

# LOCAL AND GLOBAL RESULTS FOR WAVE MAPS I

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ABSTRACT. We consider the initial value problem for wave-maps corresponding to constant coefficient second order hyperbolic equations in  $n + 1$  dimensions,  $n \geq 4$ . We prove that this problem is globally well-posed for initial data which is small in the homogeneous Besov space  $\dot{B}_{n/2}^{2,1} \times \dot{B}_{n/2-1}^{2,1}$ . Our second result deals with more regular solutions; it essentially says that if in addition the initial data is in  $H^s \times H^{s-1}$ ,  $s > n/2$ , then the solutions stay bounded in the same space. In part II of this work we shall prove that the same result holds in dimensions  $n = 2, 3$ .

## 1. INTRODUCTION

Let  $(M, g)$  be a compact Riemannian manifold without boundary. A wave map is a continuous function from the Minkowski space  $R^n \times R$  into  $M$ ,

$$\phi : R^n \times R \rightarrow M,$$

which is locally a critical point for the functional

$$F(\phi) = \int_{R^{n+1}} \langle \partial^i \phi, \partial_i \phi \rangle_g dt dx .$$

In local coordinates in  $M$  the equations for  $\phi$  can be written as

$$\square \phi^\alpha + \Gamma_{jk}^\alpha(\phi) \partial^i \phi^j \partial_i \phi^k = 0, \tag{1.1}$$

where  $\Gamma_{jk}^\alpha$  are the Riemann-Christoffel symbols. To formulate the corresponding initial value problem consider the space-like surface  $\Sigma = \{t = 0\}$  and add to the equation (1.1) the initial condition

$$\phi = f_0, \quad \phi_t = f_1 \quad \text{in } \Sigma. \tag{1.2}$$

Typically one chooses the initial data in a Sobolev space,  $(f_0, f_1) \in H^s(R^n) \times H^{s-1}(R^n)$ . Then the interesting question is what is the lowest value of  $s$  for which this equation is locally well-posed. A simple scaling analysis leads to the critical exponent

$$s_c = \frac{n}{2} .$$

This roughly implies that one could hope to obtain a local well-posedness result only for  $s \geq s_c = n/2$ . The exponent  $s = n/2$  is particularly interesting since rescaling preserves the homogeneous  $\dot{H}^{n/2} \times \dot{H}^{n/2-1}$  norm of the initial

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data; hence, for this  $s$  a local well-posedness result (say, for small data) will imply its global counterpart.

Naively one could say that this is essentially an equation of the form

$$\square u = |\nabla u|^2.$$

However, what enables one to obtain good results for this problem is the special form of the nonlinearity. Namely, the square of the gradient only appears in expressions of the form

$$Q_0(u, v) = \partial^i u \partial_i v = u_t v_t - u_x v_x.$$

With this notation, the equation can be rewritten as

$$\square \phi^\alpha + \Gamma_{jk}^\alpha(\phi) Q_0(\phi^j, \phi^k) = 0. \quad (1.3)$$

This special quadratic form exhibits certain cancellation properties in estimates and is called a null-form. One simple way to see this cancellation is in the following decomposition of  $Q_0$ :

$$Q_0(u, v) = \square(uv) - u\square v - v\square u, \quad (1.4)$$

which is used later in the article.

Taking advantage of the null structure of the nonlinearity Klainerman was able to prove global well-posedness for small smooth data [7] in dimension  $n \geq 3$ , (see also Sideris [14] for related results, including in dimension  $n = 2$ ). Recently counterexamples to global well-posedness for large data in dimension  $n \geq 3$  were also found, see [13], [3].

On the local well-posedness side Klainerman-Machedon [10] proved local well-posedness in  $H^s \times H^{s-1}$  for  $s > n/2$  in dimension  $n \geq 3$ . The same result in dimension 2 was later proved by Klainerman-Selberg [11]. One should note that the difficulty decreases with the dimension, so the problem in 2 dimensions is the hardest.

Hence, the most interesting remaining problem is to study well-posedness in the scale-invariant setting, i.e. when  $s = n/2$ . Since  $H^{n/2}$  does not embed into  $L^\infty$ , one does not expect the solution  $\phi$  to be continuous therefore any approach to this problem must take into account the global geometric properties of the target manifold (e.g. completeness, curvature, etc).

