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LOCAL ATTRACTIVITY AND STABILITY RESULTS FOR HYBRID FUNCTIONAL NONLINEAR FRACTIONAL INTEGRAL EQUATIONS

Bapurao C. Dhage[1], Shyam B. Dhage[2] and Dnyaneshwar V. Mule[3]

[1]Kasubai, Gurukul Colony, Ahmedpur-413 515
Dist: Latur, Maharashtra, India
e-mail: bcdhage@gmail.com

[2]Kasubai, Gurukul Colony, Ahmedpur-413 515
Dist: Latur, Maharashtra, India
e-mail: sbdhage4791@gmail.com

[3]Kasubai, Gurukul Colony, Ahmedpur-413 515
Dist: Latur, Maharashtra, India
e-mail: dnyaneshmule56@gmail.com

Abstract. We prove a couple of local asymptotic stability results for a hybrid functional nonlinear fractional integral equations under weaker Lipschitz and compactness type conditions. It is shown that comparable solutions of the considered hybrid functional nonlinear fractional integral equation are uniformly locally ultimately attractive and asymptotically stable on unbounded intervals of real line. We claim that our results are new and rely on a measure theoretic fixed point theorem of Dhage (2014).

1. Introduction

The object of this paper is to discuss local attractivity and asymptotic stability results for comparable solutions of the following functional nonlinear fractional integral equation (in short FIE)

$$x(t) = f(t, x(\alpha(t))) + \frac{1}{\Gamma(q)} \int_0^{\beta(t)} (t-s)^{q-1} g(s, x(\gamma(s)))\, ds,\ t \in \mathbb{R}_+, \tag{1.1}$$

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where $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\alpha, \beta, \gamma : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions, $1 \leq q < 2$ and Γ is the Euler gamma function.

By a *solution* of the FIE (1.1) we mean a function $x \in C(\mathbb{R}_+, \mathbb{R})$ that satisfies the equation (1.1), where $C(\mathbb{R}_+, \mathbb{R})$ is the space of continuous real-valued functions on $\mathbb{R}_+$.

Observe that the above integral equation in question has rather general form and includes several classes of functional, integral and functional integral equations considered in the literature (cf. [1, 3, 5, 6] and references therein). Let us also mention that the functional integral equation considered in [3, 5] is a special case of the equation (1.1), where $\alpha(t) = \beta(t) = \gamma(t) = t$.

In this paper, we prove a couple of results on the existence and uniform local attractivity of solutions for the above functional nonlinear fractional integral equation. Our investigations will be carried out in the Banach space of real functions which are defined, continuous and bounded on the right half real axis $\mathbb{R}_+$. The main tool used in our considerations is the technique of partially measures of noncompactness and the fixed point result established in Dhage [5]. The measure of noncompactness used in this paper allows us not only to obtain the existence of solutions of the mentioned functional integral equation but also to characterize the solutions in terms of uniform local ultimate attractivity. This assertion means that all possible comparable solutions of the nonlinear fractional integral equation in question are locally uniformly attractive in the sense of notion defined in the following section.

2. Auxiliary results

Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. We frequently need the concept of regulatory of E in what follows. It is known that E is *regular* if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. Then following definitions have been introduced in Dhage [4] which are frequently used in the subsequent part of this paper.

A subset S of E is called *partially bounded* if every chain C in S is bounded. Again S is called *uniformly partially bounded* if all chains in S are bounded with a unique constant.

Note that every bounded subset of a partially ordered normed linear space is uniformly partially bounded and uniformly partially bounded set in E is partially bounded, but the converse implications may not be held.

Definition 2.1. A mapping $\mathcal{T} : E \to E$ is called *isotonic* or *monotonic* if it is either monotone nondecreasing or non-increasing, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ or $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$.

Definition 2.2. (Dhage [6, 7]) A mapping $\mathcal{T} : E \to E$ is called *partially continuous* at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E. T is called *partially bounded* if $\mathcal{T}(C)$ is a bounded subset of E for all totally ordered sets or chains C in E.

If C is a chain in E then the symbols $\overline{C}$ stands for the order-closure of C in E defined by $\overline{C} = \inf C \cup C \cup \sup C$ provided $\inf C$ and $\sup C$ exist. The $\sup C$ is an element $z \in E$ such that for every $\epsilon > 0$ there exists a $c \in C$ such that $d(c, z) < \epsilon$ and $x \leq z$ for all $x \in C$. Similarly, $\inf C$ is defined in the same way. Then $\overline{C}$ is again a chain, called the closed chain in E. Thus, $\overline{C}$ is the intersection of all closed chains containing C. Moreover, we denote by $\mathcal{P}_{cl}(E)$, $\mathcal{P}_{bd}(E)$, $\mathcal{P}_{rcp}(E)$, $\mathcal{P}_{ch}(E)$, $\mathcal{P}_{bd,ch}(E)$, $\mathcal{P}_{rcp,ch}(E)$ the family of all nonempty and closed, bounded, relatively compact, chains, bounded chains and relatively compact chains of E respectively.

We accept the following definition of partially measure of noncompactness in partially ordered normed linear spaces given in Dhage [5].

Definition 2.3. A mapping $\mu^p : \mathcal{P}_{bd,ch}(E) \to \mathbb{R}_+ = [0, \infty)$ is said to be a partially measure of noncompactness in E if it satisfies the following conditions:

1^o $\emptyset \neq (\mu^p)^{-1}(\{0\}) \subset \mathcal{P}_{rcp,ch}(E)$,
2^o $\mu^p(\overline{C}) = \mu^p(C)$,
3^o μ^p is nondecreasing, i.e., if $C_1 \subset C_2 \Rightarrow \mu^p(C_1) \leq \mu^p(C_2)$, and
4^o If $\{C_n\}$ is a sequence of closed chains from $\mathcal{P}_{bd,ch}(E)$ such that $C_{n+1} \subset C_n$ $(n = 1, 2, ...)$ and if $\lim\limits_{n\to\infty} \mu^p(C_n) = 0$, then the intersection set $\overline{C}_\infty = \bigcap_{n=1}^{\infty} C_n$ is nonempty.

