

# LOCAL BOUNDARY REPRESENTATIONS OF LOCALLY $C^*$ -ALGEBRAS

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**ABSTRACT.** We initiate a study of non-commutative Choquet boundary for spaces of unbounded operators. We define the notion of local boundary representations for local operator systems in locally  $C^*$ -algebras and prove that local boundary representations provide an intrinsic invariant for a particular class of local operator systems. An appropriate analog of purity of local completely positive maps on local operator systems is used to characterize local boundary representations for local operator systems in Frechet locally  $C^*$ -algebras.

## 1. INTRODUCTION

The notion of locally  $C^*$ -algebras was introduced by Atushi Inoue [15] to study algebras of unbounded operators on a Hilbert space. In the literature, locally  $C^*$ -algebras have been studied by several authors under different names like pro- $C^*$ -algebras,  $O^*$ -algebras,  $LCM^*$ -algebras, and multinormed  $C^*$ -algebras. Effros and Webster [12] initiated a study of the locally convex version of operator spaces called the *local operator spaces*. In 2008, A. Dosiev [11] realized local operator spaces as subspaces of the locally  $C^*$ -algebra  $\mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$  of unbounded operators on a quantized domain  $\mathcal{E}$  with its union space  $\mathcal{D}$ . Also, Dosiev introduced *local operator systems* as the unital self adjoint subspaces of  $\mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$ . Based upon the local positivity concept in locally  $C^*$ -algebra, Dosiev [11] proved Stinespring representation theorem for local completely positive maps and Arveson extension theorem for local completely positive maps on Frechet local operator systems. Recently, the minimality of Stinespring representation was identified by Bhat, Anindya, and Santhoshkumar [8], and showed that minimal Stinespring's representation is unique up to unitary equivalence.

The extremal theory concerning Choquet boundary of subalgebras of function algebras play an important role in numerous areas of classical analysis. Let  $X$  be a locally compact Hausdorff space and  $C(X)$  be the algebra of all continuous functions on  $X$ . Given a uniform algebra  $\mathcal{U} \subseteq C(X)$  and a point  $x_0 \in X$ . If the evaluation functional corresponds to  $x_0$  admits a unique completely positive extension from  $\mathcal{U}$  to  $C(X)$ , then we say that the point  $x_0$  is in the *Choquet boundary*[7] of  $\mathcal{U}$ . The non-commutative analog of this notion called the *boundary representations* of linear subspaces in  $C^*$ -algebras was introduced by Arveson [2] and studied extensively by him in [4, 5]. The objects boundary representations are intrinsic invariants for operator systems (and operator algebras), and provide

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a context for showing the existence of non-commutative Silov boundary. The articles [9, 10, 14, 18] are also worth mentioning in this context. There is plenty of literature on generalizing the notion of boundary representations to different contexts [1, 13, 19]. This article initiate a study of non-commutative Choquet boundary in the context of locally  $C^*$ -algebras and a related extremal notion of purity of local completely positive maps.

This paper is organized as follows. In section 2, we recall necessary background material and results that are required throughout. Section 3 deals with certain elementary results on local completely contractive(local CC) maps and local completely positive(local CP) maps. We obtain a locally convex version of the Arveson extension theorem for unital local CC-maps. That is, a unital local CC-map from a subspace  $\mathcal{M}$  of a locally  $C^*$ -algebra  $\mathcal{A}$  can be extended to a local CP-map on  $\mathcal{A}$ . In Section 4, a suitable notion of *irreducible representations* of locally  $C^*$ -algebras is introduced using the idea of commutants. We show that an irreducible Stinespring representation of a Frechet locally  $C^*$ -algebra is minimal. The concept of *pure local CP-maps* on local operator systems are introduced and proved that a local CP-map on a locally  $C^*$ -algebra is a pure local CP-map if and only if its minimal Stinespring representation is irreducible. In Section 5, we introduce *local boundary representations* of locally  $C^*$ -algebras and prove that local boundary representations provide an intrinsic invariant for local operator systems. In the case of Frechet locally  $C^*$ -algebras, we characterize local boundary representations using the notions of pure local CP-maps and a couple of other new notions.

## 2. PRELIMINARIES

**2.1. Locally  $C^*$ -algebras.** Let  $\mathcal{A}$  be a unital  $*$ -algebra with unit  $1_{\mathcal{A}}$ . A seminorm  $p$  on  $\mathcal{A}$  is said to be sub-multiplicative, if  $p(1_{\mathcal{A}}) = 1$  and  $p(ab) \leq p(a)p(b)$  for every  $a, b \in \mathcal{A}$ . A sub-multiplicative seminorm  $p$  satisfies the condition  $p(a^*a) = p(a)^2$  for every  $a \in \mathcal{A}$ , is called a  $C^*$ -seminorm. Let  $(\Lambda, \leq)$  be a directed poset. A family of seminorms  $\mathcal{P} = \{p_{\alpha} : \alpha \in \Lambda\}$  on  $\mathcal{A}$  is called an upward filtered family, if  $\alpha \leq \beta$  in  $\Lambda$ , then  $p_{\alpha}(a) \leq p_{\beta}(a)$  for every  $a \in \mathcal{A}$ . A *locally  $C^*$ -algebra*  $\mathcal{A}$  is a  $*$ -algebra together with an upward filtered family of  $C^*$ -seminorms  $\mathcal{P}$  on  $\mathcal{A}$  such that  $\mathcal{A}$  is complete with respect to the locally convex topology generated by the family  $\mathcal{P}$ .

Throughout this article,  $\mathcal{A}$  always denote a locally  $C^*$ -algebra with a prescribed family of  $C^*$ -seminorms  $\{p_{\alpha} : \alpha \in \Lambda\}$ . Let  $I_{\alpha} = \{a \in \mathcal{A} : p_{\alpha}(a) = 0\}$  and  $\mathcal{A}_{\alpha}$  be the quotient  $C^*$ -algebra  $\mathcal{A}/I_{\alpha}$  with the  $C^*$ -norm induced by  $p_{\alpha}$ . Denote the canonical quotient  $*$ -homomorphism from  $\mathcal{A}$  to  $\mathcal{A}_{\alpha}$  by  $\pi_{\alpha}$ . Note that for  $\alpha \leq \beta$  in  $\Lambda$ , there is a canonical  $*$ -homomorphism  $\pi_{\alpha\beta} : \mathcal{A}_{\beta} \rightarrow \mathcal{A}_{\alpha}$  where  $\pi_{\alpha\beta}(a + I_{\beta}) = a + I_{\alpha}$  and that satisfies  $\pi_{\alpha\beta}\pi_{\beta} = \pi_{\alpha}$ . Then one can identify  $\mathcal{A}$  as the the inverse limit of the projective system  $\{\mathcal{A}_{\alpha}, \pi_{\alpha,\beta} : \alpha, \beta \in \Lambda\}$  of  $C^*$ -algebras [21].

**2.2. Local positive elements.** Anar Dosiev [11] introduced the notions of local hermitian and local positivity in locally  $C^*$ -algebras. An element  $a \in \mathcal{A}$  is called *local hermitian* if  $a = a^* + x$  for some  $x \in \mathcal{A}$  such that  $p_{\alpha}(x) = 0$  for some  $\alpha \in \Lambda$

and an element  $a \in \mathcal{A}$  is called *local positive* if  $a = b^*b + x$  for some  $b, x \in \mathcal{A}$  such that  $p_\alpha(x) = 0$  for some  $\alpha \in \Lambda$ . In this case, we call  $a$  is  $\alpha$ -hermition (and  $\alpha$ -positive, respectively). We use  $a \geq_\alpha 0$  to denote  $a$  is  $\alpha$ -positive. A direct computation shows that  $a \geq_\alpha 0$  in  $\mathcal{A}$  if and only if the  $\pi_\alpha(a) \geq 0$  in the  $C^*$ -algebra  $\mathcal{A}_\alpha$ .

**2.3. Local operator systems and local CP-maps.** Let  $\mathcal{A}$  be a locally  $C^*$ -algebra. For a linear subspace  $S$  of  $\mathcal{A}$  denote  $S^* = \{x^* : x \in S\}$ . We say  $S$  is *self adjoint* if  $S = S^*$ . A *local operator system* in  $\mathcal{A}$  is a unital self adjoint linear subspace of  $\mathcal{A}$ . An element  $a$  in a local operator system  $S$  is *local positive* if  $a$  is local positive in  $\mathcal{A}$ . Consider another locally  $C^*$ -algebra  $\mathcal{B}$  with the associated family of seminorms  $\{q_l : l \in \Omega\}$ . Let  $S_1$  and  $S_2$  be local operator systems in  $\mathcal{A}$  and  $\mathcal{B}$  respectively. A linear map  $\phi : S_1 \rightarrow S_2$  is said to be *local positive*, if for each  $l \in \Omega$  there corresponds  $\alpha \in \Lambda$  such that  $\phi(a) \geq_l 0$  whenever  $a \geq_\alpha 0$  in  $S_1$ . The map  $\phi$  is said to be *local bounded*, if for each  $l \in \Omega$  there exists an  $\alpha \in \Lambda$  and  $C_{l\alpha} > 0$  such that  $q_l(\phi(a)) \leq C_{l\alpha} p_\alpha(a)$  for all  $a \in S_1$ . If  $C_{l\alpha}$  can be chosen to be 1, then we say that  $\phi$  is *local contractive*. For  $n \in \mathbb{N}$ , let  $M_n(\mathcal{A})$  denotes the set of all  $n \times n$  matrices over  $\mathcal{A}$ . Naturally  $M_n(\mathcal{A})$  is a locally  $C^*$ -algebra with the defining family of seminorms  $\{p_\alpha^n : \alpha \in \Lambda\}$ , where  $p_\alpha^n([a_{ij}]) = \|\pi_\alpha^{(n)}([a_{ij}])\|_\alpha$  for  $[a_{ij}]$  in  $M_n(\mathcal{A})$ . We use  $\phi^{(n)}$  to denote the  $n$ -amplification of the map  $\phi$ , that is,  $\phi^{(n)} : M_n(S_1) \rightarrow M_n(S_2)$  defined by  $\phi^{(n)}([a_{ij}]) = [\phi(a_{ij})]$  for  $[a_{ij}]$  in  $M_n(S_1)$ . The map  $\phi$  is called *local completely bounded*(local CB-map) if for each  $l \in \Omega$ , there exists  $\alpha \in \Lambda$  and  $C_{l\alpha} > 0$  such that  $q_l^n([\phi(a_{ij})]) \leq C_{l\alpha} p_l^n([a_{ij}])$ , for every  $n \in \mathbb{N}$ . If  $C_{l\alpha}$  can be chosen to be 1, then we say  $\phi$  is *local completely contractive*(local CC-map). The map  $\phi$  is called *local completely positive*(local CP-map) if for each  $l \in \Omega$ , there exists  $\alpha \in \Lambda$  such that  $\phi^{(n)}([a_{ij}]) \geq_l 0$  in  $M_n(S_2)$  whenever  $[a_{ij}] \geq_\alpha 0$  in  $M_n(S_1)$ .