Our aim here is to prove a slightly weaker result, namely that the wave maps equation is locally well-posed for initial data in the homogeneous Besov space  $\dot{B}_{\frac{n}{2}}^{2,1} \times \dot{B}_{\frac{n}{2}-1}^{2,1}$ , which is a bit smaller than  $\dot{H}^{\frac{n}{2}} \times \dot{H}^{\frac{n}{2}-1}$ . Nevertheless, the scaling for the two spaces is the same, therefore we can still make the connection between the local and the global problem and prove a global result for small data. One advantage in using the Besov spaces is the embedding

$$\dot{B}_{\frac{n}{2}}^{2,1} \subset L^\infty,$$

which guarantees that in this case the solution  $\phi$  is continuous. Then we can study the problem locally with respect to the target manifold  $M$ . Consequently, the geometrical properties of  $M$  become irrelevant in this context.

Our main result is

**Theorem 1.** *a) There exist  $C, D > 0$  so that for any initial data satisfying*

$$|(f_0, f_1)|_{\dot{B}_{\frac{n}{2}}^{2,1} \times \dot{B}_{\frac{n}{2}-1}^{2,1}} < C, \quad (1.5)$$

*there exists a global solution  $\phi$  to (1.1), (1.2) satisfying*

$$|(\phi(t), \phi_t(t))|_{\dot{B}_{\frac{n}{2}}^{2,1} \times \dot{B}_{\frac{n}{2}-1}^{2,1}} \leq D,$$

*which is the unique limit of smooth solutions. Furthermore, the solution depends Lipschitz on the initial data.*

*b) Assume in addition that  $(f_0, f_1) \in H^s \times H^{s-1}$ . Then*

$$|(\phi(t), \phi_t(t))|_{H^s \times H^{s-1}} \leq c|(f_0, f_1)|_{H^s \times H^{s-1}}. \quad (1.6)$$

For large data we can take advantage of the finite speed of propagation and observe that the data is in effect small if restricted to a ball of sufficiently small radius. Consequently we can use Theorem 1 to obtain

**Theorem 2.** *For any initial data  $(f_0, f_1) \in \dot{B}_{\frac{n}{2}}^{2,1} \times \dot{B}_{\frac{n}{2}-1}^{2,1}$  there exists  $T > 0$  and a solution  $\phi$  to (1.1), (1.2) satisfying*

$$(\phi, \phi_t) \in C([0, T]; \dot{B}_{\frac{n}{2}}^{2,1} \times \dot{B}_{\frac{n}{2}-1}^{2,1}),$$

*which is the unique limit of smooth solutions. Furthermore, the solution depends continuously on the initial data.*

Of course in this case the existence time  $T$  fully depends on the initial data, and not only on its size.

## 2. FUNCTION SPACES

A critical part in the proof of our result is the choice of the space-time function spaces where we look for the solutions. There are two ingredients required in the definition of our spaces, namely the structure of the dyadic pieces and the Besov type norm used for the summation of the dyadic pieces.

We use the notations  $(x, t)$  for the space-time variables and  $(\xi, \tau)$  for the corresponding Fourier variables. Then the symbol of the wave operator  $\square$  is

$$p(\xi, \tau) = \tau^2 - \xi^2.$$

Denote by  $w$  the normalized symbol of the wave operator

$$w(\xi, \tau) = |(\xi, \tau)|^{-1} p(\xi, \tau).$$

Let  $\chi$  be a smooth cutoff function supported in  $[1, 4]$  so that

$$\sum_{j \in \mathbb{N}} \chi(2^{-j} x) = 1.$$

Define now dyadic cutoff symbols with respect to frequency

$$a_\lambda(\xi, \tau) = \chi\left(\frac{|(\xi, \tau)|}{\lambda}\right),$$

and with respect to the distance to the cone,

$$b_\mu(\xi, \tau) = \chi\left(\frac{|w(\xi, \tau)|}{\mu}\right),$$

Define also the multiplier

$$\tilde{b}_\mu(\xi, \tau) = \sum_{j \geq -4} b_{2^{-j}\mu}(\xi, \tau),$$

which cuts off a  $4\mu$  neighbourhood of the cone.

