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LOCAL ASYMPTOTIC STABILITY FOR NONLINEAR STATE FEEDBACK DELAY SYSTEMS[1]

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This paper considers the problem of output control of nonlinear delay systems by means of state delayed feedback. In previous papers, through the use of a suitable formalism, standard output control problems, such as output regulation, trajectory tracking, disturbance decoupling and model matching, have been solved for a class of nonlinear delay systems. However, in general an output control scheme does not guarantee internal stability of the system. Some results on this issue are presented in this paper. It is proved that if the system owns a certain Lipschitz property in a suitable neighborhood of the origin, and the initial state is inside such neighborhood, then when the output is driven to zero by means of a static state feedback the system state asymptotically goes to zero. Theoretical results are supported by computer simulations performed on a nonlinear delay systems that is unstable in open loop.

1. INTRODUCTION

Although the topic of analysis and control of linear delay systems has been widely investigated in the past years (see e.g. [1–3, 7, 14–16]), only in recent years some authors focused their attention to analysis and control of nonlinear delay systems [4–7, 10–13]. In papers [4–6], thanks to the introduction of a suitable mathematical formalism, in which a central role is played by the concept of *delay relative degree*, the problem of output control of nonlinear delay systems was solved for an interesting class of nonlinear delay systems. This is the class of *minimum phase delay systems*, that is the class of delay systems that have delay relative degree and stable zero dynamics. The formalism proposed in [4–7] allows to overcome the mathematical difficulties due to the simultaneous presence of nonlinearity in the differential equations and of a state space of infinite dimension, that characterizes delay systems. The control law presented in [4] forces the input-output mapping to be linear and removes the effect of the delay. It is a function of the actual and past values of the state and of the past values of the input (delayed feedback). As a consequence, the output and its derivatives until order $r-1$, where r is the system delay relative degree, can be easily controlled.

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In this paper we consider the problem of the so-called zero-dynamics, that is the behavior of the state when the output and its $r-1$ derivatives are driven to zero and kept to zero by a feedback law. Conditions are presented that guarantee, when the output and its derivatives until order $r-1$ go to zero, that the state asymptotically goes to zero too. Among the conditions, the delay relative degree r must be equal to the dimension n of the state variables and the function $\Phi(\cdot)$, that gives the output and its derivatives until order $n-1$ from the state variables and their past values, must be invertible with respect to the state variables, and the inverse function must be Lipschitz, in a suitable neighborhood of the origin, with a coefficient smaller than 1.

This particular Lipschitz condition is locally verified by a large class of nonlinear delay systems. For such systems the problem of output control with internal stability is solved, provided that the initial state is in a suitable neighborhood of the origin. Simulation results are reported on a nonlinear delay system that is unstable in open loop.

The paper is organized as follows. In Section 2 the necessary notations are reported. In Section 3 the main results of the paper are reported: the problem of local asymptotic stability is formulated and solved. In Section 4 an example of application is presented with simulation results. Conclusions follow in Section 5.

2. PRELIMINARIES

In this section notations and definitions presented in [4], extensively used throughout this paper, are briefly reported. The control system under investigation is described by the following equation

$$\dot{x}(t) = f(x(t), x(t-\Delta)) + g(x(t), x(t-\Delta))\, u(t), \tag{2.1}$$

$$y(t) = h(x(t)), \quad t \geq 0, \tag{2.2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}$, the vector functions f and g are C^∞ with respect to both arguments, and h is a C^∞ scalar function. The model description is completed by the knowledge of the function $x(\tau)$, $\tau \in [-\Delta, 0]$, which represents the initial state in the classical infinite dimensional description of delay systems. Let $x_{i\Delta}(t) = x(t - i\Delta)$ and $u_{i\Delta}(t) = u(t - i\Delta)$, for $i = 0, 1, 2, \ldots$. Note that $x_{i\Delta}$ is defined for $t \geq (i-1)\,\Delta$, while $u_{i\Delta}$ is defined for $t \geq i\Delta$. In the following we shall omit the time dependence, when it does not cause confusion.