**2.4. Representations of locally C\*-algebras.** Let  $H$  be a complex Hilbert space and  $\mathcal{D}$  be a dense subspace of  $H$ . A *quantized domain* in  $H$  is a triple  $\{H, \mathcal{E}, \mathcal{D}\}$ , where  $\mathcal{E} = \{H_l : l \in \Omega\}$  is an upward filtered family of closed subspaces of  $H$  such that the union space  $\mathcal{D} = \bigcup_{l \in \Omega} H_l$  is dense in  $H$ . In short, we say  $\mathcal{E}$  is a quantized domain in  $H$  with its union space  $\mathcal{D}$ . A quantized domain  $\mathcal{E}$  is called a *quantized Frechet domain* if  $\mathcal{E}$  is a countable family.

Corresponding to a quantized domain  $\mathcal{E} = \{H_l : l \in \Omega\}$  we can associate an upward filtered family  $\mathcal{P} = \{P_l : l \in \Omega\}$  of projections in  $B(H)$  where  $P_l$  is the orthogonal projection of  $H$  onto the closed subspace  $H_l$ .

**The space  $\mathcal{C}_\mathcal{D}^*(\mathcal{D})$ .** Let us denote  $L(\mathcal{D})$  by the set of all linear operators on the linear subspace  $\mathcal{D}$ . The set of all *noncommutative continuous functions* on a quantized domain  $\mathcal{E}$  is defined as

$$\mathcal{C}_\mathcal{D}(\mathcal{E}) = \{T \in L(\mathcal{D}) : TP_l = P_lTP_l \in B(H), \text{ for all } l \in \Omega\}.$$

Note that  $\mathcal{C}_\mathcal{D}(\mathcal{E})$  is an algebra and if  $T \in L(\mathcal{D})$ , then

$$T \in \mathcal{C}_\mathcal{D}(\mathcal{E}) \text{ if and only if } T(H_l) \subseteq H_l \text{ and } T|_{H_l} \in B(H_l) \text{ for all } l \in \Omega.$$

The  $*$ -algebra of all noncommutative continuous functions on a quantized domain  $\mathcal{E}$  is defined as

$$\mathcal{C}_{\mathcal{E}}^*(\mathcal{D}) = \{T \in \mathcal{C}_{\mathcal{D}}(\mathcal{E}) : P_l T \subseteq T P_l, \text{ for all } l \in \Omega\}.$$

Note that  $\mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$  is a unital subalgebra of  $\mathcal{C}_{\mathcal{D}}(\mathcal{E})$ . For more details about the adjoint of operators in  $\mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$  refer [11, Proposition 3.1]. For  $T \in L(\mathcal{D})$ , it is easy to see that  $T \in \mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$  if and only if for all  $l \in \Omega$

$$T(H_l) \subseteq H_l, \quad T|_{H_l} \in B(H_l) \text{ and } T(H_l^\perp \cap \mathcal{D}) \subseteq H_l^\perp \cap \mathcal{D}.$$

Now, define  $q_l : \mathcal{C}_{\mathcal{E}}^*(\mathcal{D}) \rightarrow \mathbb{R}$  by  $q_l(T) = \|T|_{H_l}\|$  for all  $T \in \mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$ . Then  $\mathcal{Q} = \{q_l : l \in \Omega\}$  is an upward filtered family of  $C^*$ -seminorms on  $\mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$ . Also,  $\mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$  is complete with respect to the locally convex topology generated by the family  $\mathcal{Q}$ . Hence  $\mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$  is a locally  $C^*$ -algebra.

We use  $\mathcal{CPCC}_{loc}(S, \mathcal{C}_{\mathcal{E}}^*(\mathcal{D}))$  to denotes the class of all local completely positive and local completely contractive maps from a local operator system  $S$  to  $\mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$ .

**Stinespring's theorem for local CP-maps.** A locally convex version (or an unbounded version) of the celebrated Stinespring's dilation theorem is appeared in the work of A.Dosiev [11, Theorem 5.1].

**Theorem 2.1.** [11, Theorem 5.1] *Let  $\phi \in \mathcal{CPCC}_{loc}(\mathcal{A}, \mathcal{C}_{\mathcal{E}}^*(\mathcal{D}))$ . Then there exists a Hilbert space  $H^\phi$  and a quantized domain  $\mathcal{E}^\phi = \{H_\alpha^\phi : \alpha \in \Lambda\}$  in  $H^\phi$  with its union space  $\mathcal{D}^\phi$ , a contraction  $V_\phi : H \rightarrow H^\phi$ , and a unital local contractive  $*$ -homomorphism  $\pi_\phi : \mathcal{A} \rightarrow \mathcal{C}_{\mathcal{E}^\phi}^*(\mathcal{D}^\phi)$  such that*

$$\phi(a) \subseteq V_\phi^* \pi_\phi(a) V_\phi \text{ and } V_\phi(H_\alpha) \subseteq H_\alpha^\phi$$

for every  $a \in \mathcal{A}$  and  $l \in \Lambda$ . Moreover, if  $\phi(1_{\mathcal{A}}) = 1_{\mathcal{D}}$ , then  $V_\phi$  is an isometry.

Any triple  $(\pi_\phi, V_\phi, \{H^\phi; \mathcal{E}^\phi; \mathcal{D}^\phi\})$  that satisfies the conditions of the Theorem 2.1 is called a *Stinespring representation* for  $\phi$ .

**Minimality of Stinespring representation for local CP-maps.** The *minimality* of the Stinespring representation was introduced and studied recently by Bhat and et al in [8]. A Stinespring representation  $(\pi_\phi, V_\phi, \{H^\phi; \mathcal{E}^\phi; \mathcal{D}^\phi\})$  of  $\phi$  is said to be *minimal*, if  $H_l^\phi = [\pi_\phi V_\phi H_l]$ , for every  $l \in \Lambda$ . They proved that given any Stinespring representation of a map  $\phi \in \mathcal{CPCC}_{loc}(\mathcal{A}, \mathcal{C}_{\mathcal{E}}^*(\mathcal{D}))$ , one can reduce it to minimal Stinespring representation, and also any two minimal Stinespring representations are unitarily equivalent in the following sense. Let  $\pi_1$  and  $\pi_2$  be two representations of the locally  $C^*$ -algebra  $\mathcal{A}$  on the quantized doamins  $\{H; \mathcal{E} = \{H_l : l \in \Omega\}; \mathcal{D}\}$  and  $\{H'; \mathcal{E}' = \{H'_l : l \in \Omega\}; \mathcal{D}'\}$ , respectively. We say  $\pi_1$  and  $\pi_1$  are unitarily equivalent if there exists a unitary  $U : H' \rightarrow H$  such that  $U(H'_l) \subseteq H_l$  and  $\pi_2(a) = U^* \pi_1(a) U|_{\mathcal{D}'}$  for all  $a \in \mathcal{A}$  and all  $l \in \Omega$ .

### 3. LOCAL POSITIVE LINEAR MAPS

In this section, we prove an analog of the Arveson extension theorem for local CC-maps on linear subspaces of  $\mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$  for a quantized Frechet domain  $\mathcal{E}$ . This result is crucial in establishing a theorem in the main section. Now, let  $S$  be a local operator system in the locally  $C^*$ -algebra  $\mathcal{A}$ . A linear functional  $f : S \rightarrow \mathbb{C}$

is an  $\alpha$ -contractive linear functional if  $|f(a)| \leq p_\alpha(a)$  for all  $a \in S$ . Note that, by Hahn-Banach extension theorem, there is an  $\alpha$ -contractive linear map  $\tilde{f} : \mathcal{A} \rightarrow \mathbb{C}$  such that  $\tilde{f}|_S = f$  and  $|\tilde{f}(a)| \leq p_\alpha(a)$  for all  $a \in \mathcal{A}$ . For  $a \in \mathcal{A}$  we define the  $\alpha$ -spectrum of  $a$  to be the spectrum of  $\pi_\alpha(a)$  in the  $C^*$ -algebra  $\mathcal{A}_\alpha$ . We use  $\sigma_\alpha(a)$  to denote the  $\alpha$ -spectrum of  $a$ .

