Revista Matemática Complutense (2021) 34:409–434
https://doi.org/10.1007/s13163-020-00354-y

Local atomic decompositions for multidimensional Hardy spaces

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Received: 17 October 2019 / Accepted: 10 March 2020 / Published online: 28 March 2020

Abstract

We consider a nonnegative self-adjoint operator L on $L^2(X)$, where $X \subseteq \mathbb{R}^d$. Under certain assumptions, we prove atomic characterizations of the Hardy space

$$H^1(L) = \left\{ f \in L^1(X) \ : \ \left\| \sup_{t>0} |\exp(-tL)f| \right\|_{L^1(X)} < \infty \right\}.$$

We state simple conditions, such that $H^1(L)$ is characterized by atoms being either the classical atoms on $X \subseteq \mathbb{R}^d$ or local atoms of the form $|Q|^{-1}\chi_Q$, where $Q \subseteq X$ is a cube (or cuboid). One of our main motivation is to study multidimensional operators related to orthogonal expansions. We prove that if two operators L_1, L_2 satisfy the assumptions of our theorem, then the sum $L_1 + L_2$ also does. As a consequence, we give atomic characterizations for multidimensional Bessel, Laguerre, and Schrödinger operators. As a by-product, under the same assumptions, we characterize $H^1(L)$ also by the maximal operator related to the subordinate semigroup $\exp(-tL^\nu)$, where $\nu \in (0, 1)$.

Keywords Hardy space · Maximal function · Local atomic decomposition · Subordinated semigroup · Bessel operator · Laguerre operator · Schrödinger operator

Mathematics Subject Classification Primary 42B30; Secondary 42B25 · 33C45 · 35J10 · 47D03

Edyta Kania-Strojec and Marcin Preisner are supported by the Grant No. 2017/25/B/ST1/00599 from National Science Centre (Narodowe Centrum Nauki), Poland. Paweł Plewa is supported by the National Science Centre of Poland, NCN Grant No. 2018/29/N/ST1/02424.

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1 Background and main results

1.1 Introduction

Let us first recall that the classical Hardy space $H^1(\mathbb{R}^d)$ can be defined by the maximal operator, i.e.

$$f \in H^1(\mathbb{R}^d) \qquad \Longleftrightarrow \qquad \sup_{t>0} |H_t f| \in L^1(\mathbb{R}^d).$$

Here and thereafter $H_t = \exp(t\Delta)$ is the heat semigroup on $\mathbb{R}^d$ given by $H_t f(x) = \int_{\mathbb{R}^d} H_t(x, y) f(y)\, dy$,

$$H_t(x, y) = (4\pi t)^{-d/2} \exp\left(-\frac{|x-y|^2}{4t}\right). \tag{1.1}$$

Among many equivalent characterizations of $H^1(\mathbb{R}^d)$ one of the most useful is the characterization by atomic decompositions proved by Coifman [4] in the one-dimensional case and by Latter [19] in the general case $d \in \mathbb{N}$. It says that $f \in H^1(\mathbb{R}^d)$ if and only if $f(x) = \sum_{k=1}^{\infty} \lambda_k a_k(x)$, where $\lambda_k \in \mathbb{C}$ are such that $\sum_{k=1}^{\infty} |\lambda_k| < \infty$ and a_k are *atoms*. By definition, a function a is *an atom* if there exists a ball $B \subseteq \mathbb{R}^d$ such that:

$$\operatorname{supp} a \subseteq B, \qquad \|a\|_\infty \leq |B|^{-1}, \qquad \int_B a(x)\, dx = 0,$$

i.e. a satisfies well-known localization, size, and cancellation conditions.

Later, Goldberg in [16] noticed that if we restrict the supremum in the maximal operator above to the range $t \in (0, \tau^2)$, with $\tau > 0$ fixed, then still the atomic characterization holds, but with additional atoms of the form $a(x) = |B|^{-1}\chi_B(x)$, where χ is the characteristic function and B is a ball of radius τ (see Sect. 2 for details).

Then, many atomic characterizations were proved for various operators including operators with Gaussian (or Davies-Gaffney) estimates, operators on spaces of homogeneous type, operators related to orthogonal expansions, Schrödinger operators, and others. The reader is referred to [1,2,6,9–11,17,21,22] and references therein.

In this paper we deal with atomic characterizations of the Hardy space H^1 for operators, such that H^1 admits atoms of local type, i.e. atoms of the form $|B|^{-1}\chi_B$. We shall consider operators defined on $L^2(X)$, where $X \subseteq \mathbb{R}^d$ with the Lebesgue measure. Our main focus will be on sums of the form $L = L_1 + \cdots + L_d$, where each L_i acts only on the variable x_i, where $x = (x_1, ..., x_d)$. For such L we look for atomic decompositions. As an application, we can take operators related to some multidimensional orthogonal expansions. Additionally we prove characterizations of H^1 by subordinate semigroups.

1.2 Notation

Let $X = (a_1, b_1) \times \cdots \times (a_d, b_d)$ be a subset of $\mathbb{R}^d$. We allow $a_j = -\infty$ and $b_j = \infty$ so that we consider products of lines, half-lines, and finite intervals. We equip X with the Euclidean metric and the Lebesgue measure. In the product case it is more convenient to use cubes and cuboids instead of balls, so denote for $z = (z_1, ..., z_d) \in X$ and $r_1, ..., r_d > 0$ the closed cuboid

$$Q(z, r_1, ..., r_d) = \{x \in X \ : \ |x_i - z_i| \leq r_i \text{ for } i = 1, ..., d\},$$

and the cube $Q(z, r) = Q(z, r, ..., r)$. We shall call such z the center of a cube/cuboid. For a cuboid Q by d_Q we shall denote the diameter of Q.

Definition 1.2 Let $\mathcal{Q}$ be a set of cuboids in X. We call $\mathcal{Q}$ *an admissible covering* of X if there exist $C_1, C_2 > 0$ such that:

1. $X = \bigcup_{Q \in \mathcal{Q}} Q$,
2. if $Q_1, Q_2 \in \mathcal{Q}$ and $Q_1 \neq Q_2$, then $|Q_1 \cap Q_2| = 0$,
3. if $Q = Q(z, r_1, ..., r_d) \in \mathcal{Q}$, then $r_i \leq C_1 r_j$ for $i, j \in \{1, ..., d\}$,
4. if $Q_1, Q_2 \in \mathcal{Q}$ and $Q_1 \cap Q_2 \neq \emptyset$, then $C_2^{-1} d_{Q_1} \leq d_{Q_2} \leq C_2 d_{Q_1}$.

Let us note that 3. means that our cuboids are almost cubes. In fact, we shall often use only cubes.

By Q^* we shall denote a slight enlargement of Q. More precisely, if $Q = (z, r_1, ..., r_d)$, then $Q^* := Q(z, \kappa r_1, ..., \kappa r_d)$, where $\kappa > 1$. Observe that if $\mathcal{Q}$ is an admissible covering of $\mathbb{R}^d$, then choosing κ close enough to 1 the family $\{Q^{***}\}_{Q \in \mathcal{Q}}$ is a finite covering of $\mathbb{R}^d$, namely

$$\sum_{Q \in \mathcal{Q}} \chi_{Q^{***}}(x) \leq C, \qquad x \in \mathbb{R}^d \tag{1.3}$$

and, for $Q_1, Q_2 \in \mathcal{Q}$,

$$Q_1^{***} \cap Q_2^{***} \neq \emptyset \quad \iff \quad Q_1 \cap Q_2 \neq \emptyset. \tag{1.4}$$

In this paper we always choose κ such that (1.3) and (1.4) are satisfied. Let us emphasize that Q and Q^* are always defined as a subset of X, not as a subset of $\mathbb{R}^d$.

Having two admissible coverings $\mathcal{Q}_1$ and $\mathcal{Q}_2$ on $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ we would like to produce an admissible covering on $\mathbb{R}^{d_1+d_2}$. However, one simply observe that products $\{Q_1 \times Q_2 \ : \ Q_1 \in \mathcal{Q}_1, Q_2 \in \mathcal{Q}_2\}$, would not produce admissible covering (in general, 3. would fail). Therefore, for the sake of this paper, let us state the following definition.

Definition 1.5 Assume that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are admissible coverings of $X_1 \subseteq \mathbb{R}^{d_1}$ and $X_2 \subseteq \mathbb{R}^{d_2}$, respectively. We define an admissible covering of $X_1 \times X_2$ in the following way. First, consider the covering $\{Q_1 \times Q_2 \ : \ Q_1 \in \mathcal{Q}_1, Q_2 \in \mathcal{Q}_2\}$. Then we further split each $Q = Q_1 \times Q_2$. Without loss of generality let us assume that $d_{Q_1} > d_{Q_2}$.

We split Q_1 into cuboids $Q_1^{[j]}$, $j = 1, ..., M$, such that all of them have diameters comparable to d_{Q_2} and satisfy 3. of Definition 1.2. Then the cuboids $Q^{[j]} = Q_1^{[j]} \times Q_2$, $j = 1, ..., M$, satisfy:

- $Q = \bigcup_{j=1}^{M} Q^{[j]}$,
- for $i, j \in \{1, ..., M\}$, $i \neq j$, we have $|Q^{[i]} \cap Q^{[j]}| = 0$,
- each $Q^{[j]}$ satisfies 3. from Definition 1.2.

Notice that $M \leq [d_{Q_1}/d_{Q_2}]^{d_1}$. We shall denote such covering by $\mathcal{Q}_1 \boxtimes \mathcal{Q}_2$. One may check that the definition above leads to an admissible covering of $X_1 \times X_2$.