The partially measure μ^p of noncompactness is called *sublinear* if it satisfies

5^o $\mu^p(C_1 + C_2) \leq \mu^p(C_1) + \mu^p(C_2)$ for all $C_1, C_2 \in \mathcal{P}_{bd,ch}(E)$, and
6^o $\mu^p(\lambda C) = |\lambda|\mu^p(C)$ for $\lambda \in \mathbb{R}$.

Remark 2.1. The family of sets described in 1^o is said to be *kernel of the measure of noncompactness* μ^p and is defined as

$$\ker \mu^p = \left\{C \in \mathcal{P}_{bd,ch}(E) \,\middle|\, \mu^p(C) = 0\right\}.$$

Clearly, ker $\mu^p \subset \mathcal{P}_{rcp,ch}(E)$. Observe that the intersection set C_∞ from condition 4^o is a member of the family ker μ^p. In fact, since $\mu^p(C_\infty) \leq \mu^p(C_n)$ for any n, we infer that $\mu^p(C_\infty) = 0$. This yields that $C_\infty \in$ ker μ^p. This simple observation will be essential in our further investigations.

Definition 2.4. A mapping $T : E \to E$ is called a partially k-set-contraction if there exists a constant $k > 0$ such that for any bounded chain C, $T(C)$ is a bounded chain and $\mu^p(T(C)) \leq k\, \mu^p(C)$.

We need the following definition in what follows.

Definition 2.5. (Dhage [5]) The order relation $\preceq$ and the metric d on a non-empty set E are said to be *compatible* if $\{x_n\}$ is a monotone, that is, monotone nondecreasing or monotone nondecreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the whole sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible.

The following applicable hybrid fixed point theorem for monotone mappings proved in Dhage [6] is the key tool for proving the main existence results of this paper.

Theorem 2.1. (Dhage [6]) *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete normed linear space $(E, \preceq, \|\cdot\|)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible. Let $\mathcal{T} : S \to S$ be a partially continuous, nondecreasing and partially k-set-contraction with $k < 1$. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq Tx_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges to x^*.*

Proof. The proof is given in Dhage [6] using the compatness of the every bounded chain under the k-set-contraction mapping T on E. Since the proof is not well-known, we give the details of it. Define a sequence $\{x_n\}$ of points in E by

$$x_{n+1} = \mathcal{T}x_n, \quad n = 0, 1, 2, \ldots. \tag{2.1}$$

Since $\mathcal{T}$ is nondecreasing and $x_0 \preceq \mathcal{T}x_0$, we have that

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots . \tag{2.2}$$

Denote

$$C_n = \overline{\{x_n, x_{n+1}, \ldots\}}$$

for $n = 0, 1, 2, \ldots$. By construction, each C_n is a bounded and closed chain in E and

$$C_n = T(C_{n-1}), \quad n = 0, 1, 2, \ldots.$$

Moreover,

$$C_0 \supset C_1 \supset \cdots \supset C_n \supset \cdots. \tag{2.3}$$

Therefore, by nondecreasing nature of μ^p we obtain

$$\begin{aligned} \mu^p(C_n) &= \mu^p(\mathcal{T}(C_{n-1})) \\ &\le k\,\mu^p(C_{n-1}) \\ &\le k^2\,\mu^p(C_{n-2}) \\ &\vdots \\ &\le k^n\,\mu^p(C_0). \end{aligned} \tag{2.4}$$

Taking the limit superior as $n \to \infty$ in the above equality (2.4), in view of Lemma 3.1 we obtain that

$$\lim_{n\to\infty} \mu^p(C_n) \le \limsup_{n\to\infty} k^n\,\mu^p(C_0) = \lim_{n\to\infty} k^n\,\mu^p(C_0) = 0. \tag{2.5}$$

Hence, by condition (4^o) of μ^p,

$$\overline{C}_\infty = \bigcap_{n=1}^{\infty} C_n \neq \emptyset \quad \text{and} \quad C_\infty \in \mathcal{P}_{rcp,ch}(E).$$

From (2.5) it follows that for every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$\mu^p(C_n) < \epsilon, \quad \forall\, n \ge n_0.$$

This shows that $\overline{C}_{n_0}$ and consequently $\overline{C}_0$ is a compact chain in E. Hence, $\{x_n\}$ has a convergent subsequence. Furthermore, since the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible, the whole sequence $\{x_n\} = \{\mathcal{T}^n x_0\}$ is convergent and converges to a point, say $x^* \in \overline{C}_0$. Finally, from the regularity of E and partial continuity of $\mathcal{T}$, we get

$$\mathcal{T}x^* = \mathcal{T}\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} \mathcal{T}x_n = \lim_{n\to\infty} x_{n+1} = x^*.$$

This completes the proof. □

Remark 2.2. The regularity of E and the partial continuity of $\mathcal{T}$ in above Theorem 2.1 may be replaced with a stronger continuity condition of the operator $\mathcal{T}$ on E.

Remark 2.3. If the set S of solutions to the above operator equation is chain, then all solutions belonging to S are comparable. Further, if $\mu^p(S) > 0$, then $\mu^p(S) = \mu^p(\mathcal{T}S) \leq \psi(\mu^p(S)) < \mu^p(S)$ which is a contradiction. Consequently, $S \in \ker \mu^p$. This simple fact has been utilized in the study of qualitative properties of dynamic systems under consideration.

