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Local Central Limit Theorem for Multi-group Curie–Weiss Models

Michael Fleermann[1] · **Werner Kirsch**[1] · **Gabor Toth**[1]

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Abstract
We study a multi-group version of the mean-field Ising model, also called Curie–Weiss model. It is known that, in the high-temperature regime of this model, a central limit theorem holds for the vector of suitably scaled group magnetisations, that is, for the sum of spins belonging to each group. In this article, we prove a local central limit theorem for the group magnetisations in the high-temperature regime.

Keywords Curie–Weiss model · Mean-field model · Local limit theorem

Mathematics Subject Classification (2020) 60F05 · 82B20

1 Introduction

The Curie–Weiss model is a model of ferromagnetism. In its classic form, there is a random vector $(X_1, \ldots, X_n)$ of binary random variables with values in the set of spin configurations $\{-1, 1\}^n$. The probability distribution of $(X_1, \ldots, X_n)$ is given by

$$\mathbb{P}\left(X_1 = x_1, \ldots, X_n = x_n\right) = Z^{-1} \exp\left(\frac{\beta}{2n}\left(\sum_{i=1}^{n} x_i\right)^2\right)$$

✉ Gabor Toth
gabor.toth@fernuni-hagen.de

Michael Fleermann
michael.fleermann@fernuni-hagen.de

Werner Kirsch
werner.kirsch@fernuni-hagen.de

1 FernUniversität in Hagen, Hagen, Germany

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for all $(x_1, \dots, x_n) \in \{-1, 1\}^n$, where Z is a normalisation constant that depends on n and β. The parameter $\beta \in [0, \infty)$ is called the inverse temperature. It induces correlation between individual spins, causing spins to align in the same direction. At low values of β ('high temperature'), the spins are 'nearly independent'. At high values of β ('low temperature'), the spins are strongly correlated. There is a critical value of $\beta = 1$, where the collective behaviour of spins changes. This is called a phase transition. The Curie–Weiss model has been well studied, and hence, the literature is far too extensive to cite here in its entirety. The model was first defined by Husimi [19] and Temperley [30]. Discussions of it can be found in Kac [20], Thompson [31], and Ellis [9]. More recently, the Curie–Weiss model has been used in the context of social and political interactions. The idea of using models from statistical mechanics to study social interactions goes back to Föllmer [14]. The Curie–Weiss model specifically was first employed in [4]. See, for example, [7,17,27,28,32] for other applications. Another area the Curie–Weiss model has found application is the study of random matrices (see [1,12,13,15,16,18]). Local limit theorems for the single-group Curie–Weiss model have been proved including rates of convergence, see [2,29].

In this article, we deal with a multi-group version of this model. This model was first introduced in [6] and [3]. Multi-group versions of the Curie–Weiss model have received much attention recently. Some references are [5,6,10,11,22–26]. This article is organised as follows: In Sect. 2, we define the multi-group Curie–Weiss model for general coupling matrices, see Definition 1. After this definition, we introduce the specific coupling matrices considered in this paper. In particular, our study is constrained to the so-called high-temperature regime as in Definition 2. For this regime, a non-local (or *global*) central limit theorem has been derived in [22], which we recite in Theorem 3. Our main result of this paper is a local version of Theorem 3 stated in Theorem 4 and proved in Sect. 3, the last section of this paper.

2 Setup and Results

Let there be $d \in \mathbb{N}$ groups with $n_\lambda \in \mathbb{N}$ spins in group $\lambda \in \{1, \dots, d\}$, and set $n := \sum_{\lambda=1}^{d} n_\lambda$. We regard each n_λ as a sequence that depends on n but suppress this dependence. The spin variables are

$$\left(X_{11}, X_{12}, \dots, X_{1n_1}, \dots, X_{d1}, X_{d2}, \dots, X_{dn_d}\right) \in \{-1, 1\}^n.$$

We assume that each of the d relative group sizes converges to a fixed proportion of the overall population:

$$\alpha_\lambda := \lim_{n\to\infty} \frac{n_\lambda}{n} \text{ and } n_\lambda \to \infty \text{ as } n \to \infty, \tag{1}$$

so that the α_λ sum to 1.

