

Pacific Journal of Mathematics

LOCAL COMPACTNESS OF FAMILIES OF CONTINUOUS POINT-COMPACT RELATIONS

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The purpose of this paper is to prove that the pointwise closure of an equicontinuous family of point-compact relations from a compact Hausdorff space to a locally compact Hausdorff uniform space is locally compact in the topology of uniform convergence. This is a generalization of a recent result of R. V. Fuller.

1. Introduction. The purpose of this paper is to study conditions under which a family of continuous point-compact relations is locally compact. Our theorem generalizes a theorem of Fuller [2]. For the most part we use the concepts and results of Smithson ([5], [6], [7]), and Michael [4].

We use the term relation where other authors use multivalued function or multifunction. If F is a relation from X to Y and $B \subset Y$, we write

$$F^{-1}(B) = \{x \in X: F(x) \cap B \neq \emptyset\}.$$

A relation F from a topological space X to a topological space Y is called *continuous* iff

- (a) $F^{-1}(A)$ is closed in X whenever A is closed in Y , and
- (b) $F^{-1}(B)$ is open in X whenever B is open in Y .

F is *point-closed* (respectively *point-compact*) iff $F(x)$ is closed (respectively compact) for each $x \in X$.

We recall three topologies defined on the collection of all non-empty subsets of a topological space X (see Michael [4]). The collection of all sets of the form $\{B \subset X: B \subset U\}$ where U is open in X , is a base for the *upper semi-finite* (u.s.f.) topology. The collection of all sets of the form $\{B \subset X: B \cap U \neq \emptyset\}$ where U is open in X , is a subbase for the *lower semi-finite* (l.s.f.) topology. The *finite* topology is the supremum of the l.s.f. and u.s.f. topologies. Equivalently, the finite topology has as a basis all sets of the form $\langle U_1, \dots, U_n \rangle = \{A \subset X: A \cap U_i \neq \emptyset, 1 \leq i \leq n \text{ and } A \subset \bigcup_{i=1}^n U_i\}$, where U_1, \dots, U_n are open in X . A relation $F: X \rightarrow Y$ is continuous if and only if the function $F: X \rightarrow P(Y)$ (the power set of Y with the finite topology) is continuous, c.f. remark following Theorem 2.7 of [7].

Let X and Y be topological spaces and let \mathcal{F} be a set of relations from X to Y . The pointwise topology \mathcal{P} [7] on \mathcal{F} has a subbase consisting of the sets of the form $\{F \in \mathcal{F}: F(x) \cap U \neq \emptyset\}$ or $\{F \in \mathcal{F}: F(x) \subset V\}$ where $x \in X$, and U, V are open in Y . We

note that the projections $\{\Pi_x: x \in X\}$ defined by $\Pi_x(F) = F(x)$ are continuous functions into the collection of all nonempty subsets of Y with the finite topology.

Let X be a topological space, (Y, \mathcal{U}) a uniform space and let \mathcal{F} be a set of relations from X to Y . For $V \in \mathcal{U}$, let $W(V) = \{(F, G) \in \mathcal{F} \times \mathcal{F}: \text{for all } x \in X, (y, G(x)) \cap V \neq \emptyset \text{ for all } y \in F(x), \text{ and } (F(x), y') \cap V \neq \emptyset \text{ for all } y' \in G(x)\}$. Let \mathcal{W} be the uniformity on \mathcal{F} generated by the collection of all such encourages as $W(V)$. The topology generated by \mathcal{W} is the topology of *uniform convergence* [5] and is denoted by \mathcal{UC} . If (Y, \mathcal{U}) is a uniform space, \mathcal{F} is called equicontinuous at $x \in X$ [5] iff for every $V \in \mathcal{U}$ there is a nbhd. U of x such that for all $F \in \mathcal{F}$,

- (a) $F(U) \subset V(F(x))$, and
- (b) $F(z) \cap V(y) \neq \emptyset$ for all $z \in U$ and for all $y \in F(x)$.

We now state a theorem of Smithson which we use in the final section.