Remark 2.4. Suppose that the order relation $\preceq$ is introduced in E with the help of an order cone $\mathcal{K}$ which is a non-empty closed set $\mathcal{K}$ in E satisfying (i) $\mathcal{K} + \mathcal{K} \subseteq \mathcal{K}$, (ii) $\lambda \mathcal{K} \subseteq \mathcal{K}$ and (iii) $\{-\mathcal{K}\} \cap \mathcal{K} = \{0\}$ (cf. [9]). Then the order relation $\preceq$ in E is defined as $x \preceq y \iff y - x \in \mathcal{K}$. The element $x_0 \in E$ satisfying $x_0 \preceq \mathcal{T}x_0$ in above Theorem 2.1 is called a lower solution of the operator equation $x = \mathcal{T}x$. If the operator equation $x = \mathcal{T}x$ has more than one lower solution and set of all these lower solutions are comparable, then the corresponding set S of solutions to above operator equation is a chain and hence all solutions in S are comparable. To see this, let x_0 and y_0 be any two lower solutions of the above operator equation such that $x_0 \preceq y_0$ and let x^* and y^* respectively be the corresponding solutions under the conditions of Theorem 2.1. Now, by definition of $\preceq$, one has $y_0 - x_0 \in \mathcal{K}$ and from monotone nondecreasing nature of T it follows that $T^n y_0 - T^n x_0 \in \mathcal{K}$. Since $\mathcal{K}$ is closed, we have that $y^* - x^* \in \mathcal{K}$ or $x^* \preceq y^*$.

For our purpose we introduce a handy tool for the partial measure of noncompactness in the space $BC(\mathbb{R}_+, \mathbb{R})$ which is useful in the study of the solutions of certain nonlinear integral equations. To define this partial measure, let us fix a nonempty and bounded chain X of the space $BC(\mathbb{R}_+, \mathbb{R})$ and a positive real number T. For $x \in X$ and $\epsilon \geq 0$ denote by $\omega^T(x, \epsilon)$ *the modulus of continuity* of the function x on the interval $[0, T]$ defined by

$$\omega^T(x, \epsilon) = \sup\{|x(t) - x(s)| : \ t, s \in [0, T], \ |t - s| \leq \epsilon\}.$$

Next, let us put

$$\begin{aligned} \omega^T(X, \epsilon) &= \sup\{\omega^T(x, \epsilon) : \ x \in X\}, \\ \omega_0^T(X) &= \lim_{\epsilon \to 0} \omega^T(X, \epsilon), \\ \omega_0(X) &= \lim_{T \to \infty} \omega_0^T(X). \end{aligned}$$

The partial ball or Hausdorff measure of noncompactness β^p is very useful in applications to nonlinear differential and integral equations and it can be shown that

$$\beta^p(X) = \frac{1}{2}\omega_0(X)$$

for all bounded chain X in $BC(\mathbb{R}_+, \mathbb{R})$. Thus ω_0 is a handy tool for β^p in $BC(\mathbb{R}_+, \mathbb{R})$.

Now, for a fixed number $t \in \mathbb{R}_+$ and a fixed bounded chain X in $BC(\mathbb{R}_+, \mathbb{R})$, let us denote

$$X(t) = \{x(t) : x \in X\}.$$

Let

$$\delta_a(X(t)) = |X(t)| = \sup\{|x(t)| : \ x \in X\} \ ,$$
$$\delta_a^T(X(t)) = \sup_{t \geq T} \delta_a(X(t)) = \sup_{t \geq T} |X(t)|$$

and

$$\delta_a(X) = \lim_{T \to \infty} \delta_a^T(X(t)) = \limsup_{t \to \infty} \ |X(t)|.$$

Again, for a fixed real number c, denote

$$X(t) - c = \{x(t) - c \ : \ x \in X\},$$
$$\delta_b(X(t)) = |X(t) - c| = \sup\{|x(t) - c| : \ x \in X\} \ ,$$
$$\delta_b^T(X(t)) = \sup_{t \geq T} \delta_b(X(t)) = \sup_{t \geq T} |X(t) - c|$$

and

$$\delta_b(X) = \lim_{T \to \infty} \delta_b^T(X(t)) = \limsup_{t \to \infty} \ |X(t) - c|.$$

Similarly, let

$$\delta_c(X(t)) = \operatorname{diam} X(t) = \sup\{|x(t) - y(t)| : \ x, y \in X\} \ ,$$
$$\delta_c^T(X(t)) = \sup_{t \geq T} \delta(X(t)) = \sup_{t \geq T} \ \operatorname{diam} X(t)$$

and

$$\delta_c(X) = \lim_{T \to \infty} \delta^T(X(t)) = \limsup_{t \to \infty} \ \operatorname{diam} \ X(t).$$

The details of the functions δ_a, δ_b and δ_c appear in Dhage [5]. Finally, let us consider the functions μ_a^p, μ_b^p and μ_c^p defined on the family of bounded chains in $BC(\mathbb{R}_+, \mathbb{R})$ by the formula

$$\mu_a^p(X) = \omega_0(X) \ + \ \delta_a(X), \tag{2.6}$$

$$\mu_b^p(X) = \omega_0(X) \ + \ \delta_b(X) \tag{2.7}$$

and

$$\mu_c^p(X) = \omega_0(X) \ + \ \delta_c(X). \tag{2.8}$$

It can be shown that the function μ_a^p, μ_b^p and μ_c^p are partially measures of noncompactness in the space $BC(\mathbb{R}_+, \mathbb{R})$. The components ω_0 and δ_a are called the characteristic values of the partially measure of noncompactness μ_a^p. Similarly, ω_0, δ_b and ω_0, δ_c are respectively the characteristic values of the partially measure of noncompactness μ_b^p and μ_c^p in $BC(\mathbb{R}_+, \mathbb{R})$.

Remark 2.5. The kernels ker μ_a^p, ker μ_b^p and ker μ_c^p consist of nonempty and bounded chains X of $BC(\mathbb{R}_+, \mathbb{R})$ such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle formed by functions from X tends to zero at infinity. This particular characteristic of ker μ_a^p, ker μ_b^p and ker μ_c^p has been useful in establishing the local attractivity and local asymptotic stability of the comparable solutions for functional integral equations.

3. Attractivity and stability results

Our considerations will be placed in the Banach space $BC(\mathbb{R}_+, \mathbb{R})$ consisting of all real functions $x = x(t)$ defined, continuous and bounded on $\mathbb{R}_+$. This space is equipped with the standard supremum norm

$$||x|| = \sup\{|x(t)| : \ t \in \mathbb{R}_+\}. \tag{3.1}$$

Define the order relation $\leq$ in $BC(\mathbb{R}_+, \mathbb{R})$ as follows. Let $x, y \in BC(\mathbb{R}_+, \mathbb{R})$. Then by $x \leq y$ we mean $x(t) \leq y(t)$ for all $t \in \mathbb{R}_+$. It is clear that $(BC(\mathbb{R}_+, \mathbb{R}), \leq, \|\cdot\|)$ is regular and the order relation $\leq$ and the norm $\|\cdot\|$ are compatible in $BC(\mathbb{R}_+, \mathbb{R})$.

