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Local and global smoothing effects for some linear hyperbolic equations with a strong dissipation

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Abstract

We consider an abstract second order linear equation with a strong dissipation, namely a friction term which depends on a power of the "elastic" operator.

In the homogeneous case, we investigate the phase spaces in which the initial value problem gives rise to a semigroup, and the further regularity of solutions. In the non-homogeneous case, we study how the regularity of solutions depends on the regularity of forcing terms, and we characterize the spaces where a bounded forcing term yields a bounded solution.

What we discover is a variety of different regimes, with completely different behaviors, depending on the exponent in the friction term.

We also provide counterexamples in order to show the optimality of our results.

Mathematics Subject Classification 2010 (MSC2010): 35L10, 35L15, 35L20.

Key words: linear hyperbolic equations, dissipative hyperbolic equations, strong dissipation, fractional damping, bounded solutions.

1 Introduction

Let H be a separable real Hilbert space. For every x and y in H, $|x|$ denotes the norm of x, and $\langle x, y\rangle$ denotes the scalar product of x and y. Let A be a self-adjoint linear operator on H with dense domain $D(A)$. We assume that A is nonnegative, namely $\langle Ax, x\rangle \geq 0$ for every $x \in D(A)$, so that for every $\alpha \geq 0$ the power $A^\alpha x$ is defined provided that x lies in a suitable domain $D(A^\alpha)$.

We consider the second order linear evolution equation

$$u''(t) + 2\delta A^\sigma u'(t) + Au(t) = f(t), \qquad t \geq 0, \tag{1.1}$$

where $\delta > 0$, $\sigma \geq 0$, and $f : [0, +\infty) \to H$, with initial data

$$u(0) = u_0, \qquad u'(0) = u_1. \tag{1.2}$$

Several wave equations fit in this abstract framework, for example

$$\begin{aligned} u_{tt} + u_t - \Delta u = f(t,x) \qquad & (A = -\Delta, \quad \sigma = 0), \\ u_{tt} - \Delta u_t - \Delta u = f(t,x) \qquad & (A = -\Delta, \quad \sigma = 1), \\ u_{tt} - \Delta u_t + \Delta^2 u = f(t,x) \qquad & (A = \Delta^2, \quad \sigma = 1/2), \end{aligned} \tag{1.3}$$

with ad hoc boundary conditions. The case $\sigma = 0$ is the standard damped wave equation, the case $\sigma = 1$ is sometimes called visco-elastic damping, the case $\sigma \in (0, 1)$ is usually referred to as structural damping or fractional damping. The case $\sigma > 1$ seems to be quite unexplored.

Mathematical models of this kind were proposed in [5], and then rigorously analyzed by many authors from different points of view. In the abstract setting, a natural idea is to set $U(t) := (u(t), u'(t))$, so that one can interpret (1.1) as a first order system

$$U'(t) + \mathcal{A}_\sigma U(t) = \mathcal{F}(t),$$

where

$$\mathcal{A}_\sigma = \begin{pmatrix} 0 & -I \\ A & 2\delta A^\sigma \end{pmatrix}$$

acts on some product space $\mathcal{H}$, usually chosen equal to $D(A^{1/2}) \times H$, and

$$\mathcal{F}(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

In this framework, several papers have been devoted to properties of semigroup generated by the operator $\mathcal{A}_\sigma$, such as analyticity (true in the case $\sigma \geq 1/2$) or Gevrey regularity (true in the case $0 < \sigma < 1/2$) . The interested reader is referred to [6, 7, 8], and to the more recent papers [10, 12, 23, 25]. In these papers A is usually assumed to be strictly positive, $\sigma \in [0, 1]$, and the phase space is $D(A^{1/2}) \times H$.

On a completely different side, the community working on dispersive hyperbolic equations considered the concrete equation (1.3) (with $f = 0$), or its σ-generalization, in the whole space $\mathbb{R}^n$ or in suitable classes of unbounded domains, obtaining L^p-L^q estimates or energy decay estimates. The interested reader is referred to [29, 19, 20, 21] and to the references quoted therein. In these papers A is just nonnegative, $\sigma \in [0, 1]$, and the phase space is once again $D(A^{1/2}) \times H$, with some L^p restrictions, but the dispersive properties of A are essential.

We conclude this brief historical survey, which is far from being complete, by mentioning the existence of some literature on nonlinear wave equations with fractional damping (see for example [26, 27] or the more recent work [9]).

In this paper we consider the abstract problem (1.1)–(1.2) in its full generality, with the aim of providing a complete picture in the whole range $\sigma \geq 0$.

We begin our study by considering the homogeneous case where $f(t) \equiv 0$. The first question we address is the choice of the phase space. Having in mind the standard setting for the non-dissipative case ($\delta = 0$), one is naturally led to consider the phase space $D(A^{1/2}) \times H$, or more generally $D(A^{\alpha}) \times D(A^{\alpha-1/2})$, namely with "gap 1/2". In the non-dissipative case, this choice has physical motivations (it is the usual "energy space"), but it is also dictated by the equation itself in the sense that the initial value problem generates a continuous semigroup in the phase space $D(A^{\alpha_0}) \times D(A^{\alpha_1})$ if and only if $\alpha_0 - \alpha_1 = 1/2$. We say that 1/2 is the "phase space gap".

Also further time-derivatives of u respect this gap, in the sense that the m-th time-derivative $u^{(m)}(t)$ lies in the space $D(A^{\alpha-(m-1)/2})$ for every $m \geq 1$ such that $\alpha \geq (m-1)/2$, and once again the exponents are optimal if A is unbounded. Thus we say that 1/2 is also the "derivative gap". We stress that in both cases the value 1/2 is chosen by the equation itself.

What about the dissipative case? In statements (1) and (2) of Theorem 2.1 we investigate phase space gaps and derivative gaps. Two different regimes appear.

- As long as $0 \leq \sigma \leq 1/2$, both the phase space gap and the derivative gap are equal to 1/2. Thus from this point of view the picture is exactly the same as in the non-dissipative case.

- For $\sigma > 1/2$ things are different. First of all, the initial value problem generates a continuous semigroup in $D(A^{\alpha_0}) \times D(A^{\alpha_1})$ if and only if $1 - \sigma \leq \alpha_0 - \alpha_1 \leq \sigma$, namely there is an interval of possible phase space gaps. This interval is always centered in 1/2, and contains also negative values when $\sigma > 1$.

  As for further time-derivatives, it turns out that $u^{(m)}(t) \in D(A^{\alpha_1-(m-1)\sigma})$ for every $m \geq 1$ such that $\alpha_1 \geq (m-1)\sigma$, which means that in this regime we now have a derivative gap equal to σ.

The second question we address in the homogeneous case is the regularity of solutions for $t > 0$, since a strong dissipation is expected to have a smoothing effect. Here three regimes appear, as shown by statements (3) and (4) of Theorem 2.1.

- For $\sigma = 0$ there is no further regularity for $t > 0$, as in the non-dissipative case.
- For $\sigma \in (0, 1)$ there is an instantaneous smoothing effect similar to parabolic equations, in the sense that $u \in C^\infty((0, +\infty), D(A^\alpha))$ for every $\alpha \geq 0$. Nevertheless, from the quantitative point of view (usually stated in terms of Gevrey spaces), this effect is actually weaker than in the parabolic case. We do not deepen this issue in the present paper, and we refer the interested reader to [24].
- For $\sigma \geq 1$ the dissipation is so strong that it prevents too much smoothing, but a new feature appears, namely $u^{(m)} \in C^0((0, +\infty), D(A^{\alpha_0+m(\sigma-1)}))$. Since $\sigma - 1 \geq 0$, this is the opposite of the classical regularity loss: the higher is the time-derivation order, the higher is the space regularity! We stress that this is true only for positive times. If we are interested in the regularity up to $t = 0$, then even for $\sigma \geq 1$ there is regularity loss in the standard direction, with derivative gap equal to σ, as already observed.

After settling the homogeneous case, we study several properties in presence of a non-trivial forcing $f(t)$. In this case each solution is the sum of the solution of the corresponding homogeneous equation with the same initial data, and the solution of the non-homogeneous equation with forcing $f(t)$ and null initial data. Thus in the non-homogenous case, by relying on the previous results, we are reduced to study the special case $u_0 = u_1 = 0$. We address three issues.

- First of all, we consider a forcing term $f \in L^\infty((0, T), H)$ and we describe the spaces of the form $D(A^\alpha)$ where the solution $u(t)$ and its derivative $u'(t)$ lie (see Theorem 2.4 and Remark 2.5).
- Then we consider a forcing term $f \in L^\infty((0, +\infty), H)$, defined and bounded for all positive times, and we characterize the spaces of the form $D(A^\alpha)$ where $u(t)$ and $u'(t)$ are (globally) bounded. The answer is given by Theorem 2.7 (see also Remark 2.9), and it is somewhat unexpected. Indeed it turns out that $u'(t)$ is always globally bounded in all the spaces of the form $D(A^\alpha)$ to which it belongs, while $u(t)$ is globally bounded in all the spaces of the form $D(A^\alpha)$ to which it belongs if and only if $\sigma \in [0, 1]$. On the contrary, when $\sigma > 1$ we have that $u(t) \in D(A^\alpha)$ for all $\alpha \leq \sigma$, but $u(t)$ is globally bounded in $D(A^\alpha)$ only for $\alpha < 1$.
- As a third issue, we apply our techniques to a somewhat different question. We consider the non-homogeneous equation (1.1) with a bounded forcing term $f \in L^\infty(\mathbb{R}, H)$ defined on the whole real line, and we ask ourselves whether there exists a solution which is globally bounded in some phase space.

The third issue above is usually referred to as *non-resonance property*, and has been studied by many authors in the concrete case of hyperbolic equations in a bounded domain Ω with linear or nonlinear local dissipation terms (see for example [1, 2, 3, 13, 14, 15, 18] and the references therein). Except when additional conditions are assumed on f, such as anti-periodicity or more regularity (see [16, 17]), all these authors had to assume, even in the case of a periodic forcing term, that the damping operator carries $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ (namely $D(A^{1/2})$ to $D(A^{-1/2})$ in the abstract setting) in a bounded manner. What was not clear is whether this is a fundamental obstruction or not. In a different direction, a linear dissipation term of the form Bu' was considered in [11], with the assumption that B carries $D(A^{1/2})$ to H in a bounded manner. The result proved in [11] is that one has the non-resonance property if and only if all solutions of the corresponding homogeneous equation decay to 0 in a uniform exponential way. Once again, it was not clear whether or not this uniform exponential decay is still a fundamental requirement for more general dissipation terms.

After thinking it over for several decades without any clear answer, it seems reasonable to consider the toy model where the damping is provided by a linear unbounded operator which does not carry $D(A^{1/2})$ to $D(A^{-1/2})$. This led us to equation (1.1) with $\sigma > 1$, which was actually the initial motivation of this paper.

The answer is somewhat surprising. Indeed in Theorem 2.11 we prove that (1.1) has the non-resonance property for every $\delta > 0$ and every $\sigma \geq 0$. For $\sigma > 1$, this non-resonance result seems completely new and makes a sharp contrast with the conclusions of [11] because for A unbounded and $\sigma > 1$ the semigroup is not exponentially stable. Moreover the bounded solution stays bounded in phase spaces which are stronger than the usual energy spaces traditionally considered in these problems, thus showing that linear overdamping improves the non-resonance property. A quite surprising phenomenon which has to be better understood by looking at different damping operators, linear and nonlinear.

Finally, all the exponents involved in our regularity and boundedness results are optimal in general, as shown by the counterexamples of Theorem 2.12. As far as we know, no such counterexamples were known before in the literature, and also most of the different strategies used in the construction seem to be new.

Most of the result obtained in this paper are based on a thorough knowledge of the asymptotic behavior, as $\lambda \to +\infty$, of the roots of the characteristic polynomial

$$x^2 + 2\delta\lambda^\sigma x + \lambda. \tag{1.4}$$

The form and asymptotic behavior of the roots is different in different ranges of σ (namely $\sigma = 0$, $\sigma \in (0, 1/2)$, $\sigma = 1/2$, $\sigma \in (1/2, 1)$, $\sigma = 1$, $\sigma > 1$), giving rise to the composite picture described in our results above.

We believe that these optimal regularity and boundedness results might provide a benchmark when looking at more general equations, for example nonlinear equations, or equations in which $A^\sigma u'(t)$ is replaced by the more general friction term $Bu'(t)$, where B is an operator "comparable" with some power A^σ (as in the original models in [5]).

This paper is organized as follows. In Section 2 we state all our main results. In Section 3 we give the proofs for the homogeneous case $f = 0$ and in Section 4 we give the proofs for the forced case. In Section 5 we exhibit some examples showing the optimality of our regularity and boundedness results in the forced case.

2 The results

Before stating our results, let us spend just a few words on the notion of solution. Weak solutions to evolution problems can be introduced in several ways, for example through density arguments as limits of classical solutions, or through integral forms or distributional formulations. All these notions are equivalent in the case of a damped wave-type equation equation such as (1.1).

Moreover, thanks to the spectral theory for self-adjoint operators (namely Fourier series or Fourier transform in concrete cases), the study of (1.1)–(1.2) reduces to the study of a suitable family of ordinary differential equations (see section 3.1 for further details). In this way one can prove the well-known results concerning existence of a unique solution for quite general initial data and forcing terms (even distributions or hyperfunctions), up to admitting that the solution takes its values in a very large Hilbert space as well (once again distributions or hyperfunctions). Thus in the sequel we say "the solution" without any further specification.

In this paper we investigate how the regularity of initial data and forcing terms affects the regularity of solutions. Let us start with the homogeneous case.

Theorem 2.1 (The homogeneous equation) *Let H be a separable Hilbert space, and let A be a self-adjoint nonnegative operator on H with dense domain $D(A)$. For every $\sigma \geq 0$, $\delta > 0$, $\alpha_0 \geq 0$, $\alpha_1 \geq 0$, we consider the unique solution to the homogeneous linear equation*

$$u''(t) + 2\delta A^{\sigma} u'(t) + Au(t) = 0, \qquad t \geq 0, \tag{2.1}$$

with initial data

$$u(0) = u_0 \in D(A^{\alpha_0}), \qquad u'(0) = u_1 \in D(A^{\alpha_1}). \tag{2.2}$$

Let us set $\gamma := \max\{1/2, \sigma\}$, and let us assume that

$$1 - \gamma \leq \alpha_0 - \alpha_1 \leq \gamma. \tag{2.3}$$

Then $u(t)$ satisfies the following regularity properties.

(1) (Regularity in the phase space) *It turns out that*

$$(u, u') \in C^0\left([0, +\infty), D(A^{\alpha_0}) \times D(A^{\alpha_1})\right). \tag{2.4}$$

(2) (Regularity of higher order derivatives up to $t = 0$) *Let $m \geq 1$ be an integer such that $\alpha_1 \geq (m-1)\gamma$. Then the m-th time-derivative $u^{(m)}(t)$ satisfies*

$$u^{(m)} \in C^0\left([0, +\infty), D(A^{\alpha_1 - (m-1)\gamma})\right). \tag{2.5}$$

(3) (Regularity for $t > 0$ when $0 < \sigma < 1$) *In this regime it turns out that*

$$u \in C^{\infty}((0, +\infty), D(A^{\alpha})) \qquad \forall \alpha \geq 0. \tag{2.6}$$

(4) (Regularity for $t > 0$ when $\sigma \geq 1$) *In this regime it turns out that*

$$u^{(m)} \in C^0\left((0,+\infty), D(A^{\alpha_0+m(\sigma-1)})\right) \qquad \forall m \in \mathbb{N}. \tag{2.7}$$

Remark 2.2 A careful inspection of the proofs reveals that the norm of the solution in the spaces appearing in (2.4) through (2.7) depends continuously on the norm of initial data in $D(A^{\alpha_0}) \times D(A^{\alpha_1})$. For example, in the case of (2.4) this means that

$$\|u(t)\|_{D(A^{\alpha_0})} + \|u'(t)\|_{D(A^{\alpha_1})} \leq C_1\left(\|u_0\|_{D(A^{\alpha_0})} + \|u_1\|_{D(A^{\alpha_1})}\right) \qquad \forall t \geq 0 \tag{2.8}$$

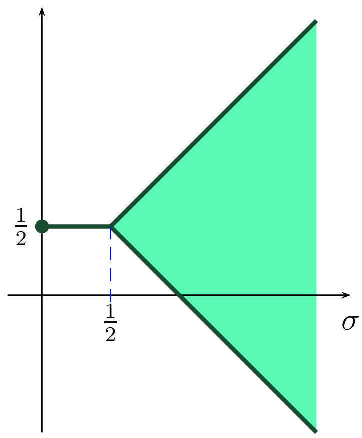
for a suitable constant $C_1 = C_1(\delta, \sigma, \alpha_0, \alpha_1)$, while in in the case of (2.6) this means that for every $m \in \mathbb{N}$ one has that

$$\left\|u^{(m)}(t)\right\|_{D(A^{\alpha})} \leq \frac{C_2}{t^{C_3}}\left(\|u_0\|_{D(A^{\alpha_0})} + \|u_1\|_{D(A^{\alpha_1})}\right) \qquad \forall t > 0$$

for suitable constants C_2 and C_3, both depending on δ, σ, α_0, α_1, α and m. We spare the reader from these standard details.

Remark 2.3 Statement (1) of Theorem 2.1 and estimate (2.8) are equivalent to saying that equation (2.1) generates a continuous semigroup in $D(A^{\alpha_0}) \times D(A^{\alpha_1})$ provided that inequality (2.3) is satisfied. In addition, a simple inspection of the proof reveals that, when A is unbounded, inequality (2.3) is also a necessary condition for equation (2.1) to generate a continuous semigroup in $D(A^{\alpha_0}) \times D(A^{\alpha_1})$.

Thus (2.3) describes all possible values of the phase space gap $\alpha_0 - \alpha_1$. These values are represented as a function of σ by the shaded region in the following picture.