Instead of a single inverse temperature parameter, there is a coupling matrix that describes the spin interactions. We will denote this matrix as

$$J := \left(J_{\lambda\mu}\right)_{\lambda,\mu=1,\dots,d} \in \mathbb{R}^{d\times d}.$$

Every spin in group λ interacts with every spin in group μ with a strength given by the coupling constant $J_{\lambda\mu}$.

Just as in the single-group model, there is a Hamiltonian function that assigns to each spin configuration $(x_1, \ldots, x_n)$ a certain energy level:

$$\mathbb{H}\left(x_{11}, \ldots, x_{dn_d}\right) := -\frac{1}{2n} \sum_{\lambda,\mu=1}^{d} J_{\lambda\mu} \sum_{i=1}^{n_\lambda} \sum_{j=1}^{n_\mu} x_{\lambda i} x_{\mu j}. \tag{2}$$

As we can see from the definition of $\mathbb{H}$, it suffices to consider symmetric J, for otherwise we can replace J by $\frac{J+J^T}{2}$, leaving the Hamiltonian unchanged.

Definition 1 The Curie–Weiss measure $\mathbb{P}$, which gives the probability of each of the 2^n spin configurations, is defined by

$$\mathbb{P}\left(X_{11} = x_{11}, \ldots, X_{dn_d} = x_{dn_d}\right) := Z^{-1} \exp\left(-\mathbb{H}\left(x_{11}, \ldots, x_{dn_d}\right)\right) \tag{3}$$

for all $x_{\lambda i} \in \{-1, 1\}$ and Z is a normalisation constant which depends on n and J.

We distinguish two different classes of coupling matrices:

1. Homogeneous coupling matrices $J = (\beta)_{\lambda,\mu=1,\ldots,d}$, where all entries are equal to the same constant $\beta \geq 0$.
2. Heterogeneous coupling matrices $J = (J_{\lambda,\mu})_{\lambda,\mu=1,\ldots,d}$, which we assume to be positive definite.

For each of the two classes of coupling matrices, the model has three 'temperature regimes', which are characterised by the coupling constants and the group sizes. These regimes are called the high-temperature, the critical, and the low-temperature regime. In each regime, the spins behave differently, and the limiting distribution for large n is different in each case. This paper is exclusively concerned with the high-temperature regime which will be defined below (cf. Definition 2). For details on the other regimes, see [22]. There, it is also pointed out that without the assumption of positive definiteness in the case of heterogeneous coupling matrices J, the high-temperature regime may be empty.

If the coupling matrix is homogeneous, then the high-temperature regime is characterised by $\beta < 1$.

For heterogeneous coupling matrices, the characterisation of the high-temperature regime is somewhat more complicated. As an initial parameter space, we define

$$\Phi := \left\{(\alpha_1, \ldots, \alpha_d) \mid \alpha_1, \ldots, \alpha_d \geq 0, \sum_{\lambda=1}^{d} \alpha_\lambda = 1\right\}$$
$$\times \{J \mid J \text{ is a } d \times d \text{ positive definite matrix}\},$$

containing all possible combinations of asymptotic relative group sizes $(\alpha_1, \ldots, \alpha_d)$ as in (1) and coupling matrices.

We define

$$\boldsymbol{\alpha} := \operatorname{diag}(\alpha_1, \ldots, \alpha_d),$$

where 'diag' stands for a diagonal matrix with the entries given between parentheses, and

$$H := J^{-1} - \boldsymbol{\alpha}. \tag{4}$$

Note that this definition of a multi-group Curie–Weiss model reduces to the classical single-group model if we set $d = 1$, since then $n_1 = n$ and $J = \beta$. See also Remark 9.

The parameter space Φ is partitioned into three regimes (for the details, see [22]).

Definition 2 The 'high-temperature regime' for heterogeneous coupling matrices is the set of parameters

$$\Phi_h := \{\phi \in \Phi | H \text{ is positive definite}\}.$$

In the high-temperature regime, a multivariate central limit theorem holds for the normalised sums of spins in each group. For a proof, see, for example, [22].