Definition 2.1. Assume that for system (2.1), (2.2) there exists an integer r such that, for every X in an open set $\Omega_r \in \mathbb{R}^{n(r+1)}$, the following conditions are verified

$$L_G L_F^k H(X) = 0, \quad k = 0, 1, \ldots, r-2, \tag{2.3}$$

$$L_G L_F^{r-1} H(X) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \neq 0, \tag{2.4}$$

where

$$X = \begin{bmatrix} x \\ x_\Delta \\ \vdots \\ x_{r\Delta} \end{bmatrix}, \quad \overline{X} = \begin{bmatrix} x \\ x_\Delta \\ \vdots \\ x_{(r-1)\,\Delta} \end{bmatrix}, \tag{2.5}$$

$$F(X) = \begin{bmatrix} f(x, x_\Delta) \\ f(x_\Delta, x_{2\Delta}) \\ \vdots \\ f(x_{(r-1)\,\Delta}, x_{r\Delta}) \end{bmatrix}, \tag{2.6}$$

$$G(X) = \operatorname{diag}\left\{g(x, x_\Delta), \ldots, g(x_{(r-1)\,\Delta}, x_{r\Delta})\right\}, \tag{2.7}$$

$$\begin{aligned} H(X) &= h(x), \\ L_F^0 H(X) &= H(X), \\ L_F^k H(X) &= \left(\frac{d}{d\overline{X}} L_F^{k-1} H\right) F(X), \\ L_G L_F^k H(X) &= \left(\frac{d}{d\overline{X}} L_F^k H\right) G(X). \end{aligned} \tag{2.8}$$

Then, we say that the system has delay relative degree equal to r in Ω_r.

If $\Omega_r = I\!R^{n(r+1)}$, we say that the system has global delay relative degree equal to r.

By denoting $U = [u \quad u_\Delta \quad \cdots \quad u_{(r-1)\,\Delta}]^{\mathrm{T}}$, the term $L_G L_F^{r-1} H(X)\, U$ can be expanded in the form

$$L_G L_F^{r-1} H(X)\, U = \Gamma(X) u + m(X, u_\Delta, \ldots, u_{(r-1)\,\Delta}), \tag{2.9}$$

where $\Gamma(X)$ is defined as

$$\Gamma(X) = L_G L_F^{r-1} H(X) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{2.10}$$

and $m(X, u_\Delta, \ldots, u_{(r-1)\,\Delta})$ is as a consequence. Note that from condition (2.4) it is $\Gamma(X) \neq 0$ for $X \in \Omega_r$.

It is not difficult to check that for systems having delay relative degree equal to r in Ω_r it is

$$\begin{aligned} y^{(k)}(t) &= L_F^k H(X), \quad k = 0, 1, \ldots, r-1, \\ y^{(r)}(t) &= L_F^r H(X) + \Gamma(X)\, u + m(X, u_\Delta, \ldots, u_{(r-1)\,\Delta}). \end{aligned} \tag{2.11}$$

From (2.12), it is easily seen that the feedback control law

$$u = \frac{-L_F^r H(X) - m(X, u_\Delta, \ldots, u_{(r-1)\,\Delta}) + \nu}{\Gamma(X)} \tag{2.12}$$

imposes the following linear input-output map with respect to the new input ν

$$y^{(r)}(t) = \nu(t). \tag{2.13}$$

Note that vector X is defined for $t \geq (r-1)\,\Delta$, and therefore the control law (2.12) can be applied starting from time instant $(r-1)\,\Delta$. The control law (2.12) is a function of the actual and past values of the state and of the past values of the input, and therefore it is called delayed feedback.

