

# LOCAL BEHAVIOR OF FRACTIONAL $p$ -MINIMIZERS

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ABSTRACT. We extend the De Giorgi-Nash-Moser theory to nonlocal, possibly degenerate integro-differential operators.

## 1. INTRODUCTION

The aim of this paper is to develop localization techniques in order to establish regularity results for nonlocal integro-differential operators and minimizers of fractional order  $s \in (0, 1)$  and summability  $p > 1$ . Let  $\Omega$  be a bounded domain and let  $g$  be a function in the fractional Sobolev space  $W^{s,p}(\mathbb{R}^n)$ . We shall prove general local regularity estimates for the minimizers  $u$ , where  $u$  is minimizing the functional

$$(1.1) \quad \mathcal{F}(v) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y) |v(x) - v(y)|^p dx dy,$$

over the class of functions  $\{v \in W^{s,p}(\mathbb{R}^n) : v = g \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}$ . Here  $K$  is a suitable kernel of order  $(s, p)$  with just measurable coefficients, see (2.1). It is standard to show, which is in fact our Theorem 2.3 below, that minimizers can be equivalently characterized by the following class of integro-differential problems

$$(1.2) \quad \begin{cases} \mathcal{L}u = 0 & \text{in } \Omega, \\ u = g & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where the operator  $\mathcal{L}$  is defined by

$$\mathcal{L}u(x) = P.V. \int_{\mathbb{R}^n} K_{\text{sym}}(x, y) |u(x) - u(y)|^{p-2} (u(x) - u(y)) dy, \quad x \in \mathbb{R}^n;$$

the symbol  $P.V.$  means “in the principal value sense” and  $K_{\text{sym}}$  is the symmetric part of  $K$  defined as  $K_{\text{sym}}(x, y) = (K(x, y) + K(y, x))/2$ . We immediately refer to Section 2 for the precise assumptions on the involved quantities.

To simplify, one can keep in mind the model case when the kernel  $K(x, y)$  coincides with  $|x - y|^{-(n+sp)}$ , though in such a case the difficulties arising from having merely

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measurable coefficients disappear; that is, the function  $u$  reduces to the solution of the following problem

$$(1.3) \quad \begin{cases} (-\Delta)_p^s u = 0 & \text{in } \Omega, \\ u = g & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where the symbol  $(-\Delta)_p^s$  denotes the standard fractional  $p$ -Laplacian operator.

Recently, a great attention has been focused on the study of problems involving fractional Sobolev spaces and corresponding nonlocal equations, both from a pure mathematical point of view and for concrete applications, since they naturally arise in many different contexts. For an elementary introduction to this topic and for a quite extensive list of related references we refer to [7].

However, for what concerns regularity and related results for this kind of operators when  $p \neq 2$ , the theory seems to be rather incomplete. Nonetheless, some partial results are known. Firstly, we would like to cite the higher regularity contributions in the case when  $s$  is close to 1 proven in the recent interesting paper [1]; see, also, [12]. Secondly, the analysis in the papers [2] and [13] considers the special case when  $p$  is suitably large - thus falling in the Morrey embedding case when concerning regularity. See also [8] for some basic results for fractional  $p$ -eigenvalues.

On the contrary, when  $p = 2$  and  $K(x, y) = |x - y|^{-n-2s}$ , that is the case of the well-known fractional Laplacian operator  $(-\Delta)^s$ , the situation simplifies notably. Although having been a classical topic in Functional and Harmonic Analysis as well as in Partial Differential Equations for a long time, in the last years the growing interest for such operator has become really significant and many important results for the minimizer of (1.1) have been achieved. For what concerns the main topic in the present paper, i.e., the local behavior of the fractional minimizers, it is worth mentioning the very relevant contributions for the case  $p = 2$  by Silvestre ([26]), Servadei & Valdinoci ([25]), and Kassmann ([10, 11]). In particular, among other results the last author proves Hölder regularity and a Harnack inequality “revisited” in the right form taking into account the nonlocality of the fractional Laplacian operator; we refer also to [9] to discover how the classic Harnack inequality fails in the fractional framework.

In the present paper, we will deal with a larger class of operators whose kernel  $K$  is not necessarily symmetric, with only measurable coefficients, and, above all, satisfying fractional differentiability for any  $s \in (0, 1)$  and  $p$ -summability for any  $p > 1$ . For this, we will have to handle not only the usual nonlocal character of such fractional operators, but also the difficulties given by the corresponding nonlinear behavior. As a consequence, we can make use neither of the powerful framework provided by the Caffarelli-Silvestre  $s$ -harmonic extension ([3]) nor of various tools as, e. g., the sharp 3-commutators estimates introduced in [4] to deduce the regularity of weak fractional harmonic maps, the strong barriers and density estimates in [23, 24], the commutator and energy estimates in [20, 21], and so on. Indeed, the aforementioned tools seem not to be trivially adaptable to a nonlinear framework; also, increasing difficulties are due to the non-Hilbertian structure of the involved fractional Sobolev spaces  $W^{s,p}$  when  $p$  is different than 2.

We will have to work carefully in order to obtain the needed local estimates. For this, we want to underline that a special quantity will be fundamental throughout the whole paper: the *nonlocal tail* of a function  $v \in W^{s,p}(\mathbb{R}^n)$  in the ball  $B_R(x_0) \subset \mathbb{R}^n$ , given by

$$(1.4) \quad \text{Tail}(v; x_0, R) := \left[ R^{sp} \int_{\mathbb{R}^n \setminus B_R(x_0)} |v(x)|^{p-1} |x - x_0|^{-(n+sp)} dx \right]^{\frac{1}{p-1}}.$$

Note that the above number is finite by Hölder's inequality whenever  $v \in L^q(\mathbb{R}^n)$ ,  $q \geq p - 1$ , and  $R > 0$ . As expected, the way how the nonlocal tail will be managed is a key-point in the present extended local theory. We believe that this is a general fact that will have to be taken into account in other results and extensions in the nonlinear fractional framework.

We are now ready to introduce our main results. The first one describes the local boundedness.

**Theorem 1.1 (Local boundedness).** *Let  $p \in (1, \infty)$ , let  $u \in W^{s,p}(\mathbb{R}^n)$  be a weak solution to problem (1.2) and let  $B_r \equiv B_r(x_0) \subset \Omega$ . Then the following estimate holds true*

$$(1.5) \quad \sup_{B_{r/2}(x_0)} u \leq \delta \text{Tail}(u_+; x_0, r/2) + c \delta^{-\frac{(p-1)n}{sp^2}} \left( \int_{B_r(x_0)} u_+^p dx \right)^{\frac{1}{p}},$$

where  $\text{Tail}(u_+; x_0, r/2)$  is defined in (1.4),  $u_+ = \max\{u, 0\}$  is the positive part of the function  $u$ ,  $\delta \in (0, 1]$ , and the constant  $c$  depends only on  $n, p, s$  and on the structural constants  $\lambda, \Lambda$  defined in (2.1).

The parameter  $\delta$  allows interpolation between the local and nonlocal terms. Armed with the Logarithmic Lemma and the Caccioppoli estimate with tail introduced below, together with the deduced local boundedness, we can prove our main result, that is, the Hölder continuity theorem.

**Theorem 1.2 (Hölder continuity).** *Let  $p \in (1, \infty)$  and let  $u \in W^{s,p}(\mathbb{R}^n)$  be a solution to problem (1.2). Then  $u$  is locally Hölder continuous in  $\Omega$ . In particular, there are positive constants  $\alpha$ ,  $\alpha < sp/(p - 1)$ , and  $c$ , both depending only on  $n, p, s, \lambda, \Lambda$ , such that if  $B_{2r}(x_0) \subset \Omega$ , then*

$$\text{osc}_{B_\varrho(x_0)} u \leq c \left( \frac{\varrho}{r} \right)^\alpha \left[ \text{Tail}(u; x_0, r) + \left( \int_{B_{2r}(x_0)} |u|^p dx \right)^{\frac{1}{p}} \right]$$

holds whenever  $\varrho \in (0, r]$ .

The theorem above provides an extension of classical analogous results by De Giorgi-Nash-Moser ([5, 19, 18]) to the nonlocal, nonlinear framework. It also extends the recent aforementioned result by Kassmann ([10]) to the case  $p \neq 2$ . Moreover, it is worth noticing that in the linear case studied in [10] a further boundedness assumption is required, which is now for free thanks to Theorem 1.1.

In the proof of the Hölder continuity the following *logarithmic estimate* plays the key role. We state it in the introduction as we think that it might be extremely useful also in other contexts.

**Lemma 1.3 (Logarithmic Lemma).** *Let  $p \in (1, \infty)$ . Let  $u \in W^{s,p}(\mathbb{R}^n)$  be a weak solution to problem (1.2) such that  $u \geq 0$  in  $B_R \equiv B_R(x_0) \subset \Omega$ . Then the following estimate holds for any  $B_r \equiv B_r(x_0) \subset B_{R/2}(x_0)$  and any  $d > 0$ ,*

$$(1.6) \quad \int_{B_r} \int_{B_r} K(x, y) \left| \log \left( \frac{d + u(x)}{d + u(y)} \right) \right|^p dx dy \\ \leq c r^{n-sp} \left\{ d^{1-p} \left( \frac{r}{R} \right)^{sp} [\text{Tail}(u_-; x_0, R)]^{p-1} + 1 \right\},$$

where  $\text{Tail}(u_-; x_0, R)$  is defined in (1.4),  $u_- = \max\{-u, 0\}$  is the negative part of the function  $u$ , and  $c$  depends only on  $n, p, s, \lambda$  and  $\Lambda$ .

Then, we will show that the fractional  $p$ -minimizers, equivalently the weak solutions to the Euler-Lagrange equation associated to (1.1), satisfy the following nonlocal Caccioppoli-type inequalities.