For each group $\lambda \in \{1, \ldots, d\}$, we define $S_\lambda := \sum_{i=1}^{n_\lambda} X_{\lambda i}$ to be the sum of all spins belonging to that group. In this article, we show a local limit theorem for the normalised magnetisation vector

$$\left(\frac{S_1}{\sqrt{n_1}}, \cdots, \frac{S_d}{\sqrt{n_d}}\right)$$

in the high-temperature regime.

Theorem 3 *In the high-temperature regime, we have*

$$\left(\frac{S_1}{\sqrt{n_1}}, \ldots, \frac{S_d}{\sqrt{n_d}}\right) \xrightarrow[n\to\infty]{} \mathcal{N}((0, \ldots, 0), C) \quad \textit{weakly},$$

where $\mathcal{N}[(0, \ldots, 0), C]$ is a zero-mean multivariate normal distribution with positive definite covariance matrix $C := I + \sqrt{\boldsymbol{\alpha}}\Sigma\sqrt{\boldsymbol{\alpha}}$, where the matrix Σ depends on the class of coupling matrices:

$$\Sigma = \begin{cases} \left(\frac{\beta}{1-\beta}\right)_{\lambda,\mu=1,\ldots,d}, & \textit{if } J \textit{ is homogeneous,} \\ H^{-1}, & \textit{if } J \textit{ is heterogeneous.} \end{cases}$$

We shall write ϕ_C for the density function of $\mathcal{N}[(0, \ldots, 0), C]$, and we set

$$\boldsymbol{S}^n := \left(\frac{S_1}{\sqrt{n_1}}, \ldots, \frac{S_d}{\sqrt{n_d}}\right).$$

For a given $n \in \mathbb{N}$ and group λ, $\frac{S_\lambda}{\sqrt{n_\lambda}}$ takes values on the grid $\frac{n_\lambda+2\mathbb{Z}}{\sqrt{n_\lambda}}$. Hence, the vector $\left(\frac{S_1}{\sqrt{n_1}}, \ldots, \frac{S_d}{\sqrt{n_d}}\right)$ takes values on the grid

$$\mathcal{L}_n := \prod_{\lambda=1}^{d} \frac{n_\lambda + 2\mathbb{Z}}{\sqrt{n_\lambda}}.$$

We show that the central limit theorem—Theorem 3—can be strengthened to a multivariate local limit theorem:

Theorem 4 *In the high-temperature regime, it holds:*

$$\sup_{x \in \mathcal{L}_n} \left| \frac{\prod_{\lambda=1}^{d} \sqrt{n_\lambda}}{2^d} \mathbb{P}\left(\boldsymbol{S}^n = x\right) - \phi_C(x) \right| \xrightarrow[n\to\infty]{} 0.$$

3 Proof

The proof of Theorem 4 is structured as follows: Since we prove the local central limit theorem using characteristic functions, we state Lemmas 5, 6, and 7 to provide the tools necessary. Then, we bound the local error in the local central limit theorems by quantities which are actually independent of the specific location, see (5) and (6). While the upper bound in (6) is trivial to handle, the upper bound in (5) requires a deeper analysis, which is initiated by Proposition 8, allowing us to express the Curie–Weiss distribution as a mixture of Rademacher distributions. The characteristic functions of the latter are bounded via a Taylor expansion below inequality (7), which in combination with a concentration inequality result in Proposition 8 establishes a large part of our analysis. The remainder of the proof consists of constructing an integrable majorant below inequality (8) and then using a bound on the characteristic functions of discrete distributions established in Lemma 5.

Lemma 5 *Let $Y := (Y_1, \ldots, Y_d)$ be a random vector with values on the grid $\prod_{\lambda=1}^{d} (v_\lambda + w_\lambda \mathbb{Z})$ and characteristic function φ, defined by $\varphi(t) := \mathbb{E}\left(\exp\left(it \cdot Y\right)\right), t \in \mathbb{R}^d$. Then, the following two properties hold:*

1. *φ is periodic, i.e. for all $t \in \mathbb{R}^d, k_1, \ldots, k_d \in \mathbb{Z}$,*

$$\varphi\left(t + 2\pi\left(\frac{k_1}{w_1}, \cdots, \frac{k_d}{w_d}\right)\right) = \varphi(t)$$