Defining the vector function

$$\Phi(x, x_\Delta, \ldots, x_{(r-1)\Delta}) = \begin{bmatrix} H(X) \\ L_F^1 H(X) \\ \vdots \\ L_F^{r-1} H(X) \end{bmatrix}, \tag{2.14}$$

the output derivatives up to order $r-1$ for $t \geq (r-1)\,\Delta$ can be written as

$$\begin{bmatrix} y(t) \\ \vdots \\ y^{(r-1)}(t) \end{bmatrix} = \Phi(X(t)). \tag{2.15}$$

Given the linearized input-output mapping (2.13), output regulation, tracking and model matching can be easily performed. For example, by suitably choosing a row vector K, the input

$$\nu = K\Phi(X) \tag{2.16}$$

drives the output and its $r-1$ derivatives to zero with any chosen decay rate.

For nonlinear systems of the form (2.1), (2.2) having global delay relative degree equal to r, the zero dynamics is called to be the state evolution of the feedback system

$$\begin{gathered} \dot{x}(t) = f(x(t), x_\Delta(t)) + g(x(t), x_\Delta(t))\, u(t), \\ t \geq t_0 \geq (r-1)\,\Delta, \end{gathered} \tag{2.17}$$

where variable $x(t)$ for $t \leq t_0$ is such that the output and its first $r-1$ derivatives are zero at t_0 ($y(t_0) = y^{(1)}(t_0) = \cdots = y^{(r-1)}(t_0) = 0$), and $u(t)$ is the feedback input that obtains $y^{(r)}(t) = 0$ for $t \geq t_0$ (it can be computed setting $\nu \equiv 0$ in (2.12)).

Nonlinear systems of the form (2.1), (2.2) are said to be *minimum phase* if they have stable zero-dynamics, extending in this way the terminology generally adopted with reference to nonlinear undelayed systems.

To conclude this section, let

$$z(t) = \begin{bmatrix} y(t) \\ \vdots \\ y^{(r-1)}(t) \end{bmatrix} = \Phi\Big(x(t), \ldots, x(t-(r-1)\,\Delta)\Big). \tag{2.18}$$

We say that the system (2.1), (2.2) is *globally delay observable* if it has global delay relative degree $r = n$ and there exists the inverse Φ^{-1} of function Φ with respect to x, that is

$$x(t) = \Phi^{-1}\Big(z(t), x(t-\Delta), \ldots, x(t-(n-1)\,\Delta)\Big). \tag{2.19}$$

3. LOCAL ASYMPTOTIC STABILITY

From here on we assume that system (2.1), (2.2) has delay relative degree r equal to n, so that when control law (2.12) and (2.16) is applied, the dynamics of variable z is described by the linear equation

$$\dot{z}(t) = (A_b + B_b K) z(t), \quad \text{for } t \geq (n-1)\,\Delta, \tag{3.1}$$

where (A_b, B_b) is a Brunowsky controllable pair. It is an easy matter to compute K such to assign eigenvalues to matrix $A_b + B_b K$, that has a companion structure. We will assume that K is such to assign real eigenvalues λ_i, $i = 1, \ldots, n$ such that

$$\lambda_n < \cdots < \lambda_1 < 0, \tag{3.2}$$

($-K$ is the vector of the coefficients of the polynomial that has the chosen eigenvalues as roots).

Remark 3.1. It is important to stress that, differently from the undelayed case, the fact that the relative degree is $r = n$ and $z(t)$ goes to zero in general does not imply convergence of $x(t)$ to zero. Extra assumptions are needed to achieve such implication, as explained in the following theorem.

Theorem 3.2. Let system (2.1), (2.2) be globally delay observable, with feedback control law (2.12), (2.16), with eigenvalues (3.2). Assume there exists a positive constant γ such that, if $\|x_{i\Delta}\| < \gamma$, $i = 1, \ldots, n-1$, then

$$\|\Phi^{-1}(z, x_\Delta, x_{2\Delta}, \ldots, x_{(n-1)\,\Delta})\| \leq \alpha \|z\| + \beta \left\| \begin{bmatrix} x_\Delta \\ \vdots \\ x_{(n-1)\,\Delta} \end{bmatrix} \right\|, \tag{3.3}$$

with $\alpha > 0$ and $0 \leq \beta < \frac{1}{(n-1)}$.