**Theorem 1.4 (Caccioppoli estimates with tail).** *Let  $p \in (1, \infty)$  and let  $u \in W^{s,p}(\mathbb{R}^n)$  be a weak solution to problem (1.2). Then, for any  $B_r \equiv B_r(x_0) \subset \Omega$  and any nonnegative  $\phi \in C_0^\infty(B_r)$ , the following estimate holds true*

$$(1.7) \quad \int_{B_r} \int_{B_r} K(x, y) |w_\pm(x)\phi(x) - w_\pm(y)\phi(y)|^p dx dy \\ \leq c \int_{B_r} \int_{B_r} \bar{K}(x, y) (\max\{w_\pm(x), w_\pm(y)\})^p |\phi(x) - \phi(y)|^p dx dy \\ + c \int_{B_r} w_\pm(x) \phi^p(x) dx \left( \sup_{y \in \text{supp } \phi} \int_{\mathbb{R}^n \setminus B_r} \bar{K}(x, y) w_\pm^{p-1}(x) dx \right),$$

where  $\bar{K}(x, y) := \max\{K(x, y), K(y, x)\}$ ,  $w_\pm := (u - k)_\pm$ , and  $c$  depends only on  $p$ .

Notice that, as expected, in the nonlocal framework one has to take into account a suitable tail; see, in particular, the estimate in (5.14) below to see how the second term in the right hand-side of (1.7) is controlled by a tail as given in definition (1.4). Also, it is worth mentioning that other fractional Caccioppoli-type inequalities have been recently of nice use in different contexts (see, for instance, [16, 17, 8]), although none of them takes into account the tails.

The paper is organized as follows. In Section 2 below, we fix the notation by also providing some preliminary results. Section 3 is devoted to the proof of the Log Lemma 1.3 and the Caccioppoli estimates with tail in Theorem 1.4. In Section 4, we establish the local boundedness given by Theorem 1.1. In Section 5, we shall finally prove the Hölder continuity given by Theorem 1.2.

## 2. PRELIMINARIES

In this section, we state the general assumptions of the problem we deal with in the present paper, we fix notation, and we provide some definitions and some basic preliminary results that we will use in the following pages.

The kernel  $K : \mathbb{R}^n \times \mathbb{R}^n \rightarrow [0, \infty)$  is a measurable function such that

$$(2.1) \quad \lambda \leq K(x, y)|x - y|^{n+sp} \leq \Lambda \text{ for almost every } x, y \in \mathbb{R}^n,$$

for some  $s \in (0, 1)$ ,  $p > 1$ ,  $\Lambda \geq \lambda \geq 1$ . Notice that such assumption on  $K$  can be weakened as follows

$$\lambda \leq K(x, y)|x - y|^{n+sp} \leq \Lambda \text{ for almost every } x, y \in \mathbb{R}^n \text{ s. t. } |x - y| \leq 1,$$

$$0 \leq K(x, y)|x - y|^{n+\eta} \leq M \text{ for almost every } x, y \in \mathbb{R}^n \text{ s. t. } |x - y| > 1,$$

for some  $s, \lambda, \Lambda$  as above,  $\eta > 0$  and  $M \geq 1$ ; see, e. g., [10, 11]. For the sake of simplicity, we will keep the assumption in (2.1), since such a choice will imply no relevant differences in all the proofs in the rest of the paper.

For any  $p \in [1, \infty)$  and  $s \in (0, 1)$  we denote by  $W^{s,p}(\mathbb{R}^n)$  the fractional Sobolev space, that is

$$W^{s,p}(\mathbb{R}^n) := \left\{ v \in L^p(\mathbb{R}^n) : \frac{|v(x) - v(y)|}{|x - y|^{\frac{n}{p} + s}} \in L^p(\mathbb{R}^n \times \mathbb{R}^n) \right\};$$

i. e., an intermediary Banach space between  $L^p(\mathbb{R}^n)$  and  $W^{1,p}(\mathbb{R}^n)$ , endowed with the natural norm

$$\|v\|_{W^{s,p}(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} |v|^p dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^p}{|x - y|^{n+sp}} dx dy \right)^{\frac{1}{p}}.$$

In a similar way, it is possible to define the fractional Sobolev space  $W^{s,p}(\Omega)$  in a domain  $\Omega \subseteq \mathbb{R}^n$ . Furthermore, by saying that  $v$  belongs to  $W_0^{s,p}(\Omega)$  we mean that  $v \in W^{s,p}(\mathbb{R}^n)$  and  $v = 0$  almost everywhere in  $\mathbb{R}^n \setminus \Omega$ .

As mentioned in the introduction, we give the definition of the *nonlocal tail of a function  $v$  in the ball  $B_R(x_0)$* , a quantity which will play an important role in the rest of the paper. For any  $v \in W^{s,p}(\mathbb{R}^n)$  and  $B_R(x_0) \subset \mathbb{R}^n$ , we write

$$(2.2) \quad \text{Tail}(v; x_0, R) := \left[ R^{sp} \int_{\mathbb{R}^n \setminus B_R(x_0)} |v(x)|^{p-1} |x - x_0|^{-n-sp} dx \right]^{\frac{1}{p-1}},$$

which is a finite number by Hölder's inequality since  $v \in L^p(\mathbb{R}^n)$  and  $R > 0$ .

For any  $u, v \in W^{s,p}(\mathbb{R}^n)$  we consider the functional

$$\mathcal{E}(u, v) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y) |u(x) - u(y)|^{p-2} (u(x) - u(y))(v(x) - v(y)) dx dy.$$

Moreover, we define a linear operator  $\mathcal{L}$  such that, for any  $u$  and  $\eta$  sufficiently smooth, e. g.  $u, \eta \in C_0^\infty(\mathbb{R}^n)$ ,

$$\langle \mathcal{L}u, \eta \rangle = \mathcal{E}(u, \eta),$$

where  $\langle \cdot, \cdot \rangle$  denotes, as usual, the dual product in the distributional sense. Thus, assuming that the kernel  $K$  satisfies (2.1), for any  $u \in W^{s,p}(\mathbb{R}^n)$  we have

$$(2.3) \quad \mathcal{L}u(x) = P.V. \int_{\mathbb{R}^n} K_{\text{sym}}(x, y) |u(x) - u(y)|^{p-2} (u(x) - u(y)) dy, \quad x \in \mathbb{R}^n$$

up to a multiplicative constant; as usual, the symbol  $P.V.$  means “in the principal value sense”, and  $K_{\text{sym}}$  denotes the symmetric part of  $K$ .

Let  $\Omega$  be a bounded open set in  $\mathbb{R}^n$  and  $g \in W^{s,p}(\mathbb{R}^n)$ , we will be interested in weak solutions to the following integro-differential problems

$$(2.4) \quad \begin{cases} \mathcal{L}u = 0 & \text{in } \Omega, \\ u = g & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Notice that the boundary condition is given in the whole complement of  $\Omega$ , as usual when dealing with such nonlocal operators. A model example we have in mind is the fractional  $p$ -Laplacian, that is

$$-(-\Delta)_p^s u(x) = c(n, s, p) P.V. \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{n+sp}} dx dy,$$

with  $s \in (0, 1)$  and  $p > 1$ .

Now, let us consider in  $W^{s,p}(\mathbb{R}^n)$ , the following functional

$$(2.5) \quad \mathcal{F}(u) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y) |u(x) - u(y)|^p dx dy.$$

In view of the assumptions (2.1) on  $K$ , one can use the standard Direct Method to prove that there exists a unique  $p$ -minimizer of  $\mathcal{F}$  over all  $u \in W^{s,p}(\mathbb{R}^n)$  such that  $u(x) = g(x)$  for  $x \in \mathbb{R}^n \setminus \Omega$ . Moreover, a  $p$ -minimizer  $u$  is a weak solution to problem (2.4) and vice versa (see Theorem 2.3 below).

To specify relevant spaces, for given  $g \in W^{s,p}(\mathbb{R}^n)$ , we define the convex sets of  $W^{s,p}(\mathbb{R}^n)$  as

$$\mathcal{K}_g^\pm(\Omega) := \{v \in W^{s,p}(\mathbb{R}^n) : (g - v)_\pm \in W_0^{s,p}(\Omega)\}$$

and

$$\mathcal{K}_g(\Omega) := \mathcal{K}_g^+(\Omega) \cap \mathcal{K}_g^-(\Omega) = \{v \in W^{s,p}(\mathbb{R}^n) : v - g \in W_0^{s,p}(\Omega)\}.$$

We recall that the functions in the space  $W_0^{s,p}(\Omega)$  are defined in the whole space, since they are considered to be extended to zero outside  $\Omega$ .

We conclude this section by recalling the definition of weak sub- and supersolutions as well as weak solutions to problem (2.4).

**Definition 2.1.** Let  $g \in W^{s,p}(\mathbb{R}^n)$ . A function  $u \in \mathcal{K}_g^-$  ( $\mathcal{K}_g^+$ ) is a weak *subsolution* (*supersolution*) to problem (2.4) if

$$(2.6) \quad \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y) |u(x) - u(y)|^{p-2} (u(x) - u(y)) (\eta(x) - \eta(y)) dx dy \leq (\geq) 0$$

for every nonnegative  $\eta \in W_0^{s,p}(\Omega)$ .

A function  $u$  is a weak *solution* to problem (2.4) if it is both weak sub- and supersolution. In particular,  $u$  belongs to  $\mathcal{K}_g(\Omega)$  and satisfies

$$(2.7) \quad \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y) |u(x) - u(y)|^{p-2} (u(x) - u(y)) (\eta(x) - \eta(y)) \, dx dy = 0$$

for every  $\eta \in W_0^{s,p}(\Omega)$ .

Similarly, we recall the definition of sub- and superminimizers of (2.5). We have

**Definition 2.2.** Let  $g \in W^{s,p}(\mathbb{R}^n)$ . A function  $u \in \mathcal{K}_g^-$  is a *subminimizer* of the functional (2.5) over  $\mathcal{K}_g^-$  if  $\mathcal{F}(u) \leq \mathcal{F}(u + \eta)$  for every nonpositive  $\eta \in W_0^{s,p}(\Omega)$ . Similarly, a function  $u \in \mathcal{K}_g^+$  is a *superminimizer* of the functional (2.5) over  $\mathcal{K}_g^+$  if  $\mathcal{F}(u) \leq \mathcal{F}(u + \eta)$  for every nonnegative  $\eta \in W_0^{s,p}(\Omega)$ .

