# Local connectivity of some Julia sets containing a circle with an irrational rotation 

by<br>CARSTEN LUNDE PETERSEN

Roskilde University
Roskilde, Denmark

The Fatou set $F_{R}$ for a rational map $R: \overline{\mathbf{C}} \rightarrow \overline{\mathbf{C}}$ is the set of points $z \in \overline{\mathbf{C}}$ possessing a neighbourhood on which the family of iterates $\left\{R^{n}\right\}_{n \geqslant 0}$ is normal (in the sense of Montel). The Julia set $J_{R}=\overline{\mathbf{C}}-F_{R}$ is the complement of the Fatou set. (The monographs [CG], [ Be$],[\mathrm{St}]$ provide introductions to the theory of iteration of rational maps.)

Let $\theta \in] 0,1[-\mathbf{Q}$ be an irrational number and write it as a continued fraction

$$
\theta=\frac{1}{a_{1}+\frac{1}{a_{2}+\frac{1}{a_{3}+\frac{1}{a_{4}+\frac{1}{\ddots}}}}}
$$

where $a_{n} \in \mathbf{N}$ for each $n \geqslant 1$. The number $\theta$ is termed of constant type, or equivalently, is termed Diophantine of exponent 2, if the sequence $\left\{a_{n}\right\}_{n \in \mathbf{N}}$ is bounded.

For $\theta \in[0,1]$ define $\lambda_{\theta}=\exp (i 2 \pi \theta)$ and $P_{\theta}(z):=\lambda_{\theta} z+z^{2}$. Moreover, let $J_{P_{\theta}}$ denote the Julia set of $P_{\theta}$. The polynomial $P_{\theta}$ has a Siegel disc around the (indifferent) fixed point 0 , if and only if it is locally linearizable. That is, if there exists a local change of coordinates $\phi:(\mathbf{C}, 0) \rightarrow(\mathbf{C}, 0)$ with $\phi \circ P_{\theta}=\lambda_{\theta} \cdot \phi$. It is well known that $P_{\theta}$ has a Siegel disc around 0 for every $\theta$ of constant type (see e.g. [Sil).

Theorem A. For every $\theta$ of constant type the Julia set $J_{P_{\theta}}$ is locally connected and has zero Lebesgue measure.

The proof uses in an essential way a model $J_{\theta}$ of $J_{P_{\theta}}$. The model $J_{\theta}$ was constructed in 1986 and proved to be quasi-conformally equivalent to $J_{P_{\theta}}$ in 1987 (see [Do] for the


Fig. 1. The Julia set $J_{P_{\theta}}$ for $\theta=\frac{1}{2}(\sqrt{5}-1)$
particular result, and e.g. the monograph [LV] for the theory of quasi-conformal maps of the plane). Let us briefly discuss the model $J_{\theta}$, as it is essential in the proof. Consider the degree-three Blaschke function:

$$
f_{0}(z)=z^{2} \frac{z-3}{1-3 z}
$$

Its restriction $f_{0}: \mathbf{S}^{1} \rightarrow \mathbf{S}^{1}$ is an analytic circle homeomorphism with 1 as a fixed critical point, an inflection point of order three. In particular, $f_{0}$ has (Poincaré) rotation number 0 . For each irrational rotation number $\theta \in[0,1]$ there exists a unique $\varrho_{\theta} \in \mathbf{S}^{1}$ such that the restriction of $f_{\theta}:=\varrho_{\theta} \cdot f_{0}$ to $\mathbf{S}^{1}$ has rotation number $\theta$. We let $J_{f_{\theta}}$ denote the Julia set of $f_{\theta}$.