Then, there exist suitable positive constants δ_1 and δ_2 such that, if

$$\begin{aligned} &\|x(\tau)\| < \delta_1, \quad \tau \in [-\Delta, (n-1)\,\Delta], \\ &\|z((n-1)\,\Delta)\| < \delta_2, \end{aligned} \tag{3.4}$$

then

$$\lim_{t \to \infty} x(t) = 0. \tag{3.5}$$

Proof. Let T be a diagonalizing matrix for $A_b + B_b K$, so that

$$\operatorname{diag}(\lambda_1, \ldots, \lambda_n) = T(A_b + B_b K)T^{-1}. \tag{3.6}$$

Let $M = \|T\|\|T^{-1}\|$. From (3.1) it is for $t \geq (n-1)\,\Delta$

$$\begin{aligned} \|z(t)\| &\leq \|e^{(A_b + B_b K)(t-(n-1)\,\Delta)}\| \cdot \|z((n-1)\,\Delta)\| \\ &\leq M e^{\lambda_1 (t-(n-1)\,\Delta)} \|z((n-1)\,\Delta)\|. \end{aligned} \tag{3.7}$$

Let

$$\delta_1 = \gamma - \epsilon, \quad 0 < \epsilon < \gamma. \tag{3.8}$$

Take as δ_2 the following

$$\delta_2 = \min\left\{ \frac{\gamma - \beta(n-1)\gamma}{M\alpha e^{\lambda_1 \Delta}}, \frac{\gamma - \beta(n-1)(\gamma - \epsilon)}{\alpha M} \right\}. \tag{3.9}$$

Let the initial state and the input in $[0, (n-1)\Delta]$ such that (3.4) are satisfied with this choice of δ_1 and δ_2. We can prove that $\|x(\tau)\| < \gamma$ for $\tau \in [-\Delta, \infty)$. It is $\|x(\tau)\| < \delta_1 < \gamma$ for $\tau \in [-\Delta, (n-1)\Delta]$. For $\tau \in [(n-1)\Delta, n\Delta]$ the following inequalities hold

$$\begin{aligned} \|x(\tau)\| &\leq \alpha\|z(\tau)\| + \beta \sum_{i=1}^{n-1} \|x(\tau - i\Delta)\| \\ &< \alpha M e^{\lambda_1(\tau - (n-1)\Delta)} \|z((n-1)\Delta)\| + \beta(n-1)\delta_1 \\ &\leq \alpha M \frac{\gamma - \beta(n-1)(\gamma - \epsilon)}{\alpha M} + \beta(n-1)(\gamma - \epsilon) = \gamma. \end{aligned} \tag{3.10}$$

Now it is not difficult to prove the following implication for $i \geq n$

$$\begin{matrix} \|x(\tau)\| < \gamma, \\ \tau \in [-\Delta, i\Delta], \end{matrix} \quad \Rightarrow \quad \begin{matrix} \|x(\tau)\| < \gamma, \\ \tau \in [-\Delta, (i+1)\Delta]. \end{matrix} \tag{3.11}$$

This happens because for $\tau \in [i\Delta, (i+1)\Delta]$

$$\begin{aligned} \|x(\tau)\| &\leq \alpha\|z(\tau)\| + \beta \sum_{j=1}^{n-1} \|x(\tau - j\Delta)\| \\ &< \alpha M e^{\lambda_1(\tau - (n-1)\Delta)} \|z((n-1)\Delta)\| + \beta(n-1)\gamma \\ &\leq \alpha M e^{\lambda_1 \Delta} \delta_2 + \beta(n-1)\gamma \\ &\leq \gamma - \beta(n-1)\gamma + \beta(n-1)\gamma = \gamma. \end{aligned} \tag{3.12}$$