Finally,  $u \in \mathcal{K}_g$  is a *minimizer* of the functional (2.5) over  $\mathcal{K}_g$  if  $\mathcal{F}(u) \leq \mathcal{F}(u + \eta)$  for every  $\eta \in W_0^{s,p}(\Omega)$ .

**2.1. Notation.** Before starting with the proofs, it is convenient to fix some notation which will be used throughout the rest of the paper. Firstly, notice that we will follow the usual convention of denoting by  $c$  a general positive constant which will not necessarily be the same at different occurrences and which can also change from line to line. For the sake of readability, dependencies of the constants will be often omitted within the chains of estimates, therefore stated after the estimate. Relevant dependences on parameters will be emphasized by using parentheses; special constants will be denoted by  $c_0, c_1, \dots$

As customary, we denote by

$$B_R(x_0) = B(x_0; R) := \{x \in \mathbb{R}^n : |x - x_0| < R\}$$

the open ball centered in  $x_0 \in \mathbb{R}^n$  with radius  $R > 0$ . When not important and clear from the context, we shall use the shorter notation  $B_R := B(x_0; R)$ . We denote by  $\beta B_R$  the concentric ball scaled by a factor  $\beta > 0$ , that is  $\beta B_R := B(x_0; \beta R)$ . Moreover, if  $f \in L^1(S)$  and the  $n$ -dimensional Lebesgue measure  $|S|$  of the set  $S \subseteq \mathbb{R}^n$  is finite and strictly positive, we write

$$(2.8) \quad (f)_S := \int_S f(x) \, dx = \frac{1}{|S|} \int_S f(x) \, dx.$$

Let  $k \in \mathbb{R}$ , we denote by

$$(2.9) \quad w_+(x) := (u(x) - k)_+ = \max\{u(x) - k, 0\},$$

and

$$(2.10) \quad w_-(x) := (u(x) - k)_- = (k - u(x))_+.$$

Clearly  $w_+(x) \neq 0$  in the set  $\{x \in S : u(x) > k\}$ , and  $w_-(x) \neq 0$  in the set  $\{x \in S : u(x) < k\}$ .

Finally, since we allow non-symmetric kernels, the following definition  $\bar{K}$  given by

$$(2.11) \quad \bar{K}(x, y) = \max\{K(x, y), K(y, x)\}$$

is convenient for our purposes.

**2.2. Existence and uniqueness of the minimizers.** The proof of the existence and uniqueness for fractional minimizers is simple and it is recorded into the following.

**Theorem 2.3.** *Let  $s \in (0, 1)$  and  $p \in [1, \infty)$ , and let  $g \in W^{s,p}(\mathbb{R}^n)$ . Then there exists a minimizer  $u$  of (2.5) over  $\mathcal{K}_g$ . Moreover, if  $p > 1$ , then the solution is unique. Moreover, a function  $u \in \mathcal{K}_g$  is a minimizer of (2.5) over  $\mathcal{K}_g$  if and only if it is a weak solution to problem (2.4).*

*Proof.* The proof plainly follows by the Direct Method of Calculus of Variations. One can take any minimizing sequence  $u_j \in \mathcal{K}_g$ . Due to the assumptions on the kernel  $K$ , one can control the fractional seminorm of  $u_j$ , so that, one can find by precompactness in  $L^{p^*}$  (see, for instance, [7, Theorem 6.7]) a subsequence  $u_{j_k}$  converging pointwise a. e. to a function  $u \in \mathcal{K}_g$ . By Fatou's Lemma we deduce that  $u$  is actually a minimizer of (2.5) over  $\mathcal{K}_g$ . The uniqueness in the case  $p > 1$  follows from the strict convexity of the functional.

Furthermore, the fact that  $u$  solves the corresponding Euler-Lagrange equation follows by perturbing  $u \in \mathcal{K}_g$  with a test function in a standard way. Indeed, supposing that  $u \in \mathcal{K}_g$  is a minimizer of (2.5) over  $\mathcal{K}_g$ , take any  $\phi \in W_0^{s,p}(\Omega)$  and calculate formally

$$\begin{aligned} \left. \frac{d}{dt} \mathcal{F}(u + t\phi) \right|_{t=0} &= \left. \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y) \frac{d}{dt} |u(x) - u(y) + t(\phi(x) - \phi(y))|^p dx dy \right|_{t=0} \\ &= p \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y) |u(x) - u(y)|^{p-2} (u(x) - u(y)) (\phi(x) - \phi(y)) dx dy. \end{aligned}$$

Since  $u$  is a minimizer, the term on the left is zero and hence  $u \in \mathcal{K}_g$  is a weak solution to problem (2.4). For the converse, let  $u \in \mathcal{K}_g$  be a weak solution to problem (2.4) and take  $\phi = u - v \in W_0^{s,p}(\Omega)$ , where  $v \in \mathcal{K}_g$ . Then, by Young's inequality,

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y) |u(x) - u(y)|^{p-2} (u(x) - u(y)) (\phi(x) - \phi(y)) dx dy \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y) |u(x) - u(y)|^p dx dy \\ &\quad - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y) |u(x) - u(y)|^{p-2} (u(x) - u(y)) (v(x) - v(y)) dx dy \\ &\geq \frac{1}{p} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y) |u(x) - u(y)|^p dx dy \\ &\quad - \frac{1}{p} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y) |v(x) - v(y)|^p dx dy, \end{aligned}$$

and hence  $u$  is a minimizer of (2.5) over  $\mathcal{K}_g$ . □



## 3. FUNDAMENTAL ESTIMATES

In this section, we establish some relevant estimates that we will use in the following. We believe that these results could have their own interest in the analysis of equations involving the (nonlinear) fractional Laplacian and related nonlocal operators. The first of them states a natural extension of the well-known Caccioppoli inequality to the nonlocal framework, by showing that in such a case one can take into account a suitable tail, in order to detect deeper informations.

*Proof of Theorem 1.4.* For the sake of generality, we would point out that the present proof is also valid when  $p = 1$ .

Let  $u$  be a weak solution as in the statement. Testing (2.6) with  $\eta := w_+ \phi^p$ , where  $\phi$  is any nonnegative function in  $C_0^\infty(B_r(x_0))$ , we get

$$\begin{aligned}
(3.1) \quad 0 &\geq \int_{B_r} \int_{B_r} K(x, y) |u(x) - u(y)|^{p-2} \\
&\quad \times (u(x) - u(y)) (w_+(x) \phi^p(x) - w_+(y) \phi^p(y)) \, dx dy \\
&\quad + \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} K(x, y) |u(x) - u(y)|^{p-2} \\
&\quad \times (u(x) - u(y)) w_+(x) \phi^p(x) \, dx dy \\
&\quad - \int_{B_r} \int_{\mathbb{R}^n \setminus B_r} K(x, y) |u(x) - u(y)|^{p-2} \\
&\quad \times (u(x) - u(y)) w_+(y) \phi^p(y) \, dx dy.
\end{aligned}$$

Note that  $\eta$  is an admissible test function since truncations of functions in  $W^{s,p}(\mathbb{R}^n)$  still belong to  $W^{s,p}(\mathbb{R}^n)$ .

Let us consider the integrands of the three terms above separately. In the first term, we may assume without loss of generality that  $u(x) \geq u(y)$ ; otherwise just exchange the roles of  $x$  and  $y$  below. We have

$$\begin{aligned}
&|u(x) - u(y)|^{p-2} (u(x) - u(y)) (w_+(x) \phi^p(x) - w_+(y) \phi^p(y)) \\
&= (u(x) - u(y))^{p-1} ((u(x) - k)_+ \phi^p(x) - (u(y) - k)_+ \phi^p(y)) \\
&= \begin{cases} (w_+(x) - w_+(y))^{p-1} (w_+(x) \phi^p(x) - w_+(y) \phi^p(y)), & u(x), u(y) > k \\ (u(x) - u(y))^{p-1} w_+(x) \phi^p(x), & u(x) > k, u(y) \leq k \\ 0, & \text{otherwise} \end{cases} \\
&\geq (w_+(x) - w_+(y))^{p-1} (w_+(x) \phi^p(x) - w_+(y) \phi^p(y)).
\end{aligned}$$

For the second term in the right hand-side of the inequality in (3.1) we instead have

$$\begin{aligned}
|u(x) - u(y)|^{p-2} (u(x) - u(y)) w_+(x) &\geq -(u(y) - u(x))_+^{p-1} (u(x) - k)_+ \\
&\geq -(u(y) - k)_+^{p-1} (u(x) - k)_+ \\
&= -w_+(y)^{p-1} w_+(x),
\end{aligned}$$

and estimating further we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} K(x, y) |u(x) - u(y)|^{p-2} (u(x) - u(y)) w_+(x) \phi^p(x) \, dx dy \\
& \geq - \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} K(x, y) w_+^{p-1}(y) w_+(x) \phi^p(x) \, dx dy \\
& \geq - \left( \sup_{x \in \text{supp } \phi} \int_{\mathbb{R}^n \setminus B_r} \bar{K}(x, y) w_+^{p-1}(y) \, dy \right) \int_{B_r} w_+(x) \phi^p(x) \, dx.
\end{aligned}$$

Noticing that the third term in (3.1) can be handled exactly in the same way, we deduce from (3.1) that

$$\begin{aligned}
(3.2) \quad 0 & \geq \int_{B_r} \int_{B_r} K(x, y) |w_+(x) - w_+(y)|^{p-2} \\
& \quad \times (w_+(x) - w_+(y)) (w_+(x) \phi^p(x) - w_+(y) \phi^p(y)) \, dx dy \\
& \quad - 2 \left( \sup_{x \in \text{supp } \phi} \int_{\mathbb{R}^n \setminus B_r} \bar{K}(x, y) w_+^{p-1}(y) \, dy \right) \int_{B_r} w_+(x) \phi^p(x) \, dx.
\end{aligned}$$