Having an admissible covering $\mathcal{Q}$ of $X \subseteq \mathbb{R}^d$ we define a local atomic Hardy space $H^1_{at}(\mathcal{Q})$ related to $\mathcal{Q}$ in the following way. We say that a function $a : X \to \mathbb{C}$ is a $\mathcal{Q} - atom$ if:

(i) either there is $Q \in \mathcal{Q}$ and a cube $K \subset Q^*$, such that:

$$\operatorname{supp} a \subseteq K, \quad \|a\|_\infty \leq |K|^{-1}, \quad \int a(x)\, dx = 0;$$

(ii) or there exists $Q \in \mathcal{Q}$ such that

$$\alpha(x) = |Q|^{-1} \chi_Q(x).$$

Having $\mathcal{Q}$-atoms we define *the local atomic Hardy space related to* $\mathcal{Q}$, $H^1_{at}(\mathcal{Q})$, in a standard way. Namely, we say that a function f is in $H^1_{at}(\mathcal{Q})$ if $f(x) = \sum_k \lambda_k a_k(x)$ with $\sum_k |\lambda_k| < \infty$ and a_k being $\mathcal{Q}$-atoms. Moreover, the norm of $H^1_{at}(\mathcal{Q})$ is given by

$$\|f\|_{H^1_{at}(\mathcal{Q})} = \inf \sum_k |\lambda_k|,$$

where the infimum is taken over all possible representations of $f(x) = \sum_k \lambda_k a_k(x)$ as above. One may simply check that $H^1_{at}(\mathcal{Q})$ is a Banach space.

In the whole paper by L we shall denote a nonnegative self-adjoint operator and by $T_t = \exp(-tL)$ the heat semigroup generated by L. We shall always assume that there exists a nonnegative integral kernel $T_t(x, y)$ such that $T_t f(x) = \int_X T_t(x, y) f(y)\, dy$. Our initial definition of the Hardy space $H^1(L)$ shall be given by means of the maximal operator associated with T_t, namely

$$H^1(L) = \left\{ f \in L^1(X) \ : \ \|f\|_{H^1(L)} := \left\| \sup_{t>0} |T_t f| \right\|_{L^1(X)} < \infty \right\}.$$

Moreover, we shall consider the subordinate semigroup $K_{t,\nu} = \exp(-tL^\nu)$, $\nu \in (0, 1)$, and its Hardy space, which is defined by

$$H^1(L^\nu) = \left\{ f \in L^1(X) \ : \ \|f\|_{H^1(L^\nu)} := \left\| \sup_{t>0} \left| K_{t,\nu} f \right| \right\|_{L^1(X)} < \infty \right\}.$$

1.3 Main results

Let us assume that an admissible covering $\mathcal{Q}$ of X is given. Recall that $H_t(x,y)$ is the classical semigroup on $\mathbb{R}^d$ given in (1.1), and denote by $P_{t,\nu} = \exp(-t(-\Delta)^\nu)$ the semigroup generated by $(-\Delta)^\nu$, $\nu \in (0,1)$, and given by $P_{t,\nu} f(x) = \int_{\mathbb{R}^d} P_{t,\nu}(x,y) f(y)\,dy$. The kernel $P_{t,\nu}(x,y)$ is a transition density of the symmetric 2ν-stable Lévy process in $\mathbb{R}^d$. It is well-known that

$$0 \le P_{t,\nu}(x,y) \le C_{d,\nu} \frac{t}{\left(t^{1/\nu} + |x-y|^2\right)^{\frac{d}{2}+\nu}}, x, y \in \mathbb{R}^d, t>0,\ \ \nu \in (0,1), \quad (1.6)$$

see e.g. [18, Subsec. 2.6], [15]. Let us mention that in the particular case of $\nu = 1/2$, the semigroup $P_{t,1/2}$ is the well-known Poisson semigroup on $\mathbb{R}^d$.

Assume that an operator L is as in Sect. 1.2. Let $\nu \in (0,1)$ and suppose that $\widetilde{T}_t(x,y)$ is either $H_t(x,y)$ or $P_{t^\nu,\nu}(x,y)$. Consider the following assumptions:

$$0 \le T_t(x,y) \le C \frac{t^\nu}{\left(t + |x-y|^2\right)^{\frac{d}{2}+\nu}}, \quad x, y \in X,\ t>0, \quad (A_0')$$

$$\sup_{y \in Q^*} \int_{(Q^{**})^c} \sup_{t>0} T_t(x,y)\,dx \le C, \quad Q \in \mathcal{Q}, \quad (A_1')$$

$$\sup_{y \in Q^*} \int_{Q^{**}} \sup_{t \le d_Q^2} \left| T_t(x,y) - \widetilde{T}_t(x,y) \right| dx \le C, \quad Q \in \mathcal{Q}. \quad (A_2')$$

Theorem A *Assume that for L, T_t, and an admissible covering $\mathcal{Q}$ the conditions (A_0')–(A_2') hold. Then $H^1(L) = H^1_{at}(\mathcal{Q})$ and the corresponding norms are equivalent.*

The proof of Theorem A is standard and uses only local characterization of Hardy spaces as in [16]. For the convenience of the reader we present the proof in Sect. 3.

Our first main goal is to describe atomic characterizations for sums of the form $L_1 + \dots + L_N$, where each L_j satisfies (A_0')–(A_2') on a proper subspace. This is very useful in many cases such as multidimensional orthogonal expansions. Instead of dealing with products of kernels of semigroups, we can consider only one-dimensional kernel, but we shall need to prove slightly stronger conditions. More precisely, we consider $X_1 \times \dots \times X_N \subseteq \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_N} = \mathbb{R}^d$. Assume that L_i is an operator on $L^2(X_i)$, as in Sect. 1.2. Slightly abusing the notation we keep the symbol L_i for $I \otimes ... \otimes L_i \otimes ... \otimes I$ as the operator on $L^2(X)$ and denote

$$Lf(x) = L_1 f(x) + \dots + L_N f(x), \quad x = (x_1, ..., x_N) \in X. \quad (1.7)$$

For $x_i, y_i \in X_i$, by $T_t^{[i]}(x_i, y_i)$ we denote the kernel of $T_t^{[i]} = \exp(-tL_i)$. We shall assume that each $T_t^{[i]}(x_i, y_i)$, $i = 1, ..., N$, is nonnegative and has the upper Gaussian estimates, namely

$$0 \leq T_t^{[i]}(x_i, y_i) \leq C_i t^{-d_i/2} \exp\left(-\frac{|x_i - y_i|^2}{c_i t}\right), \quad x_i, y_i \in X_i, t > 0. \qquad (A_0)$$

Obviously, (A_0) implies (A_0') for $T_t(x, y) = T_t^{[1]}(x_1, y_1)...T_t^{[N]}(x_N, y_N)$. Moreover, we shall assume that for each $i \in \{1, ..., N\}$ there exist a proper covering $\mathcal{Q}_i$ of $\mathbb{R}^{d_i}$ such that the following generalizations of (A_1') and (A_2') hold: there exists $\gamma \in (0, 1/3)$ such that for every $\delta \in [0, \gamma)$ and every $i = 1, .., N$,

$$\sup_{y \in Q^*} \int_{(Q^{**})^c} \sup_{t>0} t^{\delta} T_t^{[i]}(x, y) dx \leq C d_Q^{2\delta}, \quad Q \in \mathcal{Q}_i, \qquad (A_1)$$

$$\sup_{y \in Q^*} \int_{Q^{**}} \sup_{t<d_Q^2} t^{-\delta} \left| T_t^{[i]}(x, y) - H_t(x, y) \right| dx \leq C d_Q^{-2\delta}, \quad Q \in \mathcal{Q}_i. \qquad (A_2)$$

Here H_t is the classical heat semigroup on $\mathbb{R}^{d_i}$, depending on the context. Now, we are ready to state our first main theorem.

Theorem B *Assume that for* $i = 1, ..., N$ *kernels* $T_t^{[i]}(x_i, y_i)$ *are related to* L_i *and suppose that for* $T_t^{[i]}(x_i, y_i)$ *together with admissible coverings* $\mathcal{Q}_i$ *the conditions* (A_0)–(A_2) *hold. If* $L = L_1 + \cdots + L_N$ *is as in* (1.7)*, then*

$$H^1(L) = H^1_{at}(\mathcal{Q}_1 \boxtimes ... \boxtimes \mathcal{Q}_N)$$

and the corresponding norms are equivalent.

Our second main goal is to characterize $H^1(L)$ by the subordinate semigroup $K_{t,\nu} = \exp(-tL^{\nu})$, for $0 < \nu < 1$. Obviously, one can try to apply Theorem A, but for many operators the subordinate kernel $K_{t,\nu}(x, y)$ is harder to analyze than $T_t(x, y)$ (e.g., in some cases a concrete formula with special functions exists for $T_t(x, y)$, but not for $K_{t,\nu}(x, y)$). However, it appears that under our assumptions (A_0)–(A_2) we obtain the characterization by the subordinate semigroup essentially for free.

Theorem C *Under the assumptions of Theorem B, for* $\nu \in (0, 1)$*, we have that*

$$H^1(L^{\nu}) = H^1_{at}(\mathcal{Q}_1 \boxtimes ... \boxtimes \mathcal{Q}_N).$$

Moreover, the corresponding norms are equivalent.

1.4 Applications

One of the goals of this paper is to verify the assumptions of Theorems B and C for various well-known operators. In this subsection we provide a list of such operators.

1.4.1 Bessel operator

For $\beta > 0$ let $L_B^{[\beta]} = -\frac{d^2}{dx} + \frac{\beta^2-\beta}{x^2}$ denote the one-dimensional Bessel operator on the positive half-line $X = (0, \infty)$ equipped with the Lebesgue measure. The semigroup $T_{B,t} = \exp(-tL_B^{[\beta]})$ is given by $T_{B,t} f(x) = \int_X T_{B,t}(x, y) f(y)\, dy$, where

$$T_{B,t}(x, y) = \frac{(xy)^{1/2}}{2t} I_{\beta-1/2}\left(\frac{xy}{2t}\right) \exp\left(-\frac{x^2+y^2}{4t}\right), x, y \in X, t > 0. \tag{1.8}$$

Here, I_τ is the modified Bessel function of the first kind. The Hardy space $H^1(L_B^{[\beta]})$ for the one-dimensional Bessel operator was studied in [2]. In Sect. 4.1 we check that the assumptions (A_0)–(A_2) are satisfied for L_B with the admissible covering

$$\mathcal{Q}_B = \left\{[2^n, 2^{n+1}] \ : n \in \mathbb{Z}\right\}$$

of $X = (0, \infty)$. This gives a slightly simpler proof of the characterizations of $H^1(L_B^{[\beta]})$ by the maximal operators of the semigroups $\exp(-tL_B^{[\beta]})$ and, also, gives a characterization by $\exp(-t(L_B^{[\beta]})^\nu)$, $0 < \nu < 1$. We have the following corollary for the multidimensional Bessel operator.