It follows that $\|x(\tau)\| < \gamma \ \forall t \geq -\Delta$, and we can conclude that $\text{maxlim}_{t\to\infty} \|x(t)\| \leq \gamma$. Moreover

$$\begin{aligned} \text{maxlim}_{t\to\infty} \|x(t)\| &\leq \alpha \, \text{maxlim}_{t\to\infty} \|z(t)\| \\ &\quad + \beta(n-1) \, \text{maxlim}_{t\to\infty} \left(\sup_{i=1,2,\ldots,n-1} \|x(t - i\Delta)\| \right) \\ &\leq \beta(n-1) \, \text{maxlim}_{t\to\infty} \|x(t)\|, \end{aligned} \tag{3.13}$$

and, being $\beta(n-1) < 1$ and $\text{maxlim}_{t\to\infty} \|x(t)\|$ a finite quantity, it follows that it must be $\text{maxlim}_{t\to\infty} \|x(t)\| = 0$, that is the thesis. □

Remark 3.3. As it can be understood through equation (3.9), the bound δ_2 required on vector z at time $t = (n-1)\Delta$ depends on the chosen eigenvalues, through the quantity $M = \|T\| \, \|T^{-1}\|$.

Remark 3.4. Boundedness of the variable $x(t)$ in the interval $[-\Delta,(n-1)\,\Delta]$ must be assumed and can not be obtained by means of the control law of the type (2.12) because this law is well defined only for $t \geq (n-1)\,\Delta$.

Remark 3.5. Thanks to relation (2.18) the condition on the boundedness of $z((n-1)\,\Delta)$ can be transformed in a suitable condition on the boundedness of x in time instants $0, \Delta, 2\Delta, \ldots, (n-1)\,\Delta$.

Remark 3.6. Assumptions (3.4) of Theorem 3.2 can be replaced by the following: there exist positive constants δ_1 and δ_2 such that

$$\begin{aligned} \|x(\tau)\| &< \delta_1, \qquad \tau \in [-\Delta,(n-2)\,\Delta], \\ \|z(\tau)\| &< \delta_2, \qquad \tau \in [(n-2)\,\Delta,(n-1)\,\Delta]. \end{aligned} \tag{3.14}$$

In this case, the boundedness condition on the variable x must be satisfied in a narrower interval, while the one on variable z needs to be verified in a delay interval rather than only in instant $(n-1)\,\Delta$.

Results similar to those reported in Theorem 3.2 can be achieved by using the hypothesis of bounded gradient, and are presented in the next theorem. In the proof the *mean value theorem* is used, that states that if $\lambda(\xi) : I\!R^m \to I\!R$ is a C^1 function, then for any pair $\xi_1, \xi_2 \in I\!R^m$ there exists a $\bar{\xi}$ on the segment from ξ_1 to ξ_2 such that

$$\lambda(\xi_2) - \lambda(\xi_1) = \frac{d\lambda(\xi)}{d\xi}\bigg|_{\xi=\bar{\xi}} (\xi_2 - \xi_1). \tag{3.15}$$

The mean value theorem is applied to the j-th component of the map Φ^{-1}, denoted simply as Φ_j^{-1}, in which vector z is intended as a parameter. It is $x_j = \Phi_j^{-1}(z, x_\Delta, \ldots, x_{(n-1)\,\Delta})$, and using the mean value theorem between vectors $(x_\Delta, \ldots$ $\ldots, x_{(n-1)\,\Delta})$ and $(0, \ldots, 0)$, one has

$$\Phi_j^{-1}(z, x_\Delta, \ldots, x_{(n-1)\,\Delta}) - \Phi_j^{-1}(z, 0, \ldots, 0) = \sum_{i=1}^{n-1} \frac{\partial \Phi_j^{-1}}{\partial x_{i\Delta}} x_{i\Delta}, \tag{3.16}$$

and therefore

$$x_j = \Phi_j^{-1}(z, 0, \ldots, 0) + \sum_{i=1}^{n-1} \frac{\partial \Phi_j^{-1}}{\partial x_{i\Delta}} x_{i\Delta}, \tag{3.17}$$

where the derivatives are computed in a point of coordinates $(qx_\Delta, \ldots, qx_{(n-1)\,\Delta})$, with $q \in [0,1]$. In the following let $x_{i,j\Delta}(t) = x_i(t - j\Delta)$, $i = 1, 2, \ldots$, $j = 0, 1, \ldots$.