Let us then consider the first term in the inequality above. If  $w_+(x) \geq w_+(y)$  and  $\phi(x) \leq \phi(y)$  in the integrand, we appeal to Lemma 3.1 below and get

$$(3.3) \quad \phi^p(x) \geq (1 - c_p \varepsilon) \phi^p(y) - (1 + c_p \varepsilon) \varepsilon^{1-p} |\phi(x) - \phi(y)|^p$$

for any  $\varepsilon \in (0, 1]$  with the constant  $c_p \equiv (p-1)\Gamma(\max\{1, p-2\})$ . Thus, by choosing

$$\varepsilon := \frac{1}{\max\{1, 2c_p\}} \frac{w_+(x) - w_+(y)}{w_+(x)} \in (0, 1]$$

we get

$$\begin{aligned}
(w_+(x) - w_+(y))^{p-1} w_+(x) \phi^p(x) & \geq (w_+(x) - w_+(y))^{p-1} w_+(x) (\max\{\phi(x), \phi(y)\})^p \\
& \quad - \frac{1}{2} (w_+(x) - w_+(y))^p (\max\{\phi(x), \phi(y)\})^p \\
& \quad - c (\max\{w_+(x), w_+(y)\})^p |\phi(x) - \phi(y)|^p
\end{aligned}$$

with  $c \equiv c(p)$ . Note above that since we assumed that  $\phi(x) \leq \phi(y)$ ,  $\max\{\phi(x), \phi(y)\} = \phi(y)$ . But when  $0 = w_+(x) \geq w_+(y) \geq 0$  or  $w_+(x) \geq w_+(y)$  and  $\phi(x) \geq \phi(y)$ , the estimate in the display above is trivial and hence we conclude that it holds also in these cases. It follows that

$$\begin{aligned}
& (w_+(x) - w_+(y))^{p-1} (w_+(x) \phi^p(x) - w_+(y) \phi^p(y)) \\
& \geq (w_+(x) - w_+(y))^{p-1} (w_+(x) (\max\{\phi(x), \phi(y)\})^p - w_+(y) \phi^p(y)) \\
& \quad - \frac{1}{2} (w_+(x) - w_+(y))^p (\max\{\phi(x), \phi(y)\})^p \\
& \quad - c (\max\{w_+(x), w_+(y)\})^p |\phi(x) - \phi(y)|^p \\
& \geq \frac{1}{2} (w_+(x) - w_+(y))^p (\max\{\phi(x), \phi(y)\})^p \\
& \quad - c (\max\{w_+(x), w_+(y)\})^p |\phi(x) - \phi(y)|^p
\end{aligned}$$

whenever  $w_+(x) \geq w_+(y)$ . If, on the other hand,  $w_+(y) > w_+(x)$  in the integrand, we may interchange the roles of  $x$  and  $y$  in the display above by analogous reasoning. Hence we arrive in all cases at

$$\begin{aligned}
 & \int_{B_r} \int_{B_r} K(x, y) |w_+(x) - w_+(y)|^{p-2} \\
 & \quad \times (w_+(x) - w_+(y))(w_+(x)\phi^p(x) - w_+(y)\phi^p(y)) \, dx dy \\
 (3.4) \quad & \geq \frac{1}{2} \int_{B_r} \int_{B_r} K(x, y) |w_+(x) - w_+(y)|^p (\max\{\phi(x), \phi(y)\})^p \, dx dy \\
 & \quad - c \int_{B_r} \int_{B_r} \bar{K}(x, y) (\max\{w_+(x), w_+(y)\})^p |\phi(x) - \phi(y)|^p \, dx dy.
 \end{aligned}$$

Observing finally that

$$\begin{aligned}
 |w_+(x)\phi(x) - w_+(y)\phi(y)|^p & \leq 2^{p-1} |w_+(x) - w_+(y)|^p (\max\{\phi(x), \phi(y)\})^p \\
 & \quad + 2^{p-1} (\max\{w_+(x), w_+(y)\})^p |\phi(x) - \phi(y)|^p
 \end{aligned}$$

and combining this with (3.2) and (3.4) concludes the proof of (1.7) for  $w_+$ .

In order to prove the estimate in (1.7) for  $w_-$ , it will suffice to proceed as above, using the function  $\eta = -w_- \phi$ , instead of  $\eta = w_+ \phi$ , as a test function in the weak formulation of problem (2.4).  $\square$

Above we made use of the following trivial but very useful small lemma.

**Lemma 3.1.** *Let  $p \geq 1$  and  $\varepsilon \in (0, 1]$ . Then*

$$|a|^p \leq |b|^p + c_p \varepsilon |b|^p + (1 + c_p \varepsilon) \varepsilon^{1-p} |a - b|^p, \quad c_p := (p - 1) \Gamma(\max\{1, p - 2\}),$$

*holds for every  $a, b \in \mathbb{R}^m$ ,  $m \geq 1$ . Here  $\Gamma$  stands for the standard Gamma function.*

*Proof.* By the triangle inequality and convexity we obtain

$$\begin{aligned}
 |a|^p & \leq (|b| + |a - b|)^p \\
 & = (1 + \varepsilon)^p \left( \frac{1}{1 + \varepsilon} |b| + \frac{\varepsilon}{1 + \varepsilon} \frac{|a - b|}{\varepsilon} \right)^p \\
 & \leq (1 + \varepsilon)^{p-1} |b|^p + \left( \frac{1 + \varepsilon}{\varepsilon} \right)^{p-1} |a - b|^p.
 \end{aligned}$$

Estimating

$$(1 + \varepsilon)^{p-1} = 1 + (p - 1) \int_1^{1+\varepsilon} t^{p-2} \, dt \leq 1 + \varepsilon(p - 1) \max\{1, (1 + \varepsilon)^{p-2}\},$$

and then iterating, to get the Gamma function bound, concludes the proof.  $\square$

*Remark 3.2.* The estimate in (1.7) holds for  $w_+$  also when  $u$  is a weak subsolution to (2.4) and for  $w_-$  when  $u$  is a weak supersolution to (2.4).

We would like to recall that, as in the classic local case, the proven *Caccioppoli estimates with tail* encode basically all the informations deriving from the minimum property of the functions  $u$  for what concerns the corresponding Hölder continuity.

Now, we prove the second main tool in the proofs in the rest of the paper, that is the *Logarithmic Lemma 1.3* which will be fundamental to handle the nonlinearity of the considered class of operators.

*Proof of Log Lemma 1.3.* Let  $d > 0$  be a real parameter and let  $\phi \in C_0^\infty(B_{3r/2})$  be such that

$$0 \leq \phi \leq 1, \quad \phi \equiv 1 \text{ in } B_r \quad \text{and} \quad |D\phi| < cr^{-1} \text{ in } B_r \subset B_{R/2}.$$

We use in the weak formulation of problem (2.4), the test function  $\eta$  defined by

$$\eta = (u + d)^{1-p} \phi^p.$$

Note that since  $u \geq 0$  in the support of  $\phi$ , the test function is well-defined. We get

$$\begin{aligned} 0 &= \int_{B_{2r}} \int_{B_{2r}} K(x, y) |u(x) - u(y)|^{p-2} (u(x) - u(y)) \\ (3.5) \quad &\times \left[ \frac{\phi^p(x)}{(u(x) + d)^{p-1}} - \frac{\phi^p(y)}{(u(y) + d)^{p-1}} \right] dx dy \\ &+ \int_{\mathbb{R}^n \setminus B_{2r}} \int_{B_{2r}} K(x, y) |u(x) - u(y)|^{p-2} \frac{u(x) - u(y)}{(u(x) + d)^{p-1}} \phi^p(x) dx dy \\ &+ \int_{B_{2r}} \int_{\mathbb{R}^n \setminus B_{2r}} K(x, y) |u(x) - u(y)|^{p-2} \frac{u(y) - u(x)}{(u(y) + d)^{p-1}} \phi^p(y) dx dy \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

If  $u(x) > u(y)$ , for the integrand of  $I_1$ , we use the inequality in Lemma 3.1, by choosing there  $a = \phi(x)$ ,  $b = \phi(y)$  and

$$\varepsilon = \delta \frac{u(x) - u(y)}{u(x) + d} \in (0, 1), \quad \text{with } \delta \in (0, 1),$$

since  $u(y) \geq 0$  for any  $y \in B_{2r} \subset B_R$ . It follows that

$$\begin{aligned} &K(x, y) |u(x) - u(y)|^{p-2} (u(x) - u(y)) \left[ \frac{\phi^p(x)}{(u(x) + d)^{p-1}} - \frac{\phi^p(y)}{(u(y) + d)^{p-1}} \right] \\ &\leq K(x, y) \frac{(u(x) - u(y))^{p-1}}{(u(x) + d)^{p-1}} \phi^p(y) \left[ 1 + c \delta \frac{u(x) - u(y)}{u(x) + d} - \left( \frac{u(x) + d}{u(y) + d} \right)^{p-1} \right] \\ &\quad + c \delta^{1-p} K(x, y) |\phi(x) - \phi(y)|^p, \end{aligned}$$

where  $c \equiv c(p)$ . Observe that the first term that appears in the right-hand side of the previous inequality can be rewritten as

$$(3.6) \quad K(x, y) \left( \frac{u(x) - u(y)}{u(x) + d} \right)^p \phi^p(y) \left[ \frac{1 - \left( \frac{u(y) + d}{u(x) + d} \right)^{1-p}}{1 - \frac{u(y) + d}{u(x) + d}} + c \delta \right] =: J_1.$$

Now, consider the real function  $t \mapsto g(t)$  given by

$$g(t) := \frac{1 - t^{1-p}}{1 - t} = -\frac{p-1}{1-t} \int_t^1 \tau^{-p} d\tau, \quad \forall t \in (0, 1).$$