Corollary 1.9 *Let $\beta_1, ..., \beta_d > 0$ and $L_B = L_B^{[\beta_1]} + \cdots + L_B^{[\beta_d]}$, be the multidimensional Bessel operator on $L^2((0, \infty)^d)$. Then, the Hardy spaces $H^1(L_B)$, $H^1(L_B^\nu)$, $\nu \in (0, 1)$, and $H^1_{at}(\mathcal{Q}_B \boxtimes ... \boxtimes \mathcal{Q}_B)$ coincide (Fig. 1). Moreover, the associated norms are comparable.*

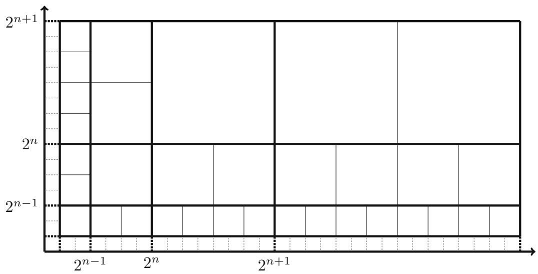

Fig. 1 The covering $\mathcal{Q}_B \boxtimes \mathcal{Q}_B$

1.4.2 Laguerre operator

Let $\alpha > -1/2$ and $L_L^{[\alpha]} = -\frac{d^2}{dx^2} + x^2 + \frac{\alpha^2 - 1/4}{x^2}$ denote the Laguerre operator on $X = (0, \infty)$. The kernels associated with the heat semigroup $T_{L,t} = \exp\left(-tL_L^{[\alpha]}\right)$ are defined by

$$T_{L,t}(x,y) = \frac{(xy)^{1/2}}{\sinh 2t} I_\alpha\left(\frac{xy}{\sinh 2t}\right) \exp\left(-\frac{\cosh 2t}{2\sinh 2t}(x^2+y^2)\right), x, y \in X, t > 0. \quad (1.10)$$

The one-dimensional version of $H^1\left(L_L^{[\alpha]}\right)$ was studied in [7]. The admissible covering is the following

$$\mathcal{Q}_L = \left\{[2^n + k2^{-n-1}, 2^n + (k+1)2^{-n-1}] \colon k = 0, \dots, 2^{2n+1} - 1, n \in \mathbb{N}\right\} \cup \left\{[2^{-n}, 2^{-n+1}] \colon n \in \mathbb{N}_+\right\},$$

see Fig. 2 for $Q_l \boxtimes Q_L$. Using methods similar to those in [7] we verify (A_0)–(A_2) in Sect. 4.2.

Corollary 1.11 *Let $\alpha_1, ..., \alpha_d > -1/2$ and $L_L = L_L^{[\alpha_1]} + \cdots + L_L^{[\alpha_d]}$, be the multidimensional Laguerre operator on $L^2((0,\infty)^d)$. Then, the Hardy spaces $H^1(L_L)$, $H^1(L_L^\nu)$, $\nu \in (0,1)$, and $H^1_{at}(\mathcal{Q}_L \boxtimes ... \boxtimes \mathcal{Q}_L)$ coincide. Moreover, the associated norms are comparable.*

1.4.3 Schrödinger operators

Let $L_S = -\Delta + V$ denote a Schrödinger operator on $\mathbb{R}^d$, where $V \in L^1_{loc}(\mathbb{R}^d)$ is a nonnegative potential. Since $V \geq 0$, we have

$$0 \leq T_{S,t}(x,y) \leq H_t(x,y), \qquad x, y \in \mathbb{R}^d, t > 0, \quad (1.12)$$

where $T_{S,t} = \exp(-tL_S)$ and $H_t = \exp(t\Delta)$, see (1.1). Following [11], for fixed V, we assume that there is an admissible covering $\mathcal{Q}_S$ of $\mathbb{R}^d$ that satisfies the following

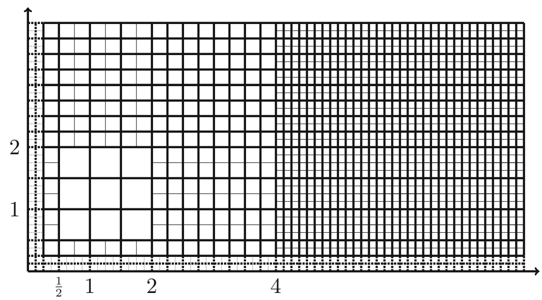

Fig. 2 The covering $\mathcal{Q}_L \boxtimes \mathcal{Q}_L$

conditions: there exist constants $\rho > 1$ and $\sigma > 0$ such that

$$\sup_{y\in Q^*}\int_{\mathbb{R}^d} T_{S,2^n d_Q^2}(x,y)\,dx \le C\rho^{-n}, \quad Q\in\mathcal{Q}_S,\ n\in\mathbb{N}, \tag{D'}$$

$$\sup_{y\in\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d} H_s(x,y)\chi_{Q^{***}}(x)V(x)\,dx\,ds \le C\left(\frac{t}{d_Q^2}\right)^{\sigma}, \quad Q\in\mathcal{Q}_S,\ t\le d_Q^2. \tag{K}$$

The Hardy spaces related to Schrödinger operators have been widely studied. It appears that for some potentials the atoms for $H^1(L_S)$ have local nature (as in our paper), but this is no longer true for other potentials. The interested reader is referred to [5,8,9,11–14,17].

In [11] the authors study potentials as above, but instead of assuming (D') they have a bit more general assumption (D), which instead of ρ^{-n} has an arbitrary summable sequence $(1+n)^{-1-\varepsilon}$ on the right-hand side of (D'). Moreover, the assumptions (D') and (K) are easy to generalize for products, see [8, Rem. 1.8]. Therefore, for Schrödinger operators Theorem B is a bit weaker than results of [11]. However, Theorem C gives additionally characterization by the semigroups $\exp\left(-tL_S^{\nu}\right)$, $0<\nu<1$, provided that the stronger assumption (D') is satisfied. Let us notice that indeed (D') is true for many examples, including L_S in dimension one with any nonnegative $V\in L^1_{loc}(\mathbb{R})$, see [5].

In Sect. 4.2 we prove that (D') and (K) imply the assumptions of Theorems B and C , which leads to the following.

Corollary 1.13 *Let L_S be given with a nonnegative $V\in L^1_{loc}(\mathbb{R}^d)$ and an admissible covering $\mathcal{Q}_S$ of $\mathbb{R}^d$. Assume that (D') and (K) are satisfied. Then the spaces $H^1(L_S)$, $H^1(L_S^{\nu})$, $\nu\in(0,1)$, and $H^1_{at}(\mathcal{Q}_S)$ coincide and the corresponding norms are equivalent.*

1.4.4 Product of local and nonlocal atomic Hardy space

As we have mentioned, all atoms on the Hardy space $H^1(\mathbb{R}^{d_1})$ satisfy cancellation condition, i.e. they are nonlocal atoms. However, if we consider the product $\mathbb{R}^d = \mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$ and the operator $L=-\Delta+L_2$, where L_2 and $\mathcal{Q}_2$ satisfies the assumptions (A_0)–(A_2) on $\mathbb{R}^{d_2}$ then the resulting Hardy space $H^1(L)$ shall have local character.

More precisely, if $\mathbb{R}^{d_1}\boxtimes\mathcal{Q}_2$ is the admissible covering that arise by splitting all the strips $\mathbb{R}^{d_1}\times Q_2$, $Q_2\in\mathcal{Q}_2$, into countable many cuboids $Q_{1,n}\times Q_2$, where $Q_{1,n}=Q(z_n,d_{Q_2})$. Then we have the following corollary (see Sect. 4.4).

Corollary 1.14 *Let $L=-\Delta+L_2$, where $-\Delta$ is the standard Laplacian on $\mathbb{R}^{d_1}$ and L_2 with an admissible covering $\mathcal{Q}_2$ of $\mathbb{R}^{d_2}$ satisfy (A_0)–(A_2). Then the spaces $H^1(L)$, $H^1(L^{\nu})$, $\nu\in(0,1)$, and $H^1_{at}(\mathbb{R}^{d_1}\boxtimes\mathcal{Q}_2)$ coincide and the corresponding norms are equivalent.*

1.5 Organization of the paper

The paper is organized in the following way. Section 2 is devoted to prove some preliminary estimates and to recall some known facts about local Hardy spaces on $\mathbb{R}^d$. In Sect. 3 we prove our main results, namely Theorems A, B, and C . In Sect. 4 we prove that the examples given in Sect. 1.4 satisfy assumptions (A_0)–(A_2). We use standard notation, i.e. C denotes some constant that can change from line to line.

2 Preliminaries

2.1 Auxiliary estimates

For an admissible covering $\mathcal{Q}$ of X let us denote for $Q \in \mathcal{Q}$ the functions $\psi_Q \in C^1(X)$ satisfying

$$0 \le \psi_Q(x) \le \chi_{Q^*}(x), \quad \left\|\psi_Q'\right\|_\infty \le C d_Q^{-1}, \quad \sum_{Q\in\mathcal{Q}} \psi_Q(x) = \chi_X(x). \tag{2.1}$$

It is easy to observe that such family $\left\{\psi_Q\right\}_{Q\in\mathcal{Q}}$ exists, provided that $\mathcal{Q}$ satisfies Definition 1.3. The family $\left\{\psi_Q\right\}_{Q\in\mathcal{Q}}$ shall be called *a partition of unity* related to $\mathcal{Q}$.

