

LOCAL AND GLOBAL EXISTENCE FOR THE COUPLED NAVIER-STOKES-POISSON PROBLEM

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Abstract. In this paper we investigate the Cauchy Problem for coupled Navier-Stokes-Poisson equation. The global existence of weak solutions in Sobolev framework is proved by using some compactification properties deduced from the Poisson equation.

1. Introduction. In this paper we deal with the existence of weak solutions for the coupled Navier-Stokes-Poisson problem. The interest for similar models is originated by the study of semiconductor devices. In the numerical simulation for semiconductor devices, the hydrodynamical models represent an acceptable compromise between accuracy and computational efficiency. Their common feature is the fact that the number of independent variables is reduced. Hydrodynamical models are obtained from the infinite hierarchy of the moment equations of the Boltzmann transport equation by suitable truncation procedures (see for instance [6], [7], [18], [19]), and from these are derived the classical continuity equation, the momentum balance equation, and the energy balance equation ([2], [3], [10], [12], [13], [24]). In the case of a 1-D Euler-Poisson model, some results have been obtained in [16], [17]. In the model system considered here, we take into account the viscosity terms but we remove the equation regarding the energy (temperature). The removal of the energy balance is not completely justified from the physical point of view, since by adding viscosity it would be reasonable to include other physical quantities leading to effects of the same order. Unfortunately, the mathematical theory for the Navier-Stokes equation has been widely developed in the isentropic case, but for the system including the energy balance equation, it is still at the initial state

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(see [15] for details). In particular we consider the following system

$$\begin{cases} \frac{\partial n}{\partial t} + \operatorname{div}(nu) = 0 \\ \frac{\partial nu}{\partial t} + \operatorname{div}(nu \otimes u) + \nabla(an^\gamma) - \mu\Delta u - \xi\nabla\operatorname{div}u = n\nabla\phi - nu \\ \Delta\phi = n - b(x) \end{cases} \quad (1.1)$$

in the region $\Omega \times (0, T)$, Ω being a bounded connected open smooth domain in \mathbb{R}^N . The previous system is supplemented with the initial conditions

$$n(x, 0) = n_0(x), \quad n(x, 0)u(x, 0) = n_0(x)u_0(x), \quad \phi(x, 0) = \phi_0(x),$$

for any $x \in \Omega$ and the Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega \times (0, T).$$

Here $n \geq 0$ denotes the electron density, $u \in \mathbb{R}^N$ the particle velocity, $\nabla\phi \in \mathbb{R}^N$ the electron field which is generated by the Coulomb force of the particles, the function $b = b(x)$ stands for the density of fixed positively charged background ions, $p = an^\gamma$, $\gamma > 1$ is the pressure density relation, and $j = nu$ will denote the electron current density.

In this paper we construct the weak solutions by using fixed point methods and compactness arguments: namely, we will consider the system

$$\begin{cases} \frac{\partial n}{\partial t} + \operatorname{div}(nu) = 0 \\ \frac{\partial nu}{\partial t} + \operatorname{div}(nu \otimes u) + \nabla(an^\gamma) - \mu\Delta u - \xi\nabla\operatorname{div}u = n\nabla\psi - nu \\ \Delta\phi = n - b(x) \end{cases} \quad (1.2)$$

with a given potential ψ . We will define the following fixed point map, from a suitable Banach space X into itself, as follows. We say that

$$A\psi = \phi \quad \text{if and only if} \quad \Delta\phi = n - b(x),$$

where n solves the first two equations of (1.2) with prescribed ψ . At this point we need to show that the map A is well-defined in suitable functional frameworks and will have some compactness properties to apply Schauder's fixed point theorem. The well-posedness of A requires to show that the map $\psi \rightarrow n(\psi)$ is well defined, namely that we can produce a solution to the problem (see [15])

$$\begin{cases} \frac{\partial n}{\partial t} + \operatorname{div}(nu) = 0 \\ \frac{\partial nu}{\partial t} + \operatorname{div}(nu \otimes u) + \nabla(an^\gamma) - \mu\Delta u - \xi\nabla\operatorname{div}u = n\nabla\psi - nu. \end{cases} \quad (1.3)$$

Of course, the uniqueness problem to (1.3) is still a major open problem (see [15], Remark 7.7, pg. 180); hence we could produce, in principle, more than one map A . Hence the lack of uniqueness for (1.3) will reflect also on our results. Moreover, we will have another

source of possible nonuniqueness which depends on the use of Schauder fixed point theory. A technical difficulty that one should keep in mind relies on the Lions theory, which assumes $\psi \in L^\infty(0, T; W^{1, 2\gamma/(\gamma-1)}(\Omega))$, while in our case ϕ is reconstructed by means of $\operatorname{div} j$ and this entails a loss of regularity with respect to ψ . We overcome this difficulty assuming some restriction on γ . The plan of the paper is as follows. In Sec. 2 we recall some of the theorems we are going to use later and we will define the kind of weak solutions we are going to build up. In Sec. 3, by using the theory of P. L. Lions [15], we investigate the existence of solutions for system (1.3). Then, in Sec. 4, we show that the map $A : X \rightarrow X$ has a fixed point, which is our solution. Because of the nature of this problem we cannot apply a contractive-type theorem but we will use a Schauder-type fixed point theorem. In order to satisfy the requirements of this theorem we will get some restrictions on the possible values of γ and on the time. Hence this will provide only a local solution for our problem. Later, by proving some uniform continuity in time of the local solutions, we will get the global existence.

2. General Framework.

2.1. *Preliminary results.* In this section we will recall the basic notations and theorems that will be used later on. We begin with the following version of the Ascoli Arzelà Theorem (see [22], pg. 135).

THEOREM 2.1. Let X be a compact metric space, Y a Banach space, and $C(X; Y)$ the Banach space of continuous functions from X to Y with the sup norm. A subset H of $C(X; Y)$ has compact closure in $C(X; Y)$ if and only if

- (i) H is equicontinuous, and
- (ii) for every $x \in X$ the set $H(x) = \{f(x) \mid f \in H\}$ has compact closure in Y .

Since in our system (1.1) we have a Poisson equation, we need the following elliptic regularity theorem [20]:

THEOREM 2.2. Consider the differential equation

$$Lu = \operatorname{div} A \nabla u = \operatorname{div} f + h, \quad \text{in } \Omega \tag{2.1}$$

where $A = A(x)$ is a complex measurable matrix such that

$$A = H + R$$

where H and R are matrices such that the inequalities

$$\begin{aligned} \lambda |\xi|^2 \leq (H\xi, \xi) \leq |\xi|^2 \quad \lambda > 0 \\ |R| \leq (1 - \theta)\lambda \quad 0 < \theta \leq 1 \end{aligned}$$

hold for all relevant x and ξ . If Ω is a bounded smooth domain, then (2.1) has a unique solution in $W_0^{1,p}(\Omega)$ for every complex vector field $f = f(x) \in L^p(\Omega)$ and every complex function $h = h(x) \in L^r(\Omega)$ with $r^* \geq p$. The solutions satisfy

$$\|\operatorname{grad} u\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)} + \|h\|_{L^r(\Omega)},$$

where C is a constant depending only on Ω , $\theta\lambda$, p , and r .

