Local C^n-operational Calculus

SHMUEL KANTOROVITZ*

Communicated by J. Schwartz

1. Let X be a Banach space, and let W be a linear manifold in X. Denote by $\mathfrak{I}(W)$ the algebra of all linear transformations of X with domain W and range contained in W. For a compact interval $\triangle$, let $C^n(\triangle)$ be the Banach algebra of all complex valued functions of class C^n on $\triangle$ ($n = 0, 1, 2, \cdots$) with the norm

$$|f|_{n,\triangle} = \sum_{j=0}^{n} \sup_{\triangle} |f^{(j)}|/j!.$$

We consider an operator (= a bounded everywhere defined linear transformation of X) T with *real* spectrum $\sigma(T)$, and we fix throughout a compact interval $\triangle = [a, b]$ which contains $\sigma(T)$ in its interior. If n is a non-negative integer and W is an invariant linear manifold for T, a *C^n-operational calculus for T on W* is an algebra homomorphism $f \to T(f)$ of $C^n(\triangle)$ into $\mathfrak{I}(W)$ with the following properties:

(i) $T(f) = I \mid W$ for $f(t) \equiv 1$;

(ii) $T(f) = T \mid W$ for $f(t) \equiv t$; and

(iii) for each $x \in W$, the mapping $f \to T(f)x$ of $C^n(\triangle)$ into X is continuous.

In (i) and (ii), I stands for the identity operator and $T \mid W$ is the restriction of T to W.

Let $\mathfrak{B}$ denote the Borel field of the real line $\mathbf{R}$. If $W \subset X$ is a linear manifold, a *generalized spectral measure on W* is a mapping $E(\cdot) : \mathfrak{B} \to \mathfrak{I}(W)$ with the following properties:

1. $E(\mathbf{R}) = I \mid W$, and
2. for each $x \in W$, $E(\cdot)x$ is a regular strongly countably additive vector measure on $\mathfrak{B}$(*cf.* [3]).

Note that, for each $x \in W$, $E(\cdot)x$ is necessarily bounded (*cf.* [1; III. 4.5]).

Let P denote the algebra of all polynomial functions on $\triangle$. For $n = 0, 1, \cdots$, we write

$$|x|_n = \sup \{ |p(T)x| ; p \in P, \ |p|_{n,\triangle} \leqq 1 \}, \qquad x \in X,$$

* Research supported by U. S. Army Research Office, Durham.

and

$$W_n = \{x \in X;\ |x|_n < \infty\} = W_n(T).$$

The following properties of $|x|_n$ and W_n are obvious:

1. $|x|_n \geqq |x|$ for all $x \in X$;
2. $(W_n, |\cdot|_n)$ is a normed linear space;
3. if S is an operator which commutes with T, then W_n is an invariant linear manifold for S, and as an operator on $(W_n, |\cdot|_n)$, S is continuous with bound $\leqq |S|$.

We can state now our main result.

Theorem. 1. *There exists a C^n-operational calculus $T(\cdot)$ for T on W_n.*

2. *W_n is "maximal" and $T(\cdot)$ is "unique" in the following sense: if W' is an invariant linear manifold for T, and $T'(\cdot)$ is a C^n-operational calculus for T on W', then $W' \subset W_n$ and $T'(\cdot)x = T(\cdot)x$ for $x \in W'$.*

3. *If X is reflexive, there exists a unique generalized spectral measure E_n on W_n with support in $\triangle = [a, b]$, (i.e., for each $x \in W_n$, the vector measure $E_n(\cdot)x$ is supported by $\triangle$) such that*

$$(*) \qquad T(f)x = \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!} (T - aI)^j x + \frac{(T - aI)^n}{n!} \int_a^b f^{(n)}(t)\, dE_n(t)x$$

for all $x \in W_n$ and $f \in C^n(\triangle)$. Furthermore, E_n commutes with every operator which commutes with T.

For $n = 0$, the first expression on the right side of (*) disappears.

2. Proof of the theorem. Part 2 is immediate: $T'(p) = p(T)|W'$ for $p \in P$, and since $T'(\cdot)x$ is continuous for each $x \in W'$, there exists a positive function $M(\cdot)$ on W' such that $|p(T)x| \leqq M(x)\, |p|_{n,\triangle}$ for all $p \in P$ and $x \in W'$, *i.e.*, $W' \subseteq W$. For $x \in W'$, the mappings $T(\cdot)x$ and $T'(\cdot)x$ clearly coincide, since they are continuous on $C^n(\triangle)$ and agree on the dense subset P.