**Lemma 3.1.** *Let  $S$  be a local operator system in a locally  $C^*$ -algebra  $\mathcal{A}$  and let  $f : S \rightarrow \mathbb{C}$  be a unital  $\alpha$ -contractive linear functional. Let  $\tilde{f}$  be a Hahn-Banach extension of  $f$  to  $\mathcal{A}$ . If  $a = x^*x + b \in S$  is an  $\alpha$ -positive element of  $\mathcal{A}$ , then  $0 \leq \tilde{f}(x^*x) \leq r_\alpha$ , where  $r_\alpha$  is the spectral radius of  $\pi_\alpha(a)$ .*

*Proof.* Assume that  $\tilde{f}(x^*x) \notin [0, r_\alpha]$ . Since a closed interval in the real line is the intersection of all closed disks containing it in the complex plane, there exists a closed disk  $D_r(\mu)$  centered at  $\mu \in \mathbb{C}$  and radius  $r$  such that  $|\tilde{f}(x^*x) - \mu| > r$  and  $[0, r_\alpha] \subseteq D_r(\mu)$ . Then  $\sigma_\alpha(x^*x - \mu 1) \subseteq D_r(0)$  as  $\sigma_\alpha(x^*x) \subseteq [0, r_\alpha] \subseteq D_r(\mu)$ . Since  $\pi_\alpha(x^*x)$  is a positive element of  $\mathcal{A}_\alpha$ ,  $\pi_\alpha(x^*x - \mu 1)$  is a normal element of  $\mathcal{A}_\alpha$ . The spectral radius and norm are same for normal elements of a  $C^*$ -algebra gives us  $\|\pi_\alpha(x^*x - \mu 1)\|_\alpha \leq r$ . Now using the fact  $\tilde{f}$  is a unital  $\alpha$ -contraction, we have

$$\begin{aligned} |\tilde{f}(x^*x) - \mu| &= |\tilde{f}(x^*x - \mu 1)| \leq p_\alpha(x^*x - \mu 1) \\ &= \|\pi_\alpha(x^*x - \mu 1)\|_\alpha \leq r. \end{aligned}$$

This is a contradiction. Hence  $\tilde{f}(x^*x) \in [0, r_\alpha]$ .  $\square$

**Theorem 3.2.** *Let  $S$  be a local operator system in a locally  $C^*$ -algebra  $\mathcal{A}$  and  $\mathcal{E}$  be a quantized domain with its union space  $\mathcal{D}$ . Let  $\phi : S \rightarrow \mathcal{C}_\mathcal{E}^*(\mathcal{D})$  be a unital local contractive map. Then  $\phi$  is a local positive map.*

*Proof.* Fix  $l \in \Omega$ . Since  $\phi$  is local contractive, there exists  $\alpha \in \Lambda$  such that  $\|\phi(a)\|_l \leq p_\alpha(a)$  for every  $a \in S$ . Let  $a \in S$  and  $a = x^*x + b$  where  $x, b \in \mathcal{A}$  and  $p_\alpha(b) = 0$  for some  $\alpha \in \Lambda$ .

We will show that  $\phi(a)|_{H_l}$  is a positive operator on  $H_l$ . Let  $h \in H_l$  with  $\|h\| = 1$ . Define  $f_h : S \rightarrow \mathbb{C}$  by  $f_h(y) = \langle \phi(y)|_{H_l} h, h \rangle$ . Then  $f_h(1) = 1$  and

$$|f_h(y)| \leq \|\phi(y)\|_l \leq p_\alpha(y).$$

Therefore, the linear functional  $f_h$  is a unital  $\alpha$ -contraction. Let  $\tilde{f}_h : \mathcal{A} \rightarrow \mathbb{C}$  be an  $\alpha$ -contractive Hahn-Banach extension of  $f_h$ . Then

$$\langle \phi(a)|_{H_l} h, h \rangle = f_h(a) = \tilde{f}_h(a) = \tilde{f}_h(x^*x) + \tilde{f}_h(b).$$

Note that,  $\tilde{f}_h(b) = 0$  as  $\tilde{f}_h$  is an  $\alpha$ -contraction and  $p_\alpha(b) = 0$ . Using Lemma 3.1 we conclude that  $\tilde{f}_h(x^*x) = \langle \phi(a)|_{H_l} h, h \rangle$  is positive. Therefore,  $\phi(a)$  is local positive and that completes the proof.  $\square$

**Remark 3.3.** *We can use the above theorem to establish the following result, which is a special case of a result in [11].*

**Theorem 3.4.** [11, Corollary 4.1] *Let  $S$  be a local operator system in a locally  $C^*$ -algebra  $\mathcal{A}$  and  $\mathcal{E}$  be a quantized domain with its union space  $\mathcal{D}$ . Let  $\phi : S \rightarrow \mathcal{C}_\mathcal{E}^*(\mathcal{D})$  be a unital linear map. Then  $\phi$  is a local CC-map if and only if  $\phi$  is a local CP-map.*

*Proof.* Let  $\phi$  be a local CC-map. Fix  $l \in \Omega$ . There exists a  $\alpha \in \Lambda$  such that  $\|\phi^{(n)}([a_{ij}])\|_l \leq p_\alpha^{(n)}([a_{ij}])$  for all  $[a_{ij}] \in M_n(S), n \in \mathbb{N}$ . From the proof of Theorem 3.2 we have  $\phi^{(n)}([a_{ij}]) \geq_l 0$  whenever  $[a_{ij}] \geq_\alpha 0$ . Thus  $\phi$  is a local CP-map.

Conversely, assume that  $\phi$  is a local CP-map. Fix  $l \in \Omega$ . There exists a  $\alpha \in \Lambda$  such that  $\phi^{(n)}(A) \geq_l 0$  whenever  $A \geq_\alpha 0$  in  $A \in M_n(S)$  and  $n \in \mathbb{N}$ . Let  $A \in M_n(S)$  such that  $p_\alpha^{(n)}(A) \leq 1$ . Then

$$\begin{bmatrix} 1_n & A \\ A^* & 1_n \end{bmatrix} \geq_\alpha 0 \text{ in } M_{2n}(S)$$

Applying the map  $\phi^{(2n)}$ , we have

$$\begin{bmatrix} I_n & \phi^{(n)}(A) \\ \phi^{(n)}(A^*) & I_n \end{bmatrix} \geq_l 0 \text{ in } M_{2n}(\mathcal{C}_\xi^*(\mathcal{D})).$$

Thus

$$\left. \begin{bmatrix} I_n & \phi^{(n)}(A) \\ \phi^{(n)}(A^*) & I_n \end{bmatrix} \right|_{H_l^n \oplus H_l^n} \geq 0 \text{ in } B(H_l^n \oplus H_l^n).$$

Equivalently  $\|\phi^n(A)|_{H_l^n}\| \leq 1$ . Hence  $\|\phi^n(A)\|_l \leq p_\alpha^{(n)}(A)$  for every  $A \in M_n(S)$ . That is,  $\phi$  is a local CC-map.  $\square$

**Theorem 3.5.** *Let  $\mathcal{A}$  be unital a locally  $C^*$ -algebra, and  $M$  be a unital subspace of  $\mathcal{A}$ . If  $\phi : M \rightarrow \mathcal{C}_\xi^*(\mathcal{D})$  be a unital local contraction, then there is a local positive extension  $\tilde{\phi}$  of  $\phi$  to  $M + M^*$  given by  $\tilde{\phi}(x + y^*) = \phi(x) + \phi(y)^*$ . Moreover,  $\tilde{\phi}$  is the only local positive extension of  $\phi$  to  $M + M^*$ .*

*Proof.* First, we will show that the map  $\tilde{\phi}$  is well-defined. Let

$$M_* = \{a \in M : a^* \in M\}.$$

Clearly,  $M_*$  is a local operator system in  $\mathcal{A}$ . Also, the map  $\phi$  is a unital local contractive map on  $M_*$ . Using Theorem 3.2 we have  $\phi$  is a local positive map. Then  $\phi$  is self adjoint on  $M_*$ , thanks to [11, Lemma 4.3]. To see  $\tilde{\phi}$  is well defined, consider  $a_1, a_2, b_1, b_2 \in M$  with  $a_1 + b_1^* = a_2 + b_2^*$ . Equivalently,  $a_1 - a_2 = (b_2 - b_1)^*$ . Thus  $b_2 - b_1 \in M_*$ . Then using the fact that  $\phi$  is self adjoint on  $M_*$ , we have

$$\begin{aligned} \phi(a_1 - a_2) &= \phi((b_2 - b_1)^*) \\ &= [\phi(b_2 - b_1)]^* \\ &= \phi(b_2)^* - \phi(b_1)^* \\ \phi(a_1) + \phi(b_1)^* &= \phi(a_2) + \phi(b_2)^*. \end{aligned}$$

Hence  $\tilde{\phi}(a_1 + b_1^*) = \tilde{\phi}(a_2 + b_2^*)$ . That is,  $\tilde{\phi}$  is well-defined.

To see  $\tilde{\phi}$  is local positive; fix  $l \in \Omega$ . By local contractivity of  $\phi$ , there exists an  $\alpha \in \Lambda$  such that  $\|\phi(a)\|_l \leq p_\alpha(a)$  for all  $a \in \mathcal{A}$ . Let  $a + b^* \in M + M^*$  be an  $\alpha$ -positive element. We will show that  $\tilde{\phi}(a + b^*)$  is local positive by showing that  $\tilde{\phi}(a + b^*)|_{H_l}$  is a positive operator on  $H_l$ . Let  $h \in H_l$  with  $\|h\| = 1$ . Define  $f : M \rightarrow \mathbb{C}$  by  $f(y) = \langle \phi(y)h, h \rangle$ . Then  $|f(y)| \leq \|\phi(y)\|_l \leq p_\alpha(y)$  for every  $y \in M$ . Using Hahn-Banach extension theorem,  $f$  extends to  $f_1 : M + M^* \rightarrow \mathbb{C}$



with  $|f_1(y)| \leq p_\alpha(y)$  for every  $y \in M + M^*$ . By Theorem 3.2 we have that  $f_1$  is local positive. Also,  $0 \leq f_1(a + b^*) = f_1(a) + \overline{f_1(b)} = f(a) + \overline{f(b)} = \langle \phi(a)h, h \rangle + \overline{\langle \phi(b)h, h \rangle} = \langle \tilde{\phi}(a + b^*)h, h \rangle$ . Hence  $\tilde{\phi}$  is local positive.