Fig. 2. The basic dynamics of $f_{\theta}$


Fig. 3. The "Julia" set $J_{\theta}$ for $\theta=\frac{1}{2}(\sqrt{5}-1)$
Each $f_{\theta}$ commutes with $\tau(z)=1 / \bar{z}$ (reflection in the unit circle). Thus all dynamical properties of $f_{\theta}$ are symmetric with respect to $\mathbf{S}^{1}$. In particular, the Julia set $J_{f_{\theta}}$ is symmetric. Moreover, the points 0 and $\infty$ are super-attractive (critical) fixed points with simply-connected immediate basins. Let $U_{0}$ be the connected component of $f_{0}^{-1}(\mathbf{D})=f_{\theta}^{-1}(\mathbf{D})$ contained in the complement of $\mathbf{D}$. The immediate basin of $\infty, \Lambda_{0}(\infty)$, is contained in $\overline{\mathbf{C}}-\left(\overline{\mathbf{D}} \cup \bar{U}_{0}\right)$.

For each irrational $\theta$ there exists a homeomorphism (unique up to postcomposition by a rigid rotation) $h_{\theta}: \mathbf{S}^{1} \rightarrow \mathbf{S}^{1}$ conjugating $f_{\theta}$ to the rigid rotation $R_{\theta}(z)=\lambda_{\theta} z$ on $\mathbf{S}^{1}$ (see [Yo1]). Let $H_{\theta}: \overline{\mathbf{D}} \rightarrow \overline{\mathbf{D}}$ denote a homeomorphism extending $h_{\theta}$. We shall suppose $H_{\theta}$ quasi-conformal if $h_{\theta}$ is quasi-symmetric (quasi-symmetric means that any two neighbouring intervals of the same length have images whose lengths are uniformly comparable).

Definition. For each irrational $\theta$ we shall define a new degree-two branched, but non-holomorphic, covering map $F_{g}: \overline{\mathbf{C}} \rightarrow \overline{\mathbf{C}}$ by

$$
F_{\theta}(z)= \begin{cases}f_{\theta}(z) & \text { if and only if }|z| \geqslant 1, \\ H_{\theta}^{-1} \circ R_{\theta} \circ H_{\theta}(z) & \text { if and only if }|z| \leqslant 1,\end{cases}
$$

and an $F_{\theta}$-invariant "Julia" set $J_{\theta}=J_{f_{\theta}}-\bigcup_{n \geqslant 0} f_{\theta}^{-n}(\mathbf{D})$. See Figure 3.
Theorem (Douady, Shishikura, Ghys, ..., 1986). If $h_{\theta}$ is quasi-symmetric, there exists a quasi-conformal homeomorphism $\phi_{\theta}: \overline{\mathbf{C}} \rightarrow \overline{\mathbf{C}}$ conjugating (the then quasi-regular map) $F_{\theta}$ to the polynomial $P_{\theta}$. The homeomorphism $\phi_{\theta}$ maps $\mathbf{D}$ onto a Siegel disc $\Delta_{\theta}$


Fig. 4. The conjugation $\phi_{\theta}$
around 0 for $P_{\theta}$, and maps $J_{\theta}$ onto the Julia set $J_{P_{\theta}}$. Furthermore, $\phi_{\theta}$ can be chosen to be conformal on the immediate basin of $\infty$ (see also Figure 4).

The above meta-theorem was given a real content a year later by M. Herman, who used inequalities, a priori real bounds, obtained by Świątec to prove the following.

Theorem (Świa̧tec-Herman, [He], 1987). An analytic circle homeomorphism with irrational rotation number $\theta$ and with one (double) critical point is quasi-symmetrically conjugate to the rigid rotation $R_{\theta}$ if and only if $\theta$ is of constant type.

Proof. See [He].
One asked if the above would help in proving Theorem A. The answer is yes and is the main concern of this paper. Note that Theorem A would follow if we knew that $J_{\theta}$ is locally connected and has Lebesgue measure 0 whenever $\theta$ is of constant type (quasi-conformal homeomorphisms map Lebesgue null sets to Lebesgue null sets). We can actually prove more than this.

Theorem B. For any $\theta \in] 0,1\left[-\mathbf{Q}\right.$ the Julia set $J_{f_{\theta}}$ and the set $J_{\theta}$ are locally connected.

Theorem C. For every $\theta$ of constant type the Lebesgue measure of $J_{\theta}$ is zero.
Theorem B gives rise to the question: Suppose that $P_{\theta}$ has a Siegel disc whose boundary is a Jordan curve containing the critical point. Does this imply that $J_{P_{\theta}}$ is locally connected?