Proposition 2.2 *Assume that T_t, and an admissible covering $\mathcal{Q}$ satisfy (A_0') and (A_1'). Let ψ_Q be a partition of unity related to $\mathcal{Q}$. Then*

$$\sup_{y\in Q^*} \int_{Q^{**}} \sup_{t>d_Q^2} T_t(x,y)dx \le C, \quad Q \in \mathcal{Q}, \tag{2.3}$$

and

$$\sup_{y\in X} \sum_{Q\in\mathcal{Q}} \int_{Q^{**}} \sup_{t\le d_Q^2} T_t(x,y) \left|\psi_Q(x) - \psi_Q(y)\right| dx \le C. \tag{2.4}$$

Proof By (A_0') we have $T_t(x,y) \le Ct^{-d/2}$. Obviously, $|Q^{**}| \le C|Q| \le Cd_Q^d$, hence

$$\sup_{y\in Q^*} \int_{Q^{**}} \sup_{t>d_Q^2} T_t(x,y)\,dx \le \int_{Q^{**}} \sup_{t>d_Q^2} t^{-d/2}\,dx \le C.$$

□

We now turn to prove (2.4). Fix $y \in X$ and $Q_0 \in \mathcal{Q}$ such that $y \in Q_0$. Denote $N(Q_0) = \left\{Q \in \mathcal{Q} \ : \ Q_0^{***} \cap Q^{***} \neq \emptyset\right\}$ (the neighbors of Q_0) . Notice that $|N(Q_0)| \leq C$, see (1.3). Then

$$\sum_{Q\in\mathcal{Q}} \int_{Q^{**}} \left[\sup_{t\leq d_Q^2} T_t(x,y)\left|\psi_Q(x)-\psi_Q(y)\right|\right] dx = \sum_{Q\in N(Q_0)} \ldots + \sum_{Q\in\mathcal{Q}\setminus N(Q_0)} \ldots =: S_1 + S_2.$$

Notice that for $Q \in N(Q_0)$ we have $d_Q \simeq d_{Q_0}$. To deal with S_1 we use (A_0') and the mean value theorem for ψ_Q,

$$\begin{aligned}
&\sum_{Q\in N(Q_0)} \int_{Q^{**}} \sup_{t\leq d_Q^2} T_t(x,y)\left|\psi_Q(x)-\psi_Q(y)\right| dx \\
&\leq C \sum_{Q\in N(Q_0)} \int_{Q^{**}} \sup_{t>0} t^{\nu}\left(t+|x-y|^2\right)^{-d/2-\nu} \frac{|x-y|}{d_Q}\, dx \\
&\leq C \sum_{Q\in N(Q_0)} d_Q^{-1} \int_{Q^{**}} |x-y|^{-d+1}\, dx \\
&\leq C|N(Q_0)| d_{Q_0}^{-1} \int_{CQ_0} |x-y|^{-d+1}\, dx \leq C.
\end{aligned}$$

To estimate S_2 we use $\left\|\psi_Q\right\|_\infty \leq 1$ and (A_1'), getting

$$\begin{aligned}
\sum_{Q\in\mathcal{Q}\setminus N(Q_0)} \int_{Q^{**}} \sup_{t\leq d_Q^2} T_t(x,y)\left|\psi_Q(x)-\psi_Q(y)\right| dx &\leq 2 \sum_{Q\in\mathcal{Q}\setminus N(Q_0)} \int_{Q^{**}} \sup_{t>0} T_t(x,y)\, dx \\
&\leq C \int_{(Q_0^{**})^c} \sup_{t>0} T_t(x,y)\, dx \leq C.
\end{aligned}$$

Lemma 2.5 *Assume that T_t satisfy (A_0'). Then, for $f \in L^1(X) + L^\infty(X)$,*

$$\|f\|_{L^1(X)} \leq \left\|\sup_{t>0} |T_t f|\right\|_{L^1(X)}.$$

The proof of the Lemma 2.5 goes by standard arguments. For the convenience of the reader we present details in Appendix.

2.2 Local Hardy spaces

In this section, we recall some classical results on local Hardy spaces, see [16]. Let $\tau > 0$ be fixed. We are interested in decomposing into atoms a function f such that

$$\left\|\sup_{t\leq\tau^2} |H_t f|\right\|_{L^1(\mathbb{R}^d)} < \infty. \tag{2.6}$$

It is known, that (2.6) holds if and only if $f(x) = \sum_k \lambda_k a_k(x)$, where $\sum_k |\lambda_k| < \infty$ and a_k are either the classical atoms or the *local atoms at scale* τ. The latter are atoms a supported in a cube Q of diameter at most τ such that $\|a\|_\infty \le |Q|^{-1}$ but we do not impose the cancellation condition. In other words one may say that this is the space $H^1_{at}(\mathcal{Q}^{\{\tau\}})$ introduced in Sect. 1.2, where $\mathcal{Q}^{\{\tau\}}$ is a covering of $\mathbb{R}^d$ by cubes with diameter τ. The next proposition states the local atomic decomoposition theorem in a version that will be suitable for us in the proof of Theorem A. This proposition can be obtained by known methods from the global characterization of the classical Hardy space $H^1(\mathbb{R}^d)$. One may also check the assumptions from a general result of Uchiyama [23, Cor. 1']. The details are left for the interested reader.

Proposition 2.7 *Let $\tau > 0$ be fixed and $\widetilde{T}_t$ denote either H_t or $P_{t^\nu,\nu}$, see* (1.1) *and* (1.6). *Then, there exists $C > 0$ that does not depend on τ such that:*

1. *For every classical atom a or an atom of the form $a(x) = |Q|^{-1}\chi_Q(x)$, where $Q = Q(z, r_1, ..., r_d)$ is such that $r_1 \simeq ... \simeq r_d \simeq \tau$ we have*

$$\left\| \sup_{t \le \tau^2} \left| \widetilde{T}_t a \right| \right\|_{L^1(\mathbb{R}^d)} \le C.$$

2. *If f is such that $\operatorname{supp} f \subseteq Q^*$, where $Q = Q(z, r_1, ..., r_d)$ is such that $r_1 \simeq ... \simeq r_d \simeq \tau$, and*

$$\left\| \sup_{t \le \tau^2} \left| \widetilde{T}_t f \right| \right\|_{L^1(Q^*)} = M < \infty,$$

then there exist sequences $\{\lambda_k\}_k$ and $\{a_k(x)\}_k$, such that $f(x) = \sum_k \lambda_k a_k(x)$, $\sum_k |\lambda_k| \le CM$, and a_k are either the classical atoms supported in Q^* or $a_k(x) = |Q|^{-1}\chi_Q(x)$.

Remark 2.8 Proposition 2.7 remains valid for many other kernels $\widetilde{T}_t$ satisfying (A'_0) and, therefore, Theorem A holds for such kernels.

3 Proofs of Theorems A, B, and C

3.1 Proof of Theorem A

Proof Recall that by the assumptions and Proposition 2.2 we also have that (2.3) and (2.4) are satisfied. We shall prove two inclusions.

First inequality: $\|f\|_{H^1(L)} \le C \|f\|_{H^1_{at}(\mathcal{Q})}$. It suffices to show that for every $\mathcal{Q}$-atom a we have $\left\|\sup_{t>0} |T_t a|\right\|_{L^1(X)} \le C$, where C does not depend on a. Let a be associated with a cuboid $Q \in \mathcal{Q}$, i.e. $\operatorname{supp} a \subset Q^*$. Recall that $\widetilde{T}_t$ is either H_t or $P_{t^\nu,\nu}$, see (1.1) and (1.6). Observe that by using (A'_1), (A'_2), (2.3), and part 1. of Proposition 2.7 we get

$$\left\| \sup_{t>0} |T_t a| \right\|_{L^1(X)} \le \left\| \sup_{t>0} |T_t a| \right\|_{L^1((Q^{**})^c)} + \left\| \sup_{t\le d_Q^2} |(T_t - \widetilde{T}_t)a| \right\|_{L^1(Q^{**})} + \left\| \sup_{t>d_Q^2} |T_t a| \right\|_{L^1(Q^{**})} + \left\| \sup_{t\le d_Q^2} |\widetilde{T}_t a| \right\|_{L^1(Q^{**})} \le C.$$

Second inequality: $\|f\|_{H^1_{at}(\mathcal{Q})} \le C \|f\|_{H^1(L)}$. Assume that $\left\|\sup_{t>0} |T_t f|\right\|_{L^1(X)} < \infty$. Let ψ_Q be a partition of unity related to $\mathcal{Q}$, see (2.1). We have $f = \sum_{Q\in\mathcal{Q}} \psi_Q f$. Denote $f_Q = \psi_Q f$ and notice that since supp $f_Q \subset Q^*$, then

$$\widetilde{T}_t f_Q = (\widetilde{T}_t - T_t) f_Q + \big(T_t f_Q - \psi_Q \cdot T_t f\big) + \psi_Q \cdot T_t f. \tag{3.1}$$

Clearly,

$$\sum_{Q\in\mathcal{Q}} \left\| \sup_{t\le d_Q^2} \left|\psi_Q T_t f\right| \right\|_{L^1(Q^{**})} \le C \left\| \sup_{t>0} |T_t f| \right\|_{L^1(X)}. \tag{3.2}$$

Using (A_2'),

$$\sum_{Q\in\mathcal{Q}} \left\| \sup_{t\le d_Q^2} \left|(\widetilde{T}_t - T_t) f_Q\right| \right\|_{L^1(Q^{**})} \le C \sum_{Q\in\mathcal{Q}} \|f_Q\|_{L^1(X)} \le C \|f\|_{L^1(X)}. \tag{3.3}$$

By (2.4),

$$\begin{aligned} &\sum_{Q\in\mathcal{Q}} \left\| \sup_{t\le d_Q^2} \left|T_t f_Q - \psi_Q \cdot T_t f\right| \right\|_{L^1(Q^{**})} \\ &\le \sum_{Q\in\mathcal{Q}} \int_X |f(y)| \int_{Q^{**}} \sup_{t\le d_Q^2} T_t(x,y) \left|\psi_Q(y) - \psi_Q(x)\right| \, dx\, dy \\ &\le C \|f\|_{L^1(X)}. \end{aligned} \tag{3.4}$$

Using (3.1)–(3.4) and Lemma 2.5 we arrive at

$$\sum_{Q\in\mathcal{Q}} \left\| \sup_{t\le d_Q^2} \left|\widetilde{T}_t f_Q\right| \right\|_{L^1(Q^{**})} \le C \left\| \sup_{t>0} |T_t f| \right\|_{L^1(X)}.$$

Now, from part 2. of Proposition 2.7 for each f_Q we obtain $\lambda_{Q,k}, a_{Q,k}$. Then

$$f = \sum_Q f_Q = \sum_{Q,k} \lambda_{Q,k} a_{Q,k}$$

and

$$\sum_Q \sum_k |\lambda_{Q,k}| \le C \sum_{Q\in\mathcal{Q}} \left\| \sup_{t\le d_Q^2} |\widetilde{T}_t f_Q| \right\|_{L^1(Q^{**})} \le C \left\| \sup_{t>0} T_t f \right\|_{L^1(X)}.$$

Finally, we notice that all the atoms $a_{Q,k}$ obtained by Proposition 2.7 are indeed $\mathcal{Q}$-atoms. □

Remark 3.5 The assumption (A_0') has only been used in Proposition 2.2. Therefore, in Theorem A one may replace the assumption (A_0') by the pair of assumptions (2.3) and (2.4).