2. *We have for all $k_1, \ldots, k_d \in \mathbb{Z}$,*

$$\left|\varphi\left(2\pi\left(\frac{k_1}{w_1}, \cdots, \frac{k_d}{w_d}\right)\right)\right| = 1$$

and for all $t \in \mathbb{R}^d$ *such that* $0 < t_\lambda < \frac{2\pi}{w_\lambda}$ *for some component* λ,

$$|\varphi(t)| < 1.$$

Proof This follows from a straightforward modification of the proof of Theorem 3.5.2 on page 140 in [8]. □

The second statement in Lemma 5 gives an upper bound for the characteristic function of a random variable on the grid, which we shall use in our calculations later on. We will use the following inversion formulas to recover distributions from their characteristic functions:

Lemma 6 *Let* $(Y_1, \ldots, Y_d)$ *be a random vector as in Lemma 5. Then, for all* $x \in \prod_{\lambda=1}^{d}(v_\lambda + w_\lambda\mathbb{Z})$,

$$\mathbb{P}((Y_1, \ldots, Y_d) = x) = \frac{\prod_{\lambda=1}^{d} w_\lambda}{(2\pi)^d} \int_{\prod_\lambda \left[-\frac{\pi}{w_\lambda}, \frac{\pi}{w_\lambda}\right]} e^{-it\cdot x}\varphi(t)\mathrm{d}t.$$

Proof See, for example, Section 3.10 in [8]. □

Lemma 7 *Let* φ *be the characteristic function of some* d*-dimensional random vector such that* φ *is Lebesgue integrable. Then,*

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-it\cdot x}\varphi(t)\mathrm{d}t$$

defines a continuous Lebesgue density function f *for the random vector.*

Proof This is Theorem 5.5 in [33]. □

Let $\varphi_{\boldsymbol{S}^n}$ be the characteristic function of $\boldsymbol{S}^n$ and $\varphi_{\mathcal{N}(C)}$ that of $\mathcal{N}((0, \ldots, 0), C)$. We use the symbol $\mathbb{E}$ as the expectation with respect to the probability measure $\mathbb{P}$ of the underlying probability space.

Let $x \in \mathcal{L}_n$. By Lemma 6, we have

$$\frac{\prod_{\lambda=1}^{d}\sqrt{n_\lambda}}{2^d}\mathbb{P}\left(\boldsymbol{S}^n = x\right) = \frac{1}{(2\pi)^d} \int_{\prod_\lambda \left[-\frac{\pi\sqrt{n_\lambda}}{2}, \frac{\pi\sqrt{n_\lambda}}{2}\right]} e^{-it\cdot x}\varphi_{\boldsymbol{S}^n}(t)\mathrm{d}t,$$

and therefore,

$$\left|\frac{\prod_{\lambda=1}^{d}\sqrt{n_\lambda}}{2^d}\mathbb{P}\left(\boldsymbol{S}^n = x\right) - \phi_C(x)\right|$$
$$= \frac{1}{(2\pi)^d}\left|\int_{\prod_\lambda \left[-\frac{\pi\sqrt{n_\lambda}}{2}, \frac{\pi\sqrt{n_\lambda}}{2}\right]} e^{-it\cdot x}\varphi_{\boldsymbol{S}^n}(t)\mathrm{d}t - \int_{\mathbb{R}^d} e^{-it\cdot x}\varphi_{\mathcal{N}(C)}(t)\mathrm{d}t\right|$$

$$\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} I_{\prod_\lambda \left[-\frac{\pi\sqrt{n_\lambda}}{2}, \frac{\pi\sqrt{n_\lambda}}{2}\right]}(t) \left|\varphi_{S^n}(t) - \varphi_{\mathcal{N}(C)}(t)\right| \mathrm{d}t \tag{5}$$

$$+ \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d \setminus \prod_\lambda \left[-\frac{\pi\sqrt{n_\lambda}}{2}, \frac{\pi\sqrt{n_\lambda}}{2}\right]} \left|\varphi_{\mathcal{N}(C)}(t)\right| \mathrm{d}t. \tag{6}$$

The term (6) converges to 0 as $n \to \infty$, since $\left|\varphi_{\mathcal{N}(C)}(t)\right|$ is integrable. Note also that expression (5) is independent of the point $x \in \mathcal{L}_n$. Thus, if we can show that (5) converges to 0, then we are done. To this end, we see by Theorem 3 that $\varphi_{S^n}(t) \to \varphi_{\mathcal{N}(C)}(t)$ pointwise. Therefore, to show that (5) converges to 0 is for the most part a matter of finding an appropriate integrable majorant, so that the theorem of dominated convergence can be applied. To construct a suitable majorant, we need to apply some properties of the multivariate Curie–Weiss distribution.