Now we can define the  $\|\cdot\|_s$  norm by

$$\|u\|_s = \sum_{\mu=2^j} \mu^s |B_\mu(D)u|_{L^2}.$$

Correspondingly we define the first dyadic building block for our function spaces, namely the space  $X_\lambda^s$  of functions with Fourier transform supported in a dyadic region in frequency  $|\xi| \in [\lambda, 4\lambda]$ , with the  $\|\cdot\|_s$  norm.

A second dyadic building block is the space  $Y_\lambda$  of functions with Fourier transform supported in a dyadic region in frequency  $|\xi| \in [\lambda, 4\lambda]$ , with norm

$$|u|_{Y_\lambda} = |u|_{L^\infty(L^2)} + \lambda^{-1} |\square u|_{L^1(L^2)}$$

(Here and below  $L^p(L^q)$  stands for  $L_t^p(L_x^q)$ ). Then we define the dyadic pieces of our function spaces at frequency  $\lambda$  as

$$F_\lambda = X_\lambda^{1/2} + Y_\lambda.$$

These dyadic blocks are on the same scale and contain the  $L^2$  solutions for the wave equation.

The norm of the space  $F$  where we will look for solutions to the wave equation with initial data in  $\dot{B}_{\frac{n}{2}}^{2,1} \times \dot{B}_{\frac{n}{2}-1}^{2,1}$  is a homogeneous 1-Besov space in frequency with norm defined by

$$|u|_F = \sum_{\lambda=2^j} \lambda^{-\frac{n}{2}} |A_\lambda u|_{F_\lambda}. \quad (2.7)$$

The power of  $\lambda$  indicates how many derivatives we add (i.e.  $n/2$ ). To be precise, the functions in the space  $F$  are uniquely determined modulo polynomials. To resolve this ambiguity we restrict the space  $F$  to functions which have limit 0 at infinity.

For larger exponents  $s > \frac{n}{2}$  define the 2- Besov spaces  $F^s$  with norms

$$|u|_{F^s}^2 = \sum_{\lambda=2^j} \lambda^{-2s} |A_\lambda u|_{F_\lambda}^2. \quad (2.8)$$

Observe that (modulo constants)  $F$  contains the solutions to the homogeneous wave equation with initial data in  $\dot{B}_{\frac{n}{2}}^{2,1} \times \dot{B}_{\frac{n}{2}-1}^{2,1}$  while  $F^s$  contains the solutions to the homogeneous wave equation with initial data in  $H^s \times H^{s-1}$ .

Applying  $\square$  to functions in  $F_\lambda$  we obtain a space we denote  $\square F_\lambda$  which equals

$$\square F_\lambda = \lambda(X_\lambda^{-1/2} + L^1(L^2)_\lambda).$$

Similarly we define the spaces  $\square F$ ,  $\square F^s$ .

Since we want to take advantage of decompositions in frequency away from the cone, the following result is useful

**Theorem 3.** a) The operators  $\tilde{B}_\mu(D)$  are bounded in  $F_\lambda$  and in  $\square F_\lambda$  uniformly in  $\mu \leq \lambda$ .

b) The operators  $(1 - \tilde{B}_\mu)(D)$  are bounded from  $Y_\lambda$  into  $\mu^{-1}L^1(L^2)$  uniformly in  $\mu \leq \lambda$ .

**Proof:** a) The statement is obvious for the  $X_\lambda^s$  spaces, therefore it suffices to show that the operators  $A_\lambda(D)B_\mu(D)$  are bounded in  $L^\infty(L^2)$  and in  $L^1(L^2)$ .

Take the Fourier transform in  $x$ . For fixed  $\xi'$  the symbol  $(a_\lambda \tilde{b}_\mu)(\tau, \xi)$  satisfies

$$|D_0^\alpha(a_\lambda \tilde{b}_\mu)(\tau, \xi)| \leq c_\alpha \mu^{-|\alpha|}$$

and is supported in the region

$$|\tau \pm |\xi|| \leq c\mu$$

therefore its time inverse Fourier transform  $K(t, \xi)$  satisfies

$$K(t, \xi) = e^{\pm it|\xi|} L(t, \xi),$$

where

$$|L(t, \xi)| \leq c_N \frac{\mu}{(1 + \mu t)^N}, \quad \text{for all } N.$$

Hence

$$K(t, \xi) \in L_t^1(L_\xi^\infty),$$

therefore the corresponding multiplier is bounded in  $L^1(L^2)$  and in  $L^\infty(L^2)$ , q.e.d.