In order to introduce further concepts used in the paper let us assume that Ω is a nonempty chain of the space $BC(\mathbb{R}_+, \mathbb{R})$. Moreover, let Q be an operator defined on Ω with values in $BC(\mathbb{R}_+, \mathbb{R})$.

Consider the operator equation of the form

$$x(t) = Qx(t), \quad t \in \mathbb{R}_+ \ . \tag{3.2}$$

Definition 3.1. We say that comparable solutions of the equation (3.2) are *locally attractive* if there exists an open ball $\mathcal{B}(x_0, r)$ in the space $BC(\mathbb{R}_+, \mathbb{R})$ such that for arbitrary comparable solutions $x = x(t)$ and $y = y(t)$ of the equation (3.2) belonging to $\overline{\mathcal{B}}(x_0, r) \cap \Omega$ we have that

$$\lim_{t \to \infty} \big[x(t) - y(t)\big] = 0 \ . \tag{3.3}$$

In the case when limit (3.2) is uniform with respect to the set $\overline{\mathcal{B}}(x_0, r) \cap \Omega$, *i.e.*, when for each $\epsilon > 0$ there exists $T > 0$ such that

$$|x(t) - y(t)| \leq \epsilon \tag{3.4}$$

for all $x, y \in \overline{\mathcal{B}}(x_0, r) \cap \Omega$ being the comparable solutions of (3.2) and for $t \geq T$, we will say that the comparable solutions of the operator equation (3.2) are *uniformly locally ultimately attractive* defined on $\mathbb{R}_+$.

Definition 3.2. We say that comparable solutions of the equation (3.2) are *locally asymptotically stable* to the line $x(t) = c$ for all $t \in \mathbb{R}_+$ if there exists an open ball $\mathcal{B}(x_0, r)$ in the space $BC(\mathbb{R}_+, \mathbb{R})$ such that for arbitrary comparable solution $x = x(t)$ of the equation (3.2) belonging to $\overline{\mathcal{B}}(x_0, r) \cap \Omega$ we have that

$$\lim_{t\to\infty} [x(t) - c] = 0 \ . \tag{3.5}$$

In the case when limit (3.2) is uniform with respect to the set $\overline{\mathcal{B}}(x_0, r) \cap \Omega$, i.e. when for each $\epsilon > 0$ there exists $T > 0$ such that

$$|x(t) - c| \leq \epsilon \tag{3.6}$$

for all $x \in \overline{\mathcal{B}}(x_0, r) \cap \Omega$ being the comparable solutions of (3.2) and for $t \geq T$, we will say that the comparable solutions of the operator equation (3.2) are *uniformly locally asymptotically stable* to the line $x(t) = c$ defined on $\mathbb{R}_+$.

The equation (1.1) will be considered under the following assumptions:

(H_1) The functions $\alpha, \beta, \gamma : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and satisfy $\alpha(t) \geq t$ and $\beta(t) \leq t$ for all $t \in \mathbb{R}_+$.

(H_2) The function $F : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $F(t) = |f(t, 0)|$ is bounded on $\mathbb{R}_+$ with

$$F_0 = \sup_{t\geq 0} F(t).$$

(H_3) There exists a constant $L > 0$ such that

$$0 \leq f(t, x) - f(t, y) \leq L(x - y)$$

for all $x, y \in \mathbb{R}$ with $x \geq y$. Moreover $L < 1$.

(H_4) $g(t, x)$ is nondecreasing in x for each $t \in J$.

(H_5) There exists an element $u \in C(J, \mathbb{R})$ such that

$$u(t) \leq f(t, u(\alpha(t))) + \frac{1}{\Gamma(q)} \int_{t_0}^{\beta(t))} (t - s)^{q-1} g(s, u(\gamma(s)))) \, ds$$

for all $t \in J$.

(H_6) There exists a function $b : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|g(t, x)| \leq b(t)$$

for $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Moreover, we assume that

$$\lim_{t\to\infty} \int_0^{\beta(t)} (t - s)^{q-1} \, b(s) \, ds = 0 \ .$$

(H_7) There exists a real number c such that $f(t, c) = c$ for all $t \in \mathbb{R}_+$.

The hypotheses (H_1) through (H_7) are standard and have been widely used in the literature on nonlinear differential and integral equations. The hypothesis (H_3) is considered recently in Nieto and Lopez [12]. Now we formulate the main existence results for the integral equation (1.1) under above mentioned natural conditions.

Theorem 3.1. *Assume that the hypotheses* (H_1) *through* (H_6) *hold. Then the functional FIE* (1.1) *has at least one solution* x^* *in the space* $BC(\mathbb{R}_+,\mathbb{R})$ *and the sequence* $\{x_n\}$ *of successive approximations defined by*

$$\begin{aligned} x_n(t) &= f(t, x_{n-1}(\alpha(t))) \\ &\quad + \frac{1}{\Gamma(q)} \int_0^{\beta(t)} (t-s)^{q-1}\, g(s, x_{n-1}(\gamma(s)))\, ds, \qquad t \in \mathbb{R}_+, \end{aligned} \tag{3.7}$$

for each $n \in \mathbb{N}$ *with* $x_0 = u$ *converges to* x^**. Moreover, the comparable solutions of the FIE* (1.1) *are uniformly locally ultimately attractive defined on* $\mathbb{R}_+$.