Admissible phase space gaps $\alpha_0 - \alpha_1$ as a function of σ

We stress that for $0 \leq \sigma \leq 1/2$ the only admissible value is $1/2$, which is typical of hyperbolic problems. When $\sigma > 1/2$ there is an interval of possible phase space gaps, centered in $1/2$, which contains also negative values when $\sigma > 1$.

This implies that (1.1) always generates a semigroup on $D(A^{1/2}) \times H$, or more generally on $D(A^{\alpha+1/2}) \times D(A^{\alpha})$, but for $\sigma > 1/2$ there are always many other possible choices. Just to give some extremal examples,

- equation (2.1) with $\sigma = 1$ generates a semigroup on $D(A) \times H$ or $H \times H$,
- equation (2.1) with $\sigma = 2$ generates a semigroup on $D(A^2) \times H$ or $H \times D(A)$ (note that the latter has a negative phase space gap, namely the time-derivative is more regular than the function itself).

Now we proceed to the non-homogeneous case. The first question we address is the regularity of solutions. It is well-known that the space regularity of solutions to non-homogeneous linear equations depends both on the space and on the time regularity of the forcing term $f(t)$, in such a way that a higher time-regularity compensates a lower space-regularity. A typical example of this philosophy is Proposition 4.1.6 in [4].

A full understanding of this interplay between time and space regularity in the case of equations with strong dissipation is probably an interesting problem, which could deserve further investigation. Here, for the sake of brevity, we limit ourselves to forcing terms with minimal space-regularity (just in H), and bounded with respect to time.

Theorem 2.4 (Non-homogeneous equation – Regularity) *Let H be a separable Hilbert space, and let A be a self-adjoint nonnegative operator on H with dense domain $D(A)$. Let $T > 0$, and let $f \in L^\infty((0,T), H)$ be a bounded forcing term.*

For every $\sigma \geq 0$ and $\delta > 0$, we consider the unique solution $u(t)$ of the non-homogeneous linear equation

$$u''(t) + 2\delta A^\sigma u'(t) + Au(t) = f(t) \tag{2.9}$$

in $[0,T]$, with null initial data

$$u(0) = 0, \qquad u'(0) = 0. \tag{2.10}$$

Then $u(t)$ satisfies the following properties.

(1) (Case $\sigma = 0$) *In this regime it turns out that*

$$u \in C^0([0,T], D(A^{1/2})) \cap C^1([0,T], H).$$

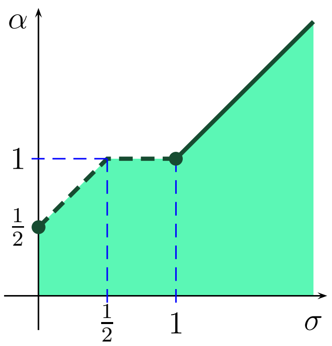
(2) (Case $0 < \sigma < 1$) *In this regime it turns out that*

$$u \in C^0([0,T], D(A^{\min\{\sigma+1/2,1\}-\varepsilon})) \cap C^1([0,T], D(A^{\sigma-\varepsilon})) \qquad \forall \varepsilon \in (0,\sigma].$$

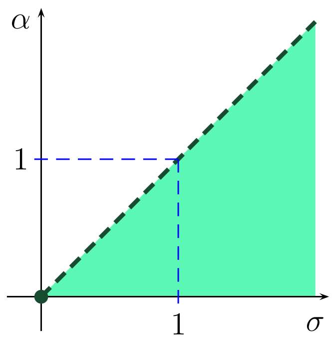
(3) (Case $\sigma \geq 1$) *In this regime it turns out that*

$$u \in C^0([0,T], D(A^\sigma)) \cap C^1([0,T], D(A^{\sigma-\varepsilon})) \qquad \forall \varepsilon \in (0,\sigma]. \tag{2.11}$$

Remark 2.5 The following two pictures sum up the conclusions of Theorem 2.4. The shaded regions represent the pairs (σ, α) for which it happens that u is continuous with values in $D(A^\alpha)$ and u' is continuous with values in $D(A^\alpha)$. The boundary is included when represented by a continuous line or a dot, and excluded when represented by a dashed line.



u continuous in $D(A^\alpha)$



u' continuous in $D(A^\alpha)$

As expected, $u(t)$ is always more regular than $u'(t)$. One could call "smoothing gap" the difference between the exponents of the spaces where $u(t)$ and $u'(t)$ lie. Thus the smoothing gap is equal to $1/2$ (as usual in hyperbolic problems) for $\sigma \in [0, 1/2]$, then it is equal to $1 - \sigma$ for $\sigma \in (1/2, 1)$, and finally it is 0 for $\sigma \geq 1$, when in any case $u(t)$ lies in the limit space $D(A^\sigma)$ while $u'(t)$ does not.

In the following result we investigate the dashed lines of the pictures above, showing that in those limit cases we have at least that $u(t) \in D(A^\alpha)$ or $u'(t) \in D(A^\alpha)$ for almost every time.

Theorem 2.6 (Non-homogeneous equation – Limit cases) *Let H be a separable Hilbert space, and let A be a self-adjoint nonnegative operator on H with dense domain $D(A)$, let $T > 0$, and let $f \in L^2((0, T), H)$.*

For every $\sigma \geq 0$ and $\delta > 0$, we consider the unique solution $u(t)$ of the non-homogeneous linear equation (2.9) in $[0, T]$, with null initial data (2.10).

Then $u(t)$ satisfies the following regularity properties, depending on σ.

(1) (Case $\sigma \geq 0$) *For every admissible value of σ it turns out that*

$$u' \in L^2((0, T), D(A^\sigma)). \tag{2.12}$$

(2) (Case $\sigma \in [0, 1]$) *In this regime it turns out that*

$$u \in L^2\left((0, T), D(A^{\min\{\sigma+1/2,1\}})\right). \tag{2.13}$$

The next result concerns the boundedness of solutions. We assume that the forcing term $f(t)$ is defined for every $t \geq 0$ and globally bounded, and we characterize the spaces where the solution is globally bounded. In this case we need to assume that the operator A is coercive, because if not there are trivial counterexamples (just think to the case where A is the null operator and f is constant).

Theorem 2.7 (Non-homogeneous equation – Global boundedness) *Let H be a separable Hilbert space, and let A be a self-adjoint nonnegative operator on H with dense domain $D(A)$. Let us assume that A is coercive, namely there exists a constant $\nu > 0$ such that $\langle Au, u\rangle \geq \nu|u|^2$ for every $u \in D(A)$. Let $f \in L^\infty((0, +\infty), H)$ be a globally bounded forcing term.*

For every $\sigma \geq 0$ and $\delta > 0$, we consider the unique solution $u(t)$ of the non-homogeneous linear equation (2.9) in $[0, +\infty)$, with null initial data (2.10).

Then $u(t)$ is bounded in the following spaces, depending on σ.

(1) (Case $\sigma = 0$) *In this regime it turns out that*

$$(u(t), u'(t)) \text{ is bounded in } D(A^{1/2}) \times H.$$

(2) (Case $0 < \sigma < 1$) *In this regime it turns out that*

$$(u(t), u'(t)) \text{ is bounded in } D(A^{\min\{\sigma+1/2,1\}-\varepsilon}) \times D(A^{\sigma-\varepsilon}) \quad \forall \varepsilon \in (0, \sigma].$$

(3) (Case $\sigma = 1$) *In this regime it turns out that*

$$(u(t), u'(t)) \text{ is bounded in } D(A) \times D(A^{1-\varepsilon}) \quad \forall \varepsilon \in (0, 1].$$

(4) (Case $\sigma > 1$) *In this regime it turns out that*

$$(u(t), u'(t)) \text{ is bounded in } D(A^{1-\varepsilon}) \times D(A^{\sigma-\varepsilon}) \quad \forall \varepsilon \in (0, 1].$$

Remark 2.8 In the proof of Theorems 2.4, 2.6 and 2.7 we actually show also that the norm of the solution (in the given spaces) depends continuously on the norm of f. Just to give an example, in the case of (2.11) this means that

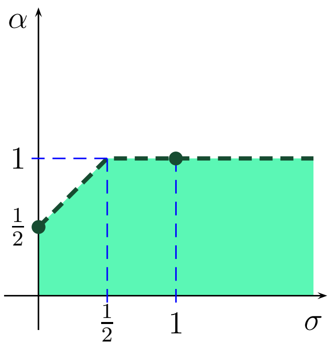
$$\|u(t)\|_{D(A^\sigma)} \leq C_1\|f\|_{L^\infty((0,T),H)} \qquad \forall t \in [0, T],$$

for a suitable constant $C_1 = C_1(\delta, \sigma)$, and

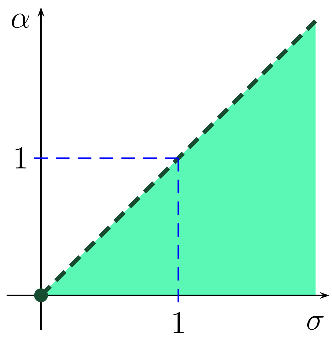
$$\|u'(t)\|_{D(A^{\sigma-\varepsilon})} \leq C_2\|f\|_{L^\infty((0,T),H)} \qquad \forall t \in [0, T],$$

for a suitable constant $C_2 = C_2(\delta, \sigma, \varepsilon)$.

Remark 2.9 The following two pictures sum up the conclusions of Theorem 2.7, namely the spaces where $u(t)$ and $u'(t)$ are globally bounded when the forcing term is globally bounded.



u bounded in $D(A^\alpha)$



u' bounded in $D(A^\alpha)$

Comparing with Remark 2.5, we see that the regularity and boundedness diagrams of $u'(t)$ coincide for all $\sigma \geq 0$, while the regularity and boundedness diagrams of $u(t)$ coincide only for $\sigma \in [0,1]$. In other words, for $\sigma > 1$ the solution is more regular, but the estimates in the stronger norms diverge as $t \to +\infty$. When $\sigma > 1$ and $1 \leq \alpha \leq \sigma$ one can obtain estimates of the form

$$|A^\alpha u(t)| \leq C_{\alpha,\sigma} \|f\|_{L^\infty((0,+\infty),H)} \cdot \begin{cases} t^{(\alpha-1)/(\sigma-1)} & \text{if } 1 < \alpha \leq \sigma, \\ \log(1+t) & \text{if } \alpha = 1. \end{cases} \tag{2.14}$$

We refer to Remark 4.2 for further details. We observe also that $\sigma = 1$ plays a special rôle in the boundedness diagrams, being the unique exponent for which there is global boundedness of $u(t)$ in $D(A)$.

Remark 2.10 Regularity and boundedness properties of $u''(t)$, or more generally of further time-derivatives of $u(t)$, can be easily deduced from the regularity and boundedness properties of the other three terms in equation (1.1). Thus we see that there is no value of σ for which a forcing term $f \in L^\infty((0,T),H)$ is enough to guarantee that all terms in the left-hand side of (1.1) make sense individually as elements of H. Therefore, solutions are always to be intended as weak solutions. We observe also that the terms $Au(t)$ and $A^\sigma u'(t)$ are in the same spaces if and only if $1/2 \leq \sigma < 1$.

For the next result we consider equation (1.1) with a forcing term $f \in L^\infty(\mathbb{R},H)$. We look for a solution $u(t)$ which is bounded in some sense for every $t \in \mathbb{R}$. As before, we restrict ourselves to coercive operators, because if not the existence of such a solution is in general false (a simple example being when $A = 0$ and f is constant). Whenever $0 \leq \sigma \leq 1$, the homogeneous system is exponentially damped in the standard energy space, and in this case the classical result is existence and uniqueness of a bounded solution on the line which attracts exponentially all solutions as t tends to $+\infty$. On the other hand when $\sigma > 1$ and A is for instance diagonal and unbounded, exponential damping is no longer satisfied. However in this case a slight modification of the tools used to prove Theorem 2.7 will give us not only the existence of a bounded solution on the line, but the properties on the line that we had previously on the half-line.

In order to state properly the next result on the line, we need to introduce briefly some notation. First, given any Hilbert space X and any possibly unbounded closed interval J, the space of all functions $f : J \to X$ which are continuous and bounded, endowed with the uniform norm on J, will be denoted from now on by $C^0_b(J,X)$. Then, following Bochner's definition, a function $f \in C^0_b(\mathbb{R},X)$ will be called almost periodic with values in X iff the set of translates

$$\bigcup_{\alpha \in \mathbb{R}} \{f(t+\alpha)\}$$

is precompact in the space $C^0_b(\mathbb{R},X)$. The Banach space of such functions, endowed by the topology of $C^0_b(\mathbb{R},X)$, is denoted by $AP(\mathbb{R},X)$. Any $f \in AP(\mathbb{R},X)$ can be represented by a formal expansion on the complexified extension of X of the form

$$f \sim \sum_{j \in \mathbb{N}} f_j e^{\mu_j t},$$

where the real numbers μ_j are defined by the property that they belong to those numbers μ for which

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(t) e^{-\mu t} dt \neq 0.$$

The set of all such real numbers μ depends on f, is always countable and is denoted by $\exp(f)$. A very important property of almost periodic function is the following: if X, Y are two real Hilbert spaces and $\mathcal{C} \in L(C_b(\mathbb{R}, X), C_b(\mathbb{R}, Y))$ is a bounded linear operator then it is immediate, using Bochner's definition, to see that

$$\forall f \in AP(\mathbb{R}, X), \quad \mathcal{C}f \in AP(\mathbb{R}, Y) \quad \text{with} \quad \exp(\mathcal{C}f) \subseteq \exp(f).$$

For more details on these questions, the construction of the mean-value, the proof that the set $\exp(f)$ is countable and the exact meaning of the formal expansion, we refer to [22]. A typical almost periodic numerical function is the sum of two periodic functions with incommensurable periods. Such objects often appear in the mechanics of vibrating systems, and sometimes infinite sums naturally impose their presence, for instance when studying the energy conservative vibrations of continuous media. For classical applications and historical comments, cf. e.g. [1].

Theorem 2.11 (Existence and properties of the bounded solution) *Let H and A be as in Theorem 2.7 (in particular the operator A is assumed to be coercive). Let $f \in L^\infty(\mathbb{R}, H)$ be a globally bounded forcing term.*

Then, for every $\sigma \geq 0$ and every $\delta > 0$, equation (1.1) admits a unique global solution which is continuous and bounded in the energy space $D(A^{1/2}) \times H$. This solution is strongly asymptotic in the energy space to any solution with initial data in $D(A^{1/2}) \times H$.

Moreover, this solution satisfies the following regularity and boundedness properties.

(1) *The pair $(u(t), u'(t))$ is continuous and bounded in the same spaces as those of Theorem 2.7, namely*

- *in $D(A^{1/2}) \times H$ if $\sigma = 0$,*
- *in $D(A^{\min\{\sigma+1/2,1\}-\varepsilon}) \times D(A^{\sigma-\varepsilon})$ for every $\varepsilon \in (0, \sigma]$ if $0 < \sigma < 1$,*
- *in $D(A) \times D(A^{1-\varepsilon})$ for every $\varepsilon \in (0, 1]$ if $\sigma = 1$,*
- *in $D(A^{1-\varepsilon}) \times D(A^{\sigma-\varepsilon})$ for every $\varepsilon \in (0, 1]$ if $\sigma > 1$.*

(2) *If in addition f is almost periodic with values in H, then (u, u') is almost periodic with values in the spaces mentioned above with $\exp(u) \subseteq \exp(f)$. Finally, if f is periodic the bounded solution is periodic as well, with the same minimal period. If $\sigma > 1$, the periodic solution is continuous and bounded also in the limit space $D(A)$.*

Finally, we show that all previous results are optimal. Note that in the counterexamples below we always produce forcing terms which are not just bounded, but also continuous. This shows that a time-continuous external force does not make the solution more space-regular than a time-bounded external force.

Theorem 2.12 (Counterexamples) *Let H be a Hilbert space, and let A be a linear operator on H with domain $D(A)$. Let us assume that the spectrum of A contains an unbounded sequence of positive eigenvalues. Then we have the following conclusions.*

(1) (Case $\sigma = 0$) *For every sequence $\{t_n\} \subseteq (0, +\infty)$ there exists $f \in C_b^0([0, +\infty), H)$ such that the unique global solution $u(t)$ of problem (2.9)–(2.10) satisfies*

$$u(t_n) \notin D(A^{1/2+\varepsilon}) \qquad \forall \varepsilon > 0, \quad \forall n \in \mathbb{N}, \tag{2.15}$$

$$u'(t_n) \notin D(A^{\varepsilon}) \qquad \forall \varepsilon > 0, \quad \forall n \in \mathbb{N}. \tag{2.16}$$

(2) (Case $0 < \sigma < 1$) *For every σ in this range, and every sequence $\{t_n\} \subseteq (0, +\infty)$, there exists $f \in C_b^0([0, +\infty), H)$ such that the unique global solution $u(t)$ of problem (2.9)–(2.10) satisfies*

$$u(t_n) \notin D(A^{\min\{\sigma+1/2,1\}}) \qquad \forall n \in \mathbb{N},$$

$$u'(t_n) \notin D(A^{\sigma}) \qquad \forall n \in \mathbb{N}.$$

(3) (Case $\sigma \geq 1$) *For every σ in this range, and every sequence $\{t_n\} \subseteq (0, +\infty)$, there exists $f \in C_b^0([0, +\infty), H)$ such that the unique global solution $u(t)$ of problem (2.9)–(2.10) satisfies*

$$u(t) \notin D(A^{\sigma+\varepsilon}) \qquad \forall \varepsilon > 0, \quad \forall t > 0, \tag{2.17}$$

$$u'(t_n) \notin D(A^{\sigma}) \qquad \forall n \in \mathbb{N}. \tag{2.18}$$

(4) (Case $\sigma > 1$ – Unboundedness) *For every σ in this range, there exists $f \in C_b^0([0, +\infty), H)$, and a sequence $t_n \to +\infty$, such that the unique global solution $u(t)$ of problem (2.9)–(2.10) satisfies*

$$\lim_{n \to +\infty} |Au(t_n)| = +\infty. \tag{2.19}$$

Remark 2.13 In most counterexamples we obtained that u or u' does not belong to a given space for a sequence of times. This cannot be improved by showing the same for all times, at least when the counterexample refers to the dashed lines in the continuity diagrams of Remark 2.5. Indeed, in those cases we already observed that u or u' lie in the spaces of the dashed lines for almost every time (see Theorem 2.6).