b) We need to prove that the multiplier with symbol  $g_{\lambda, \mu} = \lambda^{-1} a_\lambda (1 - \tilde{b}_\mu) p^{-1}$  has norm at most  $O(\mu^{-1})$  in  $L^1(L^2)$ . In this case the symbol  $g_{\lambda, \mu}(\tau, \xi)$  satisfies

$$|D_0^\alpha g_{\lambda, \mu}(\tau, \xi)| \leq c_\alpha (\mu + |\tau \pm |\xi||)^{-1-|\alpha|},$$

therefore its time inverse Fourier transform  $K(t, \xi)$  satisfies

$$K(t, \xi) = e^{\pm it|\xi|} L(\mu t, \xi),$$

where

$$|L(s, \xi)| \leq c_N s^{-\epsilon} (1 + |s|)^{-N}.$$

Hence

$$|K(t, \xi)|_{L_t^1(L_\xi^\infty)} \leq c\mu^{-1},$$

which again implies the desired conclusion.

In the estimates in the proof of Theorems 1,2 it is essential to have good  $L_t^p(L_x^q)$  embeddings for our function spaces. It is easier to (non-uniquely)

relabel the  $L^p$  spaces following the same convention as in [12], namely therefore we are going to (non-uniquely) relabel the  $L^p$  spaces in  $\mathbb{R}^m$  with two indices,

$$L^{[p,s]} := L^r, \quad \frac{1}{p} - \frac{s}{m} = \frac{1}{r}.$$

In terms of the Sobolev embeddings, this means that

$$W^{s,p} \subset L^r.$$

The desired embeddings can all be derived from the Strichartz estimates for solutions to the homogeneous wave equation. Thus, let  $u$  solve  $\square u = 0$  in  $R^{n+1}$  with initial data  $u(0) = u_0$  and  $u_t(0) = u_1$ . Then the estimates we need are the energy estimate

$$|u|_{L^\infty(L^2)} \leq c(|u_0|_{L^2} + |u_1|_{H^{-1}}), \quad (2.9)$$

and the  $L^2(L^p)$  estimate, namely

$$|u|_{L^2(L^{[2,s+\frac{1}{2}]})} \leq c(|u_0|_{H^s} + |u_1|_{H^{s-1}}), \quad \frac{n+1}{2(n-1)} < s < \frac{n-1}{2}. \quad (2.10)$$

Then for the dyadic pieces  $F_\lambda$  of our spaces we shall use the following embeddings,

**Theorem 4.** *The following embeddings hold uniformly in  $\lambda$ :*

- a) (energy)  $F_\lambda \subset L^\infty(L^2)$
- b)  $\lambda^{-\frac{n}{2}} F_\lambda \subset L^\infty$
- c) (Pecher)  $\lambda^{-s} F_\lambda \subset L^2(L^{[2,s+\frac{1}{2}]})$ ,  $\frac{n+1}{2(n-1)} < s < \frac{n-1}{2}$
- d)  $\lambda^{-\frac{n-1}{2}} F_\lambda \subset L^2(L^\infty)$

Parts (a) and (c) follow from (2.9), respectively (2.10). To see this for the  $X_\lambda^{1/2}$  component of  $F_\lambda$  it suffices to foliate the Fourier space with cones which are  $\tau$  translates of the characteristic cone  $\tau^2 - \xi^2 = 0$ . For the  $Y_\lambda$  component of  $F_\lambda$ , on the other hand, one needs to use the variation of parameters formula. These embeddings are all given in a more general context in [15]. Similar embeddings were used in [12].

Parts (b) and (d) follow from (a) and (c) by Sobolev embeddings, with the observation that while the sharp embeddings into  $L^\infty$  fail in general, they are nevertheless true on dyadic pieces.

To solve the inhomogeneous wave equation define the operator  $V$  by  $Vf = \phi$  iff

$$\begin{cases} \square \phi = f \\ \phi_0(0) = 0 \\ \partial_t \phi_0 = 0 \end{cases}.$$

The following result describes the necessary regularity properties of  $V$ :

**Theorem 5.** *The operator  $V$  maps  $\square F$  into  $F$ .*

**Proof:** Let  $f = \square u$ , with  $u \in F$ . Then  $u - Vf$  solves the homogeneous wave equation and has the same Cauchy data as  $u$ . But  $u \in F$  therefore its Cauchy data  $(u(0), u_t(0))$  is in  $B_{\frac{n}{2}}^{2,1} \times B_{\frac{n}{2}-1}^{2,1}$ . This shows that  $u - Vf \in F$ , therefore  $Vf \in F$ .