Let the Rademacher distribution $\mathcal{R}_m$ with parameter $m \in \mathbb{R}$ be defined on $\{\pm 1\}$ by the probability of the event $\{1\}$ equal to $\frac{1+\bar{m}}{2}$, setting $\bar{m} := \tanh m$.

We use the de Finetti representation of the Curie–Weiss measure (see [22]):

Proposition 8 *The distribution of the multi-group Curie–Weiss model has the following representation: For any spin configuration $\left(x_{11}, \ldots, x_{dn_d}\right)$, we have*

$$\mathbb{P}\left(X_{11}{=}x_{11}, \ldots, X_{dn_d}{=}x_{dn_d}\right) = \int_{\mathbb{R}^d} P_m\left(X_{11} = x_{11}, \ldots, X_{dn_d} = x_{dn_d}\right) \mu_{J,n}(\mathrm{d}m),$$

where P_m is the product measure of Rademacher distributions with parameters m_λ for all spins belonging to group λ. $\mu_{J,n}$ is a probability measure defined by the Lebesgue density function

$$f_{J,n}(m) \propto \exp\left(-n\left(\frac{1}{2} m^T J^{-1} m - \sum_\lambda \frac{n_\lambda}{n} \ln\cosh m_\lambda\right)\right), \quad m \in \mathbb{R}^d.$$

In the high-temperature regime, $\mu_{J,n}$ has an asymptotic concentration property, such that for all $\delta > 0$ there is a $D > 0$ with the property

$$\mu_{J,n}\left(\mathbb{R}^d \setminus [-\delta, \delta]^d\right) < \exp(-Dn)$$

for large enough n.

Remark 9 If we set $d = 1$, then the probability density $f_{J,n}$ above is proportional to

$$\exp\left(-n\left(\frac{1}{2\beta} m^2 - \ln\cosh m\right)\right), \quad m \in \mathbb{R}.$$

From this expression, we obtain an alternative probability density defined on $[-1, 1]$ proportional to

$$\frac{\exp\left(-\frac{n}{2}\left(\frac{1}{\beta}\left(\frac{1}{2}\ln\frac{1+t}{1-t}\right) + \ln\left(1 - t^2\right)\right)\right)}{1 - t^2}, \quad t \in [-1, 1]$$

by the substitution $t := \tanh m$. See Section 5.2 in [21].

Let $\varphi_{\mathcal{R}(m)}$ be the characteristic function of the Rademacher distribution with parameter m, and let E_m be the expectation under the distribution $\mathcal{R}_m$.

Now we deal with expression (5). We pick some $0 < \delta < \pi/2$ and partition the set $\prod_\lambda \left[-\frac{\pi\sqrt{n_\lambda}}{2}, \frac{\pi\sqrt{n_\lambda}}{2}\right] = \prod_\lambda \left[-\delta\sqrt{n_\lambda}, \delta\sqrt{n_\lambda}\right] \dot\cup B_n =: A_n \dot\cup B_n$ for each $n \in \mathbb{N}$.