To show  $\tilde{\phi}$  is unique; let  $\psi : M + M^* \rightarrow \mathbb{C}$  be a local positive extension of  $\phi$ . The map  $\psi$  is self adjoint by [11, Lemma 4.3]. Then the following computation shows that  $\psi = \tilde{\phi}$ .

$$\begin{aligned} \psi(a + b^*) &= \psi(a) + \psi(b^*) = \psi(a) + \psi(b)^* \\ &= \phi(a) + \phi(b)^* = \tilde{\phi}(a + b^*). \end{aligned}$$

□

Let  $\mathcal{F}$  be a quantized Frechet domain with its union space  $\mathcal{O}$ . A. Dosiev [11, Theorem 8.2] proved the analog of Arveson's extension theorem for unital local CP-maps from local operator systems into  $\mathcal{C}_{\mathcal{F}}^*(\mathcal{O})$ . Using the above theorem we deduce an analog of Arveson extension theorem for local CC-maps on subspaces of locally C\*-algebras. A locally C\*-algebra  $\mathcal{A}$  is called Frechet locally C\*-algebra if there is a local isometrical \*-homomorphism  $\mathcal{A} \rightarrow \mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$  for some quantized Frechet domain  $\mathcal{E}$  with its union space  $\mathcal{D}$ .

**Theorem 3.6.** *Let  $\mathcal{F}$  be a quantized Frechet domain and  $\mathcal{A}$  be a Frechet locally C\*-algebra. Let  $M$  be a unital linear subspace of  $\mathcal{A}$  and  $\phi : M \rightarrow \mathcal{C}_{\mathcal{F}}^*(\mathcal{O})$  be a unital local CC-map. Then  $\phi$  has a local CP-extension to  $\mathcal{A}$ .*

*Proof.* Since  $\phi$  is local CC-map, by Theorem 3.5 there is a local CP-map  $\tilde{\phi} : M + M^* \rightarrow \mathcal{C}_{\mathcal{F}}^*(\mathcal{O})$ . Then by Dosiev-Arveson extension theorem [11, Theorem 8.2]  $\tilde{\phi}$  extended to a local CP-map on  $\mathcal{A}$ . □

#### 4. IRREDUCIBLE REPRESENTATIONS AND PURE LOCAL CP-MAPS

By a representation of a locally C\*-algebra  $\mathcal{A}$  we always mean a local contractive \*-homomorphism from  $\mathcal{A}$  into  $\mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$  for some quantized domain  $\mathcal{E}$ .

**Definition 4.1.** *Let  $\pi : \mathcal{A} \rightarrow \mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$  be a representation. The commutant of  $\pi(\mathcal{A})$  is denoted by  $\pi(\mathcal{A})'$  and is defined as*

$$\pi(\mathcal{A})' = \{T \in B(H) : T\pi(a) \subseteq \pi(a)T, \text{ for all } a \in \mathcal{A}\}$$

**Definition 4.2.** *A representation  $\pi : \mathcal{A} \rightarrow \mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$  is said to be irreducible if*

$$\pi(\mathcal{A})' \cap \mathcal{C}_{\mathcal{E}}^*(\mathcal{D}) = \mathbb{C}I_{\mathcal{D}}$$

The following result is crucial in our discussions.

**Theorem 4.3.** *Let  $\mathcal{E}$  be a quantized Frechet domain. Let  $\phi \in \mathcal{CPCC}_{loc}(\mathcal{A}, \mathcal{C}_{\mathcal{E}}^*(\mathcal{D}))$  and  $(\pi, V, \{H'; \mathcal{E}'; \mathcal{D}'\})$  be a Stinespring representation of the map  $\phi$ . If  $\pi$  is irreducible, then  $(\pi, V, \{H'; \mathcal{E}'; \mathcal{D}'\})$  is a minimal Stinespring representation for the map  $\phi$ .*

*Proof.* If possible assume that there exists an  $l_1 \in \mathbb{N}$  such that  $[\pi(\mathcal{A})VH_{l_1}] \neq H'_{l_1}$ . Since  $V(H_{l_1}) \subseteq H'_l$  and  $H'_l$  is invariant for  $\pi(a)$ , for every  $a \in \mathcal{A}$ , we must have

$$[\pi(\mathcal{A})VH_{l_1}] \subsetneq H'_{l_1}.$$

Let  $l_0 = \min\{l \in \mathbb{N} : \pi(\mathcal{A})VH_l \neq H'_l\}$ . Take  $P$  to be the orthogonal projection of  $H'$  onto the closed subspace  $[\pi(\mathcal{A})VH_{l_0}]$ . We claim that  $P \in \pi(\mathcal{A})' \cap \mathcal{C}_{\mathcal{E}'}^*(\mathcal{D}')$ . First, we prove that  $P \in \mathcal{C}_{\mathcal{E}'}^*(\mathcal{D}')$ .

To see  $P(H'_l) \subseteq H'_l$ ; let  $l \in \mathbb{N}$ . If  $l \geq l_0$ , then as  $\mathcal{E}'$  is an upward filtered family and  $P$  is a projection we must have  $P(H'_l) \subseteq [\pi(\mathcal{A})VH_{l_0}] \subsetneq H'_{l_0} \subseteq H'_l$ . If  $l < l_0$ , then the choice of  $l_0$  gives us  $H'_l = [\pi(\mathcal{A})VH_l]$ . Then, to show  $P(H'_l) \subseteq H'_l$  it is enough to show that  $P(\pi(\mathcal{A})VH_l) \subseteq H'_l$ . Since  $l < l_0$  and  $P$  is a projection with range  $[\pi(\mathcal{A})VH_{l_0}]$ , we have  $H_l \subseteq H_{l_0}$ . Thus,

$$\pi(\mathcal{A})VH_l \subseteq \pi(\mathcal{A})VH_{l_0}$$

$$P(\pi(\mathcal{A})VH_l) = \pi(\mathcal{A})VH_l \subseteq H'_l.$$

Hence  $P(H'_l) \subseteq H'_l$  for every  $l \in \mathbb{N}$ .

Note that, as  $P(H'_l) \subseteq H'_l$  and  $P$  is a projection we have  $P|_{H'_l} \in B(H'_l)$ .

Now, we show that  $P(H_l^\perp \cap \mathcal{D}') \subseteq H_l^\perp \cap \mathcal{D}'$ . For  $x \in H_l^\perp \cap \mathcal{D}'$  and  $y \in H'_l$  we need to show that  $\langle Px, y \rangle = 0$ . If  $l < l_0$ , then we have  $H'_l = [\pi(\mathcal{A})VH_l]$ . Since  $H'_l = [\pi(\mathcal{A})VH_l] \subseteq [\pi(\mathcal{A})VH_{l_0}]$ , we have  $Px = x$  for every  $x \in H'_l$ . Then it follows that

$$\langle Px, y \rangle = \langle x, Py \rangle = \langle x, y \rangle = 0.$$

If  $l \geq l_0$ , then  $[\pi(\mathcal{A})VH_{l_0}] \subsetneq H'_l$ . Thus  $H_l^\perp \cap \mathcal{D}' \subseteq [\pi(\mathcal{A})VH_{l_0}]^\perp$ . It follows that  $Px = 0$  for all  $x \in H_l^\perp \cap \mathcal{D}'$ . Therefore  $\langle Px, y \rangle = 0$  for all  $y \in H'_l$ . Hence  $P \in \mathcal{C}_{\mathcal{E}'}^*(\mathcal{D}')$ .

To see  $P \in \pi(\mathcal{A})'$ , let  $a \in \mathcal{A}$ . First, we observe that  $P\pi(a)h' = \pi(a)h'$  whenever  $h' \in [\pi(\mathcal{A})VH_{l_0}]$ . As the restriction of  $\pi(a)$  to  $H'_{l_0}$  is a bounded operator on  $H'_{l_0}$ , it is enough to consider  $h'$  in the dense subspace  $\text{span}(\pi(\mathcal{A})VH_{l_0})$ . Let  $h' = \sum_{i=1}^n \pi(a_i)Vh_i$  for some  $a_i \in \mathcal{A}$ ,  $h_i \in H_{l_0}$  and  $n \in \mathbb{N}$ ,  $i = 1, 2, \dots, n$ . Then,

$$\begin{aligned} \pi(a)h' &= \pi(a)\left(\sum_{i=1}^n \pi(a_i)Vh_i\right) \\ &= \sum_{i=1}^n \pi(aa_i)Vh_i \in [\pi(\mathcal{A})VH_{l_0}]. \end{aligned}$$

It follows that  $P\pi(a)h' = \pi(a)h'$  whenever  $h' \in [\pi(\mathcal{A})VH_{l_0}]$ .