Theorem 3.7. Let system (2.1), (2.2) be globally delay observable, with feedback control law (2.12), (2.16), with eigenvalues (3.2). Let the following hypotheses be satisfied:

H1) There exists a positive α such that

$$\|\Phi^{-1}(z(t), 0, 0, \ldots, 0)\|_\infty \leq \alpha \|z(t)\|_\infty, \tag{3.18}$$

for $t \geq (n-1)\,\Delta$;

H2) there exists a positive constant γ such that if for $t \geq (n-1)\Delta$ it is $\|x_{i\Delta}\|_\infty < \gamma$, $i = 1, \ldots, n-1$, then for $j = 1, \ldots, n$, it is

$$\left\| \begin{bmatrix} \dfrac{\partial \Phi_j^{-1}}{\partial x_\Delta} & \cdots & \dfrac{\partial \Phi_j^{-1}}{\partial x_{(n-1)\Delta}} \end{bmatrix} \right\|_1 \leq \beta < 1, \tag{3.19}$$

for $t \geq (n-1)\Delta$.

Then, there exist suitable positive constants δ_1 and δ_2 such that, if

$$\begin{aligned} &\|x(\tau)\|_\infty < \delta_1, \quad \tau \in [-\Delta, (n-1)\Delta], \\ &\|z((n-1)\Delta)\|_\infty < \delta_2, \end{aligned} \tag{3.20}$$

it is

$$\lim_{t\to\infty} x(t) = 0. \tag{3.21}$$

Proof. Let $\overline{M}$ be such that, for $t \geq (n-1)\Delta$, it is

$$\|z(t)\|_\infty \leq \overline{M} e^{\lambda_1 t} \|z((n-1)\Delta)\|_\infty. \tag{3.22}$$

As in the proof of Theorem 3.2 let

$$\delta_1 = \gamma - \epsilon, \quad 0 < \epsilon < \gamma \tag{3.23}$$

and

$$\delta_2 = \min\left\{ \frac{\gamma - \beta\gamma}{\overline{M}\alpha e^{\lambda_1 \Delta}}, \frac{\gamma - \beta(\gamma - \epsilon)}{\alpha \overline{M}} \right\}. \tag{3.24}$$

Let the initial state and the input in $[0, (n-1)\Delta]$ such that (3.20) are satisfied with this choice of δ_1 and δ_2.

It can be proved that $\|x(\tau)\|_\infty < \gamma$, for $\tau \in [-\Delta, \infty)$. It is $\|x(\tau)\|_\infty < \delta_1 < \gamma$ for $\tau \in [-\Delta, (n-1)\Delta]$. In $[(n-1)\Delta, n\Delta]$ it is, exploiting equation (3.17),

$$|x_j(\tau)| \leq |\Phi_j^{-1}(z(\tau), 0, \ldots, 0)| + \sum_{i=1}^{n-1} \sum_{l=1}^{n} \left| \frac{\partial \Phi_j^{-1}}{\partial x_{l,i\Delta}} \right| |x_l(\tau - i\Delta)|. \tag{3.25}$$

From this, for $\tau \in [(n-1)\Delta, n\Delta]$ the ∞-norm of $x(\tau)$ satisfies

$$\begin{aligned} \|x(\tau)\|_\infty &\leq \|\Phi^{-1}(z(\tau), 0, \ldots, 0)\|_\infty \\ &+ \sup_{j=1,\ldots,n} \left\| \begin{bmatrix} \dfrac{\partial \Phi_j^{-1}}{\partial x_\Delta} & \cdots & \dfrac{\partial \Phi_j^{-1}}{\partial x_{(n-1)\Delta}} \end{bmatrix} \right\|_1 \sup_{\substack{l=1,\ldots,n \\ i=1,\ldots,n-1}} |x_l(\tau - i\Delta)| \\ &< \alpha \overline{M} e^{\lambda_1(\tau - (n-1)\Delta)} \|z((n-1)\Delta)\|_\infty + \beta\gamma \\ &< \alpha \overline{M} e^{\lambda_1(\tau - (n-1)\Delta)} \delta_2 + \beta\gamma \leq \gamma. \end{aligned} \tag{3.26}$$