We have that  $g$  is an increasing function in  $t$ , since

$$t \mapsto \frac{1}{1-t} \int_t^1 \tau^{-p} d\tau$$

is a decreasing function (recall that  $p > 1$ ). Thus

$$g(t) \leq -(p-1) \quad \forall t \in (0, 1).$$

Moreover, if  $t \leq 1/2$ , then

$$g(t) \leq -\frac{p-1}{2^p} \frac{t^{1-p}}{1-t}.$$

Therefore, if

$$t = \frac{u(y) + d}{u(x) + d} \in (0, 1/2];$$

that is,

$$u(y) + d \leq \frac{u(x) + d}{2},$$

then, since  $(u(x) - u(y))(u(y) + d)^{p-1}/(u(x) + d)^p \leq 1$ , we get

$$(3.7) \quad J_1 \leq K(x, y) \left( c\delta - \frac{p-1}{2^p} \right) \left[ \frac{u(x) - u(y)}{u(y) + d} \right]^{p-1} \phi^p(y),$$

Choosing

$$(3.8) \quad \delta = \frac{p-1}{2^{p+1}c},$$

we get

$$J_1 \leq -K(x, y) \frac{p-1}{2^{p+1}} \left[ \frac{u(x) - u(y)}{u(y) + d} \right]^{p-1}$$

If, on the other hand,

$$t = \frac{u(y) + d}{u(x) + d} \in (1/2, 1),$$

that is,

$$u(y) + d > \frac{u(x) + d}{2},$$

then

$$J_1 \leq K(x, y) [c\delta - (p-1)] \left[ \frac{u(x) - u(y)}{u(x) + d} \right]^p \phi^p(y),$$

and so, by the choice of  $\delta$  in (3.8), we finally get

$$(3.9) \quad J_1 \leq -K(x, y) \frac{(2^{p+1} - 1)(p-1)}{2^{p+1}} \left[ \frac{u(x) - u(y)}{u(x) + d} \right]^p \phi^p(y).$$

Furthermore, if  $2(u(y) + d) < u(x) + d$ , then

$$(3.10) \quad \left[ \log \left( \frac{u(x) + d}{u(y) + d} \right) \right]^p \leq c \left[ \frac{u(x) - u(y)}{u(y) + d} \right]^{p-1}$$

holds with  $c \equiv c(p)$ . On the other hand, if  $2(u(y) + d) \geq u(x) + d$ , recalling that we have assumed  $u(x) > u(y)$ , then

$$(3.11) \quad \left[ \log \left( \frac{u(x) + d}{u(y) + d} \right) \right]^p = \left[ \log \left( 1 + \frac{u(x) - u(y)}{u(y) + d} \right) \right]^p \leq 2^p \left( \frac{u(x) - u(y)}{u(x) + d} \right)^p,$$

where we have used

$$\log(1 + \xi) \leq \xi, \quad \forall \xi \geq 0, \quad \text{with } \xi = \frac{u(x) - u(y)}{u(y) + d} \leq \frac{2[u(x) - u(y)]}{u(x) + d}.$$

Thus, combining (3.6) with (3.7), (3.9), (3.10) and (3.11), we conclude with

$$\begin{aligned} K(x, y) |u(x) - u(y)|^{p-2} (u(x) - u(y)) & \left[ \frac{\phi^p(x)}{(u(x) + d)^{p-1}} - \frac{\phi^p(y)}{(u(y) + d)^{p-1}} \right] \\ & \leq -\frac{1}{c} K(x, y) \left[ \log \left( \frac{u(x) + d}{u(y) + d} \right) \right]^p \phi^p(y) + c \delta^{1-p} \bar{K}(x, y) |\phi(x) - \phi(y)|^p. \end{aligned}$$

Observe that when  $u(x) = u(y)$ , then the estimate above holds trivially. If, on the other hand,  $u(y) > u(x)$  we can again exchange the roles of  $x$  and  $y$  in the computations above. We finally get for  $I_1$  in (3.5)

$$(3.12) \quad \begin{aligned} I_1 & \leq -\frac{1}{c} \int_{B_{2r}} \int_{B_{2r}} K(x, y) \left| \log \left( \frac{u(x) + d}{u(y) + d} \right) \right|^p \phi^p(y) \, dx dy \\ & \quad + c \int_{B_{2r}} \int_{B_{2r}} \bar{K}(x, y) |\phi(x) - \phi(y)|^p \, dx dy \end{aligned}$$

for a constant  $c \equiv c(p)$  by the choice of  $\delta$ .

For the second contribution in (3.5), namely  $I_2$ , we can proceed as follows. First of all, notice that when  $y \in B_R$ ,  $u(y) \geq 0$  and so

$$\frac{(u(x) - u(y))_+^{p-1}}{(d + u(x))^{p-1}} \leq 1 \quad \text{for all } x \in B_{2r}, y \in B_R.$$

Moreover, when  $y \in \mathbb{R}^n \setminus B_R$ ,

$$(u(x) - u(y))_+^{p-1} \leq 2^{p-1} [u^{p-1}(x) + (u(y))_-^{p-1}] \quad \text{for all } x \in B_{2r}.$$

Therefore,

$$(3.13) \quad \begin{aligned} I_2 & \leq \int_{B_R \setminus B_{2r}} \int_{B_{2r}} K(x, y) (u(x) - u(y))_+^{p-1} (d + u(x))^{1-p} \phi^p(x) \, dx dy \\ & \quad + \int_{\mathbb{R}^n \setminus B_R} \int_{B_{2r}} K(x, y) (u(x) - u(y))_+^{p-1} (d + u(x))^{1-p} \phi^p(x) \, dx dy \\ & \leq c \int_{\mathbb{R}^n \setminus B_{2r}} \int_{B_{2r}} \bar{K}(x, y) \phi^p(x) \, dx dy \\ & \quad + c d^{1-p} \int_{\mathbb{R}^n \setminus B_R} \int_{B_{2r}} \bar{K}(x, y) (u(y))_-^{p-1} \, dx dy \end{aligned}$$

follows for  $c \equiv c(p)$ . By the assumptions on  $K$  and the fact that the support of  $\phi$  belongs to  $B_{3r/2}$ , we have

$$(3.14) \quad \int_{\mathbb{R}^n \setminus B_{2r}} \int_{B_{2r}} \bar{K}(x, y) \phi^p(x) \, dx dy \leq c \sup_{x \in B_{3r/2}} r^n \int_{\mathbb{R}^n \setminus B_{2r}} \bar{K}(x, y) \, dy \leq cr^{n-sp}$$

and

$$(3.15) \quad \begin{aligned} \int_{\mathbb{R}^n \setminus B_R} \int_{B_{2r}} \bar{K}(x, y) (u(y))_-^{p-1} \, dx dy &\leq c |B_r| \int_{\mathbb{R}^n \setminus B_R} \frac{(u(y))_-^{p-1}}{|y - x_0|^{n+sp}} \, dy \\ &\leq c \frac{r^n}{R^{sp}} [\text{Tail}(u_-; x_0, R)]^{p-1}, \end{aligned}$$

where we also used that, for any  $x \in B_r$ ,  $y \in \mathbb{R}^n \setminus B_R$  and  $2r \leq R$ ,

$$\frac{|y - x_0|}{|y - x|} \leq 1 + \frac{|x - x_0|}{|x - y|} \leq 1 + \frac{r}{R - r} \leq 2.$$

By combining (3.13) with (3.14) and (3.15), and by observing that  $I_3$  can be handled in the same way as  $I_2$ , we obtain

$$I_2 + I_3 \leq c \int_{B_{2r}} \int_{B_{2r}} \bar{K}(x, y) |\phi(x) - \phi(y)|^p \, dx dy + cr^{n-sp},$$

which, together with (3.12) in (3.5), yields

$$(3.16) \quad \begin{aligned} \int_{B_{2r}} \int_{B_{2r}} K(x, y) \left| \log \left( \frac{u(x) + d}{u(y) + d} \right) \right|^p \phi^p(y) \, dx dy \\ \leq c \int_{B_{2r}} \int_{B_{2r}} \bar{K}(x, y) |\phi(x) - \phi(y)|^p \, dx dy \\ + c d^{1-p} r^n R^{-sp} [\text{Tail}(u_-; x_0, R)]^{p-1} + cr^{n-sp}. \end{aligned}$$

Finally, in order to conclude the proof, we need the following estimate

$$(3.17) \quad \begin{aligned} \int_{B_{2r}} \int_{B_{2r}} \bar{K}(x, y) |\phi(x) - \phi(y)|^p \, dx dy &\leq cr^{-p} \int_{B_{2r}} \int_{B_{2r}} |x - y|^{-n+p(1-s)} \, dx dy \\ &\leq \frac{c}{p(1-s)} r^{-sp} |B_{2r}|, \end{aligned}$$

where we used the bound from above on the kernel  $K$  and the fact that we are assuming  $|D\phi| \leq cr^{-1}$ . The proof of (1.6) is finished.  $\square$

A first consequence of the Logarithmic Lemma is the following

**Corollary 3.3.** *Let  $p \in (1, \infty)$  and let  $u \in W^{s,p}(\mathbb{R}^n)$  be the solution to problem (2.4) such that  $u \geq 0$  in  $B_R \equiv B_R(x_0) \subset \Omega$ . Let  $a, d > 0$ ,  $b > 1$  and define*

$$v := \min \{ (\log(a + d) - \log(u + d))_+, \log(b) \}.$$

*Then the following estimate holds true, for any  $B_r \equiv B_r(x_0) \subset B_{R/2}(x_0)$ ,*

$$\int_{B_r} |v - (v)_{B_r}|^p \, dx \leq c \left\{ d^{1-p} \left( \frac{r}{R} \right)^{sp} [\text{Tail}(u_-; x_0, R)]^{p-1} + 1 \right\},$$

where  $\text{Tail}(u_-; x_0, R)$  is defined by (2.2) and  $c$  depends only on  $n, p, s, \lambda$  and  $\Lambda$ .