Proof. We seek the solutions of the FIE (1.1) in the space $E = BC(\mathbb{R}_+,\mathbb{R})$. Consider the operator Q defined on the space E by the formula

$$Qx(t) = f(t, x(\alpha(t))) + \frac{1}{\Gamma(q)} \int_0^{\beta(t)} (t-s)^{q-1}\, g(s, x(\gamma(s)))\, ds, \;\; t \in \mathbb{R}_+. \tag{3.8}$$

Observe that in view of our assumptions, for any function $x \in E$ the function Qx is continuous on $\mathbb{R}_+$. As a result, Q defines a mapping $Q : E \to E$. We show that Q satisfies all the conditions of Theorem 3.1 on E. This will be achieved in a series of following steps:

Step I. *Q is nondecreasing on E.*

Let $x, y \in E$ be such that $x \leq y$. Then by hypothesis (H_3)-(H_4), we obtain

$$\begin{aligned} Qx(t) &= f(t, x(\alpha(t))) + \frac{1}{\Gamma(q)} \int_0^{\beta(t)} (t-s)^{q-1}\, g(s, x(\gamma(s)))\, ds \\ &\leq f(t, y(\alpha(t))) + \frac{1}{\Gamma(q)} \int_0^{\beta(t)} (t-s)^{q-1}\, g(s, y(\gamma(s)))\, ds \\ &= Qy(t) \end{aligned}$$

for all $t \in \mathbb{R}_+$. This shows that Q is a nondecreasing operator on E.

Step II. *Q maps a closed and partially bounded set into itself.*

Define an open ball $\mathcal{B}(x_0, r)$, where $r = \dfrac{\|x_0\| + F_0 + V/\Gamma(q)}{1-L}$. Let X be a chain in $\overline{\mathcal{B}}(x_0, r)$ and let $x \in X$ be arbitrary. Since the function $v : \mathbb{R}_+ \to \mathbb{R}$

defined by

$$v(t) = \lim_{t\to\infty} \int_0^{\beta(t)} (t-s)^{q-1}\, b(s)\, ds \tag{3.9}$$

is continuous and in view of hypothesis (H_6), the number $V = \sup_{t\geq 0} v(t)$ exists. Moreover if $x \geq \theta$, then for arbitrarily fixed $t \in \mathbb{R}_+$ we obtain:

$$\begin{aligned}
&|x_0(t) - Qx(t)| \\
&\leq |x_0(t)| + |f(t, x(\alpha(t)))| + \frac{1}{\Gamma(q)} \int_0^{\beta(t)} (t-s)^{q-1}\, |g(s, x(s))|\, ds \\
&\leq |x_0(t)| + |f(t, x(\alpha(t))) - f(t,0)| + |f(t,0)| \\
&\quad + \frac{1}{\Gamma(q)} \int_0^{\beta(t)} (t-s)^{q-1}\, b(s)\, ds \\
&\leq |x_0(t)| + L\,|x(\alpha(t))| + F(t) + \frac{v(t)}{\Gamma(q)} \\
&\leq \|x_0\| + L\,\|x\| + F_0 + \frac{V}{\Gamma(q)} \\
&= r.
\end{aligned} \tag{3.10}$$

Similarly, if $x \leq \theta$, then it can be shown that $|x_0(t) - Qx(t)| \leq r$ for all $t \in \mathbb{R}_+$. Taking the supremum over t, we obtain $\|x_0 - Qx\| \leq r$ for all $x \in X$. This means that the operator Q transforms any bounded chain X into a bounded chain in E. More precisely, we infer that the operator Q transforms the chain X belonging to $\overline{\mathcal{B}}(x_0, r)$ into the chain $Q(X)$ contained in the ball $\overline{\mathcal{B}}(x_0, r)$. As a result, Q defines a mapping $Q : \mathcal{P}_{ch}(\overline{\mathcal{B}}(x_0, r))) \to \mathcal{P}_{ch}(\overline{\mathcal{B}}(x_0, r)))$ and that Q is partially bounded on $S = \overline{\mathcal{B}}(x_0, r)$ into itself.

Step III. *Q is partially continuous on S.*

Now we show that the operator Q is partially continuous on the ball $\overline{\mathcal{B}}(x_0, r)$. To do this, let us fix arbitrarily $\epsilon > 0$ and take $x, y \in X \subset \overline{\mathcal{B}}(x_0, r)$ such that $x \geq y$ and $||x - y|| \leq \epsilon$. Then we get:

$$\begin{aligned}
|Qx(t) - Qy(t)| &\leq \big|f(t, x(\alpha(t))) - f(t, y(\alpha(t)))\big| \\
&\quad + \frac{1}{\Gamma(q)} \left| \int_0^{\beta(t)} (t-s)^{q-1}\, g(s, x(\gamma(s)))\, ds \right. \\
&\quad \left. - \frac{1}{\Gamma(q)} \int_0^{\beta(t)} (t-s)^{q-1}\, g(s, y(\gamma(s)))\, ds \right| \\
&\leq \big|f(t, x(\alpha(t))) - f(t, y(\alpha(t)))\big|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(q)}\int_0^{\beta(t)}|(t-s)^{q-1}\,g(s,x(\gamma(s)))|\,ds\\
&+\frac{1}{\Gamma(q)}\int_0^{\beta(t)}(t-s)^{q-1}\,|g(s,y(\gamma(s)))|\,ds\\
\leq{}& L|x(\alpha(t))-y(\alpha(t))|+\frac{2}{\Gamma(q)}\int_0^{\beta(t)}(t-s)^{q-1}\,b(s)\,ds\\
\leq{}& L\|x-y\|+\frac{2}{\Gamma(q)}v(t)\\
<{}& L\,\epsilon+\frac{2}{\Gamma(q)}v(t).
\end{aligned}
$$