Our goal is to show that (5) converges to 0. We do so by showing the result separately over A_n and B_n. The following upper bound holds over A_n:

$$I_{A_n}(t)\left|\varphi_{S^n}(t)\right| \leq I_{A_n}(t)\int_{\mathbb{R}^d}\prod_\lambda\left|\varphi_{\mathcal{R}(m_\lambda)}\left(\frac{t_\lambda}{\sqrt{n_\lambda}}\right)\right|^{n_\lambda}\mu_{J,n}(\mathrm{d}m). \tag{7}$$

We calculate an upper bound for the Rademacher characteristic function:

$$\begin{aligned}
\left|\varphi_{\mathcal{R}(m)}(u)\right| &= \left|\mathrm{E}_m \exp\left(iu\left(X_{\lambda 1}-\bar{m}\right)\right)\right|\left|\exp\left(iu\bar{m}\right)\right| \\
&\leq \left|1-\left(1-\bar{m}^2\right)\frac{u^2}{2}\right| + u^2\left(1-\bar{m}^2\right)\min\left\{|u|\left(1+\bar{m}^2\right),1\right\} \\
&\leq 1-\left(1-\bar{m}^2\right)\frac{u^2}{2}+\left(1-\bar{m}^2\right)\frac{u^2}{4} \\
&\leq \exp\left(-\left(1-\bar{m}^2\right)\frac{u^2}{4}\right).
\end{aligned}$$

The first inequality follows from a Taylor expansion of the exponential function with the remainder term of order three $u^2\mathrm{E}_m\min\left\{|u|\left|X_{\lambda 1}-\bar{m}\right|^3, \left|X_{\lambda 1}-\bar{m}\right|^2\right\}$, which is smaller or equal $u^2\left(1-\bar{m}^2\right)\min\left\{|u|\left(1+\bar{m}^2\right),1\right\}$ as can be verified by direct calculation. The second inequality holds for small enough $|u|$. The third inequality for any $u \in \mathbb{R}$ is well known. Therefore,

$$\left|\varphi_{\mathcal{R}(m_\lambda)}\left(\frac{t_\lambda}{\sqrt{n_\lambda}}\right)\right| \leq \exp\left(-\left(1-\bar{m}_\lambda^2\right)\frac{t_\lambda^2}{4n_\lambda}\right),$$

and we pick some $\tau \in (0,1)$ to continue with our calculation:

$$\begin{aligned}
(7) &\leq I_{A_n}(t)\int_{\mathbb{R}^d}\exp\left(-\frac{1}{4}\sum_\lambda\left(1-\bar{m}_\lambda^2\right)t_\lambda^2\right)\mu_{J,n}(\mathrm{d}m) \\
&\leq \int_{[-\tau,\tau]^d}\exp\left(-\frac{1}{4}\sum_\lambda\left(1-\bar{m}_\lambda^2\right)t_\lambda^2\right)\mu_{J,n}(\mathrm{d}m) \\
&\quad + I_{A_n}(t)\mu_{J,n}\left(\mathbb{R}^d\backslash[-\tau,\tau]^d\right) \\
&\leq \exp\left(-\frac{1}{4}\left(1-\tanh^2\tau\right)\sum_\lambda t_\lambda^2\right)+I_{A_n}(t)\exp(-\eta n),
\end{aligned} \tag{8}$$

where the second term in the last line follows from Lemma 8. Note that $\eta > 0$.

It is clear that the first summand in (8) is integrable. For the second summand, we have

$$I_{A_n}(t)\exp(-\eta n) \leq I_{A_1}(t)\exp(-\eta) + \sum_{k=1}^{\infty} I_{A_{k+1}\setminus A_k}(t)\exp(-\eta(k+1)) =: f(t).$$

Let $\boldsymbol{\lambda}^d$ be the Lebesgue measure on $\mathbb{R}^d$. We show that the function f on the right-hand side is an integrable majorant for all $I_{A_n}(t)\exp(-\eta n)$, $n \in \mathbb{N}$:

$$\int_{\mathbb{R}^d} f(t)\mathrm{d}t = \boldsymbol{\lambda}^d(A_1)\exp(-\eta) + \sum_{k=1}^{\infty} \boldsymbol{\lambda}^d(A_{k+1}\setminus A_k)\exp(-\eta(k+1)).$$

Each summand in the series above can be bounded above by

$$\begin{aligned}\boldsymbol{\lambda}^d(A_{k+1}\setminus A_k)\exp(-\eta(k+1)) &\leq \boldsymbol{\lambda}^d(A_{k+1})\exp(-\eta(k+1)) \\ &= \boldsymbol{\lambda}^d\left(\prod_{\lambda}\left[-\delta\sqrt{(k+1)_\lambda}, \delta\sqrt{(k+1)_\lambda}\right]\right) \\ &\leq (2\delta)^d (k+1)^{\frac{d}{2}}\exp(-\eta(k+1)),\end{aligned}$$

which is summable in k.

