62. *Local Boundedness of Monotone-type Operators**)

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In this note we give a simple proof that certain monotone-type operators are locally bounded in the interior of their domains, thus generalizing a result of [1]. As special cases, we obtain the local boundedness for monotone operators from a Fréchet space to its dual and for accretive operators in a Banach space with a uniformly convex dual.

In what follows let X, Y be metrizable linear topological spaces. Further assume that Y is locally convex and complete (Fréchet space). We denote by $\langle\ ,\ \rangle$ the pairing between Y and its dual Y^*. We introduce a metric in X and denote by B_r the open ball in X with center 0 and radius $r>0$.

Let T be a mapping of X into 2^{Y^*}, with domain $D(T)=\{x \in X : Tx \neq \emptyset\}$ and graph $G(T)=\{(x, f) \in X \times Y^* ; f \in Tx\}$. Let F be a function on X to Y. Slightly generalizing a definition used in [1], we say T is F-monotone if $\langle F(x_1-x_2), f_1-f_2\rangle \geq 0$ for $(x_j, f_j) \in G(T), j=1, 2$.

Theorem. *Assume that there is $r_0>0$ such that*

(i) *F is uniformly coutinuous on B_{r_0} to Y.*

(ii) *For each $r<r_0$, $F(B_r)$ is absorbing in Y.*

(iii) *For each $u \in X$, the set $\{F(z-u)-Fz ; z \in B_{r_0}\}$ is bounded in Y.*

If $T: X \to 2^{Y^*}$ is F-monotone, then T is locally bounded at each interior point x_0 of $D(T)$, in the following sense: for each sequence $\{(x_n, f_n)\}$ in $G(T)$ with $x_n \to x_0$, $\{f_n\}$ is equicontinuous.

Examples. 1. Let $Y=X$ and $F=$identity map in X. Then F-monotonicity means monotonicity in the sense of Minty-Browder. Conditions (i) to (iii) are trivially satisfied, and the theorem shows that a monotone operator from a Fréchet space X to X^* is locally bounded in the interior of its domain (cf. [2], [3]).

2. Let X be a Banach space with X^* uniformly convex, and let $Y=X^*$ so that $Y^*=X^{**}=X$. Let F be the (normalized) duality map of X to X^*. Then F-monotonicity means accretiveness in the usual sense. It is known that F is onto X^* and uniformly continuous on any bounded set in X. Thus (i) to (iii) are satisfied, and the theorem shows

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that an accretive operator in such a space X is locally bounded in the interior of its domain (cf. [4], Section 3, where a similar result is proved under a slightly stronger assumption).

The proof of the theorem is based on the following lemma.

Lemma. *Let $\{u_n\}$ and $\{f_n\}$ be sequences in X and Y^*, respectively. Suppose $u_n \to 0$ but $\{f_n\}$ is not equicontinuous. Then for each $r < r_0$, $r > 0$, there exists $z_0 \in B_r$ such that $\langle F(z_0 - u_n), f_n \rangle \to \infty$ along a subsequence of $\{n\}$.*

Proof of Lemma. For $z \in X$ set $H_n z = F(z-u_n) - Fz$. Since $u_n \to 0$ and F is uniformly continuous on B_{r_0}, we have

(1) $$H_n z \to 0 \qquad \text{uniformly for } z \in B_r.$$

Set

(2) $$g_n = f_n / a_n, \quad a_n = \max(1, b_n), \quad b_n = \sup_{z \in B_r} |\langle H_n z, f_n \rangle|.$$

Note that b_n is finite for each fixed n, since $H_n(B_r)$ is a bounded set in Y by (iii) (see [5], p. 44). We claim that $\{g_n\}$ is not equicontinuous. This is obvious if $b_n \leq 1$ for almost all n so that $g_n = f_n$. If $b_n > 1$ for infinitely many n, on the other hand, we can choose $z_n \in B_r$ such that $|\langle H_n z_n, f_n \rangle| > b_n/2$ for those n. Then $a_n = b_n$ and $|\langle H_n z_n, g_n \rangle| > 1/2$. Since $H_n z_n \to 0$ by (1), we see that $\{g_n\}$ is not equicontinuous.

According to the uniform boundedness theorem (see [5], p. 68), it follows that there is $y_0 \in Y$ such that $\langle y_0, g_n \rangle \to \infty$ along a subsequence of $\{n\}$. Since $F(B_r)$ is absorbing by (ii), there is $z_0 \in B_r$ with $Fz_0 = cy_0$, $c > 0$. Hence $\langle Fz_0, g_n \rangle \to \infty$. On the other hand $|\langle H_n z_0, g_n \rangle| = |\langle H_n z_0, f_n \rangle| / a_n \leq b_n / a_n \leq 1$. Since $a_n \geq 1$, it follows that $\langle F(z_0 - u_n), f_n \rangle = a_n \langle Fz_0 + H_n z_0, g_n \rangle \to \infty$.

Proof of Theorem. Suppose $\{f_n\}$ is not equicontinuous. Choose $r > 0$ so small that $x_0 + B_r \subset D(T)$. According to the lemma, there exists $z_0 \in B_r$ such that

(3) $$\langle F(z_0 - (x_n - x_0)), f_n \rangle \to \infty, \qquad n \to \infty,$$

after going over to a subsequence if necessary.

Set $u_0 = x_0 + z_0 \in D(T)$ and let $h \in Tu_0$. The F-monotonicity of T implies $\langle F(u_0 - x_n), h - f_n \rangle \geq 0$. Since F is continuous at z_0, it follows that $\limsup_{n \to \infty} \langle F(u_0 - x_n), f_n \rangle \leq \langle Fz_0, h \rangle < \infty$, a contradiction to (3). This proves the theorem.

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