infty$$

[because if (u_j) is a sequence such that $u_j \in H_j^{\perp} \cap M$, then $u_j \to 0$ weakly in H; therefore $\tilde{\psi}(u_j) \to 0$]. Since $\varepsilon_j \leqq c_j < 0$, $c_j \to 0$.

(iii) Suppose that $f, g \in C(\mathbb{R}, \mathbb{R})$ are odd, satisfy (19), $G(t) > 0$ for almost all t and $G(t) \geqq -F(t) - d_3$. If $b < 0$ is a regular value of Φ, then (18) has at least k pairs of weak solutions provided $M = \Phi^{-1}(b)$ contains a compact subset of genus k. The proof is obtained by applying Corollary 4.1 to the functional $-\tilde{\psi}$. Since $-\tilde{\psi}$ may not satisfy $(PS)_0$, we must show that $c_j > 0$. Suppose $\tilde{\psi}(u_n) \to 0$. By (24), the sequence (u_n) is bounded, so we may assume that $u_n \to \bar{u}$ weakly in H. Since

$$\Phi(u_n) = \frac{1}{p}\|u_n\|^p + \Phi_1(u_n) = b < 0$$

and Φ is weakly lower semicontinuous, $\Phi(\bar{u}) \leq b < 0$. It follows that $\bar{u} \neq 0$. On the other hand, $\tilde{\psi}(u_n) \to \tilde{\psi}(\bar{u}) = 0$. So $\bar{u} = 0$. This contradiction shows that $-\tilde{\psi}$ is bounded away from 0. Therefore $c_j > 0$. A related result in a geometrically simpler situation (in which it is easy to compute k) may be found in Zeidler [23] (Proposition 11).

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