Parts 1 and 3 will be proved simultaneously. As expected, we define $T(p) = p(T) \mid W_n$ for $p \in P$. For $x \in W_n$ fixed, the mapping $p \rightarrow T(p)x$ of $(P, |\cdot|_{n,\triangle})$ into X is linear and continuous, with norm equal to $|x|_n$, and since P is dense in $C^n(\triangle)$, it extends uniquely to a continuous linear mapping $T(\cdot)x$ of $C^n(\triangle)$ into X (with norm equal to $|x|_n$). For $f \in C^n(\triangle)$ fixed, $T(f)$ is a linear transformation of X with domain W_n.

For $x \in W_n$ and x^* in the dual space X^* of X, the mapping $f \rightarrow x^*T(f)x$ is a continuous linear functional on $C^n(\triangle)$ with norm $\leqq |x^*|\, |x|_n$. It follows as in the proof of Lemma 2.8 in [2] that there exists a unique regular Borel measure $\mu(\cdot \mid x, x^*)$ on $\triangle = [a, b]$ such that

$$(1) \qquad x^*T(f)x = \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!} x^*(T - aI)^j x + \int_a^b f^{(n)}(t)\, d\mu(t \mid x, x^*)$$

for all $f \in C^n(\triangle)$, and

$$\text{var}\ \mu(\cdot\ |x, x^*) \leqq K\ |x^*|\ |x|_n \tag{2}$$

where $K = (1 + b - a)^n/n!$.

Let $m \geqq n$ be an integer, and take $f(t) = (t - a)^m$ in (1); thus, for $x \varepsilon W_n$ and $x^* \varepsilon X^*$,

$$x^*(T - aI)^m x = m(m - 1) \cdots (m - n + 1) \int_a^b (t - a)^{m-n}\, d\mu(t \mid x, x^*). \tag{3}$$

Let S be an operator commuting with T. If $x \varepsilon W_n$, then $Sx \varepsilon W_n$, and we may therefore apply (3) both to x and to Sx. Since $x^*(T - aI)^m Sx = (S^*x^*)(T - aI)^m x$, we obtain

$$\int_a^b (t - a)^k\, d\mu(t \mid Sx, x^*) = \int_a^b (t - a)^k\, d\mu(t \mid x, S^*x^*)$$

for $k = 0, 1, 2, \cdots$; hence

$$\mu(\cdot \mid Sx, x^*) = \mu(\cdot \mid x, S^*x^*) \tag{4}$$

for all $x \varepsilon W_n$ and $x^* \varepsilon X^*$.

For $p \varepsilon P$ and $x \varepsilon W_n$, we have

$$\sup \{|q(T)p(T)x|;\ q \varepsilon P,\ |q|_{n,\triangle} \leqq 1\}$$
$$\leqq \sup \{|(pq)(T)x|;\ |pq|_{n,\triangle} \leqq |p|_{n,\triangle}\} \leqq \sup \{|h(T)x|;\ h \varepsilon P,\ |h|_{n,\triangle} \leqq |p|_{n,\triangle}\},$$

i.e.,

$$|p(T)x|_n \leqq |p|_{n,\triangle}\ |x|_n, \qquad p \varepsilon P, \qquad x \varepsilon W_n. \tag{5}$$

Let g be a bounded Borel function on $\triangle$. Then, for $p \varepsilon P$, $x \varepsilon W_n$ and $x^* \varepsilon X^*$,

$$\begin{aligned} \left|\int_a^b g(t)\, d\mu(t \mid x, p(T)^*x^*)\right| &= \left|\int_a^b g(t)\, d\mu(t \mid p(T)x, x^*)\right| \\ &\leqq K\ |x^*|\ |p(T)x|_n \sup_\triangle |g| \\ &\leqq K\ |x^*|\ |x|_n\ |p|_{n,\triangle} \sup_\triangle |g|, \end{aligned}$$

where we used (2), (4) and (5). Now, for $f \varepsilon C^n(\triangle)$,

$$\begin{aligned} |x^*p(T)T(f)x| &= |p(T)^*x^*T(f)x| \\ &= \left|\sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!}\, x^*p(T)(T - aI)^j x + \int_a^b f^{(n)}(t)\, d\mu(t \mid x, p(T)^*x^*)\right| \\ &\leqq K_1\ |x^*|\ |x|_n\ |p|_{n,\triangle} \sum_{j=0}^{n-1} \frac{|f^{(j)}(a)|}{j!} + K\ |x^*|\ |x|_n\ |p|_{n,\triangle} \sup_\triangle |f^{(n)}| \end{aligned}$$