Now, consider  $h' \in \mathcal{D}'$ . Write  $h' = h'_1 + h'_2$  where  $h'_1 \in [\pi(\mathcal{A})VH_{l_0}]$  and  $h'_2 \in [\pi(\mathcal{A})VH_{l_0}]^\perp \cap \mathcal{D}'$ . It follows that  $P(h'_2) = 0$  and  $P\pi(a)h'_1 = \pi(a)h'_1$ . Then

$$\begin{aligned} \|P\pi(a)h' - \pi(a)Ph'\|^2 &= \|P\pi(a)(h'_1 + h'_2) - \pi(a)P(h'_1 + h'_2)\|^2 \\ &= \|P\pi(a)h'_1 + P\pi(a)h'_2 - \pi(a)Ph'_1 + \pi(a)Ph'_2\|^2 \\ &= \|P\pi(a)h'_2\|^2 \\ &= \langle P\pi(a)h'_2, P\pi(a)h'_2 \rangle \\ &= \langle \pi(a^*)P\pi(a)h'_2, h'_2 \rangle. \end{aligned}$$



Let  $h'_3 = P\pi(a)h'_2 \in [\pi(\mathcal{A})VH_{l_0}]$ . Then

$$\pi(a^*)h'_3 = \pi(a^*)\left(\sum_{i=1}^n \pi(a_i)Vh_i\right) = \sum_{i=1}^n \pi(a^*a_i)Vh_i \in [\pi(\mathcal{A})VH_{l_0}].$$

But  $h'_2 \in [\pi(\mathcal{A})VH_{l_0}]^\perp$  will imply that  $\langle \pi(a^*)h'_3, h'_2 \rangle = 0$ . Hence  $P\pi(a)h' = \pi(a)Ph'$  for every  $a \in \mathcal{A}$  and  $h' \in \mathcal{D}'$ . Hence  $P \in \pi(\mathcal{A})' \cap \mathcal{C}_{\mathcal{E}'}^*(\mathcal{D}')$ .

But  $P \in \pi(\mathcal{A})' \cap \mathcal{C}_{\mathcal{E}'}^*(\mathcal{D}')$  is a contradiction as  $\pi$  is irreducible, and  $P \neq 0$  and  $P \neq I_H$ . Hence  $\pi$  is a minimal Stinespring representation for  $\phi$ .  $\square$

**Remark 4.4.** *It is well known that a representation  $\theta$  of a C\*-algebra  $\mathcal{C}$  is irreducible if and only if the commutant of  $\theta(\mathcal{C})$  is trivial. If we take  $\mathcal{A}$  to be a C\*-algebra and  $\mathcal{E} = \{H\}$  in Definition 4.2, then Definition 4.2 coincides with the usual definition of irreducible representations of C\*-algebra. Also, our definition of irreducibility is motivated by the commutant considered to establish a Radon-Nikodym type theorem for local CP-maps in [8, Theorem 4.5].*

**4.1. Pure maps on local operator systems.** We introduce the notion of *pure* local completely positive maps on local operator system and study its connection with boundary representations for local operator systems. For this, we use the convexity structure of the set  $\mathcal{CPCC}_{loc}(S, \mathcal{C}_{\mathcal{E}}^*(\mathcal{D}))$ .

**Proposition 4.5.** *For a local operator system  $S$ , the set  $\mathcal{CPCC}_{loc}(S, \mathcal{C}_{\mathcal{E}}^*(\mathcal{D}))$  is a linear convex set.*

*Proof.* Let  $\phi_1, \phi_2 \in \mathcal{CPCC}_{loc}(S, \mathcal{C}_{\mathcal{E}}^*(\mathcal{D}))$  and  $0 < t < 1$ . Fix  $l \in \Omega$ . There exist  $\alpha_r, \beta_r \in \Lambda$ ,  $r = 1, 2$ , such that

$$\phi_r^{(n)}([a_{ij}]) \geq_l 0 \text{ whenever } [a_{ij}] \geq_{\alpha_r} 0 \text{ and}$$

$$\|\phi_r^{(n)}([a_{ij}])\|_l \leq p_{\beta_r}^n([a_{ij}]) \text{ for every } n \in \mathbb{N}.$$

Replace  $\phi_1$  and  $\phi_2$  by  $t\phi_1$  and  $(1-t)\phi_2$  respectively. Then, for  $\alpha = \max\{\alpha_1, \alpha_2\}$ , we have

$$t\phi_1^{(n)}([a_{ij}]) + (1-t)\phi_2^{(n)}([a_{ij}]) \geq_l 0 \text{ whenever } [a_{ij}] \geq_{\alpha} 0.$$

Thus,  $t\phi_1 + (1-t)\phi_2$  is a local CP-map. To see its local CC, take  $\beta = \max\{\beta_1, \beta_2\}$ . Then for every  $[a_{ij}] \in M_n(S)$ ,

$$\begin{aligned} \|t\phi_1^{(n)}([a_{ij}]) + (1-t)\phi_2^{(n)}([a_{ij}])\|_l &\leq \|t\phi_1^{(n)}([a_{ij}])\|_l + \|(1-t)\phi_2^{(n)}([a_{ij}])\|_l \\ &\leq tp_{\beta_1}^n([a_{ij}]) + (1-t)p_{\beta_2}^n([a_{ij}]) \\ &\leq p_{\beta}^n([a_{ij}]). \end{aligned}$$

$\square$

**Definition 4.6.** *A map  $\phi \in \mathcal{CPCC}_{loc}(S, \mathcal{C}_{\mathcal{E}}^*(\mathcal{D}))$  is called *pure* if for any map  $\psi \in \mathcal{CPCC}_{loc}(S, \mathcal{C}_{\mathcal{E}}^*(\mathcal{D}))$  such that  $\phi - \psi \in \mathcal{CPCC}_{loc}(S, \mathcal{C}_{\mathcal{E}}^*(\mathcal{D}))$ , then there is a scalar  $t \in [0, 1]$  such that  $\psi = t\phi$ .*

**Remark 4.7.** *A recent pre-print [17] also defines the notion of purity along similar lines.*

**Theorem 4.8.** *A map  $\phi \in \mathcal{CPCC}_{\text{loc}}(\mathcal{A}, \mathcal{C}_{\mathcal{E}}^*(\mathcal{D}))$  is pure if and only if  $\phi$  is of the form  $\phi(a) \subseteq V^*\pi(a)V$  for all  $a \in \mathcal{A}$ , where  $\pi$  is an irreducible representation of  $\mathcal{A}$  on some quantized domain  $\mathcal{E}'$  with its union space  $\mathcal{D}'$  and  $V \in L(\mathcal{D}, \mathcal{D}')$ ,  $V \neq 0$  and  $V(H_l) \subseteq H'_l$  for all  $l \in \Omega$ .*

*Proof.* Let  $\phi \in \mathcal{CPCC}_{\text{loc}}(\mathcal{A}, \mathcal{C}_{\mathcal{E}}^*(\mathcal{D}))$  be pure. Using [11, Theorem 5.1] we have a unital representation  $\pi : \mathcal{A} \rightarrow \mathcal{C}_{\mathcal{E}'}^*(\mathcal{D}')$  for some quantized domain  $\mathcal{E}'$  with its union space  $\mathcal{D}'$  such that  $\phi(a) \subseteq V^*\pi(a)V$  where  $V \in L(\mathcal{D}, \mathcal{D}')$  and  $V(H_l) \subseteq H'_l$  for all  $l \in \Omega$ . Clearly  $V \neq 0$ . Now, let  $T \in \pi(\mathcal{A})' \cap \mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$  with  $0 \leq T \leq I$ . Taking  $\psi(\cdot) = V^*T\pi(\cdot)V|_{\mathcal{D}}$  in [8, Theorem 4.5] we have  $\psi \leq \phi$ . As  $\phi$  is pure it follows that  $\psi = t\phi$ . Applying [8, Corollary 4.6],  $T = tI$ . Hence  $\pi$  is irreducible.

Conversely, let  $\pi$  be an irreducible representation of  $\mathcal{A}$  on some quantized domain  $\mathcal{E}'$  with its union space  $\mathcal{D}$  and  $V$  be a non zero operator in  $L(\mathcal{D}, \mathcal{D}')$  such that  $V(H_l) \subseteq H'_l$  for all  $l \in \Omega$ . To show that  $\phi(\cdot) \subseteq V^*\pi(\cdot)V$  is pure, consider  $\psi \in \mathcal{CPCC}_{\text{loc}}(\mathcal{A}, \mathcal{C}_{\mathcal{E}}^*(\mathcal{D}))$  with  $\psi \leq \phi$ . As  $\pi$  is irreducible by Theorem 4.3  $(\pi, V, \{H', \mathcal{E}', \mathcal{D}'\})$  is a minimal Stinespring representation's representation for  $\phi$ . Now, applying [8, Corollary 4.6], there exists a unique  $T \in \pi(\mathcal{A})' \cap \mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$  such that  $0 \leq T \leq I$  and  $\psi(a) \subseteq V^*T\pi(a)V$  for all  $a \in \mathcal{A}$ . Since  $\pi$  is irreducible,  $T = tI$ . It follows that  $\psi = t\phi$  and hence  $\phi$  is pure.  $\square$

**Proposition 4.9.** *Let  $S_1$  and  $S_2$  be local operator systems in a locally  $C^*$ -algebra  $\mathcal{A}$  such that  $S_1 \subseteq S_2$ . Let  $\phi : S_2 \rightarrow \mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$  be a unital local CP-map such that its a linear extreme point of  $\mathcal{CPCC}_{\text{loc}}(S_2, \mathcal{C}_{\mathcal{E}}^*(\mathcal{D}))$ . If  $\phi|_{S_1}$  is pure, then  $\phi$  is a pure.*

*Proof.* Let  $\phi_1, \phi_2 \in \mathcal{CPCC}_{\text{loc}}(S_2, \mathcal{C}_{\mathcal{E}}^*(\mathcal{D}))$  such that  $\phi = \phi_1 + \phi_2$ . Since  $\phi|_{S_1}$  is pure, there exists  $t \in (0, 1)$  such that  $\phi_1|_{S_1} = t\phi|_{S_1}$  and  $\phi_2|_{S_1} = (1-t)\phi|_{S_1}$ . The maps  $\frac{1}{t}\phi_1$  and  $\frac{1}{1-t}\phi_2$  are unital local CP-map on  $S_2$ . By Theorem 3.4 both the maps are local CC-maps. It follows that  $\frac{1}{t}\phi_1, \frac{1}{1-t}\phi_2 \in \mathcal{CPCC}_{\text{loc}}(S_2, \mathcal{C}_{\mathcal{E}}^*(\mathcal{D}))$ . Then the expression  $\phi = t\frac{1}{t}\phi_1 + (1-t)\frac{1}{1-t}\phi_2$  and the assumption  $\phi$  is linear extreme implies that  $\phi$  is pure.  $\square$

## 5. LOCAL BOUNDARY REPRESENTATIONS

In this section, we introduce the notion of local boundary representations for locally  $C^*$ -algebras and establish its connection with pure local CP-maps.