Now the following implication can be proved for $i \geq n$

$$\begin{matrix} \|x(\tau)\|_\infty < \gamma \\ \tau \in [-\Delta, i\Delta] \end{matrix} \quad \Rightarrow \quad \begin{matrix} \|x(\tau)\|_\infty < \gamma \\ \tau \in [-\Delta, (i+1)\Delta] \end{matrix} \tag{3.27}$$

This is true because for $\tau \in [i\Delta, (i+1)\Delta]$

$$\begin{aligned} \|x(\tau)\|_\infty &< \alpha\|z(\tau)\|_\infty + \beta\gamma \leq \\ &\leq \alpha\overline{M}e^{\lambda_1(\tau-(n-1)\Delta)}\|z((n-1)\Delta)\|_\infty + \beta\gamma \leq \gamma. \end{aligned} \tag{3.28}$$

Until now we have proved that $\|x(\tau)\|_\infty < \gamma$ for all $\tau \geq -\Delta$. This implies that there exists finite $\mathrm{maxlim}_{t\to\infty} \|x(t)\|_\infty$. Moreover it is

$$\begin{aligned} \mathrm{maxlim}_{t\to\infty} \|x(t)\|_\infty &\leq \beta\,\mathrm{maxlim}_{t\to\infty} \sup_{i=1,\dots,n-1} \|x(t-i\Delta)\|_\infty \\ &= \beta\,\mathrm{maxlim}_{t\to\infty} \|x(t)\|_\infty, \end{aligned} \tag{3.29}$$

and, being $\beta < 1$, it follows that $\mathrm{maxlim}_{t\to\infty} \|x(t)\|_\infty = 0$, that is the thesis. □

Remark 3.8. We want to stress that by the same hypotheses of Theorems 3.2 and 3.7, the boundedness of the state can be proved in asymptotic output tracking problems, provided that the reference output and its $n-1$ derivatives are bounded.

4. EXAMPLE

Let us consider the following nonlinear delay system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + x_1(t-\Delta)\,x_2^3(t-\Delta), \\ \dot{x}_2(t) &= x_1^3(t-\Delta) + u(t), \\ y(t) &= x_1(t), \end{aligned} \tag{4.1}$$

with $\Delta = 0.1$. For this system the delay relative degree is $r = 2$. The quantities in Definition 2.1 are in this case

$$F = \begin{bmatrix} x_2 + x_{1,\Delta}x_{2,\Delta}^3 \\ x_{1,\Delta}^3 \\ x_{2,\Delta} + x_{1,2\Delta}x_{2,2\Delta}^3 \\ x_{1,2\Delta}^3 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{4.2}$$

$$H = x_1, \quad L_F H = x_2 + x_{1,\Delta}x_{2,\Delta}^3,$$

$$\begin{aligned} L_F^2 H &= [0 \quad 1 \quad x_{2,\Delta}^3 3x_{1,\Delta}x_{2,\Delta}^2]\,F = \\ &= x_{1,\Delta}^3 + x_{2,\Delta}^3(x_{2,\Delta} + x_{1,2\Delta}x_{2,2\Delta}^3) + 3x_{1,\Delta}x_{2,\Delta}^2x_{1,2\Delta}^3, \end{aligned}$$

$$\begin{aligned} L_G H &= [1 \quad 0 \quad 0 \quad 0]\,G = [0 \quad 0] \\ L_G L_F H &= [0 \quad 1 \quad x_{2,\Delta}^3 \quad 3x_{1,\Delta}x_{2,\Delta}^2]\,G = [1 \quad 3x_{1,\Delta}x_{2,\Delta}^2], \\ & m = 3x_{1,\Delta}x_{2,\Delta}^2u_\Delta, \quad \Gamma = 1. \end{aligned}$$

The following control law brings the output to zero with a prescribed exponential rate imposed by the choice of a gain vector K

$$u = -L_F^2 H - m + K \begin{bmatrix} x_1 \\ x_2 + x_{1,\Delta}x_{2,\Delta}^3 \end{bmatrix}. \tag{4.3}$$

In the simulations presented the gain vector K has been chosen such to assign eigenvalues -1, -2 to the closed loop input/output map.