where $K_1 = \max \{1, |T - aI|, \cdots, |(T - aI)^{n-1}|\}$. Hence

$$|T(f)x|_n \leqq M\ |x|_n\ |f|_{n,\triangle}, \qquad x \varepsilon W_n, \qquad f \varepsilon C^n(\triangle) \tag{6}$$

where $M = \max \{1, |T - aI|, \cdots, |(T - aI)^{n-1}|, (1 + b - a)^n\}$. In particular,

$T(f) \in \mathfrak{J}(W_n)$ for all $f \in C^n(\triangle)$. The mapping $f \to T(f)$ of $C^n(\triangle)$ into $\mathfrak{J}(W_n)$ is clearly linear. It is also multiplicative on P. A routine argument, using (6) and the density of P in $C^n(\triangle)$, shows that the mapping is multiplicative on $C^n(\triangle)$. This completes the proof of Part 1.

Suppose now that X is reflexive, let δ be a fixed Borel subset of $\triangle$ and let $x \in W_n$. The mapping $x^* \to \mu(\delta \mid x, x^*)$ is a continuous linear functional on X^*, with norm $\leqq K \, |x|_n$. Therefore $\mu(\delta \mid x, x^*) = x^*F(\delta)x$, where $F(\delta)x$ is some element of X. The mapping $x \to F(\delta)x$ is a linear transformation of X with domain W_n, and $|F(\delta)x| \leqq K \, |x|_n$ for all δ. We will regard $F(\cdot)$ as defined on $\mathcal{B}$ (in the usual way). By Theorem IV.10.1 in [1], $F(\cdot)x$ is a strongly countably additive vector measure on $\mathcal{B}$, with support contained in $\triangle$. We now rewrite Equation (1) in the form

$$T(f)x = \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!} (T - aI)^j x + \int_a^b f^{(n)}(t) \, dF(t)x \tag{7}$$

($f \in C^n(\triangle)$, $x \in W_n$). For x and S as in (4), we have

$$x^*F(\cdot)Sx = S^*x^*F(\cdot)x = x^*SF(\cdot)x$$

for all $x^* \in X^*$, *i.e.*, $SF(\delta) = F(\delta)S$ for all $\delta \in \mathcal{B}$. Fix $x \in W_n$ and let g be a Borel function on $\mathbf{R}$, which is bounded on $\triangle$. The element $y = \int_{\mathbf{R}} g(t) \, dF(t)x$ is well-defined and the estimates following (5) show that for every $x^* \in X^*$ and $p \in P$,

$$|x^*p(T)y| \leqq K \, |x^*| \, |x|_n \, |p|_{n,\triangle} \sup_\triangle |g|,$$

i.e., $|y|_n \leqq K \, |x|_n \sup_\triangle |g|$. Thus $y = \int_{\mathbf{R}} g(t) \, dF(t)x \in W_n$. Taking in particular $g = c_\delta$, the characteristic function of $\delta \in \mathcal{B}$, we see that $F(\delta)W_n \subset W_n$, *i.e.*,

$$F(\delta) \in \mathfrak{J}(W_n)$$

(in fact, $|F(\delta)x|_n \leqq K \, |x|_n$ for all $\delta \in \mathcal{B}$ and $x \in W_n$). By the choice of $\triangle$, the point a is not in $\sigma(T)$. Define $E_n(\cdot) = n!(T - aI)^{-n}F(\cdot)$. Taking $f(t) = (t - a)^n/n!$ in (7), we see that $E_n(\mathbf{R}) = I/W_n$. This completes the proof of the theorem.

3. The Weyr characteristic.

3.1. ***Definition.*** Let $\{X_\lambda \, ; \lambda \in \Lambda\}$ be a collection of closed invariant subspaces for T. Let $T_\lambda = T \mid X_\lambda$. Write

$$w_0(\lambda) = w_0(\lambda; T) = \dim W_0(T_\lambda)$$

and $w_k(\lambda) = w_k(\lambda; T) = \dim \{W_k(T_\lambda) \mid W_{k-1}(T_\lambda)\} (k \geqq 1)$, where $W_k \mid W_{k-1}$ is the quotient space of W_k modulo W_{k-1} and "dim" denotes the algebraic dimension. The set $\{w_k(\lambda); k = 0, 1, \cdots; \lambda \in \Lambda\}$ will be called the *Weyr characteristic* of T with respect to $\{X_\lambda \, ; \lambda \in \Lambda\}$.