**Definition 5.1.** *Let  $S$  be a linear subspace of a locally  $C^*$ -algebra  $\mathcal{A}$  such that  $S$  generates  $\mathcal{A}$ . A representation  $\pi : \mathcal{A} \rightarrow \mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$  is said to have local unique extension property for  $S$  if  $\pi|_S$  has a unique local completely positive extension to  $\mathcal{A}$ , namely  $\pi$  itself.*

**Remark 5.2.** *Let  $\pi : \mathcal{A} \rightarrow \mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$  be a representation of  $\mathcal{A}$ . Then  $\pi|_S$  has just one multiplicative local CP-extension to  $\mathcal{A}$ , namely  $\pi$  itself, but in general, there may exist other local CP-extensions of  $\pi|_S$ .*

**Example 5.3.** *For a self adjoint operator  $T \in \mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$ , let  $S = \text{span}\{I, T, T^2\}$  and  $\mathcal{B}$  be the locally  $C^*$ -algebra generated by  $S$  in  $\mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$ . We show that the identity representation  $I_{\mathcal{B}}$  of  $\mathcal{B}$  has local unique extension property. Let  $\phi : \mathcal{B} \rightarrow \mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$  be a local completely positive map such that  $\phi(x) = x$  for all  $x \in S$ . Consider*

a minimal Stinespring representation  $(\pi, V, \{H'_l; \mathcal{E}'_l; \mathcal{D}'_l\})$  of  $\phi$ . To prove  $\phi = I_{\mathcal{B}}$  on  $\mathcal{B}$  it is enough to show that  $V$  is a unitary. We claim that  $V(\mathcal{D})$  is invariant for  $\pi(\mathcal{B})$ . Then by minimality  $H' = [\pi(\mathcal{B})V(\mathcal{D})] \subseteq [V(\mathcal{D})] \subseteq H'$  will imply  $V$  is a unitary. Now, to see the claim let us first show that  $\pi(T)V(\mathcal{D}) \subseteq V(\mathcal{D})$ . For that, we show that  $\pi(T)V(H_l) \subseteq V(H_l)$  for every  $l \in \Omega$ . Let  $l \in \Omega$  and  $g \in H'_l$ ,

$$\begin{aligned} \|(I - VV^*)\pi(T)VV^*g\|^2 &= \langle (I - VV^*)\pi(T)VV^*g, (I - VV^*)\pi(T)VV^*g \rangle \\ &= \langle VV^*\pi(T)(I - VV^*)\pi(T)VV^*g, g \rangle \\ &= \langle VV^*\pi(T)\pi(T)VV^*g - VV^*\pi(T)VV^*\pi(T)VV^*g, g \rangle \\ &= \langle V\phi(T^2)V^*h' - V\phi(T)\phi(T)V^*g, g \rangle \\ &= \langle VT^2V^*g - VT^2V^*h', g \rangle \\ &= 0. \end{aligned}$$

Thus  $(I - VV^*)\pi(T)VV^* = 0$  on  $H'_l$ . Since  $T$  is self adjoint,  $VV^*\pi(T)(I - VV^*) = 0$  on  $H'_l$ . These two observations and the facts  $\pi(T)|_{H'_l} \in B(H'_l)$ ,  $V|_{H_l}$  is an isometry and  $\pi(T)V(H_l) \subseteq H'_l$  will give  $\pi(T)V(H_l) \subseteq V(H_l)$ . As  $l$  is arbitrary, it follows that  $\pi(T)V(\mathcal{D}) \subseteq V(\mathcal{D})$ . To show  $\pi(\mathcal{B})V(\mathcal{D}) \subseteq V(\mathcal{D})$ , let  $T_0 \in \mathcal{B}$  and  $\mathcal{B}_T = \text{span}\{I, T, T^2, T^3, \dots\}$ . Then  $T_0 = \lim T_\lambda$ , where  $T_\lambda \in \mathcal{B}_T$ . For  $h \in H_l$ ,

$$\begin{aligned} \|\pi(T_\lambda)Vh - \pi(T_0)Vh\|_{H'_l} &= \|\pi(T_\lambda - T_0)Vh\|_{H'_l} \\ &\leq p_\alpha(T_\lambda - T_0)\|h\|_{H_l}, \end{aligned}$$

where  $\alpha$  corresponds to  $l$  in the local contractivity of  $\pi$ . As  $\{T_\lambda\}$  converges to  $T_0$ , we have  $p_\alpha(T_\lambda - T_0) \rightarrow 0$  and hence  $\{\pi(T_\lambda)Vh\}$  converges to  $\pi(T_0)Vh$  in  $H'_l$ . Therefore,

$$\pi(T_0)Vh \in [\pi(T_\lambda)Vh].$$

As  $\pi(T)$  leaves  $V(H_l)$  invariant, so is every element of  $\mathcal{B}_T$ . Then using the fact that  $VH_l$  is a closed subspace (as  $V$  is an isometry and  $H_l$  is a closed subspace),

$$\pi(T_0)Vh \in [\pi(T_\lambda)Vh] \subseteq [\pi(T_\lambda)V(H_l)] \subseteq [V(H_l)] = V(H_l).$$

Therefore  $\pi(\mathcal{B})V(H_l) \subseteq V(H_l)$  for every  $l$  and hence  $\pi(\mathcal{B})V(\mathcal{D}) \subseteq V(\mathcal{D})$ .

**Example 5.4.** Let  $K$  be an infinite dimensional separable complex Hilbert space with a complete orthonormal basis  $\{e_n : n \in \mathbb{N}\}$ . Consider  $K_n = \text{span}\{e_1, e_2, \dots, e_n\}$  and  $H_n = K \oplus K_n$ . Then  $\mathcal{E} = \{H_n : n \in \mathbb{N}\}$  is a quantized domain in the Hilbert space  $H = K \oplus K$  with union space  $\mathcal{D} = \cup\{H_n : n \in \mathbb{N}\}$ . Define  $V : H \rightarrow H$  to be the map  $V_0 \oplus 1_K$  where  $V_0 : K \rightarrow K$  be the unilateral right shift operator and  $1_K$  be the identity operator on  $K$ . Note that  $V$  is an isometry but not a unitary. Also,  $V(K \oplus K_n) \subseteq K \oplus K_n$  and

$$V((K \oplus K_n)^\perp) = V(0 \oplus K_n^\perp) = 0 \oplus K_n^\perp = (K \oplus K_n)^\perp.$$

Therefore  $V|_{\mathcal{D}} \in \mathcal{C}_\mathcal{E}^*(\mathcal{D})$ .

Consider the local operator system  $S = \text{span}\{1_{\mathcal{D}}, V|_{\mathcal{D}}, V^*\}$  in  $\mathcal{C}_\mathcal{E}^*(\mathcal{D})$  and let  $\mathcal{B}$  the locally C\*-algebra generated by  $S$  in  $\mathcal{C}_\mathcal{E}^*(\mathcal{D})$ . We claim that the inclusion map from  $S$  to  $\mathcal{C}_\mathcal{E}^*(\mathcal{D})$  have two distinct local CP-extension to  $\mathcal{B}$ . Obviously the inclusion representation  $I_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{C}_\mathcal{E}^*(\mathcal{D})$  is a local CP-extension of the inclusion map on  $S$ . Define  $\psi : \mathcal{B} \rightarrow \mathcal{C}_\mathcal{E}^*(\mathcal{D})$  by  $\psi(a) = V^*I_{\mathcal{B}}(a)V|_{\mathcal{D}}$  for all  $a \in \mathcal{B}$ . Clearly

$\psi$  is a unital local completely positive map on  $\mathcal{B}$ . For all scalars  $c_1, c_2$  and  $c_3$  we have

$$\begin{aligned}\psi(c_1 1_{\mathcal{D}} + c_2 V|_{\mathcal{D}} + c_3 V^*) &= V^*(c_1 1_{\mathcal{D}} + c_2 V|_{\mathcal{D}} + c_3 V^*)V|_{\mathcal{D}} \\ &= c_1 1_{\mathcal{D}} + c_2 V|_{\mathcal{D}} + c_3 V^*.\end{aligned}$$

Therefore  $\psi|_S = I_{\mathcal{B}}|_S$ . Now the element  $V|_{\mathcal{D}}V^* \in \mathcal{B}$ . But

$$\psi(V|_{\mathcal{D}}V^*) = V^*V|_{\mathcal{D}}V^*V|_{\mathcal{D}} = I_{\mathcal{D}} \neq V|_{\mathcal{D}}V^*.$$

That is  $\psi \neq I_{\mathcal{B}}$  on  $\mathcal{B}$ . Therefore, the irreducible representation  $I_{\mathcal{B}}$  doesn't have local unique extension property for  $S$ .

**Definition 5.5.** Let  $S$  be a linear subspace of a local  $C^*$ -algebra  $\mathcal{A}$  such that  $S$  generates  $\mathcal{A}$ . An irreducible representation  $\pi : \mathcal{A} \rightarrow \mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$  is called a local boundary representation for  $\mathcal{S}$  if  $\pi$  has local unique extension property for  $S$ .