*Proof.* By the fractional Poincaré type inequality (see, e.g., [15]) and the assumption in (2.1) for  $K$  we get

$$\int_{B_r} |v - (v)_{B_r}|^p dx \leq c r^{sp-n} \int_{B_r} \int_{B_r} K(x, y) |v(x) - v(y)|^p dx dy$$

with a constant  $c \equiv c(n, p, s, \lambda, \Lambda)$ . Now observe that  $v$  is a truncation of the sum of a constant and  $\log(u + d)$  and hence it follows that

$$\int_{B_r} \int_{B_r} K(x, y) |v(x) - v(y)|^p dx dy \leq \int_{B_r} \int_{B_r} K(x, y) \left| \log \left( \frac{u(y) + d}{u(x) + d} \right) \right|^p dx dy.$$

At this stage, in order to conclude, it just suffices to apply the estimate in (1.6).  $\square$

#### 4. LOCAL BOUNDEDNESS

In this section, we prove the local boundedness for the fractional  $p$ -minimizers of the functional (2.5), as stated in Theorem 1.1.

*Proof of Theorem 1.1.* Before starting, let us give some definitions. For any  $j \in \mathbb{N}$  and  $r > 0$  such that  $B_r(x_0) \subset \Omega$ ,

$$(4.1) \quad \begin{aligned} r_j &= \frac{1}{2}(1 + 2^{-j})r, & \tilde{r}_j &= \frac{r_j + r_{j+1}}{2}, \\ B_j &= B_{r_j}(x_0), & \tilde{B}_j &= B_{\tilde{r}_j}(x_0). \end{aligned}$$

Moreover, take

$$(4.2) \quad \begin{aligned} \phi_j &\in C_0^\infty(\tilde{B}_j), \quad 0 \leq \phi_j \leq 1, \quad \phi_j \equiv 1 \text{ on } B_{j+1}, \text{ and } |D\phi_j| < 2^{j+3}/r, \\ k_j &= k + (1 - 2^{-j})\tilde{k}, \quad \tilde{k}_j = \frac{k_{j+1} + k_j}{2}, \quad \tilde{k} \in \mathbb{R}^+ \text{ and } k \in \mathbb{R}. \end{aligned}$$

$$\tilde{w}_j = (u - \tilde{k}_j)_+ \quad \text{and} \quad w_j = (u - k_j)_+.$$

By the fractional Sobolev inequality applied to the function  $\tilde{w}_j \phi_j$ , as defined above, together with the properties of the kernel  $K$  we get

$$\left( \int_{B_j} |\tilde{w}_j(x) \phi_j(x)|^{p^*} dx \right)^{\frac{p}{p^*}} \leq c \frac{r^{sp}}{r^n} \int_{B_j} \int_{B_j} K(x, y) |\tilde{w}_j(x) \phi_j(x) - \tilde{w}_j(y) \phi_j(y)|^p dx dy,$$

recalling that  $p^* = np/(n - sp)$ , which is the critical exponent for fractional Sobolev embeddings, is well defined since we are dealing with the case when  $sp < n$ .

Using the nonlocal Caccioppoli inequality with tail given by (1.7), with  $w_+ = \tilde{w}_j$  and  $\phi = \phi_j$  there, we arrive at

$$(4.3) \quad \begin{aligned} &\left( \int_{B_j} |\tilde{w}_j(x) \phi_j(x)|^{p^*} dx \right)^{\frac{p}{p^*}} \\ &\leq c r^{sp} \int_{B_j} \int_{B_j} \bar{K}(x, y) (\max\{\tilde{w}_j(x), \tilde{w}_j(y)\})^p |\phi_j(x) - \phi_j(y)|^p dx dy \end{aligned}$$



$$+ cr^{sp} \int_{B_j} \tilde{w}_j(y) \phi_j^p(y) dy \left( \sup_{y \in \text{supp } \phi_j} \int_{\mathbb{R}^n \setminus B_j} \bar{K}(x, y) \tilde{w}_j^{p-1}(x) dx \right).$$

By the definition of  $\phi_j$  and the assumption (2.1), we obtain the following estimate for the first term in the right hand-side of the inequality above,

$$\begin{aligned} & r^{sp} \int_{B_j} \int_{B_j} \bar{K}(x, y) (\max\{\tilde{w}_j(x), \tilde{w}_j(y)\})^p |\phi_j(x) - \phi_j(y)|^p dx dy \\ & \leq c 2^{jp} \frac{r^{sp}}{r^p} \int_{B_j} w_j^p(y) \left( \int_{B_j} \frac{dx}{|x-y|^{n-p(1-s)}} \right) dy \\ & \quad + c 2^{jp} \frac{r^{sp}}{r^p} \int_{B_j} w_j^p(x) \left( \int_{B_j} \frac{dy}{|x-y|^{n-p(1-s)}} \right) dx \\ (4.4) \quad & \leq \frac{c 2^{jp}}{p(1-s)} \int_{B_j} w_j^p(x) dx. \end{aligned}$$

For the second term on the right in (4.3), we get

$$\begin{aligned} & cr^{sp} \int_{B_j} \tilde{w}_j(y) \phi_j^p(y) dy \left( \sup_{y \in \text{supp } \phi_j} \int_{\mathbb{R}^n \setminus B_j} \bar{K}(x, y) \tilde{w}_j^{p-1}(x) dx \right) \\ & \leq c 2^{j(n+sp)} r^{sp} \left( \int_{B_j} \frac{w_j^p(y)}{(\tilde{k}_j - k_j)^{p-1}} dy \right) \left( \int_{\mathbb{R}^n \setminus B_j} \frac{w_j^{p-1}(x)}{|x_0 - x|^{n+sp}} dx \right) \\ (4.5) \quad & \leq c \frac{2^{j(n+sp+p-1)}}{\tilde{k}^{p-1}} [\text{Tail}(w_0; x_0, r/2)]^{p-1} \int_{B_j} w_j^p(y) dy, \end{aligned}$$

where we have just used the definitions in (4.1)–(4.2), the facts that  $\tilde{w}_j \leq w_j^p / (\tilde{k}_j - k_j)^{p-1}$  and that  $y \in \tilde{B}_j = \text{supp } \phi_j$  and  $x \in \mathbb{R}^n \setminus B_j$  yield

$$\frac{|x - x_0|}{|x - y|} \leq \frac{|x - y| + |x_0 - x|}{|x - y|} \leq 1 + \frac{\tilde{r}_j}{r_j - \tilde{r}_j} \leq 2^{j+4}.$$

The left hand-side of (4.3) can be estimated from below as follows

$$\begin{aligned} & \left( \int_{B_j} |\tilde{w}_j(x) \phi_j(x)|^{p^*} dx \right)^{\frac{p}{p^*}} \geq (k_{j+1} - \tilde{k}_j)^{\frac{(p^*-p)p}{p^*}} \left( \int_{B_{j+1}} w_{j+1}^p(x) dx \right)^{\frac{p}{p^*}} \\ (4.6) \quad & = \left( \frac{\tilde{k}}{2^{j+2}} \right)^{\frac{(p^*-p)p}{p^*}} \left( \int_{B_{j+1}} w_{j+1}^p(x) dx \right)^{\frac{p}{p^*}}. \end{aligned}$$

By combining (4.3) with (4.4), (4.5) and (4.6), we obtain

$$\left( \frac{\tilde{k}^{1-p/p^*}}{2^{(j+2)\frac{(p^*-p)}{p^*}}} \right)^p A_{j+1}^{\frac{p^2}{p^*}} \leq c 2^{j(n+sp+p-1)} \left( \frac{1}{p(1-s)} + \frac{[\text{Tail}(w_0; x_0, r/2)]^{p-1}}{\tilde{k}^{p-1}} \right) A_j^p,$$

where we have set  $A_j := \left( \int_{B_j} w_j^p(x) dx \right)^{\frac{1}{p}}$ .

Now, by taking

$$(4.7) \quad \tilde{k} \geq \delta \text{Tail}(w_0; x_0, r/2), \quad \delta \in (0, 1],$$

we get

$$(4.8) \quad \left( \frac{A_{j+1}}{\tilde{k}} \right)^{\frac{p}{p^*}} \leq \delta^{\frac{1-p}{p}} \bar{c}^{\frac{p}{p^*}} 2^j \left( \frac{n+sp+p-1}{p} + \frac{sp}{n} \right) \frac{A_j}{\tilde{k}},$$

where  $\bar{c} = 2^{\frac{2(p^*-p)}{p}} c^{\frac{p^*}{p^2}} (1 + (p(1-s))^{-1})^{\frac{p^*}{p^2}}$ .

Setting  $\beta := sp/(n-sp) = p^*/p - 1 > 0$  and  $C := 2^{\frac{(n+sp+p-1)n}{p(n-sp)} + \frac{sp}{n-sp}} > 1$ , the estimate in (4.8) becomes

$$\frac{A_{j+1}}{\tilde{k}} \leq \delta^{\frac{(1-p)p^*}{p^2}} \bar{c} C^j \left( \frac{A_j}{\tilde{k}} \right)^{1+\beta}$$

Thus, it suffices to prove that the following estimate on  $A_0$  does hold

$$\frac{A_0}{\tilde{k}} \leq \delta^{\frac{(p-1)p^*}{\beta p^2}} \bar{c}^{-\frac{1}{\beta}} C^{-\frac{1}{\beta^2}}$$

and, by a well-known iteration argument, it will follow  $A_j \rightarrow 0$  as  $j \rightarrow \infty$ . Since

$$\frac{(p-1)p^*}{\beta p^2} = \frac{p-1}{p} \frac{n}{n-ps} \frac{n-sp}{sp} = \frac{(p-1)n}{sp^2},$$

we choose

$$\tilde{k} = \delta \text{Tail}(w_0; x_0, r/2) + \delta^{-\frac{(p-1)n}{sp^2}} H A_0, \quad \text{with } H := \bar{c}^{\frac{1}{\beta}} C^{\frac{1}{\beta^2}},$$

which is in accordance with (4.7).