3.2. Let $\Lambda = \sigma_0(T)$, the set of isolated points of $\sigma(T)$. Let $E(\lambda) = E(\lambda; T)$ be the spectral projection associated with $\lambda \in \sigma_0(T)$ (*cf.* [1, p. 573]). If T is *compact* (but not quasi-nilpotent), its Weyr characteristic (without any qualification) is understood to be taken with respect to the collection $\{E(\lambda)X;\ \lambda \in \sigma_0(T)\}$. This is so in particular if T is an arbitrary operator in finite dimensional space.

We recall that in the latter situation, the *classical* Weyr characteristic is the set $\{w_k^*(\lambda); k = 0, 1, \cdots; \lambda \in \sigma(T)\}$ where $w_k^*(\lambda)$ is equal to the number of Jordan cells of size larger than k corresponding to the eigenvalue λ.

3.3. Proposition. *For X finite dimensional, the Weyr characteristic of T coincides with the classical Weyr characteristic. In particular, the set*

$$\sigma(T) \cup \{w_k(\lambda); 0 \leqq k \leqq n; \lambda \in \sigma(T)\} \qquad (n = \dim X)$$

is a complete set of similarity invariants for T.

Proof. Let $x_1, \cdots, x_n$ be a basis of X with respect to which T has a Jordan canonical matrix representation. Thus, there exist integers $0 = n_0 < n_1 < \cdots < n_k = n$ and an enumeration $\lambda_1, \cdots, \lambda_k$ of $\sigma(T)$ (with possible repetitions) such that

$$(1) \qquad \begin{aligned} Tx_{n_j} &= \lambda_j x_{n_j} \\ Tx_q &= \lambda_j x_q + x_{q+1} \qquad n_{j-1} < q < n_j; \qquad j = 1, \cdots, k. \end{aligned}$$

Let $\lambda \in \sigma(T)$, and suppose, without loss of generality, that $\lambda_j = \lambda$ for $j = 1, \cdots, l (1 \leqq l \leqq k)$. Then $\sigma(T_\lambda) = \{\lambda\}$ and $W_0(T_\lambda)$ is the linear span of the eigenvectors $x_{n_1}, \cdots, x_{n_l}$ (*cf.* [4, p. 251]). Thus $w_0(\lambda) = l = w_0^*(\lambda)$. For $1 \leqq j \leqq l$ fixed and $n_{j-1} < q < n_j$, $(T - \lambda I)^{n_j - q + 1} x_q = 0$ but $(T - \lambda I)^m x_q \neq 0$ if $0 \leqq m \leqq n_j - q$. Therefore (*cf.* [1, p. 559, Theorem 8]) $x_q \in W_{n_j - q}(T_\lambda)$ and $x_q \notin W_{n_j - q - 1}(T_\lambda)$.

Let $[x]_h$ denote the coset containing x in $X \mid W_{h-1}(T_\lambda)$ $(h = 1, 2, \cdots)$. Then, for j and q fixed as above,

$$0 \neq [x_q]_{n_j - q} \in W_{n_j - q}(T_\lambda)/W_{n_j - q - 1}(T_\lambda).$$

It is easily seen that $[x_r]_{n_j - q} = 0$ for $q < r \leqq n_j$, and

$$[x_r]_{n_j - q} \notin W_{n_j - q}(T_\lambda)/W_{n_j - q - 1}(T_\lambda)$$

for $n_{j-1} \leqq r < q$. Thus, for $h = 1, 2, \cdots$, the quotient space $W_h(T_\lambda) \mid W_{h-1}(T_\lambda)$ has the following properties:

(i) if $n_j - h > n_{j-1}$, it contains *exactly one* coset $[x_q]_h \neq 0$ with $n_{j-1} \leqq q \leqq n_j$, namely, the coset with $n_j - q = h$, *i.e.*, with $q = n_j - h$;

(ii) if $n_j - h \leqq n_{j-1}$, it contains *no* coset $[x_q]_h$ with $n_{j-1} \leqq q \leqq n_j$.