**Remark 5.6.** The Definition 5.1 and Definition 5.5 are meaningful for local operator systems in arbitrary locally  $C^*$ -algebras. But the Arveson's extension theorem in the context of locally  $C^*$ -algebras is available only for  $\mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$  for quantized Frechet domain  $\mathcal{E}$  and thus we restrict our studies to the context of Frechet locally  $C^*$ -algebras.

Now, we show that the local boundary representations are intrinsic invariants for local operator systems. Let  $\mathcal{A}_1$  be a locally  $C^*$ -algebra and  $\mathcal{A}_2 = \mathcal{C}_{\mathcal{E}_2}^*(\mathcal{D}_2)$  be the locally  $C^*$ -algebras of all non-commutative continuous functions on a quantized Frechet domain  $\mathcal{E}_2$  with its union space  $\mathcal{D}_2$ .

**Theorem 5.7.** Let  $S_1$  and  $S_2$  be linear subspaces of  $\mathcal{A}_1$  and  $\mathcal{A}_2$  respectively. Let  $\phi : S_1 \rightarrow S_2$  be a unital surjective local completely isometric linear map. Then for every boundary representation  $\pi_1$  of  $\mathcal{A}_1$  there exists a boundary representation  $\pi_2$  of  $\mathcal{A}_2$  such that  $\pi_2 \circ \phi(a) = \pi_1(a) \forall a \in S_1$ .

*Proof.* By Theorem 3.6 we can extend  $\phi$  to a local CP-map  $\tilde{\phi} : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ . Consider the map  $\psi : S_2 \rightarrow \mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$  given by  $(\psi \circ \phi)(a) = \pi_1(a)$ . Clearly  $\psi$  is a unital local CC-map. Again by Theorem 3.6 there exists a local CP-extension of  $\psi$ , say  $\pi_2$ , where  $\pi_2 : \mathcal{A}_2 \rightarrow \mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$  such that  $(\pi_2 \circ \phi)(a) = \pi_1(a)$  for every  $a \in S_1$ . Since  $\pi_1$  is a boundary representation,  $(\pi_2 \circ \tilde{\phi})(a) = \pi_1(a)$  for every  $a \in \mathcal{A}_1$ . Note that the locally  $C^*$ -algebra generated by  $\tilde{\phi}(\mathcal{A}_1)$  is equal to  $\mathcal{A}_2$  and  $\pi_2$  is continuous for the respective topologies. Thus, to prove  $\pi_2$  is an algebra homomorphism it's enough to prove that  $\pi_2(xy) = \pi_2(x)\pi_2(y)$  for every  $x \in \tilde{\phi}(\mathcal{A}_1)$  and for all  $y \in \mathcal{A}_2$ . But in view of [11, Corollary 5.5], it's enough to prove that

$$\pi_2(x)^*\pi_2(x) = \pi_2(x^*x) \forall x \in \tilde{\phi}(\mathcal{A}_1).$$

Let  $a \in \mathcal{A}_1$ . Then using the fact that a local positive map is positive [16, Proposition 2.1], and [11, Corollary 5.5] we have, on  $\mathcal{D}$ ,

$$\begin{aligned} \pi_2(\tilde{\phi}(a))^* \pi_2(\tilde{\phi}(a)) &\leq \pi_2(\tilde{\phi}(a)^* \tilde{\phi}(a)) = \pi_2(\tilde{\phi}(a^*) \tilde{\phi}(a)) \\ &\leq \pi_2(\tilde{\phi}(a^* a)) \\ &= \pi_1(a^* a) \\ &= \pi_1(a^*) \pi_1(a) \\ &= \pi_2(\tilde{\phi}(a))^* \pi_2(\tilde{\phi}(a)). \end{aligned}$$

Therefore  $\pi_2(\tilde{\phi}(a)^* \tilde{\phi}(a)) = \pi_2(\tilde{\phi}(a))^* \pi_2(\tilde{\phi}(a))$  on  $\mathcal{D}$ . Thus  $\pi_2$  is a representation of  $\mathcal{A}_2$ . In fact we proved that any local CP-extension of  $\psi = \pi_2|_{S_2}$  to  $\mathcal{A}_2$  is multiplicative on  $\mathcal{A}_2$ . Equivalently,  $\pi_2$  has local unique extension property for  $S_2$ .

Now, note that  $\pi_1(\mathcal{A}_1) \subseteq (\pi_2 \circ \tilde{\phi})(\mathcal{A}_1) \subseteq \pi_2(\mathcal{A}_2)$ . Thus, for commutants we have  $\pi_2(\mathcal{A}_2)' \subseteq \pi_1(\mathcal{A}_1)'$ . Then the irreducibility of  $\pi_2$  follows from the irreducibility of  $\pi_1$ . This completes the proof.  $\square$

**Corollary 5.8.** *Let  $S_1$  and  $S_2$  be local operator systems of  $\mathcal{A}_1$  and  $\mathcal{A}_2$  respectively. Let  $\phi : S_1 \rightarrow S_2$  be a unital invertible local CP-map such that  $\phi^{-1}$  is also a local CP-map. Then for every boundary representation  $\pi_1$  of  $\mathcal{A}_1$  there exists a boundary representation  $\pi_2$  of  $\mathcal{A}_2$  such that  $\pi_2 \circ \phi(a) = \pi_1(a) \forall a \in S_1$ .*

**Remark 5.9.** *We expect the above theorem and consequently the corollary to be true for any Frechet locally  $C^*$ -algebras in place of  $\mathcal{A}_2 = C_{\mathcal{E}_2}^*(\mathcal{D}_2)$ .*

**5.1. Characterisation of boundary representations.** The following theorem shows that the restriction of a local boundary representation to the local operator system is a pure map.

**Theorem 5.10.** *Let  $S$  be a local operator system in a Frechet local  $C^*$ -algebra  $\mathcal{A}$  such that  $S$  generates  $\mathcal{A}$ . Let  $\mathcal{E}$  be a quantized Frechet domain with its union space  $\mathcal{D}$ , and  $\pi : \mathcal{A} \rightarrow C_{\mathcal{E}}^*(\mathcal{D})$  be a boundary representation for  $S$ . Then  $\pi|_S$  is a pure map on  $S$ .*

*Proof.* Let  $\pi_1, \pi_2 \in \mathcal{CPCC}_{\text{loc}}(S, C_{\mathcal{E}}^*(\mathcal{D}))$  such that  $\pi|_S = \pi_1 + \pi_2$ . Then by Dosiev-Arveson extension theorem [11, Theorem 8.2], each  $\pi_i$  extends to a local CPCC map on  $\mathcal{A}$ , call it  $\tilde{\pi}_i$ ,  $i = 1, 2$ . We will show that  $\tilde{\pi}_1 + \tilde{\pi}_2 \in \mathcal{CPCC}_{\text{loc}}(\mathcal{A}, C_{\mathcal{E}}^*(\mathcal{D}))$ . For that, fix  $l \in \mathbb{N}$ . Then there exists  $\alpha_i$  and  $\beta_i$  such that  $\tilde{\pi}_i(a) \geq_l 0$  whenever  $a \geq_{\alpha_i} 0$  in  $\mathcal{A}$  and  $\|\tilde{\pi}_i(b)\|_l \leq p_{\beta_i}(b)$  for every  $b \in \mathcal{A}$ . Take  $\alpha = \max\{\alpha_1, \alpha_2\}$  and  $\beta = \max\{\beta_1, \beta_2\}$ . Using the fact that the family of semi-norms  $\{p_n\}_{n \in \mathbb{N}}$  is an upward filtered family, we have  $\tilde{\pi}_i(a) \geq_l 0$  whenever  $a \geq_{\alpha} 0$  in  $\mathcal{A}$  and  $\|\tilde{\pi}_i(b)\|_l \leq p_{\beta}(b)$  for every  $b \in \mathcal{A}$ . Therefore,  $\tilde{\pi}_1 + \tilde{\pi}_2 \in \mathcal{CPCC}_{\text{loc}}(\mathcal{A}, C_{\mathcal{E}}^*(\mathcal{D}))$ .

Now, since  $\tilde{\pi}_1 + \tilde{\pi}_2|_S = \pi_1 + \pi_2 = \pi|_S$  and  $\pi$  is a boundary representation for  $S$ , we must have  $\pi(a) = \tilde{\pi}_1(a) + \tilde{\pi}_2(a)$  for every  $a \in \mathcal{A}$ . The irreducibility of  $\pi$  and the Theorem 4.8 implies that  $\pi$  is a pure map. Thus, for each  $i$ , there exist  $t_i \in [0, 1]$  such that  $\tilde{\pi}_i(a) = t_i \pi(a)$  for every  $a \in \mathcal{A}$ . It follows that  $\pi_i = t_i \pi|_S$ . Hence  $\pi|_S$  is a pure map on  $S$ .  $\square$

Now, we show that certain irreducible representations of  $\mathcal{A}$  that are pure CPCC-maps on  $S$  are local boundary representations. For this, we need to introduce a couple of new notions. Let  $S$  be a local operator system in a local  $C^*$ -algebra  $\mathcal{A}$  such that  $\mathcal{A}$  is generated by  $S$ , and let  $\pi : \mathcal{A} \rightarrow \mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$  be a representation of  $\mathcal{A}$ . We say that  $\pi$  is a *finite representation for  $S$*  if for every isometry  $V \in B(H)$  with  $V(H_l) \subseteq H_l$  for every  $l \in \Lambda$ , the condition  $\pi(x) \subseteq V^*\pi(x)V$  for every  $x \in S$  implies  $V$  is a unitary. We say that the local operator system  $S$  *separates* the irreducible representation  $\pi$  if for any irreducible representation  $\rho$  of  $\mathcal{A}$  on some quantized domain  $\mathcal{E}'$  with its union space  $\mathcal{D}' = \bigcup_{l \in \Lambda} H'_l$  and an isometry  $V$  in  $B(H, H')$  that satisfies  $V(\mathcal{H}_l) \subseteq \mathcal{H}'_l$  for every  $l \in \Lambda$  such that  $\pi(x) \subseteq V^*\rho(x)V$  for all  $x \in S$  implies that  $\pi$  and  $\rho$  are unitarily equivalent representations of  $\mathcal{A}$ .