We deduce

$$\begin{aligned} \sup_{B_{r/2}} u &\leq k + \tilde{k} \\ &= k + \delta \text{Tail}((u-k)_+; x_0, r/2) + \delta^{-\frac{(p-1)n}{sp^2}} H \left( \int_{B_r} (u-k)_+^p \right)^{\frac{1}{p}}, \end{aligned}$$

which finally gives the desired result by taking  $k = 0$ .  $\square$

*Remark 4.1.* Similarly, it is possible to prove that the weak solutions to problem (2.4) are locally bounded from below, satisfying an estimate analogous to the one in (1.5). The proof is exactly as before: one has just to work with  $\tilde{w}_j = (\tilde{k}_j - u)_+$  and  $w_j = (k_j - u)_+$  instead of the auxiliary functions defined in (4.2) and make use of the corresponding Caccioppoli estimate (1.7) for  $w_-$ .

*Remark 4.2.* We observe that the previous result also holds if  $u \in W^{s,p}(\mathbb{R}^n)$  is a weak subsolution to problem (2.4), for any  $s \in (0, 1)$  and any  $p \in [1, \infty)$ . Indeed, as noticed in Remark 3.2, the Caccioppoli inequality, that we use in the proof above, does also hold for weak subsolutions.

## 5. HÖLDER CONTINUITY

This section is devoted to the proof of the main result in this paper, that is contained in Theorem 1.2. As in the local framework, an iteration lemma is the keypoint of the proof. However, as before, we have to handle the nonlocality of the involved operator and thus a certain care is required. In the proof below, all the estimates proven in previous sections will appear.

Before starting, let us fix some notation. For any  $j \in \mathbb{N}$ , let  $0 < r < R/2$ , for some  $R$  such that  $B_R(x_0) \subset \Omega$ ,

$$r_j := \sigma^j \frac{r}{2}, \quad \sigma \in (0, 1/4] \quad \text{and} \quad B_j := B_{r_j}(x_0).$$

Moreover, let us define

$$\frac{1}{2} \omega(r_0) = \frac{1}{2} \omega\left(\frac{r}{2}\right) := \text{Tail}(u; x_0, r/2) + c \left( \int_{B_r} u_+^p dx \right)^{\frac{1}{p}},$$

with  $\text{Tail}(u; x_0, r/2)$  as in (2.2) and  $c$  as in (1.5), and

$$\omega(r_j) := \left(\frac{r_j}{r_0}\right)^\alpha \omega(r_0), \quad \text{for some } \alpha < \frac{sp}{p-1}.$$

In order to prove the Theorem 1.2, it will suffice to prove the following

**Lemma 5.1.** *Under the notation introduced above, let  $u \in W^{s,p}(\mathbb{R}^n)$  be the solution to problem (2.4). Then*

$$(5.1) \quad \text{osc}_{B_j} u \equiv \sup_{B_j} u - \inf_{B_j} u \leq \omega(r_j), \quad \forall j = 0, 1, 2, \dots$$

*Proof.* We will proceed by induction. For this, note that by the definition of  $\omega(r_0)$  and Theorem 1.1 (with  $\delta = 1$  there), the estimate in (5.1) trivially holds for  $j = 0$ .

Now, we make a strong induction assumption and assume that (5.1) is valid for all  $i \in \{0, \dots, j\}$  for some  $j \geq 0$ , and then we prove that it holds also for  $j + 1$ . We have that either

$$(5.2) \quad \frac{|2B_{j+1} \cap \{u \geq \inf_{B_j} u + \omega(r_j)/2\}|}{|2B_{j+1}|} \geq \frac{1}{2}.$$

or

$$(5.3) \quad \frac{|2B_{j+1} \cap \{u \leq \inf_{B_j} u + \omega(r_j)/2\}|}{|2B_{j+1}|} \geq \frac{1}{2}$$

must hold. If (5.2) holds, we set  $u_j := u - \inf_{B_j} u$ , and if (5.3) holds, we set  $u_j := \omega(r_j) - (u - \inf_{B_j} u)$ . In all cases we have that  $u_j \geq 0$  in  $B_j$  and

$$(5.4) \quad \frac{|2B_{j+1} \cap \{u_j \geq \omega(r_j)/2\}|}{|2B_{j+1}|} \geq \frac{1}{2}$$

holds. Moreover,  $u_j$  is a weak solution satisfying

$$(5.5) \quad \sup_{B_i} |u_j| \leq 2\omega(r_i) \quad \forall i \in \{0, \dots, j\}.$$

We now claim that under the induction assumption we have

$$(5.6) \quad [\text{Tail}(u_j; x_0, r_j)]^{p-1} \leq c \sigma^{-\alpha(p-1)} [\omega(r_j)]^{p-1},$$

where the constant  $c$  depends only on  $n, p, s$  and the difference of  $sp/(p-1)$  and  $\alpha$ , but, in particular, it is independent of  $\sigma$ . Indeed, we have

$$\begin{aligned} [\text{Tail}(u_j; x_0, r_j)]^{p-1} &= r_j^{sp} \sum_{i=1}^j \int_{B_{i-1} \setminus B_i} |u_j(x)|^{p-1} |x - x_0|^{-n-sp} dx \\ &\quad + r_j^{sp} \int_{\mathbb{R}^n \setminus B_0} |u_j(x)|^{p-1} |x - x_0|^{-n-sp} dx \\ &\leq r_j^{sp} \sum_{i=1}^j [\sup_{B_{i-1}} |u_j|]^{p-1} \int_{\mathbb{R}^n \setminus B_i} |x - x_0|^{-n-sp} dx \\ &\quad + r_j^{sp} \int_{\mathbb{R}^n \setminus B_0} |u_j(x)|^{p-1} |x - x_0|^{-n-sp} dx \\ &\leq c \sum_{i=1}^j \left( \frac{r_j}{r_i} \right)^{sp} [\omega(r_{i-1})]^{p-1}, \end{aligned}$$

where on the last line we used (5.5) and

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus B_0} |u_j(x)|^{p-1} |x - x_0|^{-n-sp} dx \\ &\leq cr_0^{-sp} \sup_{B_0} |u|^{p-1} + cr_0^{-sp} [\omega(r_0)]^{p-1} + c \int_{\mathbb{R}^n \setminus B_0} |u(x)|^{p-1} |x - x_0|^{-n-sp} dx \\ &\leq cr_1^{-sp} [\omega(r_0)]^{p-1}. \end{aligned}$$

Estimating further as

$$\begin{aligned} &\sum_{i=1}^j \left( \frac{r_j}{r_i} \right)^{sp} [\omega(r_{i-1})]^{p-1} \\ &= [\omega(r_0)]^{p-1} \left( \frac{r_j}{r_0} \right)^{\alpha(p-1)} \sum_{i=1}^j \left( \frac{r_{i-1}}{r_i} \right)^{\alpha(p-1)} \left( \frac{r_j}{r_i} \right)^{sp-\alpha(p-1)} \\ &= [\omega(r_j)]^{p-1} \sigma^{-\alpha(p-1)} \sum_{i=0}^{j-1} \sigma^{i(sp-\alpha(p-1))} \\ &\leq [\omega(r_j)]^{p-1} \frac{\sigma^{-\alpha(p-1)}}{1 - \sigma^{sp-\alpha(p-1)}} \\ &\leq \frac{4^{sp-\alpha(p-1)}}{\log(4)(sp - \alpha(p-1))} \sigma^{-\alpha(p-1)} [\omega(r_j)]^{p-1}, \end{aligned}$$

where we have used the fact that  $\sigma \leq 1/4$  and  $\alpha < sp/(p-1)$ . Hence (5.6) is proved with  $c$  depending only on  $n, p, s$  and the difference of  $sp/(p-1)$  and  $\alpha$ .

Next, consider the function  $v$  defined as follows

$$(5.7) \quad v := \min \left\{ \left[ \log \left( \frac{\omega(r_j)/2 + d}{u_j + d} \right) \right]_+, k \right\}, \quad k > 0.$$

Applying then Corollary 3.3, obviously with  $a \equiv \omega(r_j)/2$  and  $b \equiv \exp(k)$ , we get

$$\int_{2B_{j+1}} |v - (v)_{2B_{j+1}}|^p dx \leq c \left\{ d^{1-p} \left( \frac{r_{j+1}}{r_j} \right)^{sp} [\text{Tail}(u_j; x_0, r_j)]^{p-1} + 1 \right\}.$$

Thus, as a consequence of the estimate in (5.6), we arrive at

$$\int_{2B_{j+1}} |v - (v)_{2B_{j+1}}|^p dx \leq c \left\{ d^{1-p} \sigma^{sp-\alpha(p-1)} [\omega(r_j)]^{p-1} + 1 \right\}.$$

Therefore, choosing  $d = \varepsilon \omega(r_j)$  with

$$\varepsilon := \sigma^{\frac{sp}{p-1}-\alpha},$$

we get

$$(5.8) \quad \int_{2B_{j+1}} |v - (v)_{2B_{j+1}}| dx \leq c,$$

where the constant  $c$  depends only on  $n, p, s, \lambda, \Lambda$  and the difference of  $sp/(p-1)$  and  $\alpha$ .