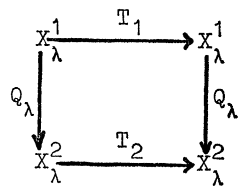
For $x_q \notin W_{h-1}$, the cosets $[x_q]_h$ are linearly independent. It follows from (i) and (ii) that $w_h(\lambda)$ is equal to the number of integers $j (1 \leqq j \leqq l)$ for which (i) holds, *i.e.*, for which $n_j - n_{j-1} > h$. But $n_j - n_{j-1}$ is the size of the j-th Jordan cell corresponding to λ. Hence $w_h(\lambda) = w_h^*(\lambda)(h = 1, 2, \cdots)$. *Q.E.D.*

3.4 Definition. Let T_1 and T_2 be operators on X. Let $\{X_\lambda^1; \lambda \in \Lambda\}$ and $\{X_\lambda^2; \lambda \in \Lambda\}$ be collections of closed invariant subspaces for T_1 and T_2 respectively, such that the sums $X^i = \sum_\lambda X_\lambda^i (i = 1, 2)$ are direct. We say that T_1 and T_2

are *quasi-similar* with respect to $\{X_\lambda^i;\ \lambda \varepsilon \Lambda;\ i = 1, 2\}$ if

(i) $T_i^* \,|(X^i)^\perp$ is quasi-nilpotent ($i = 1, 2$); and

(ii) there exist one-to-one continuous linear maps Q_λ of X_λ^1 onto $X_\lambda^2(\lambda \varepsilon \Lambda)$ such that the diagrams below commute for all $\lambda \varepsilon \Lambda$:



For compact (but not quasi-nilpotent) operators T_i with the same spectrum, quasi-similarity (without qualification) is understood to be with respect to the collection

$$\{E(\lambda; T_1)X, E(\lambda; T_2)X;\ \lambda \varepsilon \sigma_0(T_1) = \sigma_0(T_2)\}.$$

3.5 Proposition. *Let T_1 and T_2 be compact (but not quasi-nilpotent) operators with the same (real) spectrum. Then T_1 and T_2 are quasi-similar if and only if their Weyr characteristics coincide.*

Proof. This is an immediate consequence of Proposition 3.3 and of the spectral theory of compact operators (*cf.* [1] and Lemma 1.6 in [3]).

3.6. The above proposition is essentially a finite dimensional result. In fact, the usefulness of the Weyr characteristic as a criterion for similarity or quasi-similarity, seems very doubtful in "purely" infinite dimensional situations. For example, if T_1 and T_2 are hermitian quasi-similar operators on Hilbert space, then $T_i^* \,|(X^i)^\perp = T_i|\,(X^i)^\perp (i = 1, 2)$ are hermitian and quasi-nilpotent, hence equal to 0. Thus $(\sum_\lambda X_\lambda^i)^\perp \subset$ kernel $T_i = K_i$. Supposing $X_\lambda^i \perp X_\mu^i$ for $\lambda \neq \mu$, we have $X = K_i \oplus \sum_\lambda \oplus X_\lambda^i$. Since $T_i \mid X_\lambda^i (i = 1, 2)$ are hermitian, the existence of Q_λ as in Definition 3.4 implies the existence of isometries U_λ of X_λ^1 onto X_λ^2 such that $U_\lambda T_1 = T_2 U_\lambda$ on $X_\lambda^1(\lambda \varepsilon \Lambda)$. Hence there exists a partial isometry U of X intertwinning between T_1 and T_2 . However the Weyr characteristics of T_i degenerate to $\{\dim X_\lambda^i, 0, 0, \cdots;\ \lambda \varepsilon \Lambda\}(i = 1, 2)$. Obviously, these dimensions do not provide in general an efficient test for the equivalence of T_i under a partial isometry.

We should mention in this connection that an entirely different generalization of the classical Weyr characteristic has been given in [5] for a suitable class of spectral operators, and, for this class, the characteristic does provide a complete set of "semi-similarity" invariants.

References

[1] N. Dunford & J. Schwartz, *Linear operators I*, Interscience, New York, 1958.

[2] S. Kantorovitz, Classification of operators by means of their operational calculus, *Trans. Amer. Math. Soc.*, 115 (1965) 194–224.

[3] ————, A Jordan decomposition for operators in Banach space, *Trans. Amer. Math. Soc.*, **120** (1965) 526–550.

[4] ————, The semi-simplicity manifold of arbitrary operators, *Trans. Amer. Math. Soc.* **123** (1966) 241–252.

[5] A. N. Feldzamen, Semi-similarity invariants for spectral operators on Hilbert space, *Trans. Amer. Math. Soc.*, **100** (1961) 277–324.

Yale University

Date Communicated: November 18, 1966