3.2 Proof of Theorem B

Proof We shall show the following claim. If the assumptions (A_0)–(A_2) hold for $T_t^{[j]}(x_j, y_j)$ together with admissible coverings $\mathcal{Q}_j$ for $j = 1, 2$, then (A_0)–(A_2) also hold for $T_t(x, y) = T_t^{[1]}(x_1, y_1) \cdot T_t^{[2]}(x_2, y_2)$, together with $\mathcal{Q} = \mathcal{Q}_1 \boxtimes \mathcal{Q}_2$. This is enough, since by simple induction we shall get that in the general case $T_t(x, y) = T_t^{[1]}(x_1, y_1)\cdot\ldots\cdot T_t^{[N]}(x_N, y_N)$ with $\mathcal{Q}_1\boxtimes\ldots\boxtimes\mathcal{Q}_N$ satisfy (A_0)–(A_2), and, consequently, the assumptions of Theorem A will be fulfilled.

To prove the claim let $T_t^{[j]}(x_j, y_j)$ and $\mathcal{Q}_j$ satisfy (A_0)–(A_2) with γ_j for $j = 1, 2$. Let $0 < \gamma < \min(\gamma_1, \gamma_2)$ and fix $\delta \in [0, \gamma)$. Suppose that $\mathcal{Q} \ni Q \subseteq Q_1 \times Q_2$, where $Q_1 \in \mathcal{Q}_1$, $Q_2 \in \mathcal{Q}_2$, and without loss of generality we may assume that $d_{Q_1} \ge d_{Q_2}$. Hence, $Q = K \times Q_2$, where $K \subseteq Q_1$, see Definition 1.5 and Fig. 3. Denote by $z = (z_1, z_2)$ the center of $Q = K \times Q_2$. Obviously, (A_0) for the product follows from (A_0) for the factors.

Proof of (A_1) for $L_1 + L_2$. Let $y \in Q^*$. Recall that $d_Q \simeq d_K \simeq d_{Q_2} \le d_{Q_1}$. Let us write $(Q^{**})^c = S_1 \cup S_2 \cup S_3$, where

$$S_1 = (K^{**})^c \times Q_2^{**}, \quad S_2 = K^{**} \times (Q_2^{**})^c, \quad S_3 = (K^{**})^c \times (Q_2^{**})^c.$$

We start with S_1.

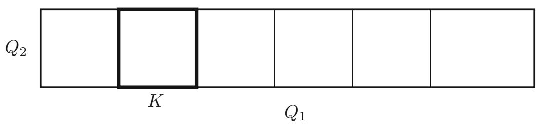

Fig. 3 Partition of $Q_1 \times Q_2$

$$\begin{aligned}\int_{S_1}\sup_{t>0} t^{\delta} T_t^{[1]}(x_1,y_1)T_t^{[2]}(x_2,y_2)\,dx &\le C\int_{(K^{**})^c}\sup_{t>0} t^{-d_1/2-1/2}\exp\left(-\frac{|x_1-y_1|^2}{ct}\right)dx_1\\ &\quad\cdot\int_{Q_2^{**}}\sup_{t>0} t^{-d_2/2+1/2+\delta}\exp\left(-\frac{|x_2-y_2|^2}{ct}\right)dx_2\\ &\le C\int_{(K^{**})^c}|x_1-z_1|^{-d_1-1}\,dx_1\cdot\int_{Q_2^{**}}|x_2-z_2|^{-d_2+1+2\delta}\,dx_2\\ &\le Cd_K^{-1}\cdot d_{Q_2}^{1+2\delta}=Cd_Q^{2\delta}.\end{aligned}$$

The set S_2 is treated similarly. To estimate S_3 recall that $\delta<\gamma$. Using (A_0) for $T_t^{[1]}(x_1,y_1)$ and (A_1) for $T_t^{[2]}(x_2,y_2)$ we arrive at

$$\begin{aligned}\int_{S_3}\sup_{t>0} t^{\delta} T_t^{[1]}(x_1,y_1)T_t^{[2]}(x_2,y_2)\,dx &\le C\int_{(K^{**})^c}\sup_{t>0} t^{-\gamma+\delta-d_1/2}\exp\left(-\frac{|x_1-y_1|^2}{ct}\right)dx_1\\ &\quad\cdot\int_{(Q_2^{**})^c}\sup_{t>0} t^{\gamma}T_t^{[2]}(x_2,y_2)\,dx_2\\ &\le Cd_K^{-2\gamma+2\delta}d_{Q_2}^{2\gamma}\le Cd_Q^{2\delta}.\end{aligned}$$

Proof of (A_2) for L_1+L_2. Let $y\in Q^*$. In this proof H_t is the classical heat semigroup on $\mathbb{R}^{d_1}$, $\mathbb{R}^{d_2}$ or on $\mathbb{R}^{d}$, depending on the context. First, notice that by (A_0), for constant $C>1$ and $i=1,2$, we have

$$\begin{aligned}&\int_{Q_i^{**}}\sup_{C^{-1}d_{Q_i}^2\le t\le Cd_{Q_i}^2} t^{-\gamma}\left|T_t^{[i]}(x_i,y_i)-H_t(x_i,y_i)\right|dx_i\\ &\le Cd_{Q_i}^{-2\gamma}\int_{Q_i^{**}} d_{Q_i}^{-d_i}\exp\left(-\frac{|x_i-y_i|^2}{cd_{Q_i}^2}\right)dx_i\\ &\le Cd_{Q_i}^{-2\gamma}.\end{aligned}\tag{3.6}$$

Using the triangle inequality,

$$\int_{Q^{**}}\sup_{t\le d_Q^2} t^{-\delta}\left|T_t(x,y)-H_t(x,y)\right|dx\le I_1+I_2,$$

where

$$\begin{aligned}I_1&=\int_{Q^{**}}\sup_{t\le d_Q^2} t^{-\delta}T_t^{[1]}(x_1,y_1)\left|T_t^{[2]}(x_2,y_2)-H_t(x_2,y_2)\right|dx,\\ I_2&=\int_{Q^{**}}\sup_{t\le d_Q^2} t^{-\delta}H_t(x_2,y_2)\left|T_t^{[1]}(x_1,y_1)-H_t(x_1,y_1)\right|dx.\end{aligned}$$

Applying (A_0) for $T_t^{[1]}(x_1, y_1)$ and (A_2) together with (3.6) for $T_t^{[2]}(x_2, y_2)$,

$$I_1 \le C \int_{K^{**}} \sup_{t \le d_Q^2} t^{\gamma-\delta} T_t^{[1]}(x_1, y_1)\, dx_1 \cdot \int_{Q_2^{**}} \sup_{t \le C d_{Q_2}^2} t^{-\gamma} \left| T_t^{[2]}(x_2, y_2) - H_t(x_2, y_2) \right| dx_2$$
$$\le C d_K^{2\gamma-2\delta} d_{Q_2}^{-2\gamma} \simeq C d_Q^{-2\delta},$$

since $0 \le \delta < \gamma < \min(\gamma_1, \gamma_2)$. Similarly, by (1.1), ($A_2$), and (3.6), we have

$$I_2 \le C \int_{Q_2^{**}} \sup_{t \le d_Q^2} t^{\gamma-\delta} H_t(x_2, y_2)\, dx_2 \cdot \int_{Q_1^{**}} \sup_{t \le C d_{Q_1}^2} t^{-\gamma} \left| T_t^{[1]}(x_1, y_1) - H_t(x_1, y_1) \right| dx_1$$
$$\le C d_Q^{2\gamma-2\delta} d_{Q_1}^{-2\gamma} \le C d_Q^{-2\delta},$$

since $d_{Q_1} \ge d_{Q_2} \simeq d_Q$. □

3.3 Proof of Theorem C

Proof For $\nu \in (0, 1)$ the subordination formula introduced by Bochner [3] states that

$$P_{t^\nu,\nu}(x, y) = \int_0^\infty H_{ts}(x, y) d\mu_\nu(s), \tag{3.7}$$

and

$$K_{t^\nu,\nu}(x, y) = \int_0^\infty T_{ts}(x, y) d\mu_\nu(s), \tag{3.8}$$

where μ_ν is a probability measure defined by the means of the Laplace transform $\exp(-x^\nu) = \int_0^\infty \exp(-xs) d\mu_\nu(s)$. By inverting the Laplace transform one obtains that $d\nu(s) = g_\nu(s)\, ds$ with

$$0 \le g_\nu(s) = \int_0^\infty \exp\left(ws \cos\theta_\nu + w^\nu \cos\theta_\nu\right) \sin\left(sw \sin\theta_\nu - w^\nu \sin\theta_\nu + \theta_\nu\right) dw, \quad s > 0,$$

where $\theta_\nu = \frac{\pi}{1+\nu} \in (\frac{\pi}{2}, \pi)$, see [25, Rem. 1]. Notice that $\cos\theta_\nu < 0$ and, therefore,

$$g_\nu(s) \le \left| \int_0^{s^{-1}} \dots dw \right| + \left| \int_{s^{-1}}^\infty \dots dw \right| \le \int_0^{s^{-1}} dw + \int_{s^{-1}}^\infty \exp(ws \cos\theta_\nu)\, dw \le C s^{-1}. \tag{3.9}$$