THEOREM 1.1. ([5]). *If \mathcal{F} is an equicontinuous family of point-compact relations from a compact space X to a uniform space Y , then on \mathcal{F} , $\mathcal{P} = \mathcal{UC}$.*

For further details and a survey the reader is referred to Smithson [7].

2. Local compactness of a space of relations. We begin by proving two lemmas.

LEMMA 2.1. *Let F be a point-compact relation from X to Y , and let $A = \{(\{x\}, F(x)): x \in X\}$ be compact in $P(X) \times P(Y)$, where each of $P(X)$, $P(Y)$ has the finite topology. Then F is a compact subset of $X \times Y$.*

Proof. Let \mathcal{O} be an open cover of F in $X \times Y$. For each $x \in X$, $\{x\} \times F(x)$ is compact; so there is a finite subcollection $V_i^x \times U_i^x$, $1 \leq i \leq n$ of \mathcal{O} which covers the set. We can assume that $x \in V_i^x$ for each i and that $F(x) \cap U_i^x \neq \emptyset$. $\langle V_1^x, \dots, V_n^x \rangle \times \langle U_1^x, \dots, U_n^x \rangle$ is an open set in $P(X) \times P(Y)$. For each $x \in X$, we obtain such a set, and this leads to an open cover of A . Since A is compact, there is a finite subcover $\langle V_1^{x_i}, \dots, V_{n_i}^{x_i} \rangle \times \langle U_1^{x_i}, \dots, U_{n_i}^{x_i} \rangle$, $1 \leq i \leq k$. Finally, $\{V_j^{x_i} \times U_j^{x_i}: 1 \leq j \leq n_i, 1 \leq i \leq k\}$ is a cover of F and so F is compact.

LEMMA 2.2. *If F is a continuous relation from X to Y , then the function $g: X \rightarrow P(X) \times P(Y)$ defined by $g(x) = (\{x\}, F(x))$ is continuous. ($P(X)$ and $P(Y)$ both have finite topology.)*

Proof. Let $\langle V \rangle \times \langle U_1, \dots, U_n \rangle$ be a basic open nbhd. of $(\{x\}, F(x))$. The function $F: X \rightarrow P(Y)$ is continuous, hence there exists a nbhd. $N \subset V$ of x such that $F(N) \subset \langle U_1, \dots, U_n \rangle$. Clearly, $g(N) \subset \langle V \rangle \times \langle U_1, \dots, U_n \rangle$.

From the above lemmas it follows that if X is compact, Y is T_2 and F is a continuous point-compact relation, then F is a compact subset of $X \times Y$.

The proof of the following lemma is straightforward.

LEMMA 2.3. *Let \mathcal{F} be a family of relations from a topological space X to a topological space Y . Then the l.s.f. topology on \mathcal{F} is contained in \mathcal{P} .*

LEMMA 2.4. *Let X be compact Hausdorff, (Y, \mathcal{V}) a uniform space, and \mathcal{F} a family of continuous point-compact relations from X to Y . Then on \mathcal{F} the u.s.f. topology is smaller than \mathcal{UC} .*

Proof. If $F \in \mathcal{F}$, then F is a compact subset of $X \times Y$. Suppose $F \subset N$ is an open subset of $X \times Y$. Let \mathcal{U} be the (unique) uniformity on X . Then from [3] page 199, it follows that there exist $U \in \mathcal{U}$, $V \in \mathcal{V}$ such that $F \subset \cup \{U(x) \times V(y) : x \in X, y \in F(x)\} \subset N$. Then $W(V)[F] \subset N$, thus completing the proof.

LEMMA 2.5. *Let \mathcal{F} be an equicontinuous family of relations from a T_1 -space X to a uniform space (Y, \mathcal{V}) . Then on \mathcal{F} , $\mathcal{P} \subset$ the finite topology.*

Proof. Let $[x, U] = \{G \in \mathcal{F} : G(x) \subset U\}$, where $x \in X$ and U is open in Y . If $F \in [x, U]$, then $N = \langle X \times U \cup (X - \{x\}) \times Y \rangle$ is a nbhd. of F in the finite topology, and $F \in N \subset [x, U]$. Suppose $F \in M = \{G \in \mathcal{F} : G(x) \cap W \neq \emptyset\}$ where $x \in X$ and W is open in Y . If $F(x) \subset W$, then the above method works, and so we assume that $F(x) \not\subset W$. Let $p \in F(x) \cap W$ and let $V \in \mathcal{V}$ such that $\overline{V^2(p)} \subset W$. Since \mathcal{F} is equicontinuous at x , there is a nbhd. U of x such that for all $T \in \mathcal{F}$, $T(U) \subset V(T(x))$. Now $F \in \langle U \times [V(p)]^0, U \times (Y - \overline{V^2(p)}), U \times W, (X - \{x\}) \times Y \rangle \subset M$, which completes the proof.