As $\left|\varphi_{\mathcal{N}(C)}(t)\right|$ is integrable as well, we have thus found that $I_{A_n}(t)\left|\varphi_{S^n}(t)-\varphi_{\mathcal{N}(C)}(t)\right|$ has an integrable majorant. By Theorem 3, $\left|\varphi_{S^n}(t)-\varphi_{\mathcal{N}(C)}(t)\right| \to 0$ pointwise as $n \to \infty$, so we conclude that the integral of $I_{A_n}(t)\left|\varphi_{S^n}(t)-\varphi_{\mathcal{N}(C)}(t)\right|$ over $\mathbb{R}^d$ converges to 0 as $n \to \infty$.

We proceed with the integrand over the set B_n:

$$\begin{aligned}I_{B_n}(t)\left|\varphi_{S^n}(t)\right| &\leq I_{B_n}(t)\int_{\mathbb{R}^d}\prod_{\lambda}\left|\varphi_{\mathcal{R}(m_\lambda)}\left(\frac{t}{\sqrt{n_\lambda}}\right)\right|^{n_\lambda}\mu_{J,n}(\mathrm{d}m) \\ &\leq I_{B_n}(t)\int_{\mathbb{R}^d}(\theta(m))^n\mu_{J,n}(\mathrm{d}m), \qquad (9)\end{aligned}$$

where the existence of

$$\theta(m) = \max_{t\in B_n, \lambda=1,\ldots,d}\left|\varphi_{\mathcal{R}(m_\lambda)}(t_\lambda)\right| < 1$$

is a consequence of Lemma 5. We continue with the calculation of an upper bound:

$$(9) \leq I_{B_n}(t)\int_{[-\tau,\tau]^d}(\theta(m))^n\mu_{J,n}(\mathrm{d}m) + I_{B_n}(t)\mu_{J,n}\left(\mathbb{R}^d\setminus[-\tau,\tau]^d\right).$$

On the interval $[-\tau,\tau]$, θ is bounded away from 1:

$$s := \sup_{m\in[-\tau,\tau]^d}\theta(m) < 1.$$

With this final upper bound for $I_{B_n}(t)\,|\varphi_{S^n}(t)|$, we see that

$$I_{B_n}(t)\,|\varphi_{S^n}(t)|\,\mathrm{d}t \le I_{B_n}(t)\left(s^n + \exp(-\eta n)\right)\mathrm{d}t.$$

For the last expression, we can construct an integrable majorant in the same manner as for the second summand in (8) because $s^n + \exp(-\eta n)$ converges exponentially to 0.