We recall the following interpolation theorem ([4], pg. 310):

THEOREM 2.3. Let $f \in L^p(\Omega) \cap W^{2,r}(\Omega)$ with $1 \leq p \leq \infty$. Then $f \in W^{1,q}(\Omega)$ where $\frac{1}{q} = \frac{1}{2} \left(\frac{1}{p} + \frac{1}{r} \right)$ and

$$\|f\|_{W^{1,q}(\Omega)} \leq c \|f\|_{W^{2,r}(\Omega)}^{1/2} \|f\|_{L^p(\Omega)}^{1/2}. \tag{2.2}$$

In particular, if $p = q = r$, then

$$\|f\|_{W^{1,p}(\Omega)} \leq c \|f\|_{W^{2,p}(\Omega)}^{1/2} \|f\|_{L^p(\Omega)}^{1/2}. \tag{2.3}$$

2.2. Statement of the problem. We will look for global weak solution of the following system

$$\begin{cases} \frac{\partial n}{\partial t} + \operatorname{div}(nu) = 0 \\ \frac{\partial nu}{\partial t} + \operatorname{div}(nu \otimes u) + \nabla(an^\gamma) - \mu \Delta u - \xi \nabla \operatorname{div} u = n \nabla \phi - nu \\ \Delta \phi = n - b(x) \end{cases} \tag{2.4}$$

in $\Omega \times (0, T)$, Ω a bounded connected open smooth domain in \mathbb{R}^N , endowed with the initial conditions

$$\begin{aligned} n(x, 0) &= n_0(x) \\ n(x, 0)u(x, 0) &= n_0(x)u_0(x) = j_0(x) \\ \phi(x, 0) &= \phi_0(x) \quad \text{for any } x \in \Omega \end{aligned} \tag{2.5}$$

and the Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega \times (0, T). \tag{2.6}$$

Here the viscosity parameters satisfy $\mu > 0$, $\mu + \xi > 0$, moreover $a > 0$ and the adiabatic constant $\gamma > 1$, $n \geq 0$, $u \in \mathbb{R}^N$, $\phi \in \mathbb{R}$. We assume the following conditions hold:

- (A.1) $\phi_0 \in L^1(0, T; W^{1,2\gamma/(\gamma-1)}(\Omega))$, $\phi = \bar{\phi}$ on $\partial\Omega \times (0, T)$, $\bar{\phi} = 0$,
- (A.2) $b = b(x) \in L^1(\Omega)$,
- (A.3) $n_0 \geq 0$ a.e. in Ω , $n_0 \in L^1(\Omega) \cap L^\gamma(\Omega)$,
- (A.4) $j_0 \in L^{2\gamma/(\gamma+1)}(\Omega)$, $j_0 = 0$ a.e. on $\{n_0 = 0\}$,
- (A.5) $|j_0|^2/n_0 \in L^1(\Omega)$ defined to be 0 on $\{n_0 = 0\}$ and $n_0 \neq 0$.

The conditions (A.1), (A.2), (A.3), (A.4), (A.5) are required to fit into the Lions framework [15]. At this stage, we are not imposing any further restrictions. We introduce here the notion of weak solution we are going to build and use.

DEFINITION 2.4. By a weak solution of (2.4), we mean (n, u, ϕ) satisfying (2.4) in the sense of distributions such that

$$n \in L^\infty(0, T; L^\gamma(\Omega)) \cap C([0, T]; L^p(\Omega)) \quad \text{for } 1 \leq p < \gamma, n \geq 0 \text{ a.e.} \tag{2.7}$$

$$\nabla u \in L^2(0, T; L^2(\Omega)), \quad n|u|^2 \in L^\infty(0, T; L^1(\Omega)) \tag{2.8}$$

$$j = nu \in C([0, T], L^{2\gamma/(\gamma+1)}(\Omega) - \text{weak}). \tag{2.9}$$

$$\begin{aligned}
 \phi &\in L^1(0, T; W^{1,2\gamma/(\gamma-1)}(\Omega)) && \text{if } N \geq 2 && (2.10) \\
 \phi &\in L^1(0, T; W^{1,2}(\Omega)) && \text{if } N = 2 \text{ and } \gamma > 2 \\
 \phi &\in L^1(0, T; W^{1,3/2}(\Omega)) && \text{if } N = 3 \text{ and } \gamma > 6.
 \end{aligned}$$

The above notion of weak solution is very natural in this frame since it simply follows from the energy identity which we will obtain in the next section.

3. Existence for the Navier-Stokes equations. This section is devoted to the analysis of the first two equations of the system (2.4), namely, the Navier-Stokes equations

$$\frac{\partial n}{\partial t} + \operatorname{div}(nu) = 0 \tag{3.1}$$

$$\frac{\partial(nu_i)}{\partial t} + \operatorname{div}(nuu_i) + \partial_i(an^\gamma) - \mu\Delta u_i - \xi\partial_i \operatorname{div}u = n\nabla\psi - nu_i, \tag{3.2}$$

$1 \leq i \leq N$, in Ω , a bounded connected open smooth domain in \mathbb{R}^N , where $\mu > 0$, $\mu + \xi > 0$, $a > 0$, and $\gamma > 1$. Moreover, in this case $\psi = \psi(x, t)$ is fixed; it is a given function corresponding to the force terms on $\Omega \times (0, T)$ for some fixed $T \in (0, \infty)$. We assume the following Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{3.3}$$

and the initial conditions

$$n|_{t=0} = n_0 \quad \text{in } \Omega, \quad n_0 u|_{t=0} = j_0 \quad \text{in } \Omega. \tag{3.4}$$

In this section we report the main ideas from P. L. Lions’s book [15] with some modifications in order to take into account the presence of the electric field and the collision term nu .

3.1. *Estimates for the Navier-Stokes equations.* We begin with some a priori estimates and compactness results.

3.1.1. *A priori estimate.* We specify better our initial condition; namely, we assume that n_0, j_0 satisfy

$$\begin{cases}
 n_0 \geq 0 \quad \text{a.e. in } \Omega, \quad n_0 \in L^1(\Omega) \cap L^\gamma(\Omega), \quad j_0 \in L^{2\gamma/(\gamma+1)}(\Omega), \\
 j_0 = 0 \quad \text{a.e. on } \{n_0 = 0\} \\
 |j_0|^2/n_0 \quad \text{defined to be 0 on } \{n_0 = 0\} \in L^1(\Omega) \text{ and } n_0 \neq 0,
 \end{cases} \tag{3.5}$$

and for the moment we take $\psi \in L^1(0, T; W^{1,2\gamma/(\gamma-1)}(\Omega))$. From Eq. (3.1) we get that n remains non-negative for all $t > 0$ and that $\int_\Omega n(t)dx$ is independent of $t \geq 0$. In addition, multiplying (3.2) by u , we deduce that

$$\begin{aligned}
 n \frac{\partial |u|^2/2}{\partial t} + nu \cdot \nabla \frac{|u|^2}{2} - \mu\Delta \frac{|u|^2}{2} - \xi \operatorname{div}(u \operatorname{div}u) + \mu |Du|^2 + \xi (\operatorname{div}u)^2 \\
 + \frac{\partial}{\partial t} \left(\frac{a}{\gamma-1} n^\gamma \right) + \operatorname{div} \left(u \frac{a\gamma}{\gamma-1} n^\gamma \right) = nu \cdot \nabla\psi - nu^2.
 \end{aligned} \tag{3.6}$$

Integrating (3.6) with respect to x and remembering the boundary condition on u , we deduce that

$$\frac{d}{dt} \int_{\Omega} \left\{ n \frac{|u|^2}{2} + \frac{a}{\gamma - 1} n^{\gamma} \right\} dx + \int_{\Omega} \mu |Du|^2 + \xi (\operatorname{div} u)^2 dx = \int_{\Omega} nu \nabla \psi dx - \int_{\Omega} nu^2 dx \tag{3.7}$$

and obviously $\int_{\Omega} (\operatorname{div} u)^2 dx = \int_{\Omega} \partial_i u_j \partial_j u_i dx \leq \int_{\Omega} |Du|^2 dx$. Since $\mu > 0$, $\mu + \xi > 0$, we deduce from (3.7) that we have for some $\nu > 0$

$$\frac{d}{dt} \int_{\Omega} \left\{ n \frac{|u|^2}{2} + \frac{a}{\gamma - 1} n^{\gamma} \right\} dx + \nu \int_{\Omega} |Du|^2 \leq \int_{\Omega} nu \cdot \nabla \psi dx - \int_{\Omega} nu^2 dx. \tag{3.8}$$