Hence, by virtue of hypothesis (B_6), we infer that there exists $T>0$ such that $v(t)\leq\frac{\epsilon}{2/\Gamma(q)}$ for $t\geq T$. Thus, for $t\geq T$ we derive that

$$
|Qx(t)-Qy(t)|<(L+1)\epsilon. \tag{3.11}
$$

Further, let us assume that $t\in[0,T]$. Similarly, evaluating as above we get:

$$
\begin{aligned}
&|Qx(t)-Qy(t)|\\
&\leq|f(t,x(\alpha(t)))-f(t,y(\alpha(t)))|\Big|\\
&\quad+\frac{1}{\Gamma(q)}\int_0^{\beta(t)}(t-s)^{q-1}\left[|g(s,x(\gamma(s)))-g(s,y(\gamma(s)))|\right]ds\\
&\leq L\ |x(\alpha(t))-y(\alpha(t))|\\
&\quad+\frac{1}{\Gamma(q)}\int_0^{t}(t-s)^{q-1}\Big[|g(s,x(\gamma(s)))-g(s,y(\gamma(s)))|\Big]\,ds\\
&<\ \epsilon+\frac{T^q}{\Gamma(q+1)}\,\omega_r^T(g,\epsilon)\ ,
\end{aligned}\tag{3.12}
$$

where we have denoted

$$
\begin{aligned}
\omega_r^T(g,\epsilon)=\sup\{&|g(s,x)-g(s,y)|:\\
&t,s\in[0,T],\ x,y\in[-r,r],\ |x-y|\leq\epsilon\}.
\end{aligned}
$$

Obviously, in view of continuity of β, we have that $T\leq T<\infty$. Moreover, from the uniform continuity of the function $g(s,x)$ on the set $[0,T]\times[-r,r]$ we derive that $\omega_r^T(g,\epsilon)\to 0$ as $\epsilon\to 0$. Now, linking (3.11), (3.12) and the above established facts we conclude that the operator Q maps partially continuously the ball $\overline{\mathcal{B}}(x_0,r)$ into itself.

Step IV. *Q is a k-set-contraction w.r.t. the characteristic value* ω_0.

Further on let us take a chain X belonging to the ball $\mathcal{B}(x_0,r)$. Next, fix arbitrarily $T>0$ and $\epsilon>0$. Let us choose $x\in X$ and $t_1,t_2\in[0,T]$ with

$|t_2 - t_1| \leq \epsilon$. Without loss of generality we may assume that $x(\alpha(t_1)) \geq x(\alpha(t_2))$. Then, taking into account our assumptions, we get:

$$\begin{aligned}
&|Qx(t_1) - Qx(t_2)| \\
&\leq \big|f(t_1, x(\alpha(t_1))) - f(t_2, x(\alpha(t_2)))\big| \\
&\quad + \left|\frac{1}{\Gamma(q)}\int_0^{\beta(t_1)} (t_1 - s)^{q-1}\, g(s, x(\gamma(s)))\, ds \right. \\
&\quad \left. -\frac{1}{\Gamma(q)}\int_0^{\beta(t_2)} (t_2 - s)^{q-1}\, g(s, x(\gamma(s)))\, ds \right| \\
&\leq \big|f(t_1, x(\alpha(t_1))) - f(t_2, x(\alpha(t_2)))\big| \\
&\quad + \left|\frac{1}{\Gamma(q)}\int_0^{\beta(t_1)} (t_1 - s)^{q-1}\, g(s, x(\gamma(s)))\, ds \right. \\
&\quad \left. -\frac{1}{\Gamma(q)}\int_0^{\beta(t_1)} (t_2 - s)^{q-1}\, g(s, x(\gamma(s)))\, ds \right| \\
&\quad + \left|\frac{1}{\Gamma(q)}\int_0^{\beta(t_1)} (t_2 - s)^{q-1}\, g(s, x(\gamma(s)))\, ds \right. \\
&\quad \left. -\frac{1}{\Gamma(q)}\int_0^{\beta(t_2)} (t_2 - s)^{q-1}\, g(s, x(\gamma(s)))\, ds \right| \\
&\leq \big|f(t_1, x(\alpha(t_1))) - f(t_2, x(\alpha(t_2)))\big| \\
&\quad + \frac{1}{\Gamma(q)}\int_0^{\beta(t_1)} \left|(t_1 - s)^{q-1} - (t_2 - s)^{q-1}\right| |g(s, x(\gamma(s)))|\, ds \\
&\quad + \left|\frac{1}{\Gamma(q)}\int_{\beta(t_2)}^{\beta(t_1)} (t_2 - s)^{q-1}\, |g(s, x(\gamma(s)))|\, ds\right| \\
&\leq \big|f(t_1, x(\alpha(t_1))) - f(t_2, x(\alpha(t_2)))\big| \\
&\quad + \frac{1}{\Gamma(q)}\int_0^{T} \left|(t_1 - s)^{q-1} - (t_2 - s)^{q-1}\right| b(s)\, ds \\
&\quad + \frac{G_r^T}{\Gamma(q)}|\beta(t_1) - \beta(t_2)|,
\end{aligned} \tag{3.13}$$

where

$$G_r^T = \sup\{|g(t, s, x)| : t \in [0, T], s \in [0, T], x \in [-r, r]\}$$

which does exists in view of the fact that the function $g(t, s, x) = (t–s)^{q-1} g(s, x)$ is continuous on compact $[0, T] \times [0, T] \times [-r, r]$.

Now, from (3.13) we obtain,

$$\begin{aligned}
|Qx(t_2) - Qx(t_1)| &\leq \big| f(t_1, x(\alpha(t_1))) - f(t_2, x(\alpha(t_1))) \big| \\
&\quad + L\,|x(\alpha(t_1)) - x(\alpha(t_2))| \\
&\quad + \frac{1}{\Gamma(q)} \int_0^T |(t_1 - s)^{q-1} - (t_2 - s)^{q-1}|\, b(s)\, ds \\
&\quad + \frac{G_r^T}{\Gamma(q)} |\beta(t_1) - \beta(t_2)| \\
&\leq L\,\omega^T(x, \omega^T(\alpha, \epsilon)) + \omega_r^T(f, \epsilon) \\
&\quad + \frac{1}{\Gamma(q)} \int_0^T |(t_1 - s)^{q-1} - (t_2 - s)^{q-1}|\, b(s)\, ds \\
&\quad + \frac{G_r^T}{\Gamma(q)} \omega^T(\beta, \epsilon),
\end{aligned} \tag{3.14}$$

where we have denoted

$$\omega^T(\alpha, \epsilon) = \sup\{|\alpha(t_2) - \alpha(t_1)| : \ t_1, t_2 \in [0, T], \ |t_2 - t_1| \leq \epsilon, \},$$

$$\omega^T(v, \epsilon) = \sup\{|v(t_2) - v(t_1)| : \ t_1, t_2 \in [0, T], \ |t_2 - t_1| \leq \epsilon, \}$$

and

$$\begin{aligned}
\omega_r^T(f, \epsilon) = \sup\{&|f(t_2, x) - f(t_1, x)| : \\
&t_1, t_2 \in [0, T], \ |t_2 - t_1| \leq \epsilon, \ x \in [-r, r]\} \ .
\end{aligned}$$