Finally, we point out that a careful inspection of the proof reveals that in our counterexample the norm in (2.19) blows-up logarithmically with respect to t_n. This proves the optimality of our growth estimates (2.14) in the limit case $\alpha = 1$.

Remark 2.14 The regularity of the bounded solution of Theorem 2.11 is also optimal. Indeed it is enough to consider a forcing term which is 0 for $t \leq 0$ and equal to one of the forcing terms of Theorem 2.12 for $t \geq 0$. It is clear that with this choice the bounded solution is 0 for $t \leq 0$ (in particular $u(0) = u'(0) = 0$), and equal to the corresponding solution in Theorem 2.12 for positive times. This yields the required regularity loss.

It is possible to show that our regularity statements are optimal also in the case of periodic forcing terms. The idea is to take the above forcing terms, which are always defined in a bounded time interval, and to extend them by periodicity. For the sake of shortness, we do not work out the details.

3 Proofs in the homogeneous case

3.1 Notation

Throughout this Section, H denotes a separable Hilbert space, and A is any self-adjoint nonnegative linear operator on H. According to the spectral theorem (see for example Theorem VIII.4 of [28]), there exist a measure space (M, μ) with μ a finite measure, a unitary operator $\Psi : H \to L^2(M, \mu)$, and a real valued nonnegative function $\lambda(\xi)$, defined for almost every $\xi \in M$, such that

- $u \in D(A)$ if and only if $\lambda(\xi)[\Psi(u)](\xi) \in L^2(M, \mu)$,
- $[\Psi(Au)](\xi) = \lambda(\xi)[\Psi(u)](\xi)$ for almost every $\xi \in M$.

In other words, H can be identified with $L^2(M, \mu)$, and under this identification the operator A becomes the multiplication operator by $\lambda(\xi)$. In the sequel, for the sake of simplicity, we write $\widehat{u}(\xi)$ instead of $[\Psi(u)](\xi)$. We also think of $\widehat{u}(\xi)$ as the "component" of u with respect to some $\xi \in M$. One can work with these "components" exactly as with Fourier series or Fourier transforms. For example, for every $\alpha \geq 0$ it is true that

$$D(A^\alpha) := \left\{u \in H : [\lambda(\xi)]^\alpha \cdot \widehat{u}(\xi) \in L^2(M, \mu)\right\},$$

and the components of $A^\alpha u$ are just $[\lambda(\xi)]^\alpha \cdot \widehat{u}(\xi)$. Moreover, $u(t)$ turns out to be a solution to (1.1) if and only if the components $\widehat{u}(t, \xi)$ of $u(t)$ and the components $\widehat{f}(t, \xi)$ of $f(t)$ satisfy the ordinary differential equation

$$\widehat{u}''(\xi, t) + 2\delta\,[\lambda(\xi)]^\sigma \cdot \widehat{u}'(t, \xi) + \lambda(\xi)\widehat{u}(t, \xi) = \widehat{f}(t, \xi) \tag{3.1}$$

for almost every $\xi \in M$, where as usual primes denote differentiation with respect to t.

In the proof of our regularity results we can always assume that the operator A is unbounded, because if not all regularity statements in Theorems 2.1, 2.4, and 2.7 are trivial. We can also assume that multipliers $\lambda(\xi)$ are large enough. Indeed the linearity of the equation implies that, given any threshold $\Lambda > 0$, every solution of (1.1)–(1.2) is the sum of two solutions, one corresponding to components with respect multipliers

$\lambda(\xi) < \Lambda$ (and this solution is regular), and one corresponding to components with respect to multipliers $\lambda(\xi) \geq \Lambda$.

In order to simplify the notation, in the sequel we always write "for every $\xi \in M$" instead of "for almost every $\xi \in M$", and when we write a supremum over M we actually mean an essential supremum.

3.2 Roots of the characteristic polynomial

The behavior of solutions of (3.1) depends on the roots of the characteristic polynomial

$$x^2 + 2\delta\lambda(\xi)^\sigma x + \lambda(\xi). \tag{3.2}$$

We call the roots $-x_1(\xi)$ and $-x_2(\xi)$ in order to emphasize that they are negative real numbers, or complex numbers with negative real part. The asymptotic behavior of the roots as $\lambda(\xi) \to +\infty$ is different in the following three regimes.

- *Subcritical dissipation.* If $0 \leq \sigma < 1/2$, or $\sigma = 1/2$ and $\delta < 1$, then for $\lambda(\xi)$ large enough the roots of (3.2) are complex conjugate numbers of the form

$$-x_1(\xi) = -a(\xi) + ib(\xi), \qquad -x_2(\xi) = -a(\xi) - ib(\xi), \tag{3.3}$$

  with

$$a(\xi) := \delta\lambda(\xi)^\sigma, \qquad b(\xi) := \left(\lambda(\xi) - \delta^2\lambda(\xi)^{2\sigma}\right)^{1/2}.$$

  As $\lambda(\xi) \to +\infty$ it turns out that

$$b(\xi) \sim \begin{cases} \lambda(\xi)^{1/2} & \text{if } \sigma < 1/2, \\ \lambda(\xi)^{1/2}(1-\delta^2)^{1/2} & \text{if } \sigma = 1/2. \end{cases}$$

- *Critical dissipation.* If $\sigma = 1/2$ and $\delta = 1$, then for every $\lambda(\xi) \geq 0$ the characteristic polynomial (3.2) has a unique root

$$-x_1(\xi) = -x_2(\xi) = -\lambda(\xi)^{1/2} \tag{3.4}$$

  with multiplicity 2.

- *Supercritical dissipation.* If $\sigma > 1/2$, or $\sigma = 1/2$ and $\delta > 1$, then for $\lambda(\xi)$ large enough the characteristic polynomial (3.2) has two distinct real roots

$$\begin{aligned} -x_1(\xi) &= -\delta\lambda(\xi)^\sigma - \left(\delta^2\lambda(\xi)^{2\sigma} - \lambda(\xi)\right)^{1/2}, \\ -x_2(\xi) &= -\delta\lambda(\xi)^\sigma + \left(\delta^2\lambda(\xi)^{2\sigma} - \lambda(\xi)\right)^{1/2}, \end{aligned} \tag{3.5}$$

  so that for $\sigma > 1/2$ it turns out that

$$x_1(\xi) \sim 2\delta\lambda(\xi)^\sigma \qquad \text{and} \qquad x_2(\xi) \sim \frac{1}{2\delta}\lambda(\xi)^{1-\sigma}$$

  as $\lambda(\xi) \to +\infty$, while for $\sigma = 1/2$ it turns out that

$$x_1(\xi) = \left(\delta + \sqrt{\delta^2 - 1}\right)\lambda(\xi)^{1/2} \qquad \text{and} \qquad x_2(\xi) = \left(\delta - \sqrt{\delta^2 - 1}\right)\lambda(\xi)^{1/2}.$$

3.3 A basic lemma

Every solution of (3.1) with $f \equiv 0$ can be written as a finite sum of terms of the form $z_0(\xi) \cdot c(\xi) \cdot g(t,\xi)$, where $z_0(\xi)$ is one of the initial conditions, $c(\xi)$ is a coefficient depending only on the roots of the characteristic polynomial (3.2), and $g(t,\xi)$ is one of the fundamental solutions of the same equation (depending once again on the roots of the characteristic polynomial). For this reason we investigate the regularity of these objects.

Lemma 3.1 *Let H, A, (M,μ), $\lambda(\xi)$ be as in section 3.1. Let $z_0(\xi)$ and $c(\xi)$ be two measurable real function on M, and let $g(t,\xi)$ be a function defined in $[0,+\infty) \times M$ which is of class C^∞ with respect to t and measurable with respect to ξ.*

Let us assume that there exists $P \geq 0$ such that

$$S_P := \int_M \lambda(\xi)^{2P} z_0(\xi)^2 \, d\mu(\xi) < +\infty, \tag{3.6}$$

and there exists $Q \geq 0$ such that

$$C_Q := \sup_{\xi \in M} \lambda(\xi)^Q |c(\xi)| < +\infty. \tag{3.7}$$

Then the following statements hold true.

(1) Let us assume that there exists a constant G_0 such that

$$|g(t,\xi)| \leq G_0 \qquad \forall (t,\xi) \in [0,+\infty) \times M. \tag{3.8}$$

Then the product

$$\widehat{z}(t,\xi) := z_0(\xi) \cdot c(\xi) \cdot g(t,\xi) \qquad \forall (t,\xi) \in [0,+\infty) \times M$$

defines a function $\widehat{z}$ which corresponds, under the usual identification of $L^2(M,\mu)$ with H, to a function $z \in C^0([0,+\infty), D(A^{P+Q}))$.

(2) More generally, let us assume that there exist an integer $m \geq 0$, and real constants R (not necessarily positive) and $G_m \geq 0$ such that $P + Q - mR \geq 0$ and the m-th time-derivative $g^{(m)}(t,\xi)$ of $g(t,\xi)$ satisfies

$$|g^{(m)}(t,\xi)| \leq G_m \lambda(\xi)^{mR} \qquad \forall (t,\xi) \in [0,+\infty) \times M. \tag{3.9}$$

Then z is m times differentiable (for example with values in H), and actually its m-th time-derivative $z^{(m)}(t)$ satisfies

$$z^{(m)} \in C^0\left([0,+\infty), D(A^{P+Q-mR})\right).$$

(3) Let us assume that there exist a measurable positive function $\eta(\xi)$ in M, a sequence $\{\Gamma_m\} \subseteq [0, +\infty)$, and constants $R \in \mathbb{R}$, $S > 0$, and $M_S > 0$ such that

$$|g^{(m)}(t, \xi)| \leq \Gamma_m \lambda(\xi)^{mR} e^{-\eta(\xi)t} \qquad \forall (m, t, \xi) \in \mathbb{N} \times [0, +\infty) \times M, \tag{3.10}$$

and

$$\lambda(\xi)^S \leq M_S \eta(\xi) \qquad \forall \xi \in M. \tag{3.11}$$

Then it turns out that $z \in C^\infty((0, +\infty), D(A^\alpha))$ for every $\alpha \geq 0$.

Proof

Statement (1) Due to (3.7) and (3.8) it turns out that

$$\begin{aligned} \lambda(\xi)^{2P+2Q} \cdot |\widehat{z}(t, \xi)|^2 &= \lambda(\xi)^{2P} z_0(\xi)^2 \cdot \lambda(\xi)^{2Q} |c(\xi)|^2 \cdot |g(t, \xi)|^2 \\ &\leq C_Q^2 \cdot G_0^2 \cdot \lambda(\xi)^{2P} z_0(\xi)^2. \end{aligned} \tag{3.12}$$

Therefore, from (3.6) it follows that

$$\int_M \lambda(\xi)^{2P+2Q} \cdot |\widehat{z}(t, \xi)|^2 \, d\mu(\xi) \leq C_Q^2 \cdot G_0^2 \cdot S_P < +\infty,$$

which is equivalent to saying that $z(t) \in D(A^{P+Q})$ for every $t \geq 0$. Since (3.12) is uniform in time, the continuity of $A^{P+Q} z(t)$ with respect to t follows from the continuity of $g(t, \xi)$ with respect to t and Lebesgue's theorem.

Statement (2) The m-th time-derivative $\widehat{z}^{(m)}(t, \xi)$ of $\widehat{z}(t, \xi)$ exists because $g(t, \xi)$ is m times differentiable. If (3.9) holds true, then

$$\begin{aligned} \lambda(\xi)^{2P+2Q-2mR} \cdot |\widehat{z}^{(m)}(t, \xi)|^2 &= \lambda(\xi)^{2P} z_0(\xi)^2 \cdot \lambda(\xi)^{2Q} c(\xi)^2 \cdot \lambda(\xi)^{-2mR} |g^{(m)}(t, \xi)|^2 \\ &\leq C_Q^2 \cdot G_m^2 \cdot \lambda(\xi)^{2P} z_0(\xi)^2. \end{aligned}$$

Therefore, from (3.6) it follows that

$$\int_M \lambda(\xi)^{2P+2Q-2mR} \cdot |\widehat{z}^{(m)}(t, \xi)|^2 \, d\mu(\xi) \leq C_Q^2 \cdot G_m^2 \cdot S_P < +\infty,$$

which is equivalent to saying that $z^{(m)}(t) \in D(A^{P+Q-mR})$. The continuity with values in the same space follows from the continuity of $g^{(m)}(t, \xi)$ with respect to t and Lebesgue's theorem as in the first statement.

Statement (3) For every $\beta > 0$ there exists a constant K_β such that $e^{-x} \leq K_\beta x^{-\beta}$ for every $x > 0$. Therefore assumption (3.10) implies that

$$\lambda(\xi)^{-2mR} \cdot \left|g^{(m)}(t,\xi)\right|^2 \leq \Gamma_m^2 e^{-2\eta(\xi)t} \leq \Gamma_m^2 \cdot K_\beta^2 \cdot \frac{1}{[\eta(\xi)t]^{2\beta}} \qquad \forall (t,\xi) \in (0,+\infty) \times M.$$

Keeping (3.11) into account, it follows that

$$\lambda(\xi)^{-2mR+2\beta S} \cdot \left|g^{(m)}(t,\xi)\right|^2 \leq \Gamma_m^2 \cdot K_\beta^2 \cdot \left(\frac{\lambda(\xi)^S}{\eta(\xi)}\right)^{2\beta} \frac{1}{t^{2\beta}} \leq \frac{\Gamma_m^2 \cdot K_\beta^2 \cdot M_S^{2\beta}}{t^{2\beta}},$$

and finally

$$\begin{aligned} \lambda(\xi)^{2P+2Q-2mR+2\beta S} \cdot \left|\widehat{z}^{(m)}(t,\xi)\right|^2 &= \lambda(\xi)^{2Q} c(\xi)^2 \cdot \lambda(\xi)^{-2mR+2\beta S} \left|g^{(m)}(t,\xi)\right|^2 \\ &\quad \cdot \lambda(\xi)^{2P} z_0(\xi)^2 \\ &\leq C_Q^2 \cdot \frac{\Gamma_m^2 \cdot K_\beta^2 \cdot M_S^{2\beta}}{t^{2\beta}} \cdot \lambda(\xi)^{2P} z_0(\xi)^2 \end{aligned}$$

for every $t > 0$ and every $\xi \in M$. As in the first two statements, we integrate with respect to ξ and what we obtain is equivalent to saying that $z^{(m)}(t) \in D(A^{P+Q-mR+\beta S})$ for every $t > 0$. Also the continuity of $z^{(m)}(t)$ with values in the same space follows as in the first two statements.

Since $S > 0$, the exponent $P + Q - mR + \beta S$ can be made arbitrarily large by choosing β large enough, and this is enough to establish that $z \in C^\infty((0,+\infty), D(A^\alpha))$ for every $\alpha \geq 0$. $\square$

3.4 Proof of Theorem 2.1

Let $\widehat{u}_0(\xi)$ and $\widehat{u}_1(\xi)$ denote the components of u_0 and u_1. It is easy to express the components $\widehat{u}(t,\xi)$ of the solution in terms of $\widehat{u}_0(\xi)$, $\widehat{u}_1(\xi)$, and of the roots of the characteristic polynomial (3.2). We distinguish three cases depending on the discriminant of (3.2).

Supercritical dissipation Let us consider the case where $\sigma > 1/2$, or $\sigma = 1/2$ and $\delta > 1$, so that for λ large enough the roots of the characteristic equation are given by (3.5). The solution $u(t)$ of (2.1)–(2.2) is the sum of four functions $v_1(t)$, $v_2(t)$, $w_1(t)$, $w_2(t)$ whose components are

$$\widehat{v}_1(t,\xi) = -\widehat{u}_0(\xi) \cdot \frac{x_2(\xi)}{x_1(\xi) - x_2(\xi)} \cdot e^{-x_1(\xi)t}, \qquad \widehat{v}_2(t,\xi) = \widehat{u}_0(\xi) \cdot \frac{x_1(\xi)}{x_1(\xi) - x_2(\xi)} \cdot e^{-x_2(\xi)t},$$

$$\widehat{w}_1(t,\xi) = -\widehat{u}_1(\xi) \cdot \frac{1}{x_1(\xi) - x_2(\xi)} \cdot e^{-x_1(\xi)t}, \qquad \widehat{w}_2(t,\xi) = \widehat{u}_1(\xi) \cdot \frac{1}{x_1(\xi) - x_2(\xi)} \cdot e^{-x_2(\xi)t}.$$

The regularity of these four functions follows quite easily from Lemma 3.1, applied with straightforward choices of the functions $z_0(\xi)$, $c(\xi)$, $g(t,\xi)$. We skip the elementary checks, but we sum up the results in the table below. The first columns show the values of P, Q, R, S for which the assumptions are satisfied. Then in the second column from the right we write the optimal value of α for which the m-th derivative of the function in that row lies in $C^0([0,+\infty),D(A^\alpha))$ (according to statement (2) of Lemma 3.1 this value is $P+Q-mR$, provided of course that it is nonnegative). Finally, in the last column we state the values of σ for which the corresponding function is in $C^\infty((0,+\infty),D(A^\alpha))$ for every $\alpha\geq 0$ (according to statement (3) of Lemma 3.1 this condition is always equivalent to $S>0$).

	P	Q	R	S	m-th deriv. in $D(A^\alpha)$	C^∞ in $D(A^\infty)$
$\widehat{v}_1(t,\xi)$	α_0	$2\sigma-1$	σ	σ	$\alpha_0+2\sigma-1-m\sigma$	always
$\widehat{v}_2(t,\xi)$	α_0	0	$1-\sigma$	$1-\sigma$	$\alpha_0-m(1-\sigma)$	if $\sigma<1$
$\widehat{w}_1(t,\xi)$	α_1	σ	σ	σ	$\alpha_1+\sigma-m\sigma$	always
$\widehat{w}_2(t,\xi)$	α_1	σ	$1-\sigma$	$1-\sigma$	$\alpha_1+\sigma-m(1-\sigma)$	if $\sigma<1$

The regularity of $u(t)$ is the minimal regularity of the four summands. Thus we obtain the following results.