Taking into account the condition on ψ , we have

$$\begin{aligned} \int_{\Omega} nu \nabla \psi dx &\leq \|nu\|_{L^{2\gamma/(\gamma+1)}(\Omega)} \|\nabla \psi\|_{L^{2\gamma/(\gamma-1)}(\Omega)} \\ &\leq \|\sqrt{n}\|_{L^{2\gamma}(\Omega)} \|\sqrt{n}u\|_{L^2(\Omega)} \|\nabla \psi\|_{L^{2\gamma/(\gamma-1)}(\Omega)}. \end{aligned}$$

Setting

$$E(t) = \int_{\Omega} \left\{ n \frac{|u|^2}{2} + n^{\gamma} \right\} dx$$

and integrating on t , we have the following energy inequality

$$E(t) + \nu \int_0^t \int_{\Omega} |Du|^2 dt dx + \int_0^t \int_{\Omega} nu^2 dt dx \leq E(0) + \int_0^t E(s) ds. \tag{3.9}$$

Applying Gronwall’s lemma, we get

$$E(t) \leq ce^{ct}.$$

In this way we have proved the following

THEOREM 3.1. Suppose (n, u) is a weak solution of (3.1), (3.2). Suppose that (A.1), (A.2), (A.3), (A.4), (A.5) hold. Then $n \in L^{\infty}(0, T; L^{\gamma}(\Omega))$, $\nabla u \in L^2(0, T; L^2(\Omega))$, $n|u|^2 \in L^1([0, T] \times \Omega)$, $nu \in C([0, T], L^{2\gamma/(\gamma+1)} - weak)$.

Finally, we remark that since we have Dirichlet boundary conditions, we get also $u \in L^2(0, T; H_0^1(\Omega))$.

3.1.2. Compactness results. Existence results often follow from the analysis of the convergence of sequences of solutions and of the passage to the limits inside the equations. In this section we briefly report the results of P. L. Lions [15] on the analysis of the convergence and of the compactness of solutions. So we suppose to have a sequence of weak solutions (n^k, u^k) of (3.1) and (3.2) with ψ replaced by ψ^k and some initial conditions

$$n^k|_{t=0} = n_0^k \quad \text{in } \Omega, \quad n_0^k u^k|_{t=0} = j_0^k \quad \text{a.c in } \Omega. \tag{3.10}$$

We always assume that n_0^k, j_0^k satisfies (3.5) for all $k \geq 1$, and that $(n_0^k)_{k \geq 1}$ is bounded in $L^1(\Omega) \cap L^{\gamma}(\Omega)$, $(j_0^k)_{k \geq 1}$ is bounded in $L^{2\gamma/(\gamma+1)}(\Omega)$, and $(|j_0^k|^2/n_0^k)_{k \geq 1}$ is bounded in $L^1(\Omega)$. By the energy inequality we can assume that

- $(n^k)_{k \geq 1}$ is bounded in $L^{\infty}(0, T; L^1 \cap L^{\gamma}(\Omega))$,
- $(n^k |u^k|^2)_{k \geq 1}$ is bounded in $L^{\infty}(0, T; L^1(\Omega))$, and

- $(u^k)_{k \geq 1}$ is bounded in $L^2(0, T; H_0^1(\Omega))$.

Without loss of generality (extracting subsequences if necessary), we can assume that

- $n^k \rightharpoonup n$ weakly in $L^\gamma(\Omega \times (0, T))$ and $n \in L^\infty(0, T; L^1 \cap L^\gamma(\Omega))$ $n \geq 0$ a.e.),
- $u^k \rightharpoonup u$ weakly in $L^2(0, T; H_0^1(\Omega))$,
- $\sqrt{n^k} \rightharpoonup \sqrt{n}$ weakly in $L^{2\gamma}(\Omega \times (0, T))$,
- $\sqrt{n^k} u^k \rightharpoonup v$ weakly in $L^2(\Omega \times (0, T))$ and $v \in L^\infty(0, T; L^2(\Omega))$,
- $n^k u^k \rightharpoonup j$ weakly in $L^{2\gamma/(\gamma+1)}(\Omega)$ and $j \in L^\infty(0, T; L^{2\gamma/(\gamma+1)}(\Omega))$,
- $n^k u_i^k u_j^k \rightharpoonup e_{ij}$ in the sense of measures on $\Omega \times (0, T)$ and e_{ij} is a bounded measure on Ω for almost all $t \in (0, T)$ which is bounded uniformly in $t \in (0, T)$,
- $n^k \nabla \psi^k \rightharpoonup \overline{n \nabla \psi}$ weakly in $L^\gamma(\Omega \times (0, T))$.

In order to get compactness it is necessary to assume some a priori bounds on n^k that don't follow from the energy estimate. These bounds will be proved in the next section when we deal with the existence question. We thus assume that for some fixed $q \geq 2$, $q < \gamma$, $s > N/2$

$$(n^k)_{k \geq 1} \text{ is bounded in } L^q(0, T; L^q(K)) \cap L^\infty(0, T; L^s(\Omega)) \quad K \subset \Omega, \text{ compact set.} \tag{3.11}$$

Now we can report the following theorem [15]:

THEOREM 3.2. (1) We always have: $v = \sqrt{n}u$, $j = nu$, $e_{ij} = nu_i u_j$ a.e in $\Omega \times (0, T)$ for all $1 \leq i, j \leq N$.

(2) If, in addition to the above assumptions, we assume that n_0^k converges in $L^1(\Omega)$ to n_0 , then (n, u) is a weak solution of (3.1), (3.2) satisfying the initial conditions (3.4) and we have

$$n^k \longrightarrow n \quad \text{in } C([0, T]; L^p(\Omega)) \cap L^r(K \times (0, T)) \tag{3.12}$$

for all $1 \leq p < s$, $1 \leq r < q$,

$$\begin{aligned} n^k u^k &\longrightarrow nu \quad \text{in } L^p(0, T; L^r(\Omega)) \text{ for all } 1 \leq p < \infty, 1 \leq r < 2\gamma/(\gamma + 1), \\ u^k &\longrightarrow u \quad \text{in } L^p(\Omega \times (0, T)) \cap \{n > 0\} \text{ and in } L^2(\Omega \times (0, T)) \cap \{n > \delta\} \end{aligned} \tag{3.13}$$

for all $1 \leq p < 2$, for all $\delta > 0$,

$$n^k u_i^k u_j^k \longrightarrow nu_i u_j \quad \text{in } L^p(0, T; L^1(\Omega)) \tag{3.14}$$

for all $1 \leq p < \infty$, for all $1 \leq i, j \leq N$.

One of the main tools in the proof of Theorem 3.2 is the use of renormalized solutions of (3.1), (3.2); namely, $u \in L^2(0, T; H_{loc}^1(\Omega))$, $n \in L_{loc}^2(\Omega)$ satisfying (3.1), (3.2) are renormalized solutions of (3.1), (3.2) if they satisfy

$$\frac{\partial \beta(n)}{\partial t} + \text{div}(u\beta(n)) + (\text{div}u)[\beta(n) - \beta'(n)n] = 0$$

for any $\beta \in C^1([0, \infty))$ such that

$$\exists C \geq 0, \quad \text{for all } t \geq 0, \quad |\beta'(t)| \leq C(1 + t)^\alpha$$

with $\alpha = (q - 2)/2$, and also

$$\beta(n^k) \{(\mu + \xi) \text{div}u^k - a(n^k)^\gamma\} \rightharpoonup \bar{\beta} \{(\mu + \xi) \text{div}u - a\bar{p}\} \quad \text{in } \mathcal{D}'.$$

where $\bar{\beta}$, \bar{p} are the weak limits of $(n^k)^\gamma$, $\beta(n^k)$ respectively. The use of the assumption (3.11) is also important in passing into the limit in the terms such as $n^k[u_j^k R_{ij}](\dot{n}^k u_{ij}^k)$, where the operator $R_{ij} = (-\Delta)^{-1} \partial_{ij}$ is bounded in all L^r spaces for $1 < r < \infty$.