From the above estimate we derive the following one:

$$\begin{aligned}
\omega^T(Q(X), \epsilon) &\leq L\,\omega^T(X, \omega^T(\alpha, \epsilon)) + \omega_r^T(f, \epsilon) \\
&\quad + \frac{1}{\Gamma(q)} \int_0^T |(t_1 - s)^{q-1} - (t_2 - s)^{q-1}|\, b(s)\, ds \\
&\quad + \frac{G_r^T}{\Gamma(q)} \omega^T(\beta, \epsilon).
\end{aligned} \tag{3.15}$$

Observe that $\omega_r^T(f, \epsilon) \to 0$ and $|(t_1 - s)^{q-1} - (t_2 - s)^{q-1}| \to 0$ as $\epsilon \to 0$, which is a simple consequence of the uniform continuity of the functions f and $(t-s)^{q-1}$ on the sets $[0, T] \times [-r, r]$ and $[0, T] \times [0, T]$ respectively. Moreover, from the uniform continuity of α, β on $[0, T]$, it follows that $\omega^T(\alpha, \epsilon) \to 0$, $\omega^T(\beta, \epsilon) \to 0$ as $\epsilon \to 0$. Thus, linking the established facts with the estimate (3.15) we get

$$\omega_0^T(Q(X)) \leq L\omega_0^T(X) \ .$$

Consequently, we obtain

$$\omega_0(Q(X)) \leq L\ \omega_0(X) \ . \tag{3.16}$$

Step V. *Q is a k-set-contraction w.r.t. characteristic value δ_c.*

Now, taking into account our assumptions, for arbitrarily fixed $t \in \mathbb{R}_+$ and for $x, y \in X$ with $x \geq y$, we deduce the following estimate:

$$\begin{aligned} |(Qx)(t) - (Qy)(t)| &\leq |f(t, x(\alpha(t))) - f(t, y(\alpha(t)))| \\ &\qquad + 2\left(\frac{1}{\Gamma(q)} \int_0^{\beta(t)} (t-s)^{q-1} b(s)\, ds\right) \\ &\leq L|x(\alpha(t)) - y(\alpha(t))| + \frac{2v(t)}{\Gamma(q)}. \end{aligned}$$

From the above inequality it follows that

$$\operatorname{diam}(QX(t)) \leq L \operatorname{diam}(X(\alpha(t))) + \frac{2v(t)}{\Gamma(q)}$$

for each $t \in \mathbb{R}_+$. Therefore, taking limit superior over $t \to \infty$, we obtain

$$\begin{aligned} \delta_c(QX) &= \limsup_{t\to\infty} \operatorname{diam}(Q(X(t))) \\ &\leq L \limsup_{t\to\infty} \operatorname{diam}(X(\alpha(t))) \\ &\leq L \limsup_{t\to\infty} \operatorname{diam}(X(t)) \\ &= L\delta_c(X). \end{aligned} \tag{3.17}$$

Step VI. *Q is a partially k-set-contraction on S.*

Further, using the measure of noncompactness μ_c^p defined by the formula (2.8) and keeping in mind the estimates (3.16) and (3.17), we obtain

$$\begin{aligned} \mu_c^p(QX) &= \omega_0(QX) + \delta_c(QX) \\ &\leq L\,\omega_0(X) + L\,\delta_c(X) \\ &= L\,\mu_c^p(X). \end{aligned}$$

This shows that Q is a partially nonlinear k-set-contraction on S with $k = L < 1$. Again, by hypothesis (H_5), there exists an element $x_0 = u \in S$ such that $x_0 \leq Qx_0$, that is, x_0 is a lower solution of the FIE (1.1) defined on $\mathbb{R}_+$. Thus Q satisfies all the conditions of Theorem 2.1 on S. Hence we apply it to the operator equation $Qx = x$ and deduce that the operator Q has a fixed point x^* in the ball $\overline{\mathcal{B}}(x_0, r)$. Obviously x^* is a solution of the functional integral equation (1.1) and the sequence $\{x_n\}$ of successive approximations defined by

$$\begin{aligned} x_n(t) &= f(t, x_{n-1}(\alpha(t))) \\ &\qquad + \frac{1}{\Gamma(q)} \int_0^{\beta(t)} (t-s)^{q-1}\, g(s, x_{n-1}(\gamma(s)))\, ds, \qquad t \in \mathbb{R}_+, \end{aligned}$$

for each $n \in \mathbb{N}$ converges to x^*. Moreover, taking into account that the image of every chain X under the operator Q is again a chain $Q(X)$ contained in the ball $\overline{\mathcal{B}}(x_0, r)$ we infer that the set $\mathcal{F}(Q)$ of all fixed points of Q is contained in $\overline{\mathcal{B}}(x_0, r)$. If the set $\mathcal{F}(Q)$ contains all comparable solutions of the equation (1.1), then we conclude from Remark 2.3 that the set $\mathcal{F}(Q)$ belongs to the family ker μ_c^p. Now, taking into account the description of sets belonging to ker μ_c^p (given in Section 2) we deduce that all comparable solutions of the equation (1.1) are uniformly locally ultimately attractive on $\mathbb{R}_+$. This completes the proof. □

Theorem 3.2. *Assume that the hypotheses* (H_1) *through* (H_7) *hold. Then the functional FIE* (1.1) *has at least one solution x^* in the space $BC(\mathbb{R}_+, \mathbb{R})$ and the sequence $\{x_n\}$ of successive approximations defined by* (3.7) *converges to x^*. Moreover, the comparable solutions of the equation* (1.1) *are uniformly locally ultimately asymptotically stable to the line $x(t) = c$ defined on $\mathbb{R}_+$.*