**Theorem 5.11.** *Let  $S$  be a local operator system in a local  $C^*$ -algebra  $\mathcal{A}$  such that  $\mathcal{A}$  is generated by  $S$ . Then, an irreducible representation  $\pi : \mathcal{A} \rightarrow \mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$  is a local boundary representations for  $S$  if and only if the following conditions hold;*

- (i)  $\pi|_S$  is a pure map on  $S$
- (ii) Every local CP-extension of  $\pi|_S$  to  $\mathcal{A}$  is a linear extreme point of  $\mathcal{CPCC}_{\text{loc}}(\mathcal{A}, \mathcal{C}_{\mathcal{E}}^*(\mathcal{D}))$
- (iii)  $\pi$  is a finite representation for  $S$
- (iv)  $S$  separates  $\pi$ .

*Proof.* Let  $\pi$  be an irreducible representation of  $\mathcal{A}$ . Assume that  $\pi$  is a local boundary representation for  $S$ . Then the statement (i) follows by Theorem 5.10.

(ii): Since  $\pi$  is a local boundary representation, there is only one local CP-extension of  $\pi|_S$  to  $\mathcal{A}$ , namely  $\pi$  itself. Let  $\phi_1, \phi_2 \in \mathcal{CPCC}_{\text{loc}}(\mathcal{A}, \mathcal{C}_{\mathcal{E}}^*(\mathcal{D}))$  such that  $\pi = \phi_1 + \phi_2$ . Then  $\pi|_S = \phi_1|_S + \phi_2|_S$ . But  $\pi|_S$  is pure by statement (i). Thus  $\phi_1|_S = t\pi|_S$  and  $\phi_2|_S = (1-t)\pi|_S$  for some  $t \in [0, 1]$ . If  $0 < t < 1$ , then  $\pi|_S = \frac{1}{t}\phi_1|_S$  and  $\pi|_S = \frac{1}{1-t}\phi_2|_S$ . Now the maps  $\frac{1}{t}\phi_1$  and  $\frac{1}{1-t}\phi_2$  on  $\mathcal{A}$  are unital local CP-extensions of  $\pi|_S$ . But  $\pi$  is a boundary representation for  $S$  would imply that  $\pi = \frac{1}{t}\phi_1$  and  $\pi = \frac{1}{1-t}\phi_2$  on  $\mathcal{A}$ . That is,  $\pi$  is a linear extreme point of  $\mathcal{CPCC}_{\text{loc}}(\mathcal{A}, \mathcal{C}_{\mathcal{E}}^*(\mathcal{D}))$ .

(iii): Consider an isometry  $V$  on  $H$  such that  $\pi(x) \subseteq V^*\pi(x)V$  for every  $x \in S$  and  $V(H_l) \subseteq H_l$  for every  $l \in \Lambda$ . Then  $\phi(a) := V^*\pi(a)V|_{\mathcal{D}}$  for all  $a \in \mathcal{A}$  is a unital local CP-extension of  $\pi|_S$ . As  $\pi$  is a local boundary representation we must have  $\pi(a) = V^*\pi(a)V|_{\mathcal{D}}$  for all  $a \in \mathcal{A}$ . We claim that  $V \in \pi(\mathcal{A})' \cap \mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$ . Clearly  $V$  is bounded and  $V(H_l) \subseteq H_l \forall l$ . Let  $x \in H_l^\perp \cap \mathcal{D}$ . Since  $\pi$  is irreducible, by Theorem 4.3  $(\pi, V, \{H, \mathcal{E}, \mathcal{D}\})$  is a minimal Stinespring for  $\pi$ . Then by [8, Lemma 4.2],  $Vx = \pi(1)Vx \in H_l^\perp$ . It follows that  $Vx \in H_l^\perp \cap \mathcal{D}$  as  $V(H_l) \subseteq H_l$ . Thus  $V(H_l^\perp \cap \mathcal{D}) \subseteq H_l^\perp \cap \mathcal{D}$  and hence  $V \in \mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$ . To see  $V \in \pi(\mathcal{A})'$ ; first note that  $\text{dom}(V\pi(a)) = \mathcal{D} \subseteq \text{dom}(\pi(a)V)$  for all  $a \in \mathcal{A}$ . Let  $h \in \mathcal{D}$  and  $a \in \mathcal{A}$ .



$$\begin{aligned}
& \|V\pi(a)h - \pi(a)Vh\|^2 \\
&= \langle V\pi(a)h - \pi(a)Vh, V\pi(a)h - \pi(a)Vh \rangle \\
&= \|V\pi(a)h\|^2 - \langle \pi(a)Vh, V\pi(a)h \rangle - \langle V\pi(a)h, \pi(a)Vh \rangle + \|\pi(a)Vh\|^2 \\
&= \|\pi(a)h\|^2 - \langle V^*\pi(a)Vh, \pi(a)h \rangle - \langle \pi(a)h, V^*\pi(a)Vh \rangle + \|\pi(a)Vh\|^2 \\
&= \|\pi(a)h\|^2 - \langle \pi(a)h, \pi(a)h \rangle - \langle \pi(a)h, \pi(a)h \rangle + \|\pi(a)Vh\|^2 \\
&= \|\pi(a)Vh\|^2 - \|\pi(a)h\|^2 = \langle \pi(a)Vh, \pi(a)Vh \rangle - \langle \pi(a)h, \pi(a)h \rangle \\
&= \langle V^*\pi(a^*)\pi(a)Vh, h \rangle - \langle \pi(a)^*\pi(a)h, h \rangle \\
&= \langle \pi(a^*a)h, h \rangle - \langle \pi(a^*a)h, h \rangle = 0.
\end{aligned}$$

Therefore  $V\pi(a) \subseteq \pi(a)V$  for every  $a \in \mathcal{A}$  and hence  $V \in \pi(\mathcal{A})' \cap \mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$ . By the irreducibility of  $\pi$  implies  $V = \lambda I_H$ ,  $\lambda \in \mathbb{C}$ . Thus, the isometry  $V$  is a unitary. Hence  $\pi$  is a finite representation for  $S$ .

(iv): Assume that  $\rho$  is an irreducible representation of  $\mathcal{A}$  on some quantized domain  $\mathcal{E}'$  with its union space  $\mathcal{D}' = \bigcup_{l \in \Lambda} H'_l$  and an isometry  $V$  in  $B(H, H')$  that satisfies  $V(\mathcal{H}_l) \subseteq \mathcal{H}'_l$  for every  $l \in \Lambda$  such that  $\pi(x) \subseteq V^*\rho(x)V$  for all  $x \in S$ . As  $\pi$  is a local boundary representation for  $S$ , it follows that  $\pi(a) \subseteq V^*\rho(a)V$  for all  $a \in \mathcal{A}$ . Here  $\pi$  and  $\rho$  are irreducible representations of  $\mathcal{A}$ . By Theorem 4.3 the Stinespring representations  $(\pi, I_H, \{H, \mathcal{E}, \mathcal{D}\})$  and  $(\rho, V, \{H', \mathcal{E}', \mathcal{D}'\})$  are minimal for  $\pi$ . Then [8, Theorem 3.4] will imply that  $\pi$  and  $\rho$  are unitarily equivalent. Hence  $S$  separate  $\pi$ .

Conversely assume that the irreducible representation  $\pi$  satisfies all the four conditions. Let  $\phi : \mathcal{A} \rightarrow \mathcal{C}_{\mathcal{E}}^*(\mathcal{D})$  be a local CP-map such that  $\phi(a) = \pi(a)$  for every  $a \in S$ . By condition (ii),  $\phi$  is a linear extreme point of  $\mathcal{CPCC}_{\text{loc}}(\mathcal{A}, \mathcal{C}_{\mathcal{E}}^*(\mathcal{D}))$ . Then statement (i) and Proposition 4.9 will imply that  $\phi$  is a pure map in  $\mathcal{CPCC}_{\text{loc}}(\mathcal{A}, \mathcal{C}_{\mathcal{E}}^*(\mathcal{D}))$ . If  $\{\omega; V; \{K, \mathcal{F}, \mathcal{O}\}\}$  is a minimal Stinespring representation for  $\phi$ , then by Theorem 4.8  $\omega$  is irreducible. Also,

$$\pi(a) = \phi(a) = V^*\omega(a)V|_{\mathcal{D}} \text{ for all } a \in S.$$

As  $\pi$  separates  $S$ ,  $\pi$  and  $\omega$  are unitarily equivalent. Let  $U : K \rightarrow H$  be a unitary such that  $U(\mathcal{O}) \subseteq \mathcal{D}$  and

$$\omega(a) = U^*\pi(a)U|_{\mathcal{D}} \text{ for all } a \in S.$$

Then

$$\pi(a) = V^*U^*\pi(a)UV|_{\mathcal{D}} \text{ for all } a \in S.$$

Since  $\pi$  is a finite representation and  $UV$  is an isometry on  $H$ , we have  $UV$  is a unitary. Thus  $V = U^*(UV)$  is also a unitary. Therefore  $\phi(a) = V^*\pi(a)V|_{\mathcal{D}}$  on  $\mathcal{A}$  is a representation of  $\mathcal{A}$  which coincides with  $\pi$  on  $S$ . Therefore  $\phi(a) = \pi(a)$  for all  $a \in \mathcal{A}$  and hence  $\pi$  is a local boundary representation for  $S$ .  $\square$

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