### 3. PROOF OF THEOREM 1

Suppose that  $D$  is small. Then, due to the embedding  $B_{\frac{n}{2}}^{2,1} \subset L^\infty$  the solution  $\phi$  should be small (modulo constants) therefore we can work in the domain of a local map, i.e. work directly on the equation (1.1).

First set up the problem so that we can use a fixed point argument. Denote by  $\phi_0$  the solution to the wave equation with the same initial data as  $\phi$ . Then the equation (1.1) can be rewritten as

$$\phi = \phi_0 + VN(\phi) \quad (3.11)$$

where  $N$  represents the nonlinear term in (1.1).

We want to solve (3.11) in the space  $F$  defined above. We know that  $\phi_0$  is in  $F$  modulo constants. To remove the ambiguity change the local coordinates (translation) in  $M$  so that  $\phi_0$  has limit 0 at  $\infty$ .

Now we need to know first that the the correct mapping properties hold, i.e.

$$VN : F \rightarrow F \quad (3.12)$$

Secondly, we need to know that the right hand side has a small Lipschitz constant in  $F$ .

Since  $N$  is at least quadratic the second property follows from the first. Hence, if (3.12) holds then for small  $\phi_0$  (in  $F$ ) we can use a fixed point argument to solve (3.11); this will also give the Lipschitz dependence of the solution on  $\phi_0$ , therefore on the initial data.

It remains to prove (3.12). By Theorem 5 it suffices to show that

$$N : F \rightarrow \square F \quad (3.13)$$

But due to (1.4) the nonlinear term can be written as

$$N(\phi) = \Gamma(\phi)(\phi \square \phi + \square(\phi^2))$$

Taking into account this form of the nonlinearity, (3.13) will follow from

**Proposition 1.** .

- a)  $F$  is an algebra.
- b)  $F \cdot \square F \subset \square F$

**Proof:**

a) Since  $F$  is a 1-Besov space, it suffices to look at the product of two given dyadic pieces and show that

$$\lambda^{-\frac{n}{2}} F_\lambda \cdot \mu^{-\frac{n}{2}} F_\mu \subset F. \quad (3.14)$$

We need to consider two cases, based on the relative size of  $\mu$  and  $\lambda$ .

a1) If  $\mu \approx \lambda$  then by Theorem 4 (a),(d) we get

$$\lambda^{-n} F_\lambda \cdot F_\lambda \subset \lambda^{-\frac{n+1}{2}} L^2 \subset F.$$

If we now use the fact that the product has Fourier transform supported in a  $\lambda$ -ball then (3.14) follows.

a2) If  $\mu \ll \lambda$  then the Fourier transform of the product is supported in the  $\lambda$  region, therefore we need to show that

$$\mu^{-\frac{n}{2}} F_\mu \cdot F_\lambda \subset F_\lambda. \quad (3.15)$$

(here to keep the notation simple we allow for a small increase in the size of the support of the Fourier transform of  $F_\lambda$  functions). Now we use Theorem 3 to decompose the  $F_\lambda$  elements into a component with Fourier transform supported within a distance  $\mu$  from the cone and a component at least  $O(\mu)$  away from the cone,

$$F_\lambda = \tilde{B}_\mu F_\lambda + (1 - \tilde{B}_\mu) F_\lambda. \quad (3.16)$$

To prove (3.15) for the first component of (3.16) we use Theorem 4(a,d) to get

$$\mu^{-\frac{n}{2}} F_\mu \cdot \tilde{B}_\mu F_\lambda \subset \mu^{-1/2} L^2(L^\infty)_\mu \cdot L^\infty(L^2)_\lambda = \mu^{-1/2} L^2_{\lambda,\mu} \subset X_\lambda^{1/2} \subset F_\lambda.$$

In this case the Fourier transform of the product is supported within distance  $\mu$  from the cone, which is crucial for the third step.