- From the second column from the right with $m=0$ we deduce that

$$u\in C^0\left([0,+\infty),D(A^{\min\{\alpha_0,\alpha_1+\sigma\}})\right),$$

  while for $m\geq 1$ we deduce that

$$u^{(m)}\in C^0\left([0,+\infty),D(A^{\min\{\alpha_0+\sigma-1,\alpha_1\}-(m-1)\sigma})\right).$$

  Since $1-\sigma\leq\alpha_0-\alpha_1\leq\sigma$, this proves (2.4) and (2.5) in the supercritical regime.

- For $\sigma<1$ all four functions are in $C^\infty((0,+\infty),D(A^\alpha))$ for every $\alpha\geq 0$, so that the same is true for their sum $u(t)$. This proves (2.6) in the supercritical regime.

- For $\sigma\geq 1$ we obtain that the m-th derivatives of $\widehat{v}_2(t,\xi)$ and $\widehat{w}_2(t,\xi)$ exist in the space $C^0([0,+\infty),D(A^{\alpha_0+m(\sigma-1)}))$. Since the other two functions are always in $C^\infty((0,+\infty),D(A^\alpha))$ for every $\alpha\geq 0$, this proves (2.7).

Critical dissipation Let us consider the case where $\sigma=1/2$ and $\delta=1$, so that the roots of the characteristic polynomial are given by (3.4). In this case the solution $u(t)$ of (2.1)–(2.2) is the sum of three functions $v_1(t)$, $v_2(t)$, $w(t)$ whose components are

$$\widehat{v}_1(t,\xi)=\widehat{u}_0(\xi)\cdot 1\cdot e^{-\lambda(\xi)^{1/2}t},\qquad \widehat{v}_2(t,\xi)=\widehat{u}_0(\xi)\cdot 1\cdot\lambda(\xi)^{1/2}t\,e^{-\lambda(\xi)^{1/2}t},$$

$$\widehat{w}(t,\xi) = \widehat{u}_1(\xi) \cdot \frac{1}{\lambda(\xi)^{1/2}} \cdot \lambda(\xi)^{1/2} t\, e^{-\lambda(\xi)^{1/2} t}.$$

The regularity of these three functions follows from Lemma 3.1 as before. We point out that we choose $g(t,\xi) := \lambda(\xi)^{1/2} t\, e^{-\lambda(\xi)^{1/2} t}$ in the case of $\widehat{v}_2(t,\xi)$ and $\widehat{w}(t,\xi)$. We sum up the results in the table below.

	P	Q	R	S	m-th deriv. in $D(A^\alpha)$	C^∞ in $D(A^\infty)$
$\widehat{v}_1(t,\xi)$	α_0	0	1/2	1/2	$\alpha_0 - m/2$	always
$\widehat{v}_2(t,\xi)$	α_0	0	1/2	1/2	$\alpha_0 - m/2$	always
$\widehat{w}(t,\xi)$	α_1	1/2	1/2	1/2	$\alpha_1 + 1/2 - m/2$	always

The regularity of $u(t)$ is the minimal regularity of the three summands. Looking at the second column from the right, and taking into account that in this case $\alpha_0 = \alpha_1 + 1/2$, we obtain that

$$u^{(m)} \in C^0([0,+\infty), D(A^{\alpha_0 - m/2})) \qquad \forall m \in \mathbb{N}.$$

This establishes (2.4) and (2.5) in the critical regime.

Moreover, since $S > 0$, all three functions are in $C^\infty((0,+\infty), D(A^\alpha))$ for every $\alpha \geq 0$, which proves (2.6) in the critical regime.

Subcritical dissipation Let us consider the case where $\sigma < 1/2$, or $\sigma = 1/2$ and $\delta < 1$, so that for λ large enough the roots of the characteristic polynomial are given by (3.3). The solution $u(t)$ of (2.1)–(2.2) is the sum of three functions $v_1(t)$, $v_2(t)$, $w(t)$ whose components are

$$\widehat{v}_1(t,\xi) = \widehat{u}_0(\xi) \cdot 1 \cdot e^{-a(\xi)t} \cos(b(\xi)t), \qquad \widehat{v}_2(t,\xi) = \widehat{u}_0(\xi) \cdot \frac{a(\xi)}{b(\xi)} \cdot e^{-a(\xi)t} \sin(b(\xi)t),$$

$$\widehat{w}(t,\xi) = \widehat{u}_1(\xi) \cdot \frac{1}{b(\xi)} \cdot e^{-a(\xi)t} \sin(b(\xi)t).$$

The regularity of these three functions follows from Lemma 3.1 as in the previous cases. We sum up the results in the table below. We just point out that the growth of derivatives of $g(t,\xi)$, represented by the parameter R, is due to the terms $b(\xi)$ coming from the trigonometric part. Since $b(\xi) \sim \lambda(\xi)^{1/2}$, we have that $R = 1/2$, and this is the reason why the derivative gap is always 1/2 in the subcritical regime.

	P	Q	R	S	m-th deriv. in $D(A^\alpha)$	C^∞ in $D(A^\infty)$
$\widehat{v}_1(t,\xi)$	α_0	0	1/2	σ	$\alpha_0 - m/2$	if $\sigma > 0$
$\widehat{v}_2(t,\xi)$	α_0	$1/2 - \sigma$	1/2	σ	$\alpha_0 + 1/2 - \sigma - m/2$	if $\sigma > 0$
$\widehat{w}(t,\xi)$	α_1	1/2	1/2	σ	$\alpha_1 + 1/2 - m/2$	if $\sigma > 0$

The regularity of $u(t)$ is the minimal regularity of the three summands. Since $\alpha_0 = \alpha_1 + 1/2$ also in the subcritical case, as before we obtain that

$$u^{(m)} \in C^0([0, +\infty), D(A^{\alpha_0 - m/2})) \qquad \forall m \in \mathbb{N},$$

which proves (2.4) and (2.5) in the subcritical regime.

Moreover, all three functions are in $C^\infty((0, +\infty), D(A^\alpha))$ for every $\alpha \geq 0$ provided that $S > 0$, hence when $\sigma > 0$. This proves (2.6) in the subcritical regime. □

4 Proofs in the non-homogeneous case

In this section we keep the notation of Section 3.

4.1 A lemma adapted to the forced case

Solutions of (3.1) with null initial data can always be written as integrals of $\widehat{f}(t, \xi)$ multiplied by some convolution kernel containing exponential terms. Let us address the regularity of these objects.

Lemma 4.1 *Let H, A, (M, μ), $\lambda(\xi)$ be as in section 3.1. Let us assume that $\lambda(\xi) \geq 1$ for every $\xi \in M$. Let $T > 0$, and let $f \in L^\infty((0, T), H)$ be a bounded function whose components we denote by $\widehat{f}(t, \xi)$. Let $y(\xi)$ be a measurable real function on M, let $\eta(\xi)$ be a measurable positive function on M, and let $\psi(t, \xi)$ be a function defined in $[0, T] \times M$ which is continuous with respect to t and measurable with respect to ξ.*

Then the following statements hold true.

(1) Let us assume that there exists a constant M_1 such that

$$|y(\xi)| \cdot |\psi(t, \xi)| \leq M_1 \qquad \forall (t, \xi) \in [0, T] \times M. \tag{4.13}$$

Then the integral

$$\widehat{z}(t, \xi) := y(\xi) \int_0^t e^{-\eta(\xi)(t-s)} \psi(t - s, \xi) \widehat{f}(s, \xi)\, ds \tag{4.14}$$

defines a function $\widehat{z}$ which corresponds, under the usual identification of $L^2(M, \mu)$ with H, to a function $z \in C^0([0, T], H)$.

(2) If in addition to (4.13) we assume also the existence of real numbers $\alpha \geq 0$, $b \in [0, 1)$, $c \geq 0$, $M_{\alpha,b,c}$ such that

$$\lambda(\xi)^\alpha \cdot |y(\xi)| \cdot |\psi(t, \xi)| \leq M_{\alpha,b,c} \min\{\eta(\xi)^b, \eta(\xi)^c\} \qquad \forall (t, \xi) \in [0, T] \times M, \tag{4.15}$$

then it turns out that $z \in C^0([0,T], D(A^\alpha))$. *Moreover, there exists a constant* $K_{b,c}$, *depending only on* b *and* c, *such that*

$$|A^\alpha z(t)| \leq K_{b,c} \cdot M_{\alpha,b,c} \cdot \|f\|_{L^\infty((0,T),H)} \cdot \int_0^t \min\left\{\frac{1}{s^b}, \frac{1}{s^c}\right\} ds \tag{4.16}$$

for every $t \in [0,T]$, *and more generally*

$$|A^\alpha z(t)| \leq K_{b,c} \cdot M_{\alpha,b,c} \cdot \|f\|_{L^p((0,T),H)} \cdot \left(\int_0^t \min\left\{\frac{1}{s^{qb}}, \frac{1}{s^{qc}}\right\} ds\right)^{1/q} \tag{4.17}$$

for every $t \in [0,T]$, *and every pair of real numbers* (p,q) *such that*

$$p > 1, \qquad q > 1, \qquad \frac{1}{p} + \frac{1}{q} = 1, \qquad qb < 1. \tag{4.18}$$

(3) As a special case, let us assume that there exists M_2 *such that*

$$|\psi(t,\xi)| \leq M_2 \qquad \forall (t,\xi) \in [0,T] \times M,$$

and that there exist $Q \geq 0$, $S \in \mathbb{R}$, *and real numbers* M_3, M_4, M_5 *such that*

$$\lambda(\xi)^Q |y(\xi)| \leq M_3, \qquad \text{and} \qquad 0 < M_4 \leq \frac{\eta(\xi)}{\lambda(\xi)^S} \leq M_5 \tag{4.19}$$

for every $\xi \in M$.

Then z *is continuous and bounded in the spaces shown in the following table (it is intended that* $0 < \varepsilon \leq Q + S$ *when needed).*

	z *continuous in*	z *bounded in*
$S > 0$	$D(A^{Q+S-\varepsilon})$	$D(A^{Q+S-\varepsilon})$
$S = 0$	$D(A^Q)$	$D(A^Q)$
$S < 0$	$D(A^Q)$	$D(A^{Q+S-\varepsilon})$

Here "z *continuous in* $D(A^\alpha)$*" means that* $z \in C^0([0,T], D(A^\alpha))$ *and* $|A^\alpha z(t)|$ *satisfies an estimate such as (4.16) for some* $b < 1$ *and* $c \geq 0$, *while "*z *bounded in* $D(A^\alpha)$*" means that in addition* $c > 1$, *so that the right-hand side of (4.16) is bounded independently of* t.

Proof Due to the usual identification of H with $L^2(M,\mu)$, it turns out that

$$\left\|\widehat{f}(t,\xi)\right\|_{L^2(M,\mu)} = |f(t)| \qquad \text{for almost every } t \in (0,T). \tag{4.20}$$

Statement (1) The $L^2(M,\mu)$ norm of the integral defining $\widehat{z}(t,\xi)$ in (4.14) is less than or equal to the integral of the norm of the integrand. Therefore, from (4.13) it follows that

$$\begin{aligned}
\|\widehat{z}(t,\xi)\|_{L^2(M,\mu)} &\leq \int_0^t \left\| y(\xi)e^{-\eta(\xi)(t-s)}\psi(t-s,\xi)\widehat{f}(s,\xi)\right\|_{L^2(M,\mu)} ds \\
&\leq M_1 \int_0^t \left\|\widehat{f}(s,\xi)\right\|_{L^2(M,\mu)} ds \\
&\leq M_1 \cdot t \cdot \|f\|_{L^\infty((0,T),H)}
\end{aligned}$$

for every $t \in [0,T]$, which proves that $\widehat{z} \in L^\infty\left((0,T), L^2(M,\mu)\right)$. The continuity with values in the same space follows from the continuity of the map

$$t \to e^{-\eta(\xi)(t-s)}\psi(t-s,\xi)$$

and Lebesgue's theorem.

Statement (2) With a simple variable change, we write $\widehat{z}(t,\xi)$ in the form

$$\widehat{z}(t,\xi) = y(\xi)\int_0^t e^{-\eta(\xi)s}\psi(s,\xi)\widehat{f}(t-s,\xi)\,ds. \tag{4.21}$$

Then we set

$$K_{b,c} := \max\left\{e^{-x}\cdot\max\{x^b,x^c\} : x \geq 0\right\} \tag{4.22}$$

and, for every $\xi \in M$ and every $0 \leq s \leq t \leq T$, we consider the function

$$\widehat{\varphi}(s,t,\xi) := \lambda(\xi)^\alpha y(\xi)e^{-\eta(\xi)s}\psi(s,\xi)\widehat{f}(t-s,\xi).$$

We claim that for every $t \in (0,T)$ the estimate

$$\|\widehat{\varphi}(s,t,\xi)\|_{L^2(M,\mu)} \leq K_{b,c}\cdot M_{\alpha,b,c}\cdot\min\left\{\frac{1}{s^b},\frac{1}{s^c}\right\}\cdot|f(t-s)| \tag{4.23}$$

holds true for almost every $s \in (0,t)$. Indeed from (4.22) it follows that

$$e^{-x} \leq K_{b,c}\min\left\{\frac{1}{x^b},\frac{1}{x^c}\right\} \qquad \forall x > 0,$$

hence

$$e^{-\eta(\xi)s} \leq K_{b,c}\min\left\{\frac{1}{\eta(\xi)^b s^b},\frac{1}{\eta(\xi)^c s^c}\right\} \leq K_{b,c}\cdot\frac{1}{\min\{\eta(\xi)^b,\eta(\xi)^c\}}\cdot\min\left\{\frac{1}{s^b},\frac{1}{s^c}\right\}$$

for every $s \geq 0$. Thus from (4.15) it follows that

$$\begin{aligned} |\widehat{\varphi}(s,t,\xi)| &\leq K_{b,c} \cdot \frac{\lambda(\xi)^\alpha |y(\xi)|}{\min\{\eta(\xi)^b, \eta(\xi)^c\}} \cdot \min\left\{\frac{1}{s^b}, \frac{1}{s^c}\right\} \cdot |\psi(s,\xi)| \cdot |\widehat{f}(t-s,\xi)| \\ &\leq K_{b,c} \cdot M_{\alpha,b,c} \cdot \min\left\{\frac{1}{s^b}, \frac{1}{s^c}\right\} \cdot |\widehat{f}(t-s,\xi)|. \end{aligned}$$

The coefficient of $|\widehat{f}(t-s,\xi)|$ is independent of ξ. Therefore, if we integrate with respect to ξ and we exploit (4.20), we obtain (4.23).

Now we are ready to prove (4.16). Indeed from (4.21) we obtain that

$$|A^\alpha z(t)| = \|\lambda(\xi)^\alpha \widehat{z}(t,\xi)\|_{L^2(M,\mu)} = \left\| \int_0^t \widehat{\varphi}(s,t,\xi)\, ds \right\|_{L^2(M,\mu)} \leq \int_0^t \|\widehat{\varphi}(s,t,\xi)\|_{L^2(M,\mu)}\, ds$$

for every $t \in [0,T]$. Plugging (4.23) into the last term we easily obtain (4.16).

Let us assume now that p and q satisfy (4.18). Then Hölder's inequality gives

$$\begin{aligned} \int_0^t \|\widehat{\varphi}(s,t,\xi)\|_{L^2(M,\mu)}\, ds &\leq K_{b,c} \cdot M_{\alpha,b,c} \int_0^t \min\left\{\frac{1}{s^b}, \frac{1}{s^c}\right\} |f(t-s)|\, ds \\ &\leq K_{b,c} \cdot M_{\alpha,b,c} \left(\int_0^t \min\left\{\frac{1}{s^{qb}}, \frac{1}{s^{qc}}\right\} ds \right)^{1/q} \cdot \left(\int_0^t |f(t-s)|^p\, ds \right)^{1/p}, \end{aligned}$$

which easily implies (4.17).

For the time being we just proved that $z \in L^\infty((0,T), D(A^\alpha))$. Now we want to prove that z is actually continuous with values in the same space. To this end, we show that $A^\alpha z(t)$ is the uniform limit of continuous functions. Let us set

$$M_n := \{\xi \in M : \lambda(\xi) \leq n\},$$

and let us set $\chi_n(\xi) = 1$ if $\xi \in M_n$ and $\chi_n(\xi) = 0$ otherwise. It is well-known that $\mu(M \setminus M_n) \to 0$ as $n \to +\infty$. Let us set

$$\widehat{z}_n(t,\xi) := \widehat{z}(t,\xi)\chi_n(\xi), \qquad \widehat{f}_n(t,\xi) := \widehat{f}(t,\xi)\chi_n(\xi). \tag{4.24}$$

Due to the boundedness of $\lambda(\xi)$ in M_n, the same argument of statement (1) proves that $\lambda(\xi)^\alpha \cdot \widehat{z}_n(t,\xi) \in C^0([0,T], L^2(M,\mu))$, which is equivalent to saying that $\widehat{z}_n$ corresponds to a function $z_n \in C^0([0,T], D(A^\alpha))$. Moreover, from Lebesgue's theorem it follows that

$$\widehat{f}_n \to \widehat{f} \quad \text{in } L^p((0,T), L^2(M,\mu))$$

for every $p \geq 1$ (but not necessarily for $p = +\infty$). Once again, this is equivalent to saying that the sequence $\widehat{f}_n(t,\xi)$ represents the components of a sequence of functions $f_n \in L^\infty((0,T), H)$ such that

$$f_n \to f \quad \text{in } L^p((0,T), H) \tag{4.25}$$

Let us choose p large enough so that all assumptions (4.18) are satisfied. By the same argument as before, we obtain

$$|A^\alpha(z(t)-z_n(t))| \leq K_{b,c}\cdot M_{\alpha,b,c}\left(\int_0^t \min\left\{\frac{1}{s^{qb}},\frac{1}{s^{qc}}\right\}ds\right)^{1/q}\cdot\|f-f_n\|_{L^p((0,T),H)}$$

for every $t\in[0,T]$, hence by (4.25) we can conclude that

$$\lim_{n\to+\infty}\sup_{t\in[0,T]}|A^\alpha(z(t)-z_n(t))| = 0.$$

This is equivalent to saying that $z_n(t)\to z(t)$ uniformly in $C^0([0,T],D(A^\alpha))$, and the result follows since a uniform limit of continuous functions is continuous.