3.2. *Existence results.* In this section we are going to prove the existence of a solution (n, u) of (3.1), (3.2).

3.2.1. *A priori bounds.* We always assume that n_0, j_0 satisfy

$$\begin{cases} n_0 \in L^1(\Omega) \cap L^\gamma(\Omega) & n_0 \geq 0 \text{ a.e. in } \Omega, \quad n_0 \neq 0, \\ j_0 \in L^{2\gamma/(\gamma+1)}(\Omega), & j_0 = 0 \text{ a.e. on } \{n_0 = 0\}, \\ |j_0|^2/n_0 \in L^1(\Omega) & \text{defined to be 0 on } \{n_0 = 0\}. \end{cases} \tag{3.15}$$

Following [15], in this section we take a more general hypothesis on ψ :

$$\nabla\psi \in L^1(0, T; L^{2\gamma/(\gamma-1)}(\Omega)) + L^{2/(1+\alpha)}(0, T; L^r(\Omega)), \tag{3.16}$$

where $\alpha = (2 - \gamma)_+, \frac{1}{r} + \frac{1}{\gamma} (1 - \frac{\alpha}{2}) + \frac{\alpha}{2} + \frac{1-\alpha}{q} = 1$, and $q = \frac{2N}{N-2}$ if $N \geq 3$, q is arbitrary in $[2, \infty)$ if $N = 2$. Then from the energy estimate we have the natural a priori bounds

$$u \in L^2(0, T; H_0^1(\Omega)), \quad n \in L^\infty(0, T; L^\gamma(\Omega)), \quad n|u|^2 \in L^1([0, T] \times \Omega). \tag{3.17}$$

In the last section the assumption (3.11) was important. Here, we want to derive this crucial a priori estimate on n .

THEOREM 3.3. Let (n, u) be a solution of (3.1), (3.2). We assume that $\nabla\psi \in L^1(0, T; L^{3/2})$ if $N = 3$ and $\gamma > 6$ and $\nabla\psi \in L^1(0, T; L^2)$ if $N = 2$ and $\gamma > 2$. We assume in addition that $n \in L^{\bar{p}}(K \times (0, T))$ with $\bar{p} = \max(p, 2)$ and $p = \gamma + \frac{2}{N}\gamma - 1$, K being an arbitrary compact set in Ω . Then, n is bounded in $L^p(K \times (0, T))$ in terms of bounds on the data only.

REMARK 3.4. It is possible to provide some explanation for the exponent p occurring in the above result. Indeed, on one hand $n \in L_t^\infty(L_x^{\bar{p}})$, and on the other hand, if we simply expect n^γ and $nu \otimes u$ to have the same integrability, we deduce that $n^{\gamma-1}$ should belong to $L_t^1(L_x^{N/(N-2)})$ since $u \in L_t^2(L_x^{N/(N-2)})$ by Sobolev embeddings. These two bounds imply by interpolation that $n \in L_{t,x}^p$, with $p = \gamma + \frac{2}{N}\gamma - 1$.

We will give only a sketch of the proof of Theorem 3.3, emphasizing the parts in which some modifications are needed because of the collision term nu .

Proof. There is nothing to prove if $\gamma \leq \frac{N}{2}$, since in that case, $p < \gamma - 1$ and $n \in L^\infty(0, T; L^1 \cap L^\gamma)$. Therefore, in all that follows, we assume that $\gamma > \frac{N}{2}$. Since we are in Dirichlet boundary conditions we have to localize our problem. In order to do so, we introduce for any compact set $K \subset \Omega$ a cut-off function $\varphi \in C_0^\infty(\Omega)$ such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on K , and $\varphi \geq 0$ on Ω . We then apply the divergence on Eq. (3.2):

$$\frac{\partial}{\partial t} \operatorname{div}(nu_i) + \partial_i \partial_j (nu_i u_j) - (\mu + \xi) \Delta \operatorname{div} u + \Delta(an^\gamma) = \operatorname{div}(n \nabla \psi) - \operatorname{div}(nu_i).$$

Multiplying by φ we obtain

$$\begin{aligned} \frac{\partial}{\partial t} (\varphi \operatorname{div}(nu_i)) + \partial_i \partial_j (\varphi nu_i u_j) - (\mu + \xi) \Delta (\varphi \operatorname{div} u) + \Delta (a \varphi n^\gamma) = & \operatorname{div}(\varphi n \nabla \psi) \\ & - \varphi \operatorname{div}(nu_i) + F. \end{aligned}$$

where F is given by

$$F = (\partial_{ij}\varphi)nu_iu_j + 2\partial_i\varphi\partial_j(nu_iu) - (\mu + \xi)\Delta\varphi(\operatorname{div}u) + a\Delta\varphi n^\gamma - 2(\mu + \xi)\nabla\varphi \cdot \nabla\operatorname{div}u + 2a\nabla\varphi \cdot \nabla n^\gamma - n\nabla\psi\nabla\varphi. \tag{3.18}$$

Now, applying the operator Δ^{-1} and multiplying the above equation by n^θ , $\theta = p - \gamma$, we obtain

$$\begin{aligned} a\varphi n^{\gamma+\theta} &= \frac{\partial}{\partial t}[n^\theta(-\Delta)^{-1}(\varphi\operatorname{div}(nu))] + \operatorname{div}[un^\theta(-\Delta)^{-1}(\varphi\operatorname{div}(nu))] - n^\theta(-\Delta)^{-1}F \\ &\quad - n^\theta(-\Delta)^{-1}\operatorname{div}(\varphi n\nabla\psi) + n^\theta(-\Delta)^{-1}(\varphi\operatorname{div}(nu)) + (\mu + \xi)\varphi(\operatorname{div}u)n^\theta \\ &\quad + n^\theta\{R_iR_j(\varphi nu_iu_j) - u \cdot \nabla(-\Delta)^{-1}(\varphi\operatorname{div}(nu))\} \\ &\quad + (\theta - 1)(\operatorname{div}u)n^\theta(-\Delta)^{-1}(\varphi\operatorname{div}(nu)), \end{aligned} \tag{3.19}$$

where $R_iR_j = (-\Delta)^{-1}\partial_i\partial_j$. We consider first when $N \geq 3$. If $N \geq 3$, we get an $L^1_{x,t}$ bound on $\varphi n^{\gamma+\theta} = \varphi n^p$ and thus on $\int_K n^p dx$. In fact, by integrating (3.19) over $\Omega \times (0, T)$, we deduce that

$$\begin{aligned} \int_0^T \int_\Omega n^p dx dt &\leq C + C \int_\Omega n^\theta(-\Delta)^{-1}(\varphi\operatorname{div}(nu)) dx \\ &\quad + \int_0^T dt \int_\Omega dx |\operatorname{div}u| n^\theta (1 + |(-\Delta)^{-1}\operatorname{div}(nu)|) \\ &\quad + C \int_0^T dt \int_\Omega dx n^\theta |R_iR_j(\varphi nu_iu_j) - u_iR_iR_j(\varphi nu_j)| \\ &\quad + C \int_0^T dt \int_\Omega dx n^\theta |(-\Delta)^{-1}\operatorname{div}(\varphi\nabla\psi)| \\ &\quad + \int_0^T dt \int_\Omega dx n^\theta (-\Delta)^{-1}F + n^\theta(-\Delta)^{-1}(\varphi\operatorname{div}(nu)). \end{aligned} \tag{3.20}$$