Assume that T_t and $\mathcal{Q}$ satisfy (A_0)–(A_2). Then, Theorem C follows from Theorem A, provided that we prove (A_0')–(A_2') for $K_{t^\nu,\nu}$ and $\mathcal{Q}$. First, notice that (A_0') for $K_{t^\nu,\nu}$ follows from (3.8) and (A_0) for T_t. Coming to (A_1'), let $Q \in \mathcal{Q}$ and $y \in Q^*$. Since μ_ν is a probability measure, using (3.8) and (A_1') for T_t, we obtain

$$\int_{(Q^{**})^c} \sup_{t>0} K_{t^\nu,\nu}(x,y)\,dx = \int_{(Q^{**})^c} \sup_{t>0} \int_0^\infty T_{st}(x,y)d\mu_\nu(s)\,dx$$
$$\leq \int_0^\infty \int_{(Q^{**})^c} \sup_{t>0} T_{st}(x,y)\,dx\,d\mu_\nu(s) \leq C.$$

Having (A'_1) proved, we turn to (A'_2). By (3.7)–(3.9), and (A_2) for T_t, we have

$$\int_{Q^{**}} \sup_{t\leq d_Q^2} \left|K_{t^\nu,\nu}(x,y) - P_{t^\nu,\nu}(x,y)\right|\,dx$$
$$= \int_{Q^{**}} \sup_{t\leq d_Q^2} \left|\int_0^\infty \left(T_u(x,y) - H_u(x,y)\right)\, g_\nu(u/t)\,\frac{du}{t}\right|\,dx$$
$$\leq C \int_{Q^{**}} \sup_{t\leq d_Q^2} \int_0^\infty |T_u(x,y) - H_u(x,y)|\,(u/t)^{-1}\,\frac{du}{t}\,dx$$
$$\leq C \int_{Q^{**}} \int_0^{d_Q^2} |T_u(x,y) - H_u(x,y)|\,\frac{du}{u}\,dx$$
$$+C \int_{Q^{**}} \int_{d_Q^2}^\infty |T_u(x,y) - H_u(x,y)|\,\frac{du}{u}\,dx$$
$$\leq C \int_0^{d_Q^2} u^{-1+\delta} \int_{Q^{**}} \sup_{u\leq d_Q^2} u^{-\delta}\,|T_u(x,y) - H_u(x,y)|\,dx\,du$$
$$+C \int_{Q^{**}} \int_{d_Q^2}^\infty u^{-d/2-1}\,du\,dx$$
$$\leq C d_Q^{-2\delta} \int_0^{d_Q^2} u^{-1+\delta}\,du + C d_Q^d d_Q^{-d} \leq C.$$

This ends the proof of Theorem C. □

Remark 3.10 It is worth to notice, that in the proof of (A'_2) for the subordinate semigroup $K_{t,\nu}$ we needed (A_2) for T_t, not only (A'_2).

4 Applications

In this section for simplicity, we use the same notation $T_t(x,y)$ for the integral kernels of semigroups generated by different operators.

4.1 Bessel operator

Let us start with the following asymptotics of the Bessel function I_τ,

$$I_\tau(x) = C_\tau x^\tau + O(x^{\tau+1}),\ \text{for } x\sim 0, \tag{4.1}$$

$$I_\tau(x) = (2\pi x)^{-1/2} e^x + O(x^{-3/2} e^x), \ \text{for } x \sim \infty, \tag{4.2}$$

see e.g. [24, pp. 203–204].

Proposition 4.3 *Let $X = (0, \infty)$ and $\beta > 0$. Then (A_0)–(A_2) hold for $L_B^{[\beta]}$ with $\mathcal{Q}_B$.*

Proof We shall use similar ideas to those of [2]. The proof of (A_0) is well-known and follows almost directly from (1.8), (4.1) and (4.2). We skip the details. Let $\gamma \in (0, \min(1/2, \beta/2))$ and $\delta \in [0, \gamma)$. Take $\mathcal{Q}_B \ni Q = [2^n, 2^{n+1}]$, for some $n \in \mathbb{Z}$, and fix $y \in Q^*$.

Proof of (A_1). Notice that $y \simeq d_Q \simeq 2^n$. We have

$$\int_{(Q^{**})^c} \sup_{t>0} t^\delta T_t(x, y)\, dx \le \int_0^\infty \sup_{t>xy} t^\delta T_t(x, y)\, dx + \int_{(Q^{**})^c} \sup_{t \le xy} t^\delta T_t(x, y)\, dx =: I_1 + I_2.$$

Using (1.8) and (4.1), we obtain

$$\begin{aligned} I_1 &\le C \int_0^\infty \sup_{t>xy} (xy)^\beta t^{\delta-\beta-1/2} \exp\left(-\frac{x^2+y^2}{4t}\right) dx \\ &\le C \int_0^\infty (xy)^\beta (x^2+y^2)^{\delta-\beta-1/2}\, dx \\ &= C y^{2\delta} \int_0^\infty x^\beta (x^2+1)^{\delta-\beta-1/2}\, dx \le C d_Q^{2\delta}, \end{aligned}$$

where in the last inequality we used the fact that $2\delta < \beta$.

Denote $z = 3 \cdot 2^{n-1}$ (the center of Q). By (1.8) and (4.2),

$$\begin{aligned} I_2 &\le C \int_{(Q^{**})^c} \sup_{t \le xy} t^{\delta-1/2} \exp\left(-\frac{|x-y|^2}{4t}\right) dx \\ &\simeq C \int_{(Q^{**})^c} \sup_{t \le xy} t^{\delta-1/2} \exp\left(-\frac{|x-z|^2}{ct}\right) dx \\ &\le C \int_0^{2^n} \sup_{t>0} t^{\delta-1/2} \exp\left(-\frac{z^2}{c_1 t}\right) dx + C \int_{2^{n+1}}^\infty \sup_{t \le xy} t^{\delta-1/2} \exp\left(-\frac{x^2}{c_2 t}\right) dx \\ &\le C z^{2\delta-1} 2^n + C \int_{2^{n+1}}^\infty (xy)^{\delta-1/2} \exp\left(-\frac{x}{c_2 y}\right) dx \\ &\le C d_Q^{2\delta}. \end{aligned}$$

Proof of (A_2). Now observe that if $y \in Q^*$ and $x \in Q^{**}$, then $x \simeq y \simeq d_Q$. Therefore, $\frac{xy}{2t} \ge c$, when $t \le d_Q^2$. Using (1.8), (4.2), and $\delta < 1/2$, we arrive at

$$\int_{Q^{**}} \sup_{t \le d_Q^2} t^{-\delta} \left|T_t(x, y) - H_t(x, y)\right| dx$$

$$
\begin{aligned}
&\leq \int_{Q^{**}} \frac{\sqrt{xy}}{2} \sup_{t\leq d_Q^2} t^{-1-\delta} \exp\left(-\frac{x^2+y^2}{4t}\right) \left| I_{\beta-\frac{1}{2}}\left(\frac{xy}{2t}\right) - \frac{e^{\frac{xy}{2t}}}{\sqrt{\frac{\pi xy}{t}}} \right| dx \\
&\leq C \int_{Q^{**}} \sup_{t\leq d_Q^2} t^{1/2-\delta}(xy)^{-1} \exp\left(-\frac{|x-y|^2}{4t}\right) dx \\
&\leq C d_Q^{1-2\delta} \cdot d_Q^{-2} \cdot d_Q \leq C d_Q^{-2\delta}.
\end{aligned}
$$

□

4.2 Laguerre operator

Using the asymptotic estimates for the Bessel function (4.1) and (4.2) in formula (1.10), one can obtain

$$
T_t(x,y) \leq C t^{-1/2} \exp\left(-c\frac{|x-y|^2}{t}\right) e^{-ctxy} \min(1, (xy/t)^{\alpha+1/2}), \qquad x, y \in X, t > 0, \tag{4.4}
$$

see [7, Eq. (2.12) and (2.13)].

Proposition 4.5 *Let $X = (0,\infty)$ and $\alpha > -1/2$. Then (A_0)–(A_2) hold for $L_L^{[\alpha]}$ with $\mathcal{Q}_L$.*

Proof We shall use similar estimates to those of [7]. Note that (A_0) follows immediately from (4.4). Let us fix positive constants $\gamma < \min(1/4, \alpha/2+1/4)$ and $\delta \in [0,\gamma)$. Fix $Q \in \mathcal{Q}_L$ and $y \in Q^*$.

Proof of (A_1). We write

$$
\int_{(Q^{**})^c} \sup_{t>0} t^{\delta} T_t(x,y)\, dx = \int_{(Q^{**})^c \cap (0,d_Q)} \ldots + \int_{(Q^{**})^c \cap (d_Q,\infty)} \ldots =: I_1 + I_2.
$$

Since $|x-y| \geq C d_Q$ and $\delta < 1/2$, we have

$$
\begin{aligned}
I_1 &\leq C \int_{(Q^{**})^c \cap (0,d_Q)} \sup_{t>0} t^{\delta-1/2} \exp\left(-\frac{|x-y|^2}{ct}\right) dx \\
&\leq C \int_{(Q^{**})^c \cap (0,d_Q)} |x-y|^{2\delta-1}\, dx \\
&\leq C d_Q^{2\delta-1} d_Q \leq C d_Q^{2\delta}.
\end{aligned}
$$

In order to estimate I_2 we consider two cases depending on the localization of Q.

Case 1: $Q = [2^{-n}, 2^{-n+1}]$, $n \in \mathbb{N}_+$. In this case $y \simeq d_Q = 2^{-n}$. Observe that if $x \in (Q^{**})^c \cap (d_Q, \infty)$, then $|x - y| \sim x$ and

$$\sup_{t>0} t^{\delta} T_t(x, y) \leq C \sup_{t>0} t^{\delta-1/2} \left(\frac{xy}{t}\right)^{\alpha+1/2} \exp\left(-\frac{x^2}{ct}\right) \\ \leq C d_Q^{\alpha+1/2} x^{2\delta-\alpha-3/2}.$$

Therefore, $I_2 \leq C d_Q^{\alpha+1/2} \int_{d_Q}^{\infty} x^{2\delta-\alpha-3/2}\, dx \leq C d_Q^{2\delta}$, since $\delta \leq \alpha/2 + 1/4$.