Statement (3) All conclusions of this statement follow from (4.16) with suitable choices of the parameters b and c, which we list below.

Assumption	b	c	Conclusion
$S>0$	$1-\varepsilon/S$	2	z continuous and bounded in $D(A^{Q+S-\varepsilon})$
$S=0$	0	2	z continuous and bounded in $D(A^Q)$
$S<0$	0	0	z continuous in $D(A^Q)$
$S<0$	0	$1-\varepsilon/S$	z bounded in $D(A^{Q+S-\varepsilon})$

The verification of (4.13) and (4.15) in all these cases, with the value of the exponent α given in the conclusion, is a straightforward check. We just point out that in the case $S>0$ and $S=0$ one can replace $c=2$ with any $c>1$, and that in the last line one has that $c>1$ because $S<0$. $\square$

Remark 4.2 Lemma 4.1 is stated as a local result in a bounded interval $[0,T]$, but it is designed to provide also global-in-time estimates when f is defined for all positive times. Let us examine for example the third statement of Lemma 4.1. When $S<0$ and $\alpha\leq Q$, one can always apply estimate (4.16) with $b=0$ and $c=(\alpha-Q)/S$. If in addition $\alpha<Q+S$ we obtain $c>1$, which gives the boundedness as stated. If $Q+S\leq\alpha\leq Q$ (remember that $S<0$) we obtain that $0\leq c\leq 1$. In this case the right-hand side of (4.16) is not bounded independently of t, but its growth can be explicitly estimated as follows

$$|A^\alpha z(t)| \leq C\cdot\|f\|_{L^\infty((0,T),H)}\cdot\begin{cases} t^{(Q+S-\alpha)/S} & \text{if } Q+S<\alpha\leq Q,\\ \log(1+t) & \text{if } \alpha=Q+S,\end{cases}$$

where C is a suitable constant independent of f and t. This estimate, applied with suitable values of Q and S (those exploited in the proof of Theorems 2.4 and 2.7), leads to (2.14).

4.2 Proof of Theorem 2.4 and Theorem 2.7

It is easy to express the components $\widehat{u}'(t,\xi)$ of the solution in terms of integrals involving the components $\widehat{f}(t,\xi)$ of the forcing term $f(t)$ and the fundamental solutions of the associated homogenous equation. As in the proof of Theorem 2.1, we distinguish three cases depending on the discriminant of the characteristic polynomial (3.2).

Supercritical dissipation Let us consider the case where $\sigma > 1/2$, or $\sigma = 1/2$ and $\delta > 1$, so that for λ large enough the roots of the characteristic equation are given by (3.5). The solution $u(t)$ of (2.9)–(2.10) is the sum of two functions $v_1(t)$ and $v_2(t)$ whose components are

$$\widehat{v}_i(t,\xi) = \frac{(-1)^i}{x_1(\xi) - x_2(\xi)} \int_0^t e^{-x_i(\xi)(t-s)} \widehat{f}(s,\xi)\, ds \qquad i \in \{1,2\}.$$

The regularity of these two functions follows quite easily from Lemma 4.1, applied with straightforward choices of the functions $y(\xi)$, $\eta(\xi)$, and $\psi(t,\xi) \equiv 1$. We skip the elementary checks, but we sum up the results in the table below. We distinguish the three cases $\sigma < 1$, $\sigma = 1$, $\sigma > 1$, and for each function we show the values of Q and S for which assumption (4.19) is satisfied, and the conclusions according to statement (3) of Lemma 4.1.

Assumption	Function	Q	S	Continuous in	Bounded in
$\sigma < 1$	$v_1(t)$	σ	σ	$D(A^{2\sigma-\varepsilon})$	$D(A^{2\sigma-\varepsilon})$
	$v_2(t)$	σ	$1-\sigma$	$D(A^{1-\varepsilon})$	$D(A^{1-\varepsilon})$
$\sigma = 1$	$v_1(t)$	1	1	$D(A^{2-\varepsilon})$	$D(A^{2-\varepsilon})$
	$v_2(t)$	1	0	$D(A)$	$D(A)$
$\sigma > 1$	$v_1(t)$	σ	σ	$D(A^{2\sigma-\varepsilon})$	$D(A^{2\sigma-\varepsilon})$
	$v_2(t)$	σ	$1-\sigma$	$D(A^{\sigma})$	$D(A^{1-\varepsilon})$

The best space where $u(t)$ is continuous or bounded is always the maximal space where both $v_1(t)$ and $v_2(t)$ fulfil the same property. This proves the conclusions for $u(t)$ required by Theorem 2.4 and Theorem 2.7 in the supercritical regime.

Analogously, the derivative $u'(t)$ of the solution is the sum of two functions $w_1(t)$ and $w_2(t)$, whose components are

$$\widehat{w}_i(t,\xi) = \frac{(-1)^{i+1} x_i(\xi)}{x_1(\xi) - x_2(\xi)} \int_0^t e^{-x_i(\xi)(t-s)} \widehat{f}(s,\xi)\, ds \qquad i \in \{1,2\}.$$

The regularity of $w_1(t)$ and $w_2(t)$ follows in the same way from Lemma 4.1, as shown in the following table.

Assumption	Function	Q	S	Continuous in	Bounded in
$\sigma < 1$	$w_1(t)$	0	σ	$D(A^{\sigma-\varepsilon})$	$D(A^{\sigma-\varepsilon})$
	$w_2(t)$	$2\sigma - 1$	$1 - \sigma$	$D(A^{\sigma-\varepsilon})$	$D(A^{\sigma-\varepsilon})$
$\sigma = 1$	$w_1(t)$	0	1	$D(A^{1-\varepsilon})$	$D(A^{1-\varepsilon})$
	$w_2(t)$	1	0	$D(A)$	$D(A)$
$\sigma > 1$	$w_1(t)$	0	σ	$D(A^{\sigma-\varepsilon})$	$D(A^{\sigma-\varepsilon})$
	$w_2(t)$	$2\sigma - 1$	$1 - \sigma$	$D(A^{2\sigma-1})$	$D(A^{\sigma-\varepsilon})$

As before, this is enough to prove the conclusions for $u'(t)$ required by Theorem 2.4 and Theorem 2.7 in the supercritical regime.

Critical dissipation Let us consider the case where $\sigma = 1/2$ and $\delta = 1$, so that the roots of the characteristic equation are given by (3.4). In this case the solution $u(t)$ of (2.9)–(2.10) has components

$$\widehat{u}(t,\xi) = \int_0^t (t-s)e^{-\lambda(\xi)^{1/2}(t-s)}\widehat{f}(s,\xi)\,ds.$$

The right-hand side can be written in the form of the right-hand side of (4.14) with

$$y(\xi) := \frac{1}{\lambda(\xi)^{1/2}}, \qquad \eta(\xi) := \frac{\lambda(\xi)^{1/2}}{2}, \qquad \psi(t,\xi) := \lambda(\xi)^{1/2} t \exp\left(-\frac{\lambda(\xi)^{1/2}}{2}t\right).$$

Since assumption (4.19) is satisfied with $Q = S = 1/2$, we deduce that $u(t)$ is continuous and bounded with values in $D(A^{1-\varepsilon})$, as required by Theorem 2.4 and Theorem 2.7 in the critical regime.

As for the derivative $u'(t)$, its components are

$$\widehat{u}'(t,\xi) = -\lambda(\xi)^{1/2}\int_0^t (t-s)e^{-\lambda(\xi)^{1/2}(t-s)}\widehat{f}(s,\xi)\,ds + \int_0^t e^{-\lambda(\xi)^{1/2}(t-s)}\widehat{f}(s,\xi)\,ds.$$

For the first term, we set $y(\xi) \equiv 1$, and we define $\eta(\xi)$ and $\psi(t,\xi)$ as before. For the second term, we set $y(\xi) = \psi(t,\xi) \equiv 1$ and $\eta(\xi) = \lambda(\xi)^{1/2}$. In both cases assumption (4.19) is satisfied with $Q = 0$ and $S = 1/2$, and therefore both terms are continuous and bounded in $D(A^{1/2-\varepsilon})$, as required by Theorem 2.4 and Theorem 2.7 in the critical regime.

Subcritical dissipation Let us consider the case where $\sigma \in [0, 1/2)$, or $\sigma = 1/2$ and $\delta \in (0,1)$, so that for λ large enough the roots of the characteristic equation are given by (3.3). The components of the solution $u(t)$ are

$$\widehat{u}(t,\xi) = \frac{1}{b(\xi)}\int_0^t e^{-a(\xi)(t-s)}\sin(b(\xi)(t-s))\widehat{f}(s,\xi)\,ds.$$

The right-hand side can be written in the form of the right-hand side of (4.14) with

$$y(\xi) := \frac{1}{b(\xi)}, \qquad \eta(\xi) := a(\xi), \qquad \psi(t,\xi) := \sin(b(\xi)t).$$

It is easy to check that assumption (4.19) is satisfied with $Q = 1/2$ and $S = \sigma$. Thus from Lemma 4.1 it turns out that $u(t)$ is continuous and bounded in $D(A^{\sigma+1/2-\varepsilon})$ if $\sigma > 0$, and in $D(A^{1/2})$ if $\sigma = 0$, as required by Theorem 2.4 and Theorem 2.7 in the subcritical regime.

As for the derivative $u'(t)$, its components are

$$\begin{aligned} \widehat{u}'(t,\xi) &= \int_0^t e^{-a(\xi)(t-s)}\cos(b(\xi)(t-s))\widehat{f}(s,\xi)\,ds \\ &\quad -\frac{a(\xi)}{b(\xi)}\int_0^t e^{-a(\xi)(t-s)}\sin(b(\xi)(t-s))\widehat{f}(s,\xi)\,ds. \end{aligned}$$

Once again, we apply Lemma 4.1 to both terms, with straightforward choices of the parameters. For $\sigma > 0$ we obtain that the first term is continuous and bounded in $D(A^{\sigma-\varepsilon})$ (since $Q = 0$ and $S = \sigma$), while the second term is continuous and bounded in $D(A^{1/2-\varepsilon})$ (since $Q = 1/2 - \sigma$ and $S = \sigma$). For $\sigma = 0$ we obtain that the first term is continuous and bounded in H (since $Q = 0$ and $S = 0$), while the second term is continuous and bounded in $D(A^{1/2})$ (since $Q = 1/2$ and $S = 0$). Therefore, the regularity and boundedness of $u'(t)$ is the same of the first term, and it is the same required by Theorem 2.4 and Theorem 2.7 in the subcritical regime. □

4.3 Proof of Theorem 2.6

Let us consider the function

$$E(t) := |A^{\sigma/2}u'(t)|^2 + |A^{(\sigma+1)/2}u(t)|^2.$$

An easy computation shows that

$$E'(t) = -4\delta|A^{\sigma}u'(t)|^2 + 2\langle A^{\sigma}u'(t), f(t)\rangle \leq -3\delta|A^{\sigma}u'(t)|^2 + \frac{1}{\delta}|f(t)|^2,$$

hence

$$|A^{\sigma/2}u'(t)|^2 + |A^{(\sigma+1)/2}u(t)|^2 + 3\delta\int_0^t |A^{\sigma}u'(s)|^2\,ds \leq \frac{1}{\delta}\int_0^t |f(s)|^2\,ds \qquad (4.26)$$

for every $t \in [0,T]$, which proves (2.12).

In order to prove (2.13), we can assume as always that the operator A is coercive, namely there exists a constant $\nu > 0$ such that $\langle Au, u\rangle \geq \nu|u|^2$ for every $u \in D(A)$. This allows to estimate $|A^{\alpha}u|$ with $|A^{\beta}u|$ (up to a constant) whenever $\alpha \leq \beta$. Now we distinguish two cases.

Case $\sigma \in [1/2, 1]$ An easy computation shows that

$$\begin{aligned} |Au|^2 + \frac{d}{dt}\left(\delta |A^{(\sigma+1)/2}u|^2\right) &= |A^{1/2}u'|^2 - \frac{d}{dt}\langle Au, u'\rangle + \langle f, Au\rangle \\ &\leq |A^{1/2}u'|^2 - \frac{d}{dt}\langle Au, u'\rangle + \frac{1}{2}|Au|^2 + \frac{1}{2}|f|^2, \end{aligned}$$

hence

$$\frac{1}{2}|Au(t)|^2 + \frac{d}{dt}\delta |A^{(\sigma+1)/2}u(t)|^2 \leq |A^{1/2}u'(t)|^2 + \frac{1}{2}|f(t)|^2 - \frac{d}{dt}\langle Au(t), u'(t)\rangle.$$

Integrating in $[0, t]$ we obtain that

$$\frac{1}{2}\int_0^t |Au(s)|^2\, ds \leq \int_0^t |A^{1/2}u'(s)|^2\, ds + \frac{1}{2}\int_0^t |f(s)|^2\, ds + |A^{(1-\sigma)/2}u(t)| \cdot |A^{\sigma/2}u'(t)|.$$

Now in the right-hand side we estimate $|A^{1/2}u'(t)|$ with $|A^{\sigma}u'(t)|$, and $|A^{(1-\sigma)/2}u(t)|$ with $|A^{(1+\sigma)/2}u(t)|$. This can be done because $\sigma \geq 1/2$ and the operator A can be assumed to be coercive. At this point (2.13) follows easily from (4.26).

Case $\sigma \in [0, 1/2]$ An easy computation shows that

$$\begin{aligned} |A^{\sigma+1/2}u|^2 + \frac{d}{dt}\left(\delta |A^{3\sigma/2}u|^2\right) &= |A^{\sigma}u'|^2 - \frac{d}{dt}\langle A^{2\sigma}u, u'\rangle + \langle f, A^{2\sigma}u\rangle \\ &\leq |A^{\sigma}u'|^2 - \frac{d}{dt}\langle A^{2\sigma}u, u'\rangle + \eta |A^{2\sigma}u|^2 + \frac{1}{\eta}|f|^2. \end{aligned}$$

Now we estimate $|A^{2\sigma}u(t)|$ with $|A^{\sigma+1/2}u|$. This can be done because $2\sigma \leq \sigma + 1/2$ and the operator A can be assumed to be coercive. Thus if we take η small enough we find that

$$\frac{1}{2}|A^{\sigma+1/2}u(t)|^2 + \frac{d}{dt}\left(\delta |A^{3\sigma/2}u(t)|^2\right) \leq |A^{\sigma}u'(t)|^2 + \frac{1}{\eta}|f(t)|^2 - \frac{d}{dt}\langle A^{2\sigma}u(t), u'(t)\rangle.$$

Integrating in $[0, t]$ we obtain that

$$\begin{aligned} &\frac{1}{2}\int_0^t |A^{\sigma+1/2}u(s)|^2\, ds + \delta |A^{3\sigma/2}u(t)|^2 \\ &\leq \int_0^t |A^{\sigma}u'(s)|^2\, ds + \frac{1}{\eta}\int_0^t |f(s)|^2\, ds + |\langle A^{2\sigma}u(t), u'(t)\rangle|. \end{aligned}$$

Since

$$|\langle A^{2\sigma}u(t), u'(t)\rangle| \leq |A^{3\sigma/2}u(t)| \cdot |A^{\sigma/2}u'(t)| \leq \delta |A^{3\sigma/2}u(t)|^2 + \frac{1}{4\delta}|A^{\sigma/2}u'(t)|^2,$$

we easily deduce that

$$\frac{1}{2}\int_0^t |A^{\sigma+1/2}u(s)|^2\, ds \leq \int_0^t |A^{\sigma}u'(s)|^2\, ds + \frac{1}{\eta}\int_0^t |f(s)|^2\, ds + \frac{1}{4\delta}|A^{\sigma/2}u'(t)|^2.$$

At this point (2.13) follows from (4.26). $\square$

4.4 Estimates on the whole line

The basic tool in the proof of Theorem 2.11 is the following variant of Lemma 4.1.

Lemma 4.3 *Let H, A, (M,μ), $\lambda(\xi)$ be as in section 3.1. Let us assume that $\lambda(\xi)\geq 1$ for every $\xi\in M$. Let $f\in L^{\infty}(\mathbb{R},H)$ be a bounded function whose components we denote by $\widehat{f}(t,\xi)$. Let $y(\xi)$ be a measurable real function on M, let $\eta(\xi)$ be a measurable positive function on M, and let $\psi(t,\xi)$ be a function defined in $\mathbb{R}\times M$ which is continuous with respect to t and measurable with respect to ξ.*

Then the following statements hold true.

(1) Let us assume that there exist real numbers $\alpha\geq 0$, $b\in[0,1)$, $c>1$ (this is stronger than the corresponding assumption in Lemma 4.1), $M_{\alpha,b,c}$ such that

$$\lambda(\xi)^{\alpha}\cdot|y(\xi)|\cdot|\psi(t,\xi)|\leq M_{\alpha,b,c}\min\left\{\eta(\xi)^{b},\eta(\xi)^{c}\right\}\qquad\forall(t,\xi)\in\mathbb{R}\times M.$$

Then the integral

$$\widehat{z}(t,\xi):=y(\xi)\int_{-\infty}^{t}e^{-\eta(\xi)(t-s)}\psi(t-s,\xi)\widehat{f}(s,\xi)\,ds$$

defines a function $\widehat{z}$ corresponding, under the usual identification of $L^2(M,\mu)$ with H, to a function $z\in C^0(\mathbb{R},D(A^{\alpha}))\cap L^{\infty}(\mathbb{R},D(A^{\alpha}))$.