LEMMA 2.6. *Let X be compact Hausdorff, and let (Y, \mathcal{V}) be a uniform space. Let \mathcal{F} be an equicontinuous family of point-compact relations from X to Y . If $\hat{\mathcal{F}}$ is the \mathcal{P} -closure of \mathcal{F} in the space of all point-compact relations from X to Y , then $\hat{\mathcal{F}}$ is closed in $\mathcal{C}(X \times Y)$, the space of all nonempty compact subsets of $X \times Y$ with the finite topology.*

Proof. Let (F_α) be a net in $\hat{\mathcal{F}}$ converging to $F \in \mathcal{C}(X \times Y)$. Note that the domain of F is X ; for if $x \in X - \text{dom } F$, then $\{G \in \mathcal{C}(X \times Y): G \subset X \times Y - \{x\} \times Y\}$ is an u.s.f. nbhd. of F in $\mathcal{C}(X \times Y)$, and F_α is eventually in this nbhd., a contradiction. Clearly F is point-compact. We now show that $(F_\alpha) \rightarrow F$ in \mathcal{P} . Let U be open in Y , and suppose $F(x) \subset U$. Then the set $N = \langle X \times U \cup (X - \{x\}) \times Y \rangle$ is a nbhd. of F in the finite topology on $\mathcal{C}(X \times Y)$. Since F_α is eventually in N , it follows that F_α is eventually in $[x, U]$. If we are given a nbhd. $M = \{G: G(x) \cap W = \emptyset\}$, (W open in Y) of F and $F(x) \not\subset W$, then we employ the technique used in the last part of the proof of Lemma 2.5, and use the fact that $\hat{\mathcal{F}}$ is equicontinuous ([5], Lemma 6).

We now prove the main result.

THEOREM 2.7. *Let \mathcal{F} be an equicontinuous family of point-compact relations from a compact Hausdorff space X to a locally compact Hausdorff uniform space Y . Let $\hat{\mathcal{F}}$ be the \mathcal{P} -closure of \mathcal{F} in the space of all point-compact relations from X to Y . Then $\hat{\mathcal{F}}$ is locally compact in \mathcal{UC} .*

Proof. We first note that on $\hat{\mathcal{F}}$, $\mathcal{P} \subset \mathcal{UC}$ ([5], Lemma 1). From Lemmas 2.3 and 2.4 it follows that the finite topology is contained in \mathcal{UC} , and since $\hat{\mathcal{F}}$ is equicontinuous, we have by Theorem 1.1, $\mathcal{P} = \mathcal{UC}$ on $\hat{\mathcal{F}}$. From Lemma 2.5, it follows that the finite topology equals \mathcal{UC} on $\hat{\mathcal{F}}$. Each member of $\hat{\mathcal{F}}$ is compact, and so $\hat{\mathcal{F}} \subset C(X \times Y)$. By Lemma 2.6, $\hat{\mathcal{F}}$ is closed in the finite topology on $C(X \times Y)$. Since $C(X \times Y)$ is locally compact ([4], Prop. 4.4.1), $\hat{\mathcal{F}}$ is locally compact.

If in the above theorem, each $F \in \mathcal{F}$ is a (single valued) function, then it is easy to verify that each member of $\hat{\mathcal{F}}$ is also a function. Hence a recent result of R. V. Fuller [2] on the local compactness of $\hat{\mathcal{F}}$ is a special case of the above theorem.

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Received October 27, 1973 and in revised form February 8, 1974. The second author's research was partially supported by an operating grant from the National Research Council of Canada.

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