Another interesting question is: does there exist $\theta$ for which the full Julia set $J_{f_{\theta}}$ has positive measure?

The main ingredients in proving the Herman-Świạtec Theorem are the Świa̧tec a priori real bounds (inequalities) for the ratios of closest returns of the critical point
(here 1) to itself. We shall state later the precise statement of the Świa̧tec a priori real bounds, when we have introduced the points of closest return. The main ingredient in the present proof that $J_{\theta}$ and $J_{f_{\theta}}$ are locally connected is a dynamically defined geometric construction, a "puzzle", which permits to transmit the Świątec a priori real bounds to complex bounds for the Julia sets. The puzzle is inspired by Yoccoz puzzles (see [Hu] for quadratic polynomials) and Branner-Hubbard puzzles (see $[\mathrm{BH}]$ for cubic polynomials). Knowing the "classical" puzzle constructions by Branner-Hubbard and Yoccoz, there were several mental obstacles to overcome in order to arrive at this new type of puzzle and in controlling it. One has to accept that the critical point chops up puzzle pieces giving puzzle pieces containing the critical point on the boundary. Moreover, one has to turn this phenomena into a "friend". Secondly, when estimating the size of puzzle pieces, one has to give up completely the central idea in "classical puzzles" that some annuli defined by differences of puzzle pieces map properly to each other. Thus killing the foundations of the central Grötzsch argument in proving divergence of nests. The replacement is ideas which permit to control lengths of boundaries of puzzle pieces. In implementing these ideas, we use essentially the "realness" of $f_{\theta}$, i.e. that $J_{\theta}$ contains the unit circle. The first consequence of the "realness" of $f_{\theta}$ is that we can draw arcs of finite Euclidean length in $J_{\theta}$. The second is that the Świa̧tec a priori bounds hold. These say that the closest returns of the critical point to itself essentially come geometrically. The third is that we can transform the angular contraction for inverse branches around the critical point into a hyperbolic contraction on appropriate domains.

The structure of the rest of this paper is as follows. $\S 0$ contains the red thread of the proof of spreading local connectivity from the critical point to all of the sets $J_{\theta}$ and $J_{P_{\theta}}$, together with some additional results, interesting in their own right. Moreover, it introduces the notation used in subsequent sections. $\S 1$ is essentially self-contained. It introduces the "puzzle pieces" containing the critical point on their boundary. Moreover, the results needed to prove local connectivity at the critical point are stated. $\S 2$ contains the proofs of the statements of $\S 1$ together with the necessary technical machinery to do so. It has $\S 1$ as prerequisite. $\S 3$ spreads local connectivity from 1 to all of $J_{\theta}$ and proves the theorem on zero measure. Finally $\S 4$ shows how to spread local connectivity also to all of $J_{f_{\theta}}$.

Added in revision. C. T. McMullen has proved, using the results of this paper, that the Hausdorff dimension of $J_{P_{\theta}}$ is strictly less than two whenever $\theta$ is of constant type, thus improving the measure statement of Theorem A. Moreover, he proves that the Siegel disc for $P_{\theta}$ is self-similar about the critical point, whenever $\theta$ is a quadratic irrational (such as the golden mean) (see the manuscript [Mc]). M. Lyubich has proved that $J_{\theta}$ has Lebesgue measure zero for every irrational $\theta$, thus improving Theorem C. This result
would also follow by slightly changing the proof of Theorem $C$ given in this paper. The proof by Lyubich is outlined in the preprint [Ya] by M. Yampolsky, which also outlines an alternative proof of Theorems A and B above.

Acknowledgements. The author would like to thank Institut des Hautes Études Scientifiques for its hospitality and support during the birth and writing of this paper. Especially, I would like to thank Marie-Claude Vergne for her drawings. Moreover, I would like to thank Tan Lei, Marguerite Flexor and Dierk Schleicher for helpful conversations and suggestions. Also, I would like to express my gratitude to Jean-Christophe Yoccoz for introducing me to the results of Świa̧tec and to Adrien Douady for his moral support.