We focus our attention on the bound in $L^1(\Omega \times [0, T])$ of the new term

$$n^\theta(-\Delta)^{-1}(\varphi\operatorname{div}(nu)).$$

Now we recall that n^γ and $n|u|^2$ are bounded in $L^\infty(0, T; L^1)$, while Du is bounded in $L^2(\Omega \times (0, T))$ and u (using Sobolev embeddings) is bounded in $L^2(0, T; L^{2N/N-2}(\Omega))$. In particular, using Hölder inequality, nu is bounded in $L^\infty(0, T; L^{2\gamma/\gamma+1}) \cap L^2(0, T; L^r)$ where $\frac{1}{r} = \frac{N-2}{2N} + \frac{1}{\gamma}$, and thus $n^\theta(-\Delta)^{-1}(\varphi\operatorname{div}(nu)) \in L^\infty(0, T; L^1 \cup L^s)$ where $\frac{1}{s} = \frac{\theta}{\gamma} + \frac{\gamma+1}{2\gamma} - \frac{1}{N} = \frac{1}{N} + \frac{1}{2} - \frac{1}{2\gamma} < 1$. The other terms are handled in the same way as in [15]. We deduce then from (3.19) that

$$\int_0^T \int_\Omega \varphi n^p dx dt \leq C \left[1 + \left(\int_0^T \int_\Omega n^p [|\nabla\varphi|^{p/\gamma} + |\Delta\varphi|^{p/\gamma}] dx dt \right)^{\gamma/p} \right] \tag{3.21}$$

to get

$$\int_0^T \int_\Omega \varphi n^p dx dt \leq C1 + \left(\int_0^T \int_\Omega \varphi n^p dx dt \right)^{\gamma/p} \leq C1 + T^{1-\frac{p}{\gamma}} \int_0^T \int_\Omega \varphi n^\gamma dx dt.$$

Remembering that $\varphi \equiv 1$ on $K \subset \Omega$, a compact set, we obtain the desired estimate when $n \geq 3$.

Finally we consider the case when $N = 2$. In that case we need to handle in a different way the terms $n^\theta \{R_i R_j(\varphi n u_i u_j) - u \cdot \nabla(-\Delta)^{-1}(\varphi \operatorname{div}(n u))\}$ and $[\operatorname{div} n^\theta(-\Delta)^{-1}(\varphi \operatorname{div}(n u))]$. After some calculation we have only to consider $n^\theta \{R_i R_j(\varphi n u_i u_j) - u_i R_i R_j(\varphi n u_j)\}$ and $\operatorname{div} n^\theta(-\Delta)^{-1}(\operatorname{div}(\varphi n u))$. In order to do so, we are going to multiply (3.21) by φ^m for some m to be determined later on. Here we follow [15] exactly, but we point out that to estimate the term $\{R_i R_j(\varphi n u_i u_j) - u_i R_i R_j(\varphi n u_j)\}$, we use the fact that u is bounded in $L^2(0, T; H^1)$ and thus in $L^2(0, T; BMO)$. Then, by the Coifman-Rochberg-Weiss commutator Theorem [5], we have for almost all $t \in [0, T]$

$$\|R_i R_j(\varphi n u_i u_j) - u_i R_i R_j(\varphi n u_j)\|_{L^{2p/(p+1)}} \leq C \|u\|_{BMO} \|n u\|_{L^{2p/(p+1)}}.$$

and we finally deduce that

$$\|n^\theta \{R_i R_j(\varphi n u_i u_j) - u_i R_i R_j(\varphi n u_j)\}\|_{L^{2p/(p+1)}} \leq C \|n\|_{L^{p,t}}^{1/2}. \tag{3.22}$$

Hence, if we choose m such that $mp/\theta \geq 1 + m$ and $2p \geq 1 + m$, then we have

$$\int_0^T dt \int_\Omega \varphi^{1+m} n^p dx \leq C \left(1 + \|\varphi^{1+m/p} n\|_{L^{p/\theta}(\Omega \times [0, T])}^\theta\right)$$

if we choose m such that $mp/\theta \geq 1 + m$. In conclusion, the proof is complete, provided we choose m and φ such that $\nabla \varphi^m \leq C$ holds and $1 + m \leq 2p$, $mp/\theta \geq 1 + m$. In particular we may take $m = 1$ and we conclude since $p \geq 1$ and $p = 2\gamma - 1 \geq 2\theta = 2(\gamma - 1)$. \square

3.2.2. Existence. We recall the notion of weak solution. We require n and u to satisfy (3.1), (3.2) and

$$n \in L^\infty(0, T; L^1 \cap L^\gamma(\Omega)) \cap C([0, T], L^r(\Omega)) \quad \text{for all } 1 \leq r < \gamma, n \geq 0 \text{ a.e.} \tag{3.23}$$

$$u \in L^2(0, T; H_0^1(\Omega)), \quad n|u|^2 \in L^\infty(0, T; L^1(\Omega)) \tag{3.24}$$

$$n u \in C([0, T], L^{2\gamma/(\gamma+1)}(\Omega) - \text{weak}). \tag{3.25}$$

Finally, as in Theorem 3.3, we assume that $\nabla \psi \in L^1(0, T; L^{3/2})$ if $N = 3$ and $\gamma > 6$ and $\nabla \psi \in L^1(0, T; L^2)$ if $N = 2$ and $\gamma > 2$.

THEOREM 3.5. Under the above conditions, and if we assume that $\gamma \geq \frac{3}{2}$ if $N = 2$, $\gamma \geq \frac{9}{5}$ if $N = 3$, and $\gamma > \frac{N}{2}$ if $N \geq 4$, there exists a solution (n, u) of (3.1) and (3.2) satisfying the initial conditions (3.4) and such that $n \in L^p(K \times (0, T))$ for any compact set $K \subset \Omega$, where $p = \gamma + \frac{2}{N} - 1$. In addition (n, u) satisfies the following energy inequality for almost all $t \in [0, T]$:

$$\begin{aligned} \int_\Omega n \frac{|u|^2}{2} + \frac{a}{\gamma - 1} n^\gamma dx + \int_0^t ds \int_\Omega \mu |Du|^2 + \xi (\operatorname{div} u)^2 + n u^2 dx \\ \leq \int_\Omega \frac{1}{2} \frac{|j_0|^2}{n_0} + \frac{a}{\gamma - 1} n_0^\gamma dx + \int_0^t ds \int_\Omega n u \nabla \psi dx. \end{aligned} \tag{3.26}$$

Moreover, for any $R > 0$, if

$$\|n_0\|_{L^\gamma(\Omega)} \leq R \quad \|j_0\|_{L^{2\gamma/(\gamma+1)}(\Omega)} \leq R \quad \|\psi\|_{W^{1,2\gamma/(\gamma-1)}(\Omega)} \leq R.$$

then there exists $c(R) > 0$ such that

$$\|n(\cdot, t)\|_{L^\infty(0,T;L^\gamma(\Omega))} \leq c(R) \quad \|j(\cdot, t)\|_{L^\infty(0,T;L^{2\gamma/(\gamma+1)}(\Omega))} \leq c(R). \tag{3.27}$$

REMARK 3.6. The uniqueness of solutions, the regularity of solutions, and the fact that this energy inequality (3.26) might be an equality are outstanding open questions.

REMARK 3.7. It is possible to prove that the momentum $j = nu$ of the Navier-Stokes equation has actually some time regularity with values into $L^{2\gamma/(\gamma+1)}$ endowed with its strong topology (see [8] for details).