To prove (3.15) for the second component of (3.16) we further decompose  $F_\lambda$  into  $X_\lambda^{1/2}$  and  $Y_\lambda$  and treat each case separately. Then it suffices to show that

$$\mu^{-\frac{n}{2}} F_\mu \cdot (1 - \tilde{B}_\mu) X_\lambda^{1/2} \subset X_\lambda^{1/2}. \quad (3.17)$$

and

$$\mu^{-\frac{n}{2}} F_\mu \cdot (1 - \tilde{B}_\mu) Y_\lambda \subset Y_\lambda. \quad (3.18)$$

For (3.17) observe that each of the dyadic pieces of  $(1 - \tilde{B}_\mu) X_\lambda^{1/2}$  with respect to the distance to the cone are  $L^2$  and have Fourier transform supported in a dyadic region of the form

$$|\xi| + |\tau| \approx \lambda, \quad |\xi| - |\tau| \approx d.$$

where  $\mu \ll d \leq \lambda$ . This region has thickness larger than  $\mu$ , therefore when we multiply  $(1 - \tilde{B}_\mu) X_\lambda^{1/2}$  by  $F_\mu$  the support of the Fourier transform of the product will stay roughly in the same region. Then it suffices to work on a given such dyadic piece therefore (3.17) reduces to

$$\mu^{-\frac{n}{2}} F_\mu \cdot L^2 \subset L^2.$$

which follows from the embedding  $\mu^{-\frac{n}{2}} F_\mu \subset L^\infty$  in Theorem 4(b).

The same embedding leads to the  $L^\infty(L^2)$  part of (3.18). Taking into account the definition of the  $Y_\lambda$  spaces, it then remains to show that

$$\square(\mu^{-\frac{n}{2}} F_\mu \cdot (1 - \tilde{B}_\mu) Y_\lambda) \subset L^1(L^2).$$



But

$$\square(\mu^{-\frac{n}{2}}F_\mu \cdot (1 - \tilde{B}_\mu)Y_\lambda) \subset \mu^{-\frac{n}{2}}F_\mu \cdot \square((1 - \tilde{B}_\mu)Y_\lambda) + \mu^{-\frac{n}{2}+1}F_\mu \cdot \lambda(1 - \tilde{B}_\mu)Y_\lambda.$$

(the second term contains all the combinations where at least one derivative applies to the first factor) Now by Theorem 4(b) the first term is contained in  $L^\infty \cdot L^1(L^2) = L^1(L^2)$ . On the other hand, by Theorem 4(b) for the first factor and Theorem 3(b) for the second factor the second term is in  $\mu L^\infty \cdot \mu^{-1}L^1(L^2) = L^1(L^2)$ , q.e.d.

b) Since  $F$  is a 1-Besov space it suffices to look at the product of two dyadic pieces and show that

$$\mu^{-\frac{n}{2}}F_\mu \cdot \lambda^{-\frac{n}{2}}\square F_\lambda \subset \square F.$$

Again, we need to consider all possible relative sizes of  $\lambda$  and  $\mu$ .

b1) Suppose  $\lambda \leq \mu$ . Then use Theorem 4(c) for the first term and the straightforward embedding  $\square F_\lambda \subset \lambda^{\frac{3}{2}}L^2$  for the second term to obtain

$$\begin{aligned} \mu^{-\frac{n}{2}}F_\mu \cdot \lambda^{\frac{n}{2}}\square F_\lambda &\subset \mu^{1-\frac{n}{2}}L^2(L^{[2, \frac{3}{2}]})_\mu \cdot \lambda^{\frac{3-n}{2}}L^2_\lambda \\ &\subset \mu^{1-\frac{n}{2}}L^2(L^{[2, \frac{3}{2}]})_\mu \cdot L^2(L^{[2, \frac{n-3}{2}]})_\lambda \quad (\text{Sobolev embedding}) \\ &= \mu^{1-\frac{n}{2}}L^1(L^2) \subset \square F, \end{aligned}$$

where at the last step we use the fact that the Fourier transform of the product is supported in a ball of radius  $O(\mu)$ .

b2) Suppose  $\mu \ll \lambda$ . Then we need to show that

$$\mu^{-\frac{n}{2}}F_\mu \cdot \square F_\lambda \subset \square F_\lambda.$$

First we decompose  $\square F_\lambda$  in its two components. By Theorem 4(b) we get

$$\mu^{-\frac{n}{2}}F_\mu \cdot L^1(L^2)_\lambda \subset L^1(L^2)_\lambda,$$

therefore it remains to show that

$$\mu^{-\frac{n}{2}}F_\mu \cdot X_\lambda^{-1/2} \subset \lambda^{-1}\square F_\lambda.$$