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References

1. Baik, J., Lee, J.O.: Fluctuations of the free energy of the spherical Sherrington–Kirkpatrick model with ferromagnetic interaction. Ann. Henri Poincaré **18**, 1867–1917 (2017)
2. Barbour, A.D., Röllin, A., Ross, N.: Error bounds in local limit theorems using Stein's method. Bernoulli **25**(2), 1076–1104 (2019)
3. Berthet, Quentin, Rigollet, P., Srivastava, P.: Exact recovery in the Ising blockmodel. Ann. Stat. **47**(4), 1805–1834 (2019)
4. Brock, W.A., Durlauf, S.N.: Discrete choice with social interactions. Rev. Econ. Stud. Oxford Univ. Press **68**(2), 235–260 (2001)
5. Collet, Francesca: Macroscopic limit of a bipartite Curie–Weiss model: a dynamical approach. J. Stat. Phys. **157**(6), 1301–1319 (2014)
6. Contucci, P., Gallo, I.: Bipartite mean field spin systems. Existence and solution. Math. Phys. Elec. J. **14**(1), 1–22 (2008)
7. Contucci, P., Ghirlanda, S.: Modelling society with statistical mechanics: an application to cultural contact and immigration. Qual. Quant. **41**, 569–578 (2007)
8. Durrett, R.: Probability Theory and Examples, 5th edn. Cambridge University Press, Cambridge (2019)
9. Ellis, R.: Entropy, Large Deviations, and Statistical Mechanics. Wiley, Hoboken (1985)
10. Fedele, Micaela: Rescaled Magnetization for Critical Bipartite Mean-Fields Models. J. Stat. Phys. **155**, 223–226 (2014)
11. Fedele, M., Contucci, P.: Scaling limits for multi-species statistical mechanics mean-field models. J. Stat. Phys. **144**, 1186–1205 (2011)
12. Fleermann, M.: Global and Local Semicircle Law for Random Matrices with Correlated Entries. PhD thesis. FernUniversität in Hagen, Germany (2019)
13. Fleermann, M., Heiny, J.: High-dimensional sample covariance matrices with Curie–Weiss entries. (2019) arXiv:1910.12332
14. Föllmer, H.: Random economies with many interacting agents. J. Math. Econ. **1**(1), 51–62 (1974)
15. Fleermann, M., Kirsch, W., Kriecherbauer, T.: The almost sure semicircle law for random band matrices with dependent entries. Stoch. Process. Appl. **131**, 172–200 (2021)
16. Friesen, O., Löwe, M.: A phase transition for the limiting spectral density of random matrices. Electron. J. Probab. **18**(17), 1–17 (2013)
17. Gallo, I., Barra, A., Contucci, P.: Parameter evaluation of a simple mean-field model of social interaction. Math. Models Methods Appl. Sci. **19**, 1427–1439 (2009)
18. Hochstättler, W., Kirsch, W., Warzel, S.: Semicircle law for a matrix ensemble with dependent entries. J. Theor. Probab. **29**(3), 1047–1068 (2016)

19. Husimi, K.: Statistical Mechanics of Condensation. In Proceedings of the International Conference of Theoretical Physics, pp. 531–533. Science Council of Japan, Tokyo (1953)
20. Kac, M.: Mathematical Mechanisms of Phase Transitions, in Statistical Physics: Phase Transitions and Superfluidity 1:241–305, Brandeis University Summer Institute in Theoretical Physics (1968)
21. Kirsch, W.: A Survey on the Method of Moments. Available from http://www.fernuni-hagen.de/stochastik/
22. Kirsch, W., Toth, G.: Limit Theorems for Multi-Group Curie–Weiss Models via the Method of Moments. arXiv:2102.05903
23. Kirsch, W., Toth, G.: Two groups in a Curie–Weiss model. Math. Phys. Anal. Geom. **23**, 17 (2020)
24. Kirsch, W., Toth, G.: Two groups in a Curie–Weiss model with heterogeneous coupling. J. Theor. Probab. **33**, 2001–2026 (2020)
25. Knöpfel, H., Löwe, M., Schubert, K., et al.: Fluctuation results for general block spin Ising models. J. Stat. Phys. **178**, 1175–1200 (2020)
26. Löwe, M., Schubert, K.: Fluctuations for block spin Ising models. Electron. Commun. Probab. **23**, 1–12 (2018)
27. Löwe, M., Schubert, K., Vermet, F.: Multi-group binary choice with social interaction and a random communication structure—a random graph approach. Phys. A Stat. Mech. Appl. **556**, 124735 (2020)
28. Opoku, A.A., Edusei, K.O., Ansah, R.K.: A conditional Curie–Weiss model for stylized multi-group binary choice with social interaction. J. Stat. Phys. **171**, 106–126 (2018)
29. Röllin, A., Ross, N.: Local limit theorems via Landau–Kolmogorov inequalities. Bernoulli **21**(2), 851–880 (2015)
30. Temperley, H.N.V.: The Mayer theory of condensation tested against a simple model of the imperfect gas. Proc. Phys. Soc. A **67**, 233–238 (1954)
31. Thompson, C.J.: Mathematical Statistical Mechanics. Macmillan, Basingstoke (1972)
32. Toth, G.: Correlated Voting in Multipopulation Models, Two-Tier Voting Systems, and the Democracy Deficit, PhD Thesis, Fernuniversität in Hagen. (2020) https://doi.org/10.18445/20200505-103735-0
33. Wengenroth, J.: Wahrscheinlichkeitstheorie. Walter de Gruyter (2008)