Moreover, there exists a constant $K_{b,c}$, depending only on b and c, such that

$$|A^{\alpha}z(t)|\leq K_{b,c}\cdot M_{\alpha,b,c}\cdot\|f\|_{L^{\infty}(\mathbb{R},H)}\cdot\int_{0}^{+\infty}\min\left\{\frac{1}{s^{b}},\frac{1}{s^{c}}\right\}ds\qquad\forall t\in\mathbb{R}.\qquad(4.27)$$

(2) As a special case, let us assume that $\sup\{|\psi(t,\xi)|:(t,\xi)\in\mathbb{R}\times M\}<+\infty$, and that there exist $Q\geq 0$, $S\in\mathbb{R}$, and real numbers M_3, M_4, M_5 such that (4.19) holds true for every $\xi\in M$.

Then z is continuous and bounded in the spaces shown in the following table (it is intended that $0<\varepsilon\leq Q+S$ when needed).

	z continuous and bounded in
$S>0$	$D(A^{Q+S-\varepsilon})$
$S=0$	$D(A^{Q})$
$S<0$	$D(A^{Q+S-\varepsilon})$

(3) Let us assume in addition that $\psi(t,\xi)\equiv 1$, that $f(t)$ is periodic, and in (4.19) we have that $S<0$.

Then $z(t)$ is continuous and bounded also in the limit space $D(A^{Q+S})$.

Proof First of all, with a variable change we rewrite $\widehat{z}(t,\xi)$ as

$$\widehat{z}(t,\xi) = y(\xi)\int_0^{+\infty} e^{-\eta(\xi)s}\psi(s,\xi)\widehat{f}(t-s,\xi)\,ds.$$

Now we are ready to prove our conclusions. The proof of (4.27) is analogous to the proof of (4.16), the only difference being that now the integral is over $(0,+\infty)$ instead of $(0,t)$. This gives the boundedness of $z(t)$. In order to prove the continuity, we define $z_n(t)$ and $f_n(t)$ as in (4.24), and we prove that $z_n(t)$ uniformly converges to $z(t)$ on every closed interval $[A,B]$.

Arguing as in the proof of Lemma 4.1 we obtain that

$$|A^\alpha(z(t)-z_n(t))| \leq K_{b,c}\cdot M_{\alpha,b,c}\int_0^{+\infty}\min\left\{\frac{1}{s^b},\frac{1}{s^c}\right\}|f(t-s)-f_n(t-s)|\,ds. \tag{4.28}$$

Let us fix now any $\varepsilon>0$, and let us split the integral in the right-hand side of (4.28) as an integral in some bounded interval $[0,T]$ and the integral in $[T,+\infty)$. For every $T\geq 0$ we have that

$$\int_T^{+\infty}\min\left\{\frac{1}{s^b},\frac{1}{s^c}\right\}|f(t-s)-f_n(t-s)|\,ds \leq \|f\|_{L^\infty(\mathbb{R},H)}\int_T^{+\infty}\min\left\{\frac{1}{s^b},\frac{1}{s^c}\right\}ds.$$

Therefore, if T is large enough we have that

$$K_{b,c}\cdot M_{\alpha,b,c}\int_T^{+\infty}\min\left\{\frac{1}{s^b},\frac{1}{s^c}\right\}|f(t-s)-f_n(t-s)|\,ds \leq \frac{\varepsilon}{2} \qquad \forall n\in\mathbb{N}. \tag{4.29}$$

Let us consider now the integral in $[0,T]$. Let us choose p and q in such a way that all assumptions in (4.18) are satisfied. When $t\in[A,B]$ and $s\in[0,T]$ we have that $t-s\in[A-T,B]$, hence

$$\begin{aligned}
&\int_0^T \min\left\{\frac{1}{s^b},\frac{1}{s^c}\right\}|f(t-s)-f_n(t-s)|\,ds\\
&\quad\leq \left(\int_0^T\min\left\{\frac{1}{s^{qb}},\frac{1}{s^{qc}}\right\}ds\right)^{1/q}\left(\int_0^T|f(t-s)-f_n(t-s)|^p\,ds\right)^{1/p}\\
&\quad\leq \left(\int_0^T\min\left\{\frac{1}{s^{qb}},\frac{1}{s^{qc}}\right\}ds\right)^{1/q}\|f-f_n\|_{L^p((A-T,B),H)}.
\end{aligned}$$

Since $f_n\to f$ in $L^p((A-T,B),H)$, when n is large enough we have that

$$K_{b,c}\cdot M_{\alpha,b,c}\int_0^T\min\left\{\frac{1}{s^b},\frac{1}{s^c}\right\}|f(t-s)-f_n(t-s)|\,ds \leq \frac{\varepsilon}{2}. \tag{4.30}$$

Plugging (4.29) and (4.30) into (4.28) we obtain that

$$\sup_{t\in[A,B]}|A^\alpha(z(t)-z_n(t))| \leq \varepsilon$$

provided that n is large enough. This completes the proof of the first statement.

The proof of the second statement is analogous to the proof of statement (3) of Lemma 4.1. The unique difference is that in the case $S<0$ we cannot choose $c=0$, because now we need $c>1$. Thus we choose $c=1-\varepsilon/S$ (which is larger than 1 because $S<0$), and we obtain both continuity and boundedness in the same space, namely $D(A^{Q+S-\varepsilon})$.

It remains to prove statement (3). If $\psi(t,\xi)\equiv 1$, and $f(t)$ is T_0-periodic for some $T_0>0$, then

$$\widehat{z}(t,\xi)=y(\xi)\int_0^{+\infty}e^{-\eta(\xi)s}\widehat{f}(t-s,\xi)\,ds=y(\xi)\sum_{n=0}^{\infty}\int_{nT_0}^{(n+1)T_0}e^{-\eta(\xi)s}\widehat{f}(t-s,\xi)\,ds=$$

$$=y(\xi)\sum_{n=0}^{\infty}e^{-\eta(\xi)nT_0}\int_0^{T_0}e^{-\eta(\xi)s}\widehat{f}(t-s,\xi)\,ds=\frac{y(\xi)}{1-e^{-\eta(\xi)T_0}}\int_0^{T_0}e^{-\eta(\xi)s}\widehat{f}(t-s,\xi)\,ds,$$

hence

$$\begin{aligned}\lambda(\xi)^{Q+S}|\widehat{z}(t,\xi)| &\leq \frac{\lambda(\xi)^{Q+S}|y(\xi)|}{1-e^{-\eta(\xi)T_0}}\int_0^{T_0}e^{-\eta(\xi)s}|\widehat{f}(t-s,\xi)|\,ds\\ &\leq \frac{\lambda(\xi)^{Q+S}|y(\xi)|}{1-e^{-\eta(\xi)T_0}}\int_0^{T_0}|\widehat{f}(t-s,\xi)|\,ds.\end{aligned}$$

Since $S<0$, from (4.19) we deduce that $\eta(\xi)\leq M_6$ for a suitable constant M_6, hence there exists a constant M_7 such that

$$\frac{\eta(\xi)}{1-e^{-\eta(\xi)T_0}}\leq M_7\qquad\forall\xi\in M.$$

Taking once again (4.19) into account, we deduce that there exists a constant M_8 such that

$$\frac{\lambda(\xi)^{Q+S}|y(\xi)|}{1-e^{-\eta(\xi)T_0}}=\lambda(\xi)^{Q}|y(\xi)|\cdot\frac{\lambda(\xi)^{S}}{\eta(\xi)}\cdot\frac{\eta(\xi)}{1-e^{-\eta(\xi)T_0}}\leq M_8\qquad\forall\xi\in M.$$

It follows that

$$|A^{Q+S}z(t)|\leq M_8\int_0^{T_0}|f(t-s)|\,ds\leq M_8\|f\|_{L^\infty((0,T_0),H)}T_0\qquad\forall t\in\mathbb{R},$$

which proves that z is bounded in $D(A^{Q+S})$. The continuity follows from the uniform convergence of the sequence $z_n(t)$, which can be proved exactly in the same way. □

4.5 Proof of Theorem 2.11

First we construct a bounded solution on $\mathbb{R}$ fulfilling the various boundedness and continuity requirements. The formula defining the unique bounded solution for a forced exponentially damped system is our guide since in the (M, μ) formulation, the restriction of the system on states with both components supported in $M_n := \{\xi \in M : \lambda(\xi) \leq n\}$ is exponentially damped, for every integer n. Let $\widehat{f}(t,\xi)$ denote the components of the forcing term $f(t)$. Let us define $\widehat{u}(t,\xi)$ as in the proof of Theorem 2.4 and Theorem 2.7, the only difference being that now the integration is over $(-\infty, t)$ instead of $(0, t)$. For example, in the case of supercritical dissipation, $\widehat{u}(t,\xi)$ is given by

$$-\frac{1}{x_1(\xi)-x_2(\xi)}\int_{-\infty}^{t} e^{-x_1(\xi)(t-s)}\widehat{f}(s,\xi)\,ds + \frac{1}{x_1(\xi)-x_2(\xi)}\int_{-\infty}^{t} e^{-x_2(\xi)(t-s)}\widehat{f}(s,\xi)\,ds.$$

Due to Lemma 4.3, these two integrals define two functions (whose sum is a solution to (1.1)), whose regularity is given by statement (2) of the same lemma. The conclusion is that the solution is continuous and bounded in the same spaces where it was bounded in the case of Theorem 2.7 (in other words, now there is no difference between the spaces where the solution is guaranteed to be bounded and the spaces where the solution is guaranteed to be continuous). The same arguments apply to the critical and subcritical dissipation, and to the regularity of derivatives. This completes the proof of the first statement.

Concerning uniqueness, it is a simple consequence of the fact that the only solution $v \in C_b^0(\mathbb{R}, D(A^{1/2})) \cap C_b^1(\mathbb{R}, H)$ of the homogeneous equation is 0. This follows from the fact that the restriction of the system on states with both components supported in $M_n := \{\xi \in M : \lambda(\xi) \leq n\}$ is exponentially damped, a property implying that the projections of v on all M_n are trivial. It follows immediately from uniqueness of the bounded solution that if f is periodic, the bounded solution is periodic with the same period.

It follows obviously from the previous estimates that uniqueness of the bounded solution u is reinforced by the fact that the norm of u in $C_b^0(\mathbb{R}, D(A^{1/2})) \cap C_b^1(\mathbb{R}, H)$ is bounded by a constant times the norm of f in $L^\infty(\mathbb{R}, H)$. As a consequence, if f is almost periodic with values in H, the bounded solution(u, u') is almost periodic from $\mathbb{R}$ to $D(A^{1/2}) \times H$ with $\exp(u) \subset \exp(f)$. The property of almost periodicity, for the same reason, is in fact valid also with values in all the product spaces where boundedness and continuity has been proved for the first statement.

It remains to prove that in the periodic case with $\sigma > 1$ the solution is continuous and bounded also in the limit space $D(A)$. In this case $u(t)$ is the sum of two terms, whose components are written above. The first one is continuous and bounded in $D(A^{2\sigma-\varepsilon})$ (same proof as in Theorem 2.7), and $2\sigma - \varepsilon \geq 1$ if ε is small enough. The second term fits in the framework of statement (3) of Lemma 4.3 with $Q = \sigma$ and $S = 1 - \sigma < 0$. Thus we obtain that $u(t)$ is continuous and bounded in $D(A^{Q+S})$, which is exactly $D(A)$. □

5 Counterexamples

In this section we exhibit all the counterexamples needed in the proof of Theorem 2.12. To begin with, let us fix some notations. Let $\{\lambda_k\}$ be an unbounded sequence of positive eigenvalues of A, which we can always assume to be increasing. Let $\{e_k\}$ be a sequence of corresponding eigenvectors, which we can always take with unit norm. Up to restricting to the smallest closed vector subspace containing the sequence $\{e_k\}$, we can assume that $\{e_k\}$ is an orthonormal basis of H. This allows to identify any vector $v \in H$ with the sequence $\{v_k\}$ of its components with respect to $\{e_k\}$. Under this identification, $v \in D(A^\alpha)$ if and only if

$$\sum_{k=0}^{\infty} \lambda_k^{2\alpha} v_k^2 < +\infty.$$

Different statements of Theorem 2.12 require different strategies, which now we briefly introduce.

The easiest one is the proof of (2.17), which is the only case where we obtain a solution which lacks in regularity for all positive times. In this case a constant forcing term is enough to produce the required regularity loss.

Then we pass to examples where we prove lack of regularity on a given sequence $\{t_n\}$ (possibly dense) of positive times. The underlying strategy is the same, and it consists in the following two main steps.

- In the first step we produce a solution with the required regularity loss at a given single time $T > 0$. The main point is that this can be done using a forcing term with norm as small as we want, and concentrated on the subspace of H generated by a given countable subset of the eigenvectors $\{e_n\}$.

- In the second step we begin by partitioning $\{e_n\}$ into countably many disjoint countable subsets, which thus generate a countable set of pairwise orthogonal subsets H_n of H. Then for each n we apply the result of the first step in order to obtain a forcing term $f_n(t)$, with values in H_n, which produces a solution with the required regularity loss at time $T = t_n$. Since we can take the norm of $f_n(t)$ as small as we want, we can arrange things so that the series with general term $f_n(t)$ converges. The sum $f(t)$ is the required forcing term giving rise to a solution $u(t)$ which lacks in regularity at all times of the sequence $\{t_n\}$.

Thanks to this strategy, the proof of statements (1) and (2), and of part (2.18) of statement (3), is reduced to the verification of the first step. In all cases, this in turn requires a forcing term defined as the sum of a suitable series of forcing terms, whose construction and convergence differs from case to case. When $\sigma = 0$ the series converges because the series of the norms converges. In the other two cases, the series converges because its terms have norms tending to 0 and disjoint supports, namely the different components are "activated" one by one.

Finally, we produce the unbounded solution required by statement (4). Once again, the strategy is twofold. We show that, provided that t is large enough, $|Au(t)|$ can be made larger than a given constant, even if the forcing term is smaller than a given constant. Then we conclude with an argument similar to the second step described above.

All proofs begin with careful asymptotic estimates on solutions to the family of ordinary differential equations

$$u_\lambda''(t) + 2\delta\lambda^\sigma u_\lambda'(t) + \lambda u_\lambda(t) = f_\lambda(t) \qquad \forall t \geq 0, \tag{5.1}$$

with a suitable family of forcing terms $\{f_\lambda(t)\}$, and null initial data

$$u_\lambda(0) = u_\lambda'(0) = 0. \tag{5.2}$$

In turn, the asymptotic behavior of these solutions depends on the asymptotic behavior of the roots of the characteristic polynomial (1.4). In analogy with section 3.2, these roots are denoted by $-x_{1,\lambda}$ and $-x_{2,\lambda}$. Their expressions and asymptotic behavior are the same stated in section 3.2, just with λ instead of $\lambda(\xi)$. In particular, if λ is large enough, the picture is the following.

- In the subcritical regime the characteristic roots are complex conjugate numbers of the form $-x_{1,\lambda} = -a_\lambda + ib_\lambda$ and $-x_{2,\lambda} = -a_\lambda - ib_\lambda$, with

$$a_\lambda := \delta\lambda^\sigma, \qquad b_\lambda := (\lambda - \delta^2\lambda^{2\sigma})^{1/2}, \tag{5.3}$$

  and as a consequence

$$\lim_{\lambda\to+\infty} \frac{b_\lambda}{\lambda^{1/2}} = \begin{cases} 1 & \text{if } \sigma < 1/2, \\ (1-\delta^2)^{1/2} & \text{if } \sigma = 1/2. \end{cases} \tag{5.4}$$

- In the critical regime it turns out that $-x_{1,\lambda} = -x_{2,\lambda} = -\lambda^{1/2}$.

- In the supercritical regime the characteristic roots are real numbers with

$$\lim_{\lambda\to+\infty} \frac{x_{1,\lambda}}{\lambda^\sigma} = \lim_{\lambda\to+\infty} \frac{\lambda^{1-\sigma}}{x_{2,\lambda}} = \begin{cases} 2\delta & \text{if } \sigma > 1/2, \\ \delta + (\delta^2-1)^{1/2} & \text{if } \sigma = 1/2. \end{cases} \tag{5.5}$$

5.1 Proof of (2.17) in statement (3)

Let us consider problem (5.1)–(5.2) with $f_\lambda(t) \equiv 1$. Since we are in the case $\sigma \geq 1$, when λ is large enough the solution turns out to be

$$u_\lambda(t) = \frac{1}{\lambda} + \frac{1}{\lambda} \cdot \frac{1}{x_{1,\lambda} - x_{2,\lambda}} \left(x_{2,\lambda} e^{-x_{1,\lambda}t} - x_{1,\lambda} e^{-x_{2,\lambda}t} \right). \tag{5.6}$$

Thanks to (5.5), it is not difficult to show that

$$\lim_{\lambda\to+\infty} \lambda^\sigma u_\lambda(t) = \begin{cases} t/(2\delta) & \text{if } \sigma > 1 \\ 1 - e^{-t/(2\delta)} & \text{if } \sigma = 1 \end{cases} \tag{5.7}$$

for all $t \geq 0$. In both cases, the limit is finite and different from 0 when $t > 0$.

Now let us choose a sequence a_k such that

$$\sum_{k=0}^{\infty} a_k^2 < +\infty \tag{5.8}$$

and

$$\sum_{k=0}^{\infty} \lambda_k^{2\varepsilon} a_k^2 = +\infty \qquad \forall \varepsilon > 0. \tag{5.9}$$

Let us consider the constant forcing term

$$f(t) := \sum_{k=0}^{\infty} a_k e_k.$$

The series converges because of (5.8). The corresponding solution of (2.9)–(2.10) is

$$u(t) = \sum_{k=0}^{\infty} a_k u_{\lambda_k}(t) e_k,$$

and in particular

$$\left|A^{\sigma+\varepsilon} u(t)\right|^2 = \sum_{k=0}^{\infty} \lambda_k^{2\sigma+2\varepsilon} a_k^2 |u_{\lambda_k}(t)|^2.$$

Due to (5.7), this series is equivalent to the series in (5.9) for every $t > 0$, hence it is divergent for every $t > 0$ and every $\varepsilon > 0$. This proves (2.17).