Proof. As in Theorem 3.1, we seek the solutions of the FIE (1.1) in the space $E = BC(\mathbb{R}_+, \mathbb{R})$. Define the closed bounded set $S = \overline{\mathcal{B}}(x_0, r)$ and define the operator Q on S into itself by (3.8). Then proceeding as in the Step IV of the proof of Theorem 3.1 it can be proved that

$$\omega_0(Q(X)) \leq L\, \omega_0(X)\ .$$

Next, we show that Q is k-set-contraction with respect to the characteristic value δ_a. Now, taking into account our assumptions, for arbitrarily fixed $t \in \mathbb{R}_+$ and for $x \in X$ with $x \geq c$, we deduce the following estimate:

$$\begin{aligned}|(Qx)(t) - c| &\leq |f(t, x(\alpha(t))) - f(t, c)| + \frac{1}{\Gamma(q)} \int_0^{\beta(t)} (t-s)^{q-1} b(s)\, ds \\ &\leq L\, |x(\alpha(t)) - c| + \frac{v(t)}{\Gamma(q)}.\end{aligned}$$

From the above inequality it follows that

$$|QX(t) - c| \leq L\, |X(\alpha(t)) - c| + \frac{v(t)}{\Gamma(q)}$$

for each $t \in \mathbb{R}_+$. Therefore, taking limit superior over $t \to \infty$, we obtain

$$\begin{aligned}\delta_b(QX) &= \limsup_{t\to\infty} |Q(X(t)) - c| \\ &\leq L\, \limsup_{t\to\infty}\, |X(\alpha(t)) - c| \\ &\leq L\, \limsup_{t\to\infty}\, |X(t) - c| \\ &= L\, \delta_b(X).\end{aligned} \tag{3.18}$$

Further, using the measure of noncompactness μ_b^p defined by the formula (2.7) and keeping in mind the estimates (3.16) and (3.18), we obtain

$$\begin{aligned}\mu_b^p(QX) &= \omega_0(QX) + \delta_b(QX)\\ &\leq L\,\omega_0(X) + L\,\delta_b(X)\\ &= L\,\mu_b^p(X). \end{aligned} \tag{3.19}$$

This shows that Q is a partially k-set-contraction on S with $k = L < 1$. Again, by hypothesis (H$_5$), there exists an element $x_0 = u \in S$ such that $x_0 \leq Qx_0$, that is, x_0 is a lower solution of the FIE (1.1) defined on $\mathbb{R}_+$. The rest of the proof is similar to Theorem 3.1 and now we conclude from Remark 2.3 that the set $\mathcal{F}(Q)$ belongs to the family $\ker \mu_b^p$. Now, taking into account the description of sets belonging to $\ker \mu_b^p$ (given in Section 2) we deduce that the equation (1.1) has a solution x^* and the sequence $\{x_n\}$ of successive iterations defined by (3.7) converges to x^*. Moreover, all comparable solutions of the equation (1.1) are uniformly locally ultimately asymptotically stable to the line $x(t) = c$ on $\mathbb{R}_+$. This completes the proof. □

If $c = 0$ in Theorem 3.2, we obtain the following existence result concerning the asymptotically stability of the solutions to zero and all comparable solutions if exist have the same property.

Theorem 3.3. *Assume that the hypotheses* (H$_1$) *through* (H$_7$) *hold with* $c = 0$. *Then the functional FIE* (1.1) *has at least one solution* x^* *in the space* $BC(\mathbb{R}_+, \mathbb{R})$ *and the sequence* $\{x_n\}$ *of successive approximations defined by* (3.7) *converges to* x^*. *Moreover, the comparable solutions of the equation* (1.1) *are uniformly locally ultimately asymptotically stable to* 0 *defined on* $\mathbb{R}_+$.

Remark 3.1. The conclusion of Theorems 3.1, 3.2 and 3.3 also remains true if we replace the hypothesis (H$_5$) with the following one:

(H$'_5$) There exists an element $u \in C(\mathbb{R}_+, \mathbb{R})$ such that

$$u(t) \geq f(t, u(\alpha(t))) + \frac{1}{\Gamma(q)} \int_{t_0}^{\beta(t))} (t-s)^{q-1} g(s, u(\gamma(s)))) \, ds,$$

for all $t \in \mathbb{R}_+$.

The proof under this new hypothesis is similar to Theorem 3.1, and 3.2 and now, the desired conclusion follows by an application of Theorem 3.2.

Remark 3.2. The existence theorems proved in Section 3 may be extended with appropriate modifications to the generalized nonlinear hybrid functional integral equation

$$x(t) = f(t, x(\alpha_1(t)), \ldots, x(\alpha_n(t))) + \frac{1}{\Gamma(q)} \int_0^{\beta(t)} (t-s)^{q-1} g(s, x(\gamma_1(s)), \ldots, \gamma_n(s)))\, ds, \tag{3.20}$$

for all $t \in \mathbb{R}_+$, where $\alpha_i, \beta, \gamma_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2, \ldots, n$, $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, and $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ are continuous functions.

4. Conclusion

In this paper we have been able to weaken the Lipschitz condition to partially Lipschitz condition which otherwise is considered to be a very strong condition in the existence theory for nonlinear differential and integral equations. However, we needed an additional assumption of monotonicity on the nonlinearities involved in the integral equation in order to guarantee the required characterization of attractivity of the comparable solutions. The advantage of the present approach over previous ones lies in the fact that we have been able to develop an algorithm for the solutions of the considered integral equations which otherwise is not possible via classical approach of measure of noncompactness treated in Banas and Goebel [2]. Another interesting feature of our work is that we generally need the uniqueness of the solution for predicting the behavior of the dynamic systems related to the considered nonlinear fractional integral equation, however with the present approach it possible for us to discuss the qualitative behaviour of the systems even though there exist a number of solutions. Finally, while concluding this paper we mention that the results presented here are of local nature, however analogous study can also be made for global asymptotic attractivity and stability using similar arguments with appropriate modifications and some of the results in this direction will be elsewhere.

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