Case 2: $Q \subset [2^n, 2^{n+1}]$, $n \in \mathbb{N}$. Then $y^{-1} \simeq d_Q \simeq 2^{-n}$. Recall that $\delta < 1/2$. By using the inequality $\exp(-cxyt) \leq C(xyt)^{-1}$ in (4.4), we get

$$\begin{aligned} I_2 &\leq C \int_{(Q^{**})^c \cap (d_Q, \infty)} \sup_{t>0} (xy)^{-1} t^{\delta-3/2} \exp\left(-\frac{|x-y|^2}{ct}\right) dx \\ &\leq C d_Q \int_{(Q^{**})^c \cap (d_Q, \infty)} x^{-1} |x-y|^{2\delta-3}\, dx \\ &\leq C d_Q d_Q^{2\delta-1} \int_{(Q^{**})^c \cap (d_Q, \infty)} x^{-1} |x-y|^{-2}\, dx \\ &\leq C d_Q^{2\delta} \left(\int_{(Q^{**})^c \cap \left(d_Q, d_Q^{-1}/4\right)} d_Q^{-1} y^{-2}\, dx \right. \\ &\quad \left. + \int_{(Q^{**})^c \cap \left(d_Q^{-1}/4,\ \infty\right)} d_Q |x-y|^{-2}\, dx \right) \leq C d_Q^{2\delta}. \end{aligned}$$

Proof of (A_2). For $x \in Q^{**}$, $y \in Q^*$ and $t \leq d_Q^2$, we apply an estimate that can be deduced from the proof of [7, Prop. 2.3], namely

$$|T_t(x, y) - H_t(x, y)| \leq C t^{1/2} \left(xy + (xy)^{-1}\right) \leq C t^{1/2} d_Q^{-2},$$

where the second inequality follows from the relation between d_Q and the center of Q. Thus, for $\delta < 1/2$,

$$\int_{Q^{**}} \sup_{t<d_Q^2} t^{-\delta} |T_t(x, y) - H_t(x, y)|\, dx \leq C d_Q^{-2} \int_{Q^{**}} \sup_{t<d_Q^2} t^{1/2-\delta}\, dx \leq C d_Q^{-2\delta}.$$

□

4.3 Schrödinger operator

This subsection is devoted to proving the following proposition.

Proposition 4.6 *Let $L_S = -\Delta + V$ be a Schrödinger operator with $0 \leq V \in L^1_{loc}(\mathbb{R}^d)$. Assume that for some admissible covering $\mathcal{Q}_S$ the conditions* (D') *and* (K) *hold. Then* (A_0)–(A_2) *are satisfied for L_S and $\mathcal{Q}_S$.*

Proof In the proof we use estimates similar to those in [11]. For the completeness we present all the details. As we have already mentioned in (1.12), (A_0) holds since $V \geq 0$. Let us fix a positive $\gamma < \min(\log_2 \rho, \sigma)$, where ρ and σ are as in (D') and (K), see Sect. 1.4.3. Consider $Q \in \mathcal{Q}_S$, $\delta \in [0, \gamma)$, and $y \in Q^*$.

Proof of (A_1). We have that

$$\int_{(Q^{**})^c} \sup_{t>0} t^\delta T_t(x,y)\,dx \leq \int_{(Q^{**})^c} \sup_{t \leq 4d_Q^2} t^\delta T_t(x,y)\,dx + \sum_{n\geq 2} \int_X \sup_{2^n d_Q^2 < t \leq 2^{n+1} d_Q^2} t^\delta T_t(x,y)\,dx$$
$$=: I_1 + I_2.$$

Denote by z the center of the cube Q. For $y \in Q^*$ and $x \notin Q^{**}$ we have $d_Q \leq C|x-y| \simeq |x-z|$. Using ($A_0$) we obtain that

$$\begin{aligned} I_1 &\leq C \int_{(Q^{**})^c} \sup_{t \leq 4d_Q^2} t^{-d/2+\delta} \exp\left(-\frac{|x-z|^2}{ct}\right) dx \\ &\leq C \int_{(Q^{**})^c} d_Q^{-d+2\delta} \exp\left(-\frac{|x-z|^2}{c\,d_Q^2}\right) dx \leq C d_Q^{2\delta}. \end{aligned}$$

By (A_0) and (D'),

$$\begin{aligned} I_2 &\leq \sum_{n\geq 2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{2^n d_Q^2 < t \leq 2^{n+1} d_Q^2} t^\delta T_{t-2^{n-1}d_Q^2}(x,u) T_{2^{n-1}d_Q^2}(u,y)\,du\,dx \\ &\leq C \sum_{n\geq 1} (2^n d_Q^2)^\delta \int_{\mathbb{R}^d} T_{2^n d_Q^2}(u,y) \underbrace{\int_{\mathbb{R}^d} (2^n d_Q^2)^{-d/2} \exp\left(-\frac{|x-u|^2}{c2^n d_Q^2}\right) dx}_{\leq C}\, du \\ &\leq C d_Q^{2\delta} \sum_{n\geq 1} 2^{\delta n} \rho^{-n} \leq C d_Q^{2\delta}, \end{aligned}$$

where in the last inequality we have used that $2^\delta < \rho$.

Proof of (A_2). As in [11, Lem. 3.11] we write $V = \chi_{Q^{***}} V + \chi_{(Q^{***})^c} V =: V' + V''$. The perturbation formula states that $H_t(x,y) - T_t(x,y) = \int_0^t \int_{\mathbb{R}^d} H_{t-s}(x,u) V(u) T_s(u,y)\,du\,ds$, so

$$\begin{aligned} t^{-\delta} |H_t(x,y) - T_t(x,y)| &= t^{-\delta} \int_{\mathbb{R}^d} \int_0^t H_{t-s}(x,u) V''(u) T_s(u,y)\,ds\,du \\ &\quad + t^{-\delta} \int_{\mathbb{R}^d} \int_0^{t/2} H_{t-s}(x,u) V'(u) T_s(u,y)\,ds\,du \\ &\quad + t^{-\delta} \int_{\mathbb{R}^d} \int_{t/2}^t H_{t-s}(x,u) V'(u) T_s(u,y)\,ds\,du \\ &=: I_3(x,y) + I_4(x,y) + I_5(x,y). \end{aligned}$$

For $0 < s < t \le d_Q^2$, $x \in Q^{**}$, $u \in (Q^{***})^c$, we have that $d_Q \le C|x-u|$ and

$$t^{-\delta} H_{t-s}(x,u) \le (t-s)^{-\delta} H_{t-s}(x,u) \le C d_Q^{-d-2\delta} \exp\left(-\frac{|x-u|^2}{c\, d_Q^2}\right)$$

and, consequently,

$$\begin{aligned}\int_{Q^{**}} \sup_{t\le d_Q^2} I_3(x,y)\,dx &\le C \int_{Q^{**}} \int_{\mathbb{R}^d} \int_0^\infty d_Q^{-d-2\delta} \exp\left(-\frac{|x-u|^2}{c\, d_Q^2}\right) V''(u) T_s(u,y)\,ds\,du\,dx \\ &\le C d_Q^{-2\delta} \int_{\mathbb{R}^d} \int_0^\infty V''(u) T_s(u,y)\,ds\,dz \\ &\le C d_Q^{-2\delta}.\end{aligned}$$

In the last inequality we have used equivalent form of [11, Lem. 3.10]. To estimate I_4, denote $t_j = 2^{-j} d_Q^2$ for $j \ge 1$. Notice that

$$\begin{aligned}I_{4,j}(x,y) := \sup_{t_j\le t\le t_{j-1}} I_4(x,y) &\le C \sup_{t_j\le t\le t_{j-1}} \int_{\mathbb{R}^d} \int_0^{t/2} (t-s)^{-\delta} H_{t-s}(x,u) V'(u) T_s(u,y)\,ds\,du \\ &\le C \int_0^{t_j} \int_{\mathbb{R}^d} t_j^{-d-\delta} \exp\left(-\frac{|x-u|^2}{c\,t_j}\right) V'(u) H_s(u,y)\,du\,ds.\end{aligned} \tag{4.7}$$

Using (4.7) and then applying (K) we obtain

$$\begin{aligned}\int_{Q^{**}} \sup_{t\le d_Q^2} I_4(x,y)\,dx &\le \sum_{j\ge 1} \int_{\mathbb{R}^d} \sup_{t_j\le t\le t_j} I_{4,j}(x,y)\,dx \\ &\le C \sum_{j\ge1} t_j^{-\delta} \int_{\mathbb{R}^d} \int_0^{t_j} \underbrace{\int_{\mathbb{R}^d} t_j^{-d} \exp\left(-\frac{|x-u|^2}{c\,t_j}\right) dx}_{\le C}\, V'(u) H_s(u,y)\,ds\,du \\ &\le C d_Q^{-2\delta} \sum_{j\ge1} 2^{j\delta} \left(\frac{t_j}{d_Q^2}\right)^\sigma \le C d_Q^{-2\delta} \sum_{j\ge1} 2^{-j(\sigma-\delta)} \le C d_Q^{-2\delta},\end{aligned}$$

since $\delta < \sigma$. Finally, $I_5(x,y)$ can be estimated by a similar argument. We skip the details. □

4.4 Products of local and nonlocal atomic Hardy spaces

In this section we consider operator $L = -\Delta + L_2$, where $-\Delta$ is the standard Laplacian on $\mathbb{R}^{d_1}$ and L_2 together with an admissible covering $\mathcal{Q}_2$ of $X_2 \subseteq \mathbb{R}^{d_2}$ satisfies (A_0)–(A_2). Obviously, the kernel of $\exp(-tL)$ is given by $T_t(x,y) = H_t(x_1,y_1) \cdot T_t^{[2]}(x_2,y_2)$, where $x = (x_1,x_2) \in \mathbb{R}^{d_1} \times X_2 \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} = \mathbb{R}^d$. One immediately see that $T_t(x,y)$ satisfies (A_0). Moreover, almost identical argument as in the proof of Theorem B shows that T_t with $\mathcal{Q} = \mathbb{R}^d \boxtimes \mathcal{Q}_2$ satisfies (A_1) and (A_2). The details are left to the interested reader.

Acknowledgements The authors would like to thank Jacek Dziubański, Błażej Wróbel, and the reviewers for helpful comments.