The proof of Theorem (3.5) will be made “backwards”: if Eq. B approximates Eq. A, we first explain how solutions of A can be obtained from solutions of B before explaining how to obtain the latter via solutions of C. Here we report for completeness the first step of the proof from [15].

Proof. We consider a “periodic box” $\tilde{\Omega} = \prod_{i=1}^N(a_i, b_i)$ such that $\bar{\Omega} \subset \tilde{\Omega}$; without loss of generality, we may take $a_i = 0$ and set $T_i = b_i - a_i$ for $1 \leq i \leq N$. Then we introduce $p \in C^\infty(\mathbb{R}^N)$, periodic in each x_i of period T_i for each $i \in \{1, \dots, N\}$, such that $p \equiv 0$ in $\bar{\Omega}$, $p > 0$ in $\tilde{\Omega} - \bar{\Omega}$. We then consider the following problem in $\mathbb{R}^N \times (0, T)$, where the solution $(n^\varepsilon, u^\varepsilon)$ is required to be periodic in x_i of period T_i for each i and $\varepsilon \in (0, 1]$:

$$\begin{cases} \frac{\partial n^\varepsilon}{\partial t} + \operatorname{div}(n^\varepsilon u^\varepsilon) = 0 & n^\varepsilon \geq 0 \\ \frac{\partial(n^\varepsilon u^\varepsilon)}{\partial t} + \operatorname{div}(n^\varepsilon u \otimes u^\varepsilon) + a \nabla(n^\varepsilon)^\gamma - \mu \Delta u^\varepsilon - \xi \nabla \operatorname{div} u^\varepsilon + \frac{1}{\varepsilon} p u^\varepsilon = n^\varepsilon \nabla \tilde{\psi} - n^\varepsilon u^\varepsilon. \end{cases} \tag{3.28}$$

where $\tilde{\psi}$ is an extension of ψ to $\tilde{\Omega} \times (0, T)$ by 0 and is then extended periodically to $\mathbb{R}^N \times (0, T)$. Now we assume to have a solution $(n^\varepsilon, u^\varepsilon)$ of (3.28) satisfying all of the properties listed in Theorem 3.5 and (3.26) is replaced by

$$\begin{aligned} \int_\Omega n^\varepsilon \frac{1}{2} |u^\varepsilon|^2 + \frac{a}{\gamma - 1} (n^\varepsilon)^\gamma dx + \int_0^t ds \int_{\tilde{\Omega}} \mu |Du^\varepsilon|^2 + \xi (\operatorname{div} u^\varepsilon)^2 + n^\varepsilon |u^\varepsilon|^2 + \frac{1}{\varepsilon} p u^\varepsilon dx \\ \leq \int_{\tilde{\Omega}} \frac{1}{2} \frac{|\tilde{j}_0|^2}{\tilde{n}_0} + \frac{a}{\gamma - 1} \tilde{n}_0^\gamma dx + \int_0^t ds \int_{\tilde{\Omega}} n^\varepsilon u^\varepsilon \nabla \tilde{\psi} dx. \end{aligned} \tag{3.29}$$

Of course, $n^\varepsilon|_{t=0} = \tilde{n}_0$, $n^\varepsilon u^\varepsilon|_{t=0} = \tilde{j}_0$, and $(\tilde{n}_0, \tilde{j}_0)$ are periodic extensions of (n_0, j_0) by 0. The inequality (3.29) yields bounds on $(n^\varepsilon)^\gamma$ in $L^\infty(0, T; L^1(\tilde{\Omega}))$, on $n^\varepsilon |u^\varepsilon|^2$ in $L(\tilde{\Omega} \times (0, T))$, on u^ε in $L^2(0, T; H^1(\tilde{\Omega}))$, and on $\frac{1}{\varepsilon} p u^\varepsilon$ in $L^1(\tilde{\Omega} \times (0, T))$ uniformly in ε . In addition, Theorem 3.3 yields a bound on (n^ε) in $L^p(K \times (0, T))$ for any compact set $K \subset \Omega$, where $p = \gamma + \frac{2}{N} - 1$. Extracting subsequences if necessary, we may assume that n^ε converges weakly in $L^r(0, T; L^\gamma(\tilde{\Omega}))$ for all $r \in (1, \infty)$ as ε goes to 0 to some $n \in L^\infty(0, T; L^\gamma(\tilde{\Omega})) \cap L^p(K \times (0, T))$ for any compact set $K \subset \Omega$, and we may assume that u^ε converges weakly in $L^2(0, T; H^1(\tilde{\Omega}))$ as ε goes to 0 to some u . In addition, since $\frac{1}{\sqrt{\varepsilon}} \sqrt{p} u^\varepsilon$ is bounded in $L^2(\tilde{\Omega} \times (0, T))$, $\sqrt{p} u^\varepsilon$ converges to 0 in $L^2(\tilde{\Omega} \times (0, T))$ and thus $\tilde{p} u \equiv 0$ in $\tilde{\Omega} \times (0, T)$. Therefore $u = 0$ a.e. in $(\tilde{\Omega} - \Omega) \times (0, T)$. Since Ω is smooth, this implies that $u \in L^2(0, T; H_0^1)$.

Finally, in order to conclude, we simply observe that we can apply Theorem 3.2 to the

sequences $(n^\varepsilon, u^\varepsilon)$. In fact we can apply part (1) and part (2) of Theorem 3.2. Finally from the energy inequality (3.26) we get (3.27). Now we have to show how to construct a solution for (3.28): for this we can follow [15]. □

4. Existence for the Coupled Navier-Stokes-Poisson equation.

4.1. *Local existence.* We recall we are studying the following problem:

$$\begin{cases} \frac{\partial n}{\partial t} + \operatorname{div}(nu) = 0 \\ \frac{\partial nu}{\partial t} + \operatorname{div}(nu \otimes u) + \nabla(an^\gamma) - \mu\Delta u - \xi\nabla\operatorname{div}u = n\nabla\phi - nu \\ \Delta\phi = n - b(x) \end{cases} \tag{4.1}$$

with the initial conditions (2.5), the boundary condition (2.6), and the hypotheses (A.1), (A.2), (A.3), (A.4), (A.5). We wish to solve the previous problem by using a fixed point argument on the potential ϕ : namely, we consider

$$\frac{\partial n}{\partial t} + \operatorname{div}(nu) = 0 \tag{4.2}$$

$$\frac{\partial nu}{\partial t} + \operatorname{div}(nu \otimes u) + \nabla(an^\gamma) - \mu\Delta u - \xi\nabla\operatorname{div}u = n\nabla\psi - nu \tag{4.3}$$

$$\Delta\phi = n - b(x) \tag{4.4}$$

with the potential ψ given. We assume the initial condition (2.5), the boundary conditions (2.6), the hypotheses (A.1), (A.2), (A.3), (A.4), (A.5), and the following hypothesis for all $T > 0$:

(A.6)

$$\begin{aligned} \psi &\in L^1(0, T; W^{1.2\gamma/(\gamma-1)}) && \text{if } N \geq 2. \\ \psi &\in L^1(0, T; W^{1.2}) && \text{if } N = 2 \text{ and } \gamma > 2. \\ \psi &\in L^1(0, T; W^{1.3/2}) && \text{if } N = 3 \text{ and } \gamma > 6. \end{aligned}$$