5.2 Proof of statement (1)

ODE estimates In the subcritical case the roots of the characteristic polynomial are complex conjugate numbers $-a_\lambda \pm i b_\lambda$, at least when λ is large enough. For every such λ and every $T > 0$, we consider problem (5.1)–(5.2) with

$$f_\lambda(t) := \cos\left(b_\lambda(T-t) - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left[\cos\left(b_\lambda(T-t)\right) + \sin\left(b_\lambda(T-t)\right)\right]. \tag{5.10}$$

We claim that

$$\lim_{\lambda\to+\infty} \lambda^{1/2} u_\lambda(T) = \lim_{\lambda\to+\infty} u'_\lambda(T) = \frac{\sqrt{2}}{4} \cdot \frac{1}{\delta}\left(1 - e^{-\delta T}\right). \tag{5.11}$$

Indeed the solution of (5.1)–(5.2) is

$$u_\lambda(t) = \frac{1}{b_\lambda} \int_0^t e^{-a_\lambda(t-s)} \sin(b_\lambda(t-s)) f_\lambda(s)\, ds, \tag{5.12}$$

and its derivative $u_\lambda'(t)$ is

$$u_\lambda'(t) = -a_\lambda u_\lambda(t) + \int_0^t e^{-a_\lambda(t-s)} \cos(b_\lambda(t-s)) f_\lambda(s)\, ds. \tag{5.13}$$

In the special case where $\sigma = 0$, and $f_\lambda(t)$ is given by (5.10), with the variable change $x = T - s$ we obtain that

$$\lambda^{1/2} u_\lambda(T) = \frac{\sqrt{2}}{2} \frac{\lambda^{1/2}}{b_\lambda} \int_0^T e^{-\delta x} \left(\sin^2(b_\lambda x) + \sin(b_\lambda x)\cos(b_\lambda x) \right)\, dx, \tag{5.14}$$

$$u_\lambda'(T) = -\delta u_\lambda(T) + \frac{\sqrt{2}}{2} \int_0^T e^{-\delta x} \left(\cos^2(b_\lambda x) + \sin(b_\lambda x)\cos(b_\lambda x) \right)\, dx. \tag{5.15}$$

Now we have to compute the limits as $\lambda \to +\infty$. The coefficient $\lambda^{1/2} \cdot b_\lambda^{-1}$ tends to 1 because of (5.4). As for the integrals, the coefficient b_λ in the trigonometric terms tends to $+\infty$. This produces a homogenization effect, so that

$$\lim_{\lambda \to +\infty} \int_0^T e^{-\delta x} \sin^2(b_\lambda x)\, dx = \lim_{\lambda \to +\infty} \int_0^T e^{-\delta x} \cos^2(b_\lambda x)\, dx = \frac{1}{2} \int_0^T e^{-\delta x}\, dx,$$

$$\lim_{\lambda \to +\infty} \int_0^T e^{-\delta x} \sin(b_\lambda x)\cos(b_\lambda x)\, dx = 0.$$

Plugging these limits into (5.14) and (5.15), we obtain (5.11).

Lack of regularity for a given positive time Let $T > 0$, let $\eta > 0$, let $\nu_k \to +\infty$ be any unbounded sequence of eigenvalues of A, let $\{\widehat{e}_k\}$ be a corresponding sequence of orthonormal eigenvectors, and let $\widehat{H}$ be the subspace of H generated by $\{\widehat{e}_k\}$. We claim that there exists a function $f \in C_b^0([0, +\infty), H)$ such that

$$|f(t)| \leq \eta \qquad \forall t \geq 0, \tag{5.16}$$

$$f(t) \in \widehat{H} \qquad \forall t \geq 0, \tag{5.17}$$

and such that the corresponding solution u of (2.9)–(2.10) satisfies

$$u(T) \notin D(A^{1/2+\varepsilon}) \quad \text{and} \quad u'(T) \notin D(A^\varepsilon) \qquad \forall \varepsilon > 0. \tag{5.18}$$

Indeed let us choose a sequence a_k such that

$$\sum_{k=0}^{\infty} a_k^2 \leq \eta \tag{5.19}$$

and

$$\sum_{k=0}^{\infty} \nu_k^{2\varepsilon} a_k^2 = +\infty \qquad \forall \varepsilon > 0. \tag{5.20}$$

Let us consider the family of forcing terms $f_\lambda(t)$ defined in (5.10), and the corresponding solutions $u_\lambda(t)$ of problem (5.1)–(5.2). Let us set

$$f(t) := \sum_{k=0}^{\infty} a_k f_{\nu_k}(t)\widehat{e}_k.$$

Due to (5.19), the series converges to a continuous function $f : [0, +\infty) \to H$ satisfying both (5.16) and (5.17). The corresponding solution of (2.9)–(2.10) is of course

$$u(t) = \sum_{k=0}^{\infty} a_k u_{\nu_k}(t)\widehat{e}_k,$$

and in particular

$$\left|A^{1/2+\varepsilon}u(T)\right|^2 = \sum_{k=0}^{\infty} \nu_k^{1+2\varepsilon} a_k^2 |u_{\nu_k}(T)|^2,$$

$$|A^{\varepsilon}u'(T)|^2 = \sum_{k=0}^{\infty} \nu_k^{2\varepsilon} a_k^2 |u'_{\nu_k}(T)|^2.$$

Due to (5.11), both series are equivalent to the series in (5.20), hence they diverge for every $\varepsilon > 0$. This proves (5.18).

Lack of regularity for a given sequence of times We are now ready to prove the conclusions of statement (1) of Theorem 2.12. To this end, we partition the given unbounded sequence $\{\lambda_k\}$ of eigenvalues of A into countably many disjoint (unbounded) subsequences. For example, the n-th subsequence could be that of the form $\lambda_{2^n(2k+1)}$.

Let $\{t_n\} \subseteq (0, +\infty)$ be any sequence of positive times. For every $n \in \mathbb{N}$, we apply the construction of the previous paragraph with $T := t_n$, $\eta := 2^{-n}$, and $\{\nu_k\}$ equal to the n-th subsequence of $\{\lambda_k\}$. We call H_n the subspace generated by the corresponding eigenvectors of A.

We obtain an external force $f_n \in C_b^0([0, +\infty), H)$ such that $f_n(t) \in H_n$ and $|f_n(t)| \leq 2^{-n}$ for every $t \geq 0$, and such that the corresponding solution $u_n(t)$ of (2.9)–(2.10) takes its values in H_n and satisfies

$$u_n(t_n) \notin D(A^{1/2+\varepsilon}) \quad \text{and} \quad u'_n(t_n) \notin D(A^{\varepsilon}) \qquad \forall \varepsilon > 0. \tag{5.21}$$

Finally, we define

$$f(t) := \sum_{k=0}^{\infty} f_n(t), \qquad\qquad u(t) := \sum_{k=0}^{\infty} u_n(t). \tag{5.22}$$

It is easy to see that the first series converges to a bounded continuous function $f(t)$, and that $u(t)$ is the corresponding solution of (2.9)–(2.10).

Since the spaces H_n are pairwise orthogonal, $u(t)$ cannot be more regular than its projections $u_n(t)$ into H_n, and therefore both (2.15) and (2.16) follow from (5.21).

5.3 Proof of statement (2)

Blow-up triples We say that a triple $(\sigma, \sigma_0, \sigma_1)$ of positive real numbers satisfies the blow-up condition if there exists families $\{\tau_\lambda\} \subseteq (0, +\infty)$ and $\{f_\lambda\} \subseteq C_b^0([0, +\infty), H)$ such that

$$|f_\lambda(t)| \leq 1 \qquad \forall t \geq 0, \quad \forall \lambda \geq 0, \tag{5.23}$$

$$\lim_{\lambda \to +\infty} \tau_\lambda = 0, \tag{5.24}$$

and the corresponding solutions $u_\lambda(t)$ of (5.1)–(5.2) satisfy

$$\lim_{\lambda \to +\infty} \lambda^{\sigma_0} |u_\lambda(\tau_\lambda)| \in (0, +\infty) \qquad \text{and} \qquad \lim_{\lambda \to +\infty} \lambda^{\sigma_1} |u'_\lambda(\tau_\lambda)| \in (0, +\infty). \tag{5.25}$$

We claim that the triples $(\sigma, \min\{\sigma + 1/2, 1\}, \sigma)$ satisfy the blow-up condition for every $\sigma \in (0, 1)$. The verification of this fact requires several cases.

In the case of supercritical dissipation, namely when $\sigma \in (1/2, 1)$ or $\sigma = 1/2$ and $\delta > 1$, we can take $f_\lambda(t) \equiv 1$, so that for λ large enough the solution of (5.1)–(5.2) is given by (5.6). Thus, if we take $\tau_\lambda := (x_{2,\lambda})^{-1}$ and $D := (\delta + \sqrt{\delta^2 - 1})^2$, from (5.5) we obtain that

$$\lim_{\lambda \to +\infty} \lambda u_\lambda(\tau_\lambda) = \begin{cases} 1 - e^{-1} & \text{if } \sigma > 1/2 \\ 1 + (D-1)^{-1}(e^{-D} - De^{-1}) & \text{if } \sigma = 1/2 \text{ and } \delta > 1, \end{cases}$$

and

$$\lim_{\lambda \to +\infty} \lambda^\sigma u'_\lambda(\tau_\lambda) = \begin{cases} (2\delta)^{-1} e^{-1} & \text{if } \sigma > 1/2 \\ (2\sqrt{\delta^2 - 1})^{-1}(e^{-1} - e^{-D}) & \text{if } \sigma = 1/2 \text{ and } \delta > 1. \end{cases}$$

It is not difficult to see that these limits are always finite and different from 0, which proves that $(\sigma, 1, \sigma)$ is a blow up triple in the case of supercritical dissipation with $\sigma < 1$ (the latter condition guarantees that $\tau_\lambda \to 0^+$).

In the case of critical dissipation, namely when $\sigma = 1/2$ and $\delta = 1$, we can take once again $f_\lambda(t) \equiv 1$. The solution of (5.1)–(5.2) is

$$u_\lambda(t) = \frac{1}{\lambda} - \frac{1}{\lambda} \left(\lambda^{1/2} t + 1 \right) e^{-\lambda^{1/2} t}.$$

Setting $\tau_\lambda := \lambda^{-1/2}$ we have that

$$\lim_{\lambda \to +\infty} \lambda u_\lambda(\tau_\lambda) = 1 - 2e^{-1} \qquad \text{and} \qquad \lim_{\lambda \to +\infty} \lambda^{1/2} u'_\lambda(\tau_\lambda) = e^{-1},$$

which proves that $(1/2, 1, 1/2)$ is a blow-up triple in the critical case.

In the case of subcritical dissipation, namely when $\sigma < 1/2$, or $\sigma = 1/2$ and $\delta \in (0, 1)$, we need to use a family of oscillating forcing terms. Let us assume that λ is large enough so that the roots of the characteristic polynomial are of the form $-a_\lambda \pm i b_\lambda$, and let us set

$$\tau_\lambda := \frac{W}{a_\lambda}, \qquad f_\lambda(t) := \sin(b_\lambda(\tau_\lambda - t) + \psi), \tag{5.26}$$

where $W > 0$ and $\psi \in \mathbb{R}$ are parameters to be chosen in the sequel.

As we already observed, when the roots of the characteristic polynomial are complex conjugate numbers the solution $u_\lambda(t)$ of (5.1)–(5.2) and its derivative $u'_\lambda(t)$ are given by (5.12) and (5.13), respectively. Keeping (5.26) into account, with the variable change $x = a_\lambda(\tau_\lambda - s)$ we obtain that

$$\lambda^{\sigma+1/2} u_\lambda(\tau_\lambda) = \frac{\lambda^{\sigma+1/2}}{a_\lambda b_\lambda} \left(\cos\psi \cdot S_\lambda + \sin\psi \cdot M_\lambda\right), \tag{5.27}$$

$$\lambda^{\sigma} u'_\lambda(\tau_\lambda) = -\frac{\lambda^{\sigma}}{b_\lambda} \left(\cos\psi \cdot S_\lambda + \sin\psi \cdot M_\lambda\right) + \frac{1}{\delta} \left(\cos\psi \cdot M_\lambda + \sin\psi \cdot C_\lambda\right), \tag{5.28}$$

where

$$S_\lambda := \int_0^W e^{-x} \sin^2\left(\frac{b_\lambda}{a_\lambda} x\right) dx, \qquad C_\lambda := \int_0^W e^{-x} \cos^2\left(\frac{b_\lambda}{a_\lambda} x\right) dx,$$

$$M_\lambda := \int_0^W e^{-x} \sin\left(\frac{b_\lambda}{a_\lambda} x\right) \cos\left(\frac{b_\lambda}{a_\lambda} x\right) dx.$$

Now we have to compute the limits as $\lambda \to +\infty$. The limits of the coefficients $\lambda^{\sigma+1/2}(a_\lambda b_\lambda)^{-1}$ and $\lambda^\sigma b_\lambda^{-1}$ follow easily from the explicit expressions (5.3). As for the integrals, we have to distinguish two cases.

When $\sigma < 1/2$, the coefficient $b_\lambda a_\lambda^{-1}$ in the trigonometric functions tends to $+\infty$. This produces an homogenization effect in the integrals, so that

$$\lim_{\lambda\to+\infty} S_\lambda = \lim_{\lambda\to+\infty} C_\lambda = \frac{1}{2}\int_0^W e^{-x}\, dx = \frac{1}{2}\left(1 - \frac{1}{e^W}\right), \qquad \lim_{\lambda\to+\infty} M_\lambda = 0.$$

Plugging these limits into (5.27) and (5.28), we obtain that

$$\lim_{\lambda\to+\infty} \lambda^{\sigma+1/2} u_\lambda(\tau_\lambda) = \frac{1}{2\delta}\left(1 - \frac{1}{e^W}\right)\cos\psi, \qquad \lim_{\lambda\to+\infty} \lambda^{\sigma} u'_\lambda(\tau_\lambda) = \frac{1}{2\delta}\left(1 - \frac{1}{e^W}\right)\sin\psi.$$

Both limits are different from 0 for many values of W and ψ (for example $W = 1$ and $\psi = \pi/4$), and this proves that $(\sigma, \sigma + 1/2, \sigma)$ is a blow-up triple for $\sigma \in (0, 1/2)$.

When $\sigma = 1/2$, we have that both $\lambda^{\sigma+1/2} u_\lambda(\tau_\lambda)$ and $\lambda^\sigma u'_\lambda(\tau_\lambda)$ do not depend on λ. In particular, for $\psi = \pi/2$ we obtain that

$$\lambda^{\sigma+1/2} u_\lambda(\tau_\lambda) = \frac{1}{\delta\sqrt{1-\delta^2}} \int_0^W e^{-x} \sin(Dx)\cos(Dx)\, dx,$$

$$\lambda^{\sigma} u'_\lambda(\tau_\lambda) = \frac{1}{\delta}\int_0^W e^{-x}\cos^2(Dx)\, dx - \frac{1}{\sqrt{1-\delta^2}}\int_0^W e^{-x}\sin(Dx)\cos(Dx)\, dx,$$

where $D := \delta^{-1}\sqrt{1-\delta^2}$. Now it is easy to see that both expressions are positive when $W > 0$ is small enough. This proves that $(1/2, 1, 1/2)$ is a blow-up triple also in the subcritical case.

Growth of one component in a short time Let $0 \leq A < T$, let $(\sigma, \sigma_0, \sigma_1)$ be a triple of positive real numbers satisfying the blow-up condition, and let c_0 and c_1 be the two limits in (5.25). Due to the definition of blow-up triple, there exists $\Lambda \geq 0$ such that the following conditions

$$\tau_\lambda \leq T - A, \qquad \lambda^{\sigma_0}|u_\lambda(\tau_\lambda)| \geq \frac{3}{4}c_0, \qquad \lambda^{\sigma_1}|u'_\lambda(\tau_\lambda)| \geq \frac{3}{4}c_1$$

hold true for every $\lambda \geq \Lambda$.

We claim that, for every $\lambda \geq \Lambda$, there exists $B_\lambda \in (A, T)$ and a continuous function $g_\lambda : [0, +\infty) \to \mathbb{R}$ such that

$$|g_\lambda(t)| \leq 1 \qquad \forall t \geq 0, \tag{5.29}$$

$$g_\lambda(t) = 0 \qquad \forall t \in [0, A] \cup [B_\lambda, +\infty), \tag{5.30}$$

and the unique solution $u_\lambda(t)$ of (5.1)–(5.2) with $g_\lambda(t)$ instead of $f_\lambda(t)$ satisfies

$$\lambda^{\sigma_0}|u_\lambda(T)| \geq \frac{c_0}{2} \qquad \text{and} \qquad \lambda^{\sigma_1}|u'_\lambda(T)| \geq \frac{c_1}{2}. \tag{5.31}$$

In order to prove this result, we begin by defining a piecewise continuous function $\varphi_\lambda : [0, +\infty) \to \mathbb{R}$ as

$$\varphi_\lambda(t) := \begin{cases} f_\lambda(t - (T - \tau_\lambda)) & \text{if } t \in [T - \tau_\lambda, T], \\ 0 & \text{otherwise,} \end{cases}$$

where $f_\lambda(t)$ is the function which appears in the blow-up condition. It is easy to see that the solution of the ordinary differential equation

$$v''_\lambda(t) + 2\delta\lambda^\sigma v'_\lambda(t) + \lambda v_\lambda(t) = \varphi_\lambda(t), \tag{5.32}$$

with null initial data $v_\lambda(0) = v'_\lambda(0) = 0$, is given by

$$v_\lambda(t) = \begin{cases} 0 & \text{if } t \in [0, T - \tau_\lambda], \\ u_\lambda(t - (T - \tau_\lambda)) & \text{if } t \in [T - \tau_\lambda, T], \end{cases}$$

so that

$$\lambda^{\sigma_0}|v_\lambda(T)| = \lambda^{\sigma_0}|u_\lambda(\tau_\lambda)| \geq \frac{3}{4}c_0 \qquad \text{and} \qquad \lambda^{\sigma_1}|v'_\lambda(T)| = \lambda^{\sigma_1}|u'_\lambda(\tau_\lambda)| \geq \frac{3}{4}c_1. \tag{5.33}$$

Now we approximate $\varphi_\lambda(t)$ with suitable continuous functions $\varphi_{\lambda,\varepsilon}(t)$. To this end, for every $\varepsilon \in (0, \tau_\lambda/4)$ we consider a cut-off function $\psi_\varepsilon : [0, +\infty) \to [0, 1]$ with

$$\psi_\varepsilon(t) = 0 \qquad \forall t \in [0, T - \tau_\lambda + \varepsilon] \cup [T - \varepsilon, +\infty],$$

$$\psi_\varepsilon(t) = 1 \qquad \forall t \in [T - \tau_\lambda + 2\varepsilon, T - 2\varepsilon],$$

and then we set $\varphi_{\lambda,\varepsilon}(t) := \psi_\varepsilon(t) \cdot \varphi_\lambda(t)$. Let $v_{\lambda,\varepsilon}(t)$ be the solution of (5.32), with $\varphi_{\lambda,\varepsilon}(t)$ instead of $\varphi_\lambda(t)$, and null initial data.