To continue, denote in short  $\tilde{B} \equiv 2B_{j+1}$ , and follow the path paved in [14], together with (5.4) and the definition of  $v$  given in (5.7). We obtain

$$\begin{aligned} k &= \frac{1}{|\tilde{B} \cap \{u_j \geq \omega(r_j)/2\}|} \int_{\tilde{B} \cap \{u_j \geq \omega(r_j)/2\}} k dx \\ &= \frac{1}{|\tilde{B} \cap \{u_j \geq \omega(r_j)/2\}|} \int_{\tilde{B} \cap \{v=0\}} k dx \\ &\leq \frac{2}{|\tilde{B}|} \int_{\tilde{B}} (k - v) dx = 2[k - (v)_{\tilde{B}}]. \end{aligned}$$

By integrating the preceding inequality over the set  $\tilde{B} \cap \{v = k\}$  we obtain

$$\begin{aligned} \frac{|\tilde{B} \cap \{v = k\}|}{|\tilde{B}|} k &\leq \frac{2}{|\tilde{B}|} \int_{\tilde{B} \cap \{v=k\}} [k - (v)_{\tilde{B}}] dx \\ &\leq \frac{2}{|\tilde{B}|} \int_{\tilde{B}} |v - (v)_{\tilde{B}}| dx \leq c, \end{aligned}$$

thanks to (5.8). Let us take

$$k = \log \left( \frac{\omega(r_j)/2 + \varepsilon \omega(r_j)}{3 \varepsilon \omega(r_j)} \right) = \log \left( \frac{1/2 + \varepsilon}{3 \varepsilon} \right) \approx \log \left( \frac{1}{\varepsilon} \right),$$

so that

$$\frac{|\tilde{B} \cap \{v = k\}|}{|\tilde{B}|} k \leq c$$

yields

$$(5.9) \quad \frac{|\tilde{B} \cap \{u_j \leq 2\varepsilon \omega(r_j)\}|}{|\tilde{B}|} \leq \frac{c}{k} \leq \frac{c_{\log}}{\log\left(\frac{1}{\sigma}\right)},$$

where the constant  $c_{\log}$  depends only on  $n, p, s, \lambda, \Lambda$  and the difference of  $sp/(p-1)$  and  $\alpha$  via the definition of  $\varepsilon$ .

We are now in a position to start a suitable iteration to deduce the desired oscillation reduction. First, for any  $i = 0, 1, 2, \dots$ , we define

$$\varrho_i = r_{j+1} + 2^{-i}r_{j+1}, \quad \tilde{\varrho}_i := \frac{\varrho_i + \varrho_{i+1}}{2}, \quad B^i = B_{\varrho_i}, \quad \tilde{B}^i = B_{\tilde{\varrho}_i}$$

and corresponding cut-off functions

$$\phi_i \in C_0^\infty(\tilde{B}^i), \quad 0 \leq \phi_i \leq 1, \quad \phi_i \equiv 1 \text{ on } B^{i+1}, \quad \text{and } |D\phi_i| < c\varrho_i^{-1}.$$

Furthermore, set

$$k_i = (1 + 2^{-i})\varepsilon \omega(r_j), \quad w_i := (k_i - u_j)_+,$$

and

$$A_i = \frac{|B^i \cap \{u_j \leq k_i\}|}{|B^i|} = \frac{|B^i \cap \{w_i > 0\}|}{|B^i|}.$$

The Caccioppoli inequality in (1.7) now yields

$$(5.10) \quad \begin{aligned} & \int_{B^i} \int_{B^i} K(x, y) |w_i(x)\phi_i(x) - w_i(y)\phi_i(y)|^p \, dx dy \\ & \leq c \int_{B^i} \int_{B^i} \bar{K}(x, y) (\max\{w_i(x), w_i(y)\})^p |\phi_i(x) - \phi_i(y)|^p \, dx dy \\ & \quad + c \int_{B^i} w_i(x)\phi_i^p(x) \, dx \left( \sup_{y \in \tilde{B}^i} \int_{\mathbb{R}^n \setminus B^i} \bar{K}(x, y) w_i^{p-1}(x) \, dx \right). \end{aligned}$$

Using the fractional Sobolev inequality, we estimate the term on the left below as

$$(5.11) \quad \begin{aligned} A_{i+1}^{\frac{p}{p^*}} (k_i - k_{i+1})^p &= \frac{1}{|B^{i+1}|^{\frac{p}{p^*}}} \left( \int_{B^{i+1} \cap \{u_j \leq k_{i+1}\}} (k_i - k_{i+1})^{p^*} \phi_i^{p^*}(x) \, dx \right)^{\frac{p}{p^*}} \\ &\leq \frac{1}{|B^{i+1}|^{\frac{p}{p^*}}} \left( \int_{B^i} w_i^{p^*}(x) \phi_i^{p^*}(x) \, dx \right)^{\frac{p}{p^*}} \\ &\leq c r_{j+1}^{sp-n} \int_{B^i} \int_{B^i} K(x, y) |w_i(x)\phi_i(x) - w_i(y)\phi_i(y)|^p \, dx dy. \end{aligned}$$

Recalling that  $|D\phi_i| \leq c2^i r_{j+1}^{-1}$ , the first term on the right in (5.10) can be treated as follows,

$$r_{j+1}^{sp} \int_{B^i} \int_{B^i} \bar{K}(x, y) (\max\{w_i(x), w_i(y)\})^p |\phi_i(x) - \phi_i(y)|^p \, dx dy$$

$$\begin{aligned}
 &\leq c 2^{ip} r_{j+1}^{sp} r_{j+1}^{-p} k_i^p \int_{B^i \cap \{u_j \leq k_i\}} \int_{B^i} \frac{1}{|x-y|^{-p+n+sp}} dy dx \\
 (5.12) \quad &\leq c 2^{ip} [\varepsilon \omega(r_j)]^p |B^i \cap \{u_j \leq k_i\}|.
 \end{aligned}$$

Moreover,

$$(5.13) \quad \int_{B^i} w_i(x) \phi_i^p(x) dx \leq c [\varepsilon \omega(r_j)] |B^i \cap \{u_j \leq k_i\}|$$

holds. To tackle the third integral in (5.10), we first have

$$(5.14) \quad r_{j+1}^{sp} \left( \sup_{y \in \tilde{B}^i} \int_{\mathbb{R}^n \setminus B^i} \bar{K}(x, y) w_i^{p-1}(x) dx \right) \leq c 2^{i(n+sp)} [\text{Tail}(w_i; x_0, r_{j+1})]^{p-1},$$

using

$$\inf_{y \in \tilde{B}^i} |y-x| \geq |x_0-x| \inf_{y \in \tilde{B}^i} \frac{|y-x|}{|x_0-x|} \geq 2^{-i-1} |x-x_0|$$

for all  $x \in \mathbb{R}^n \setminus B^i$  and the fact that

$$B_{r_{j+1}} \equiv B_{j+1} \subset B^i \quad \Rightarrow \quad \mathbb{R}^n \setminus B^i \subset \mathbb{R}^n \setminus B_{j+1}.$$

Recalling (5.6) and the facts that  $w_i \leq 2\varepsilon \omega(r_j)$  in  $B_j$  and  $w_i \leq |u_j| + 2\varepsilon \omega(r_j)$  in  $\mathbb{R}^n$ , we further get

$$\begin{aligned}
 &[\text{Tail}(w_i; x_0, r_{j+1})]^{p-1} \\
 &\leq c r_{j+1}^{sp} \int_{B_j \setminus B_{j+1}} w_i^{p-1}(x) |x-x_0|^{-n-sp} dx + c \left( \frac{r_{j+1}}{r_j} \right)^{sp} [\text{Tail}(w_i; x_0, r_j)]^{p-1} \\
 &\leq c \varepsilon^{p-1} \omega(r_j)^{p-1} + c \sigma^{sp} [\text{Tail}(u_j; x_0, r_j)]^{p-1} \\
 &\leq c \left( 1 + \frac{\sigma^{sp-\alpha(p-1)}}{\varepsilon^{p-1}} \right) [\varepsilon \omega(r_j)]^{p-1} \\
 &\leq c [\varepsilon \omega(r_j)]^{p-1},
 \end{aligned}$$

by the very definition of  $\varepsilon$ . Combining the estimates above, we deduce that

$$(5.15) \quad r_{j+1}^{sp} \left( \sup_{y \in \tilde{B}^i} \int_{\mathbb{R}^n \setminus B^i} \bar{K}(x, y) w_i^{p-1}(x) dx \right) \leq c 2^{i(n+sp)} [\varepsilon \omega(r_j)]^{p-1}.$$

Putting together (5.10), (5.11) (5.12), (5.13) and (5.15), we arrive at

$$A_{i+1}^{\frac{p}{p^*}} (k_i - k_{i+1})^p \leq c 2^{i(n+sp+p)} [\varepsilon \omega(r_j)]^p A_i,$$

which yields

$$A_{i+1} \leq c 2^{i[n+(2+s)p]p^*/p} A_i^{1+\beta}$$

with  $\beta := sp/(n-sp)$  by the definition of  $k_i$ 's. Now, we recall that if we prove the following estimate on  $A_0$ ,

$$(5.16) \quad A_0 = \frac{|\tilde{B} \cap \{u_j \leq 2\varepsilon \omega(r_j)\}|}{|\tilde{B}|} \leq c^{-1/\beta} 2^{-[n+(2+s)p]p^*/[p\beta^2]} =: \nu^*,$$

then we can deduce that

$$A_i \rightarrow 0 \quad \text{as} \quad i \rightarrow \infty.$$

Indeed, the condition (5.16) we can guarantee by (5.9) choosing

$$\sigma = \min\{1/4, \exp(-c_{\log}/\nu^*)\},$$

which then depends only on  $n, p, s, \lambda, \Lambda$  and the difference of  $sp/(p-1)$  and  $\alpha$ . In other words, we have shown that

$$\operatorname{osc}_{B_{j+1}} u \leq (1 - \varepsilon)\omega(r_j) = (1 - \varepsilon) \left( \frac{r_j}{r_{j+1}} \right)^\alpha \omega(r_{j+1}) = (1 - \varepsilon)\sigma^{-\alpha}\omega(r_{j+1}).$$

Taking finally  $\alpha \in \left(0, \frac{sp}{p-1}\right)$  small enough satisfying

$$\sigma^\alpha \geq 1 - \varepsilon = 1 - \sigma^{\frac{sp}{p-1} - \alpha},$$

then, clearly,  $\alpha$  depends only on  $n, p, s, \lambda, \Lambda$  and

$$\operatorname{osc}_{B_{j+1}} u \leq \omega(r_{j+1})$$

holds, proving the induction step and finishing the proof.  $\square$

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