We now define the mapping for which a fixed point will yield a solution of (4.1). Let us denote by

$$\begin{aligned} F(0, T_1; R) &= \left\{ \phi \in L^1(0, T_1, W^{1.2\gamma/(\gamma-1)}(\Omega)) \mid \|\phi - \phi_0\|_{L^1(0, T_1, W^{1.2\gamma/(\gamma-1)}(\Omega))} \leq R \right\} \\ &\text{if } N \geq 8 \text{ and } \gamma > N/2 \text{ or } 4 \leq N < 8 \text{ and } \gamma > 5N/(N+2) \text{ or } N = 3 \text{ and } 3 \leq \gamma \leq 6, \\ F(0, T_1; R) &= \left\{ \phi \in L^1(0, T_1, W^{1.2}(\Omega)) \mid \|\phi - \phi_0\|_{L^1(0, T_1, W^{1.2}(\Omega))} \leq R \right\} \\ &\text{if } N = 2 \text{ and } \gamma > 2, \text{ and} \\ F(0, T_1; R) &= \left\{ \phi \in L^1(0, T_1, W^{1.3/2}(\Omega)) \mid \|\phi - \phi_0\|_{L^1(0, T_1, W^{1.3/2}(\Omega))} \leq R \right\} \\ &\text{if } N = 3 \text{ and } \gamma > 6. \end{aligned}$$

We define a map from $F(0, T_1; R)$ into itself as follows:

$$A : F(0, T_1; R) \longrightarrow F(0, T_1; R)$$

and

$$A\psi = \phi \quad \text{if and only if} \quad \Delta\phi = n - b(x),$$

where (n, u) is the weak solution to the Navier-Stokes equation (4.2), (4.3) with a given potential ψ . The map A is well-posed, since from the previous section we have the existence of weak solutions to (4.2), (4.3) in any arbitrary time interval $[0, T]$. Now we show the existence of a fixed point which will give a local weak solution to (4.1) with the regularity defined in (2.7), (2.8), (2.9), and (2.10).

THEOREM 4.1. Suppose that (A.1), (A.2), (A.3), (A.4), (A.5), (A.6) hold. Then if $N \geq 8$ and $\gamma \geq N/2$ or $3 < N < 8$ and $\gamma > 5N/(N + 2)$ or $N = 3$ and $3 < \gamma \leq 6$, there exists $T_1 > 0$ such that the map A is continuous and compact.

Proof. From Theorem 3.5 we have that n is bounded in $L^\infty(0, T; L^\gamma(\Omega))$, while $j = nu$ is bounded in $L^\infty(0, T; L^{2\gamma/(\gamma+1)}(\Omega))$. More precisely, there exists a constant $c(R) > 0$ depending only on the initial data such that

$$\|n - n_0\|_{L^\infty(0, T; L^\gamma(\Omega))} \leq c(R) \quad \|j\|_{L^\infty(0, T; L^{2\gamma/(\gamma+1)}(\Omega))} \leq c(R).$$

From the Poisson equation (4.4) we get in the sense of distribution

$$\operatorname{div} [\nabla\phi_t] = -\operatorname{div} j \tag{4.5}$$

and applying the elliptic regularity Theorem 2.2 we have

$$\begin{aligned} \|A\psi(\cdot, t) - \phi_0(\cdot)\|_{L^s(\Omega)} &\leq \|A\psi(\cdot, t) - \phi_0(\cdot)\|_{W^{1, 2\gamma/(\gamma+1)}(\Omega)} \\ &= \|\phi(\cdot, t) - \phi_0(\cdot)\|_{W^{1, 2\gamma/(\gamma+1)}(\Omega)} \\ &\leq t\|j\|_{L^\infty(0, T; L^{2\gamma/(\gamma+1)}(\Omega))} \leq tc(R), \end{aligned}$$

where s is the Sobolev exponent of $\frac{2\gamma}{\gamma+1}$. We can also observe that

$$\Delta\phi(\cdot, t) - \Delta\phi(\cdot, 0) = n(\cdot, t) - n_0(\cdot).$$

and so, applying Theorem 2.2 again we get

$$\|A\psi(\cdot, t) - \phi_0(\cdot)\|_{W^{2, \gamma}(\Omega)} \leq \|n(\cdot, t) - n_0(\cdot)\|_{L^\gamma(\Omega)} \leq c(R).$$

Now we apply (2.2) of Theorem 2.3 to have

$$\begin{aligned} \|A\psi(\cdot, t) - \phi_0(\cdot)\|_{W^{1, 2\gamma/(\gamma-1)}(\Omega)} &\leq c\|\phi(\cdot, t) - \phi_0(\cdot)\|_{W^{2, \gamma}(\Omega)}^{1/2} \|\phi(\cdot, t) - \phi_0(\cdot)\|_{L^q(\Omega)}^{1/2} \\ &\leq c(R)t^{1/2}, \end{aligned}$$

where $q = \frac{\gamma}{\gamma-2}$ and provided that $q < s$, which is true if $N \geq 8$ and $\gamma \geq N/2$ or $3 < N < 8$ and $\gamma > 5N/(N + 2)$. We have

$$\|A\psi - \phi_0\|_{L^1(0, T_1, W^{1, 2\gamma/(\gamma-1)}(\Omega))} \leq c(R)T_1^{3/2}.$$

Now taking, for instance,

$$T_1 = \frac{1}{2} \left(\frac{R}{c(R)} \right)^{2/3},$$

we get that A is continuous from $F(0, T_1; R)$ into itself. Finally, we have to show that A is a compact map. It is sufficient to see that the range of A is relatively compact. Applying Theorem 2.2 again we have, taking $\psi \in F(0, T_1; R)$,

$$\|A\psi(\cdot, t) - A\psi(\cdot, s)\|_{W^{1,2\gamma/(\gamma+1)}} \leq c|t - s| \|j\|_{L^\infty(0, T; L^{2\gamma/(\gamma+1)}(\Omega))} \leq c(R)|t - s|,$$

so $AF(0, T_1; R) \in C^{1/2}(0, T; W^{1,2\gamma/(\gamma-1)}(\Omega))$. Moreover, since $A\psi(\cdot, t) - \phi_0(\cdot) \in W_0^{1,\gamma}(\Omega) \cap W^{2,\gamma}(\Omega)$, by using the hypotheses on γ we get $AF(0, T_1; R)$, for any fixed t , is relatively compact in $W^{1,2\gamma/(\gamma-1)}(\Omega)$. In this way, all the hypotheses of Theorem 2.1 are fulfilled and we get, finally, the compactness of the map A . \square

Now we investigate what happens in the other cases.

THEOREM 4.2. Suppose that (A.1), (A.2), (A.3), (A.4), (A.5), (A.6) hold. Then if $N = 3$ and $\gamma = 3$ or $\gamma \geq 6$, there exists $T_1 > 0$ such that the map A is continuous and compact.

Proof. If $\gamma > 6$ by hypothesis (A.6), we can take $\psi \in L^1(0, T_1; W^{1,3/2}(\Omega))$, and so by Theorem 2.2 we have

$$\begin{aligned} \|A\psi(\cdot, t) - \phi_0(\cdot)\|_{W^{1,3/2}} &\leq \|A\psi(\cdot, t) - A\phi_0(\cdot)\|_{W^{1,2\gamma/(\gamma+1)}} \leq t \|j\|_{L^\infty(0, T; L^{2\gamma/(\gamma+1)}(\Omega))} \\ &\leq t(R), \end{aligned}$$

and, moreover,

$$\|A\psi - \phi_0\|_{L^1(0, T_1; W^{1,3/2}(\Omega))} \leq c(R)T_1^2.$$

When $\gamma = 3$, by applying the Sobolev embeddings theorem and the elliptic regularity theorem, we get

$$\|\phi(\cdot, t) - \phi_0(\cdot)\|_{L^s(\Omega)} \leq t \|j\|_{L^\infty(0, T; L^{2\gamma/(\gamma+1)}(\Omega))},$$

where s is the Sobolev exponent of $\frac{2\gamma}{\gamma+1}$. Using the interpolation inequality (2.3) of Theorem 2.3 we get

$$\|\nabla\phi(\cdot, t) - \nabla\phi_0(\cdot)\|_{L^\gamma(\Omega)} \leq c\|\phi(\cdot, t) - \phi_0(\cdot)\|_{W^{2,\gamma}(\Omega)}^{1/2} \|\phi(\cdot, t) - \phi_0(\cdot)\|_{L^\gamma(\Omega)}^{1/2}.$$