Maps Φ and Φ^{-1} are given by

$$z = \begin{bmatrix} x_1 \\ x_2 + x_{1,\Delta} x_{2,\Delta}^3 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} z_1 \\ z_2 - x_{1,\Delta} x_{2,\Delta}^3 \end{bmatrix}$$

and so

$$\|x\| \le \|z\| + \|x_\Delta\|^4$$

Hypothesis (3.3) of Theorem 3.2 is satisfied with any $\gamma < 1$. In this case we have $\alpha = 1$, $\beta = \gamma^3$.

In simulations the initial state has been chosen constant

$$x(\tau) = \begin{bmatrix} 0 \\ \delta \end{bmatrix}, \quad \tau \in [-\Delta, 0].$$

Control law (4.3) can be applied starting from instant $t = \Delta$. The free evolution of the system in the interval $[0, \Delta]$ gets

$$\sup_{\tau \in [-\Delta, \Delta]} \|x(\tau)\| = \delta\sqrt{(1 + \Delta^2)}$$

and

$$\sup_{\tau \in [-\Delta, \Delta]} \|z(\tau)\| = \delta\sqrt{(1 + \Delta^2)}.$$

All simulations worked out using values of δ smaller than 2.5 have shown stable internal dynamics for the simulated system. In Figures 1, 2 the two components of the state and the control input are plotted in the case $\delta = 2$.

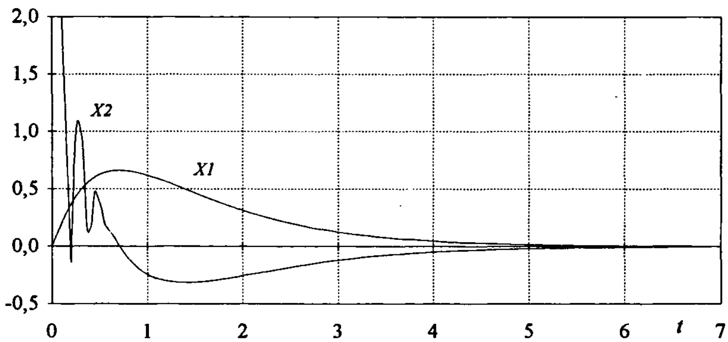

Fig. 1. Plot of state variables x_1 and x_2.

5. CONCLUSIONS

In this work the issue of internal stability for nonlinear delay systems, whose output is driven to zero by a delayed state feedback law, is investigated. The output control

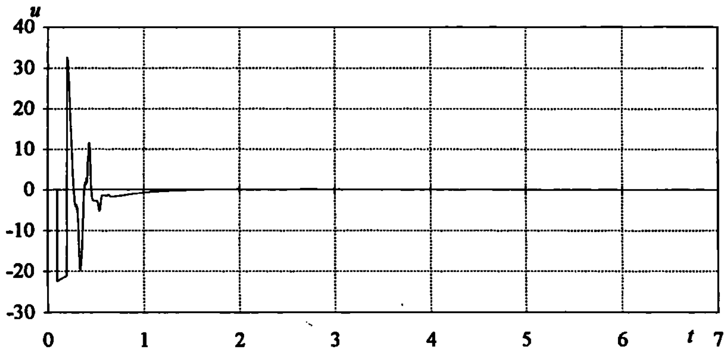

Fig. 2. Plot of control input u.

law was proposed by the authors in previous papers, in which the problem of the system *zero dynamics* was mentioned but not studied. In this paper local conditions on the system structure and on the initial state that guarantee the asymptotic stability of the closed loop system are given.

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