Appendix

This appendix is devoted to prove Lemma 2.5. This proof uses standard methods, see e.g. [20]. We present details for the sake of completeness. In fact we prove a more general Proposition 4.12, from which Lemma 2.5 follows immediately. Recall that we consider a semigroup of operators T_t that is strongly continuous on $L^2(X)$ and has integral kernel $T_t(x, y)$ satisfying (A_0'). We start with the following lemma.

Lemma 4.8 *Suppose that T_t satisfies (A_0'). There exists a sequence $\{t_n\}_n$ such that $t_n \to 0$ and for every $r > 0$ we have:*

$$\lim_{n\to\infty} \int_{|x-y|>r} T_{t_n}(x, y)\, dy = 0, \tag{4.9}$$

$$\lim_{n\to\infty} \int_{|x-y|\le r} T_{t_n}(x, y)\, dy = 1, \tag{4.10}$$

for a.e. $x \in X$.

Proof Let $\nu \in (0, 1)$ be the constant from (A_0'). Observe that

$$\begin{aligned}\int_{|x-y|>r} T_t(x, y)\, dy &\le C \int_{|x-y|>r} \frac{t^\nu}{(t + |x-y|^2)^{\frac{d}{2}+\nu}}\, dy \\ &= C \int_{|y|>\frac{r}{\sqrt{t}}} (1 + |y|^2)^{-d/2-\nu}\, dy \to 0,\end{aligned}$$

as $t \to 0$, and (4.9) is proved (for every $\{t_n\}_n$ such that $t_n \to 0$).

To show (4.10) observe that for $f \in L^2(X)$ we have $\lim_{t\to 0} T_t f$ converges to f in $L^2(X)$, so we can choose a sequence with a.e. convergence. Applying this to functions $f_n(x) = \chi_{Q(0,n)}(x)$ and using a diagonal argument we obtain a sequence $\{t_n\}_n$, which goes to 0, and such that for a.e. $x \in X$ we have

$$\lim_{n\to\infty} \int_X T_{t_n}(x, y)\, dy = 1. \tag{4.11}$$

Thus, (4.10) follows from (4.11) and (4.9). □

Proposition 4.12 *Assume that T_t satisfies (A_0') and let $f \in L^1(X) + L^\infty(X)$. There exists a sequence $\{t_n\}_n$ such that $t_n \to 0$ and for almost every $x \in X$,*

$$\lim_{n\to\infty} T_{t_n} f(x) = f(x).$$

Proof Let $\{t_n\}_n$ be the sequence from Lemma 4.8. By the Lebesgue differentiation theorem we have

$$\lim_{s\to 0} |Q(x,s)|^{-1} \int_{Q(x,s)} |f(y) - f(x)|\, dy = 0 \tag{4.13}$$

for almost every $x \in X$, since $f \in L^1(X) + L^\infty(X) \subset L^1_{\mathrm{loc}}(X)$. Consider the set A of points $x \in X$ such that we have (4.13), and, additionally, (4.9)–(4.10) hold for all rational $r > 0$. Obviously, such set has full measure. Fix $\varepsilon > 0$ and $x \in A$. We will show that $\left|T_{t_n} f(x) - f(x)\right| \le C\varepsilon$ for large $n \in \mathbb{N}$. Let $r > 0$ be a fixed rational number such that for $s < r$ we have

$$\int_{Q(x,s)} |f(y) - f(x)|\, dy \le \varepsilon\, |Q(x,s)|\,. \tag{4.14}$$

Assume that $\sqrt{t_n} < r$ for large n. Write

$$\begin{aligned} T_{t_n} f(x) - f(x) &= f(x)\left(\int_{|x-y|\le r} T_{t_n}(x,y)\, dy - 1\right) + \int_{|x-y|>r} T_{t_n}(x,y) f(y)\, dy \\ &\quad + \int_{|x-y|<\sqrt{t_n}} T_{t_n}(x,y)\,(f(y) - f(x))\, dy \\ &\quad + \int_{\sqrt{t_n}\le|x-y|\le r} T_{t_n}(x,y)\,(f(y) - f(x))\, dy \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Applying (4.10) we obtain that $|I_1| < \varepsilon$ for n large enough. To treat I_2 we consider two cases.

Case 1: $f \in L^\infty$. Using (4.9) we have that $|I_2| < \varepsilon$ for n large enough.
Case 2: $f \in L^1$. By (A_0'),

$$|I_2| \le C \int_{|x-y|>r} \frac{t_n^\nu}{(t_n + |x-y|^2)^{d/2+\nu}}\, |f(y)|\, dy \le C \frac{t_n^\nu}{(t_n + r^2)^{d/2+\nu}}\, \|f\|_{L^1(X)} < \varepsilon,$$

for t_n small enough. To estimate I_3 observe that $T_{t_n}(x,y) \le C{t_n}^{-d/2}$ and $\left|Q(x,\sqrt{t_n})\right| \simeq t_n^{d/2}$. Since $\sqrt{t_n} < r$, by applying (4.14) we obtain

$$|I_3| \le C t_n^{-d/2} \int_{|x-y|<\sqrt{t_n}} |f(y) - f(x)|\, dy < C\varepsilon.$$

To deal with I_4 let $N = \lceil \log_2(r/\sqrt{t_n}) \rceil$, so that $r \le \sqrt{t_n} 2^N \le 2r$. Define

$$S_k = \left\{ x \in X \ : \ r2^{-k} < |x-y| < r2^{-k+1} \right\}$$

for $k = 1, ..., N$. Using (A_0') and (4.14) we get

$$\begin{aligned} |I_4| &\le C t_n^{\nu} \sum_{k=1}^{N} \int_{S_k} (t_n + |x-y|^2)^{-d/2-\nu} \, |f(y) - f(x)| \, dy \\ &\le C t_n^{-d/2} \sum_{k=1}^{N} (r2^{-k}/\sqrt{t_n})^{-d-2\nu} \int_{S_k} |f(y) - f(x)| \, dy \\ &\le C \varepsilon t_n^{\nu} \sum_{k=1}^{N} (r2^{-k})^{-d-2\nu} (r2^{-k})^{d} \\ &\le C\varepsilon (\sqrt{t_n} r^{-1} 2^N)^{2\nu} \le C\varepsilon. \end{aligned}$$

□

References

1. Auscher, P., Duong, X.T., McIntosh, A.: Boundedness of banach space valued singular integral operators and hardy spaces. Unpublished preprint (2005)
2. Betancor, J.J., Dziubański, J., Torrea, J.L.: On Hardy spaces associated with Bessel operators. J. Anal. Math. **107**, 195–219 (2009)
3. Bochner, S.: Diffusion equation and stochastic processes. Proc. Nat. Acad. Sci. USA **35**, 368–370 (1949)
4. Coifman, R.R.: A real variable characterization of H^p. Studia Math. **51**, 269–274 (1974)
5. Czaja, W., Zienkiewicz, J.: Atomic characterization of the Hardy space $H^1{}_L(\mathbb{R})$ of one-dimensional Schrödinger operators with nonnegative potentials. Proc. Amer. Math. Soc. **136**(1), 89–94 (2008). (electronic)
6. Dziubański, J.: Hardy spaces associated with semigroups generated by Bessel operators with potentials. Houst. J. Math. **34**(1), 205–234 (2008)
7. Dziubański, J.: Hardy spaces for Laguerre expansions. Constr. Approx. **27**(3), 269–287 (2008)
8. Dziubański, J., Preisner, M.: On Riesz transforms characterization of H^1 spaces associated with some Schrödinger operators. Potential Anal. **35**(1), 39–50 (2011)
9. Dziubański, J., Preisner, M.: Hardy spaces for semigroups with gaussian bounds. Ann. Mat. Pura Appl. **197**(3), 965–987 (2018)
10. Dziubański, J., Preisner, M., Roncal, L., Stinga, P.R.: Hardy spaces for Fourier–Bessel expansions. J. Anal. Math. **128**, 261–287 (2016)
11. Dziubański, J., Zienkiewicz, J.: Hardy spaces H^1 for Schrödinger operators with certain potentials. Studia Math. **164**(1), 39–53 (2004)
12. Dziubański, J., Zienkiewicz, J.: Hardy spaces H^1 for Schrödinger operators with compactly supported potentials. Ann. Mat. Pura Appl. (4) **184**(3), 315–326 (2005)
13. Dziubański, J., Zienkiewicz, J.: On Hardy spaces associated with certain Schrödinger operators in dimension 2. Rev. Mat. Iberoam. **28**(4), 1035–1060 (2012)
14. Dziubański, J., Zienkiewicz, J.: A characterization of Hardy spaces associated with certain Schrödinger operators. Potential Anal. **41**(3), 917–930 (2014)
15. Głowacki, P., Hebisch, W.: Pointwise estimates for densities of stable semigroups of measures. Studia Math. **104**(3), 243–258 (1993)

16. Goldberg, D.: A local version of real Hardy spaces. Duke Math. J. **46**(1), 27–42 (1979)
17. Hofmann, S., Lu, G., Mitrea, D., Mitrea, M., Yan, L.: Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates. Mem. Amer. Math. Soc. **214**(1007), vi+78 (2011)
18. Kwaśnicki, M.: Ten equivalent definitions of the fractional Laplace operator. Fract. Calc. Appl. Anal. **20**(1), 7–51 (2017)
19. Latter, R.H.: A characterization of $H^p(R^n)$ in terms of atoms. Studia Math. **62**(1), 93–101 (1978)
20. Preisner, M.: Atomic decompositions for Hardy spaces related to Schrödinger operators. Studia Math. **239**(2), 101–122 (2017)
21. Song, L., Yan, L.: A maximal function characterization for Hardy spaces associated to nonnegative self-adjoint operators satisfying Gaussian estimates. Adv. Math. **287**, 463–484 (2016)
22. Stein, E.M.: Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III (1993)
23. Uchiyama, A.: A maximal function characterization of H^p on the space of homogeneous type. Trans. Am. Math. Soc. **262**(2), 579–592 (1980)
24. Watson, G.N.: A treatise on the theory of Bessel functions. Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1995, Reprint of the second edition (1944)
25. Yosida, K.: Fractional powers of infinitesimal generators and the analyticity of the semi-groups generated by them. Proc. Jpn. Acad. **36**, 86–89 (1960)

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