If $s \geq \gamma$ we get

$$\begin{aligned} \|\nabla\phi(\cdot, t) - \nabla\phi_0(\cdot)\|_{L^\gamma(\Omega)} &\leq c(R)^{1/2} \|\phi(\cdot, t) - \phi_0(\cdot)\|_{L^s(\Omega)} \\ &\leq c(R)t^{1/2} \|j\|_{L^\infty(0, T; L^{2\gamma/(\gamma+1)}(\Omega))} \leq c(R)t^{1/2}. \end{aligned}$$

Now provided $\gamma > \frac{2\gamma}{\gamma-1}$ and choosing t sufficiently small, we get as in the previous theorem our result. We have only to verify that $\frac{2\gamma}{\gamma-1} < \gamma < s$, and this is the case if $\gamma = 3$ (actually this is the only case). As in Theorem 4.1 we get the compactness of A . \square

Finally we investigate the case $N = 2$.

THEOREM 4.3. Suppose that (A.1), (A.2), (A.3), (A.4), (A.5), (A.6) hold. If $N = 2$ and $\gamma > 2$, there exists $T_1 > 0$ such that the map A is continuous and compact.

Proof. In the case $N = 2$ and $\gamma > 2$, we follow the same argument of Theorem 4.1 with the difference that, in this case, $s = 2\gamma$ and $q = \frac{\gamma-1}{\gamma}$ and since $\gamma > 2$ we have $q < s$.

The compactness of A follows by using Theorem 2.1 in the same way as in Theorem 4.1. □

Now we can conclude with the main result in this section:

THEOREM 4.4. Suppose that (A.1), (A.2), (A.3), (A.4), (A.5), (A.6) hold. Then if $N \geq 8$ and $\gamma > N/2$ or $4 \leq N < 8$ and $\gamma > 5N/(N + 2)$ or $N = 3$ and $\gamma \geq 3$ or $N = 2$ and $\gamma > 2$, there exists a local solution of system (2.4).

Proof. The theorem is a consequence of Theorem 3.5 and Schauder’s fixed point Theorem ([21], pg. 84). In fact, by Theorem 3.5, we get the existence of n and u in an arbitrary time interval for (4.2), (4.3). Using Theorem 4.1, Theorem 4.2, and Theorem 4.3, we can apply Schauder’s fixed point Theorem ([21], pg. 84) to the map A , getting a fixed point ϕ for the map A . Namely, there exist $\phi \in F(0, T_1; R)$ such that $\Delta\phi = n - b(x)$; that is our solution to the Cauchy problem. □

4.2. *Global existence.* From Theorem 4.4 we have the existence of a local weak solution (n, u, ϕ) in a time interval $[0, T^*)$, where T^* is the maximal time of existence. We want to prove that we can extend the weak solution of the system (4.1) to the whole interval $[0, T^*]$. From the energy estimates follows

THEOREM 4.5. Let (n, u, ϕ) be a local weak solution of system (4.1); then $j = nu$ is equibounded in $L^\infty(0, T^*; L^{2\gamma/(\gamma+1)}(\Omega))$.

Proof. Multiplying Eq. (4.3) by u and integrating with respect to x and remembering the boundary conditions on u , we deduce that we have for some $\nu > 0$

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \left\{ n \frac{|u|^2}{2} + \frac{a}{\gamma - 1} n^\gamma \right\} dx + \nu \int_{\Omega} |Du|^2 dx + \frac{3\gamma - 1}{2\gamma} \int_{\Omega} n \frac{|u|^2}{2} dx \\ & \leq \frac{\gamma + 1}{4\gamma} \int_{\Omega} n^\gamma dx + \frac{\gamma - 1}{4\gamma} \|\nabla\phi(\cdot, t)\|_{L^{2\gamma/(\gamma-1)}(\Omega)}. \end{aligned} \tag{4.6}$$

Using Gronwall’s lemma we get for any $t \in [0, T^*)$

$$\int_{\Omega} \left\{ n \frac{|u|^2}{2} + \frac{a}{\gamma - 1} n^\gamma \right\} dx \leq C_1 + C_2 \int_0^{T^*} \|\nabla\phi(\cdot, s)\|_{L^{2\gamma/(\gamma-1)}(\Omega)} ds$$

and

$$\begin{aligned} \|\dot{j}(\cdot, t)\|_{L^{2\gamma/(\gamma+1)}(\Omega)} & \leq \|n(\cdot, t)\|_{L^\gamma(\Omega)}^{\gamma/2} \|nu^2(\cdot, t)\|_{L^1(\Omega)}^{1/2} \\ & \leq C_1 + C_2 \int_0^{T^*} \|\nabla\phi(\cdot, s)\|_{L^{2\gamma/(\gamma-1)}(\Omega)} ds. \end{aligned}$$

By Theorem 2.2 and the previous estimates we get (from (4.1))

$$\begin{aligned} \frac{d}{dt} \|\nabla\phi(\cdot, t)\|_{L^{2\gamma/(\gamma+1)}(\Omega)} & \leq \|\nabla\phi_t(\cdot, t)\|_{L^{2\gamma/(\gamma+1)}(\Omega)} \leq c\|j(\cdot, t)\|_{L^{2\gamma/(\gamma+1)}(\Omega)} \\ & \leq C_1 + C_2 \sup_{t \in [0, T^*]} \|\nabla\phi(\cdot, t)\|_{L^{2\gamma/(\gamma-1)}(\Omega)}. \end{aligned}$$

By Gronwall’s lemma there exists $M > 0$ such that

$$\sup_{t \in [0, T^*]} \|\nabla\phi(\cdot, t)\|_{L^{2\gamma/(\gamma+1)}(\Omega)} \leq M$$

from which follows the equibound on $j = nu$. \square

COROLLARY 4.6. Suppose T^* is the maximal time of existence of the solutions of system (4.1) and that the hypotheses of Theorem 4.4 hold; then

$$\phi \in C^{1/2}(0, T_1; W^{1, 2\gamma/(\gamma-1)}(\Omega)) \quad \text{if } N \geq 2, \quad (4.7)$$

$$\phi \in C^{1/2}(0, T_1; W^{1, 2}(\Omega)) \quad \text{if } N = 2 \text{ and } \gamma > 2, \quad (4.8)$$

$$\phi \in Lip(0, T_1; W^{1, 3/2}(\Omega)) \quad \text{if } N = 3 \text{ and } \gamma > 6; \quad (4.9)$$

moreover, ϕ is equicontinuous in time.

Proof. Combining the equation of system (4.1) and Theorem 2.2, we have

$$\|\phi(\cdot, t) - \phi_0(\cdot, s)\|_{W^{1, 2\gamma/(\gamma+1)}(\Omega)} \leq |t - s| \|j\|_{L^\infty(0, T^*; L^{2\gamma/(\gamma+1)}(\Omega))};$$

by Theorem 4.5 and the values of γ we get the equicontinuity in time of ϕ . \square

By the previous results and by using standard continuation methods, we obtain the following globally in time existence theorem.

THEOREM 4.7. Assume that $N \geq 8$ and $\gamma > N/2$ or $4 \leq N < 8$ and $\gamma > 5N/(N+2)$ or $N = 3$ and $\gamma \geq 3$ or $N = 2$ and $\gamma > 2$. If (A.1), (A.2), (A.3), (A.4), and (A.5) hold, then the system (2.4) has a global weak solution satisfying (2.7), (2.8), (2.9), (2.10).

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