It is easy to see that $\varphi_{\lambda,\varepsilon}(t) \to \varphi_\lambda(t)$ in $L^p((0,+\infty))$ for every $p < +\infty$. This is more than enough to guarantee that $v_{\lambda,\varepsilon} \to v_\lambda$ in the energy space, hence

$$\lim_{\varepsilon\to 0^+} v_{\lambda,\varepsilon}(T) = v_\lambda(T) \qquad \text{and} \qquad \lim_{\varepsilon\to 0^+} v'_{\lambda,\varepsilon}(T) = v'_\lambda(T).$$

Keeping (5.33) into account, for $\varepsilon(\lambda)$ small enough we have that

$$\lambda^{\sigma_0}|v_{\lambda,\varepsilon(\lambda)}(T)| \geq \frac{c_0}{2} \qquad \text{and} \qquad \lambda^{\sigma_1}|v'_{\lambda,\varepsilon(\lambda)}(T)| \geq \frac{c_1}{2}.$$

Therefore, our requirements (5.29) through (5.31) are fulfilled by taking $g_\lambda(t) := \varphi_{\lambda,\varepsilon(\lambda)}(t)$ and $B_\lambda := T - \varepsilon(\lambda)$.

Lack of regularity for a given positive time Let $T > 0$, let $\eta > 0$, let $(\sigma, \sigma_0, \sigma_1)$ be a triple of positive real numbers satisfying the blow-up condition, let $\{\nu_k\}$ be any unbounded sequence of eigenvalues of A, let $\{\widehat{e}_k\}$ be a corresponding sequence of orthonormal eigenvectors, and let $\widehat{H}$ be the subspace of H generated by $\{\widehat{e}_k\}$.

We claim that there exists a function $f \in C^0_b([0,+\infty), H)$ satisfying (5.16) and (5.17), and such that the corresponding solution $u(t)$ of (2.9)–(2.10) satisfies

$$u(T) \notin D(A^{\sigma_0}) \quad \text{and} \quad u'(T) \notin D(A^{\sigma_1}). \tag{5.34}$$

In order to prove this claim, we begin by choosing a sequence $\{\omega_n\} \subseteq [0,1]$ of positive real numbers such that

$$\lim_{n\to+\infty} \omega_n = 0 \qquad \text{and} \qquad \sum_{n=0}^{\infty} \omega_n^2 = +\infty.$$

Then we choose an increasing sequence $\{k_n\}$ of positive integers, an increasing sequence $\{B_n\} \subseteq [0,T)$ of times, and a sequence $f_n : [0,+\infty) \to [0,1]$ of continuous functions such that $f_n(t) = 0$ for every $t \notin (B_{n-1}, B_n)$ (hence with disjoint supports), and such that the corresponding solutions $u_n(t)$ of (2.9)–(2.10) satisfy

$$\nu_{k_n}^{\sigma_0}|u_n(T)| \geq \frac{c_0}{2} \qquad \text{and} \qquad \nu_{k_n}^{\sigma_1}|u'_n(T)| \geq \frac{c_1}{2}.$$

As soon as we have such sequences, our claim follows with

$$f(t) := \eta \sum_{n=1}^{\infty} \omega_n f_n(t) \widehat{e}_{k_n}.$$

Indeed the series converges because its terms have disjoint supports and their norm goes to 0. The sum $f(t)$ satisfies (5.17) for trivial reasons, and satisfies (5.16) because all terms do and have disjoint supports. The corresponding solution $u(t)$ of (2.9)–(2.10) is clearly

$$u(t) := \eta \sum_{n=1}^{\infty} \omega_n u_n(t) \widehat{e}_{k_n},$$

so that

$$|A^{\sigma_0}u(T)|^2 = \eta^2 \sum_{n=1}^{\infty} \omega_n^2 \nu_{k_n}^{2\sigma_0} |u_n(T)|^2 \geq \eta^2 \frac{c_0^2}{4} \sum_{n=1}^{\infty} \omega_n^2 = +\infty,$$

$$|A^{\sigma_1}u'(T)|^2 = \eta^2 \sum_{n=1}^{\infty} \omega_n^2 \nu_{k_n}^{2\sigma_1} |u_n'(T)|^2 \geq \eta^2 \frac{c_1^2}{4} \sum_{n=1}^{\infty} \omega_n^2 = +\infty,$$

which proves (5.34).

In order to define the sequences we need, we repeatedly apply the result of the previous paragraph. First of all, we apply it with $A := 0$ and we obtain a value Λ_1 such that for every $\lambda \geq \Lambda_1$ there exists $B_\lambda \in (0, T)$ and $g_\lambda : [0, +\infty) \to [0, 1]$ satisfying (5.29) through (5.31). Now we choose a positive integer k_1 such that $\nu_{k_1} \geq \Lambda_1$, and then we set $B_1 := B_{\nu_{k_1}}$ and $f_1(t) := g_{\nu_{k_1}}(t)$.

Then we proceed by induction. Let us assume that k_{n-1}, B_{n-1} and $f_{n-1}(t)$ have been defined, and let us apply the result of the previous paragraph with $A := B_{n-1}$. Once again we obtain a value Λ_n such that for $\lambda \geq \Lambda_n$ there exists $B_\lambda \in (B_{n-1}, T)$ and $g_\lambda : [0, +\infty) \to [0, 1]$ satisfying (5.29) through (5.31). Now we choose a positive integer $k_n > k_{n-1}$ such that $\nu_{k_n} \geq \Lambda_n$, and then we set $B_n := B_{\nu_{k_n}}$ and $f_n(t) := g_{\nu_{k_n}}(t)$.

Lack of regularity for a given sequence of times In the previous paragraph we produced a solution of (2.9)–(2.10) which has the required regularity loss at a given time $T > 0$. This solution has been constructed using a given unbounded sequence of eigenvalues of A, and an external force as small as we want. Now we need to produce the same regularity loss for all times in a given sequence $\{t_n\}$.

The procedure is exactly the same as in the last paragraph of Section 5.2. We partition the sequence $\{\lambda_k\}$ into countably many disjoint subsequences, and for each $n \in \mathbb{N}$ we apply the previous result in order to obtain an external force $f_n(t)$, with norm less than or equal to 2^{-n}, for which the corresponding solution $u_n(t)$ of (2.9)–(2.10) has the required regularity loss at time t_n. The series of these external forces converges to an external force with all the properties we need. We refer to the last paragraph of Section 5.2 for the details.

5.4 Proof of (2.18) in statement (3)

The construction is analogous to the one for statement (2), the only difference being that now we have to produce a regularity loss only for the derivative.

We begin by saying that a pair (σ, σ_1) of positive real numbers satisfies the blow-up condition for the derivative if there exist families $\{\tau_\lambda\} \subseteq (0, +\infty)$ and $\{f_\lambda\} \subseteq C_b^0([0, +\infty), H)$ satisfying (5.23) and (5.24), and such that the corresponding solutions $u_\lambda(t)$ of (5.1)–(5.2) satisfy the second relation in (5.25).

Then we prove that the pairs (σ, σ) satisfy the blow-up condition for the derivative for every $\sigma \geq 1$. Indeed, if we take $f_\lambda(t) \equiv 1$, for λ large enough the solution of

(5.1)–(5.2) is given by (5.6), hence its derivative is

$$u'_\lambda(t) = \frac{1}{x_{1,\lambda} - x_{2,\lambda}} \left(e^{-x_{2,\lambda}t} - e^{-x_{1,\lambda}t}\right).$$

At this point the conclusion easily follows with $\tau_\lambda := (x_{1,\lambda})^{-1}$.

From now on the argument is exactly the same as in the proof of statement (2). First we obtain the growth of (the derivative of) one component in an arbitrary short time, then the lack of regularity at a given positive time, and finally the lack of regularity at a given sequence of positive times.

5.5 Proof of statement (4)

Preliminary integral estimate Let $\{\alpha_n\}$ be any sequence of positive real numbers such that $\alpha_n \to 0^+$. We claim that there exist an increasing sequence $\{k_n\}$ of positive integers, and an increasing sequence $\{T_n\} \subseteq [0, +\infty)$ with $T_0 = 0$ such that

$$\alpha_{k_n} \int_{T_{n-1}}^{T_n} e^{-\alpha_{k_n} x} dx \geq \frac{1}{e}\left(1 - \frac{1}{e}\right) \qquad \forall n \geq 1. \tag{5.35}$$

In order to prove this claim, we set $k_1 := 1$, and then by induction we choose $k_{n+1} > k_n$ in such a way that

$$\frac{1}{\alpha_{k_{n+1}}} \geq \frac{1}{\alpha_{k_1}} + \ldots + \frac{1}{\alpha_{k_n}} \qquad \forall n \geq 1. \tag{5.36}$$

Such a choice is possible because $\alpha_n \to 0^+$. Then we set $T_0 := 0$ and

$$T_n := \frac{1}{\alpha_{k_1}} + \ldots + \frac{1}{\alpha_{k_n}} \qquad \forall n \geq 1. \tag{5.37}$$

Due to (5.36) and (5.37) we have that $\alpha_{k_n} T_{n-1} \leq 1$ and $\alpha_{k_n}(T_n - T_{n-1}) = 1$, hence

$$\alpha_{k_n} \int_{T_{n-1}}^{T_n} e^{-\alpha_{k_n} x} dx = e^{-\alpha_{k_n} T_{n-1}} \left(1 - e^{-\alpha_{k_n}(T_n - T_{n-1})}\right) \geq \frac{1}{e}\left(1 - \frac{1}{e}\right)$$

for every $n \geq 1$, as required.

Passing any given threshold Let $M \geq 0$, let $\eta > 0$, let $\{\nu_n\}$ be any unbounded sequence of eigenvalues of A, and let $\{\widehat{e}_n\}$ be a corresponding sequence of orthonormal eigenvectors.

We claim that there exist $T > 0$, a subspace $\widehat{H}$ of H generated by a finite subset of $\{\widehat{e}_n\}$, and a function $f \in C^0_b([0, +\infty), H)$ satisfying (5.16) and (5.17), and such that the corresponding solution $u(t)$ of (2.9)–(2.10) satisfies

$$|Au(T)|^2 \geq M. \tag{5.38}$$

In other words, the external force is as small as we want and concentrated on a finite number of components, but $|Au(T)|$ exceeds a given threshold.

In order to prove the claim, let us consider the roots of the characteristic polynomial (1.4) with $\lambda = \nu_n$, and let us set for simplicity $x_{1,n} := x_{1,\nu_n}$ and $x_{2,n} := x_{2,\nu_n}$. We always assume that ν_n is large enough so that these roots are distinct real numbers. We also assume also that ν_n is large enough so that

$$\frac{\nu_n}{x_{1,n} - x_{2,n}} \leq 1, \qquad x_{1,n} \geq 1, \qquad \frac{\nu_n}{x_{1,n} - x_{2,n}} \cdot \frac{1}{x_{2,n}} \geq \frac{1}{2}. \tag{5.39}$$

This is clearly possible because of (5.5). Since $\sigma > 1$, we have also that $x_{2,n} \to 0^+$, and therefore we can apply the result of the previous paragraph with $\alpha_n := x_{2,n}$. Let k_n and T_n be the corresponding sequences for which (5.35) holds true.

Let us consider the piecewise constant function $\psi : [0, +\infty) \to \mathbb{R}$ defined by $\psi(t) := \widehat{e}_{k_n}$ for every $t \in [T_{n-1}, T_n)$ and every $n \in \mathbb{N}$. Let us choose a positive integer N large enough so that

$$\eta^2 \frac{1}{4e^2} \left(1 - \frac{1}{e}\right)^2 \cdot N \geq (2M + \eta)^2.$$

Let us set $T := T_N$ and

$$g(t) := \begin{cases} \eta\psi(T - t) & \text{if } t \in [0, T], \\ 0 & \text{if } t > T. \end{cases}$$

Let $\widehat{H}$ be the subspace of H generated by $\{\widehat{e}_{k_1}, \ldots, \widehat{e}_{k_N}\}$. The function $g(t)$ is not continuous, but it satisfies (5.16) and (5.17). The corresponding solution of (2.9)–(2.10) is

$$u(t) := v(t) + w(t) := \eta \sum_{n=1}^{N} v_n(t)\widehat{e}_{k_n} + \eta \sum_{n=1}^{N} w_n(t)\widehat{e}_{k_n},$$

with

$$v_n(t) := -\frac{1}{x_{1,k_n} - x_{2,k_n}} \int_0^t e^{-x_{1,k_n}(t-s)} \psi_{k_n}(T - s)\, ds,$$

$$w_n(t) := \frac{1}{x_{1,k_n} - x_{2,k_n}} \int_0^t e^{-x_{2,k_n}(t-s)} \psi_{k_n}(T - s)\, ds,$$

where of course $\psi_{k_n}(t)$ denotes the component of $\psi(t)$ with respect to $\widehat{e}_{k_n}$.

Let us estimate $v(T)$ and $w(T)$ separately. In order to estimate $v(T)$, we argue as in the proof of Lemma 4.1. We consider T as a parameter and we introduce the vector

$$\varphi(s) := \sum_{n=1}^{N} \varphi_n(s)\widehat{e}_{k_n}$$

with components

$$\varphi_n(s) := -\frac{\nu_{k_n}}{x_{1,k_n} - x_{2,k_n}} e^{-x_{1,k_n}(T-s)} \psi_{k_n}(T - s),$$

so that

$$|Av(T)| = \eta \left| \int_0^T \varphi(s)\, ds \right| \leq \eta \int_0^T |\varphi(s)|\, ds.$$

In order to estimate $|\varphi(s)|$, we exploit the first two conditions in (5.39) and we obtain that $|\varphi_n(s)| \leq e^{-(T-s)}$ for every $n = 1, \ldots, N$ and every $s \geq 0$. Since only one component of $\varphi(s)$ is different from 0 for each s, we conclude that $|\varphi(s)| \leq e^{-(T-s)}$, hence

$$|Av(T)| \leq \eta \int_0^T e^{-(T-s)}\, ds \leq \eta. \tag{5.40}$$

In order to estimate $w(T)$, we first observe that

$$\int_0^T e^{-x_{2,k_n}(T-s)} \psi_{k_n}(T-s)\, ds = \int_0^T e^{-x_{2,k_n} y} \psi_{k_n}(y)\, dy = \int_{T_{n-1}}^{T_n} e^{-x_{2,k_n} y}\, dy.$$

From (5.35) and the last condition in (5.39) we deduce that

$$\begin{aligned} \nu_{k_n} w_n(T) &= \frac{\nu_{k_n}}{x_{1,k_n} - x_{2,k_n}} \int_{T_{n-1}}^{T_n} e^{-x_{2,k_n} y} \psi_{k_n}(y)\, dy \\ &\geq \frac{\nu_{k_n}}{x_{1,k_n} - x_{2,k_n}} \cdot \frac{1}{x_{2,k_n}} \cdot \frac{1}{e} \left(1 - \frac{1}{e}\right) \\ &\geq \frac{1}{2e} \left(1 - \frac{1}{e}\right), \end{aligned}$$

and finally

$$|Aw(T)|^2 = \eta^2 \sum_{n=1}^{N} |w_n(T)|^2 \geq \eta^2 \frac{1}{4e^2} \left(1 - \frac{1}{e}\right)^2 \cdot N \geq (2M + \eta)^2. \tag{5.41}$$

Therefore, from (5.40) and (5.41) we conclude that

$$|Au(T)| \geq |Aw(T)| - |Av(T)| \geq 2M.$$

It remains to fix the issue that $g(t)$ is not continuous. To this end, it is enough to approximate $g(t)$ with a continuous function $f(t)$ which still satisfies (5.16) and (5.17) (to this end, it is enough to approximate from below the characteristic functions of the intervals in the definition of $\psi(t)$). If $f(t)$ is close enough to $g(t)$, for example in $L^2((0,T), H)$, then the corresponding solution of (2.9)–(2.10) is as close as we want to $u(t)$ in the energy norm, which is equivalent to the norm in $D(A)$ because only a finite number of components is involved. This proves that we can choose $f(t)$ so that the new solution satisfies (5.38).

Conclusion We construct a sequence $\{t_n\}$ of positive times, a sequence $\{H_n\}$ of (finite dimensional) pairwise orthogonal subspaces of H, and a sequence of continuous functions $f_n : [0, +\infty) \to H$ such that $|f_n(t)| \leq 2^{-n}$ and $f_n(t) \in H_n$ for every $t \geq 0$ and $n \in \mathbb{N}$, and such that the corresponding solutions $u_n(t)$ of (2.9)–(2.10) satisfy

$$|Au_n(t_n)|^2 \geq n \qquad \forall n \in \mathbb{N}.$$

The existence of such sequences follows easily from a repeated application of the result of the previous paragraph, each time with $\eta := 2^{-n}$, $M := n$, and $\{\nu_n\}$ equal to the elements of the sequence $\{\lambda_n\}$ which have not yet been used up to that point (we recall that at each step only a finite number of eigenvalues is involved).

The conclusion follows as in the previous cases by defining $f(t)$ and $u(t)$ as in (5.22). Since the subspaces H_n are pairwise orthogonal, we have that $|Au(t_n)| \geq |Au_n(t_n)| \geq n$ for every $n \in \mathbb{N}$, which proves (2.19). □

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