To achieve this we split  $X_\lambda^{-1/2}$  into a component with with Fourier transform supported within a distance  $\mu$  from the cone and a component at least  $O(\mu)$  away from the cone,

$$X_\lambda^{-1/2} = \tilde{B}_\mu X_\lambda^{-1/2} + (1 - \tilde{B}_\mu)X_\lambda^{-1/2}.$$

Since each of the dyadic pieces of  $(1 - \tilde{B}_\mu)X_\lambda^{-1/2}$  are  $L^2$  and have Fourier transform supported in a dyadic region of thickness larger than  $\mu$ , the product estimate for the second term reduces as in part (a) to

$$\mu^{-\frac{n}{2}}F_\mu \cdot (L^2)_\lambda \subset (L^2)_\lambda,$$

and follows from the embedding  $\mu^{-\frac{n}{2}}F_\mu \subset L^\infty$ .

For the first term, on the other hand, use Theorem 4(d) to get  
 $\mu^{-\frac{n}{2}} F_\mu \cdot \tilde{B}_\mu X_\lambda^{-1/2} \subset \mu^{-\frac{1}{2}} L^2(L^\infty) \cdot \mu^{\frac{1}{2}} L^2 \subset L^2(L^\infty) \cdot \lambda^{1-\frac{n}{2}} L^2 = L^1(L^2)_\lambda \subset \lambda^{-1} \square F$ .  
q.e.d.

Next we want to show that if in addition the data is more regular then the solution is globally more regular as well. We start with an estimate:

**Proposition 2.** *Suppose that*

$$|\phi|_F, |\psi|_F \leq C. \quad (3.19)$$

*Then*

$$|N(\phi) - N(\psi)|_{\square F^s} \leq c|\phi - \psi|_{F^s} (|\phi|_F + |\psi|_F) + |\phi - \psi|_F (|\phi|_{F^s} + |\psi|_{F^s}). \quad (3.20)$$

**Proof:** The conclusion would follow from the following estimates:

$$|\phi\psi|_{F^s} \leq c(|\phi|_F |\psi|_{F^s} + |\psi|_F |\phi|_{F^s}), \quad (3.21)$$

$$|\phi \square \psi|_{\square F^s} \leq c(|\phi|_F |\psi|_{F^s} + |\psi|_F |\phi|_{F^s}). \quad (3.22)$$

But these estimates are proved in the same way as in Proposition 1; the appropriate dyadic estimates are the same or simpler.

Now let  $M > 1$  and define the space  $Z = F^s \cap F$  with the norm

$$|\phi|_Z = \frac{1}{M} |\phi|_{F^s} + \frac{1}{\epsilon} |\phi|_F.$$

We claim that if  $\epsilon$  is sufficiently small, independently of  $M$ , then the map

$$\phi \rightarrow VN(\phi),$$

is a contraction from the unit ball in  $Z$  to the  $1/2$  ball in  $Z$ .

To prove that observe first that in order to use Proposition 2 it suffices to take  $\epsilon < C$  so that (3.19) holds. Then estimate

$$|VN(\phi)|_F \leq c|N(\phi)|_{\square F} \leq c|\phi|_F^2 \leq c\epsilon|\phi|_F,$$

and, by Proposition 2,

$$|VN(\phi)|_{F^s} \leq c|N(\phi)|_{\square F^s} \leq c|\phi|_F |\phi|_{F^s} \leq c\epsilon|\phi|_{F^s}.$$

Hence if  $\epsilon$  is sufficiently small then  $VN$  maps the unit ball in  $Z$  to the  $1/2$  ball in  $Z$ . The fact that  $VN$  is a contraction follows in a similar fashion.

Go back now to our fixed point problem in (3.11) and choose

$$M = 2|(f_0, f_1)|_{H^s \times H^{s-1}}.$$

Then the fix point argument works in  $Z$ . Consequently, the unique solution  $\phi$  of our problem in  $F$  is in effect in  $F^s$  and satisfies the bound

$$|\phi|_{F^s} \leq 2M.$$

Since both fixed point arguments give continuous dependence on the data, now we can argue that the solutions in  $F$  that we have produced are the unique strong limit of smooth solutions to the equation.

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