## 0. Strategy of the proof of local connectivity and further results

The definitions and structures we are about to discuss depend on $\theta \in] 0,1[-\mathbf{Q}$. We shall however only use an additional index $\theta$ in our definitions when we want to stress the dependence on $\theta$. Thus the dependence on $\theta$ is always to be assumed, if not stated explicitly otherwise.

The point $\infty$ is a super-attractive fixed point for each $f_{\theta}$ and $F_{\theta}$. The corresponding immediate basin $\Lambda_{0}(\infty)$ is simply-connected. Let $\psi=\psi_{\theta}: \Lambda_{0}(\infty) \rightarrow \overline{\mathbf{C}}-\overline{\mathbf{D}}$ denote the Riemann map conjugating $f_{\theta}$ on $\Lambda_{0}(\infty)$ to $z \mapsto z^{2}$ on $\overline{\mathbf{C}}-\overline{\mathbf{D}}$. The image by $\psi$ of the line $\left\{r e^{i 2 \pi \eta} \mid r>1\right\}$ for $\eta \in[0,1]$ shall be called the $\eta$ external ray and be denoted $R_{\eta}$. The ray $R_{\eta}$ lands if and only if $\psi$ has a continuous extension along $\left\{r e^{i 2 \pi \eta} \mid r>1\right\}$ to $e^{i 2 \pi \eta}$. The impression of the $\eta$ prime end is the set of accumulation points for sequences $\left\{\psi\left(z_{n}\right)\right\}_{n \geqslant 0}$, with $z_{n}$ converging to $e^{i 2 \pi \eta}$. In particular, the impression of the $\eta$ prime end is a singleton if and only if $\psi$ extends continuously to $e^{i 2 \pi \eta}$.

Theorem 1.3. For each $\theta \in] 0,1\left[-\mathbf{Q}\right.$ the critical point $1 \in J_{f_{\theta}}$ is in the impression of precisely two prime ends of the immediate basin of $\infty$ for $f_{\theta}$. The impressions of these two prime ends equal $\{1\}$. In particular, there are precisely two external rays landing on 1 .

The proof shall be given in $\S \S 1$ and 2.
First we describe an abstract topologial model $J_{\theta}^{\text {abs }}$ for $J_{\theta}$ and a model dynamics $F_{\theta}^{\text {abs }}$ on $J_{\theta}^{\text {abs }}$. Secondly we discuss the proof of spreading local connectivity. Before however let us mention another topological model known as the pinched disc model. The pinched disc model is well described by K. Keller in [Ke]. Our work implies that for any irrational $\theta$ the corresponding pinched disc model described by Keller is homeomorphic to $J_{\theta}$. The pinched disc model is locally connected and thus not homeomorphic to $J_{P_{\theta}}$


Fig. 5. The initial 34 first-generation drops with parent either $\mathbf{S}^{1}$ or $\partial U_{0}$
when $J_{P_{\theta}}$ is not locally connected, e.g. when $P_{\theta}$ is not linearizable on any neighbourhood of 0 . The construction of $J_{\theta}^{\mathrm{abs}}$ and $F_{\theta}^{\mathrm{abs}}$ follows on the next couple of pages.

Define a subset $J_{\theta}^{\text {skeleton }}=\bigcup_{n \geqslant 0} F_{\theta}^{-n}\left(\mathbf{S}^{1}\right)=J_{\theta} \cap \bigcup_{n \geqslant 0} f_{\theta}^{-n}\left(\mathbf{S}^{1}\right)$. The set $J_{\theta}^{\text {skeleton }}$ (or even the set $J_{f_{\theta}}^{\text {skeleton }}=\bigcup_{n \geqslant 0} f_{\theta}^{-n}\left(\mathbf{S}^{1}\right)$ ) naturally decomposes into a countable union of Jordan curves, with two such curves having at most one common point. Moreover, $J_{\theta}^{\text {skeleton }}$ is dense in $J_{\theta}=J_{f_{\theta}}-\bigcup_{n \geqslant 0} f_{\theta}^{-n}(\mathbf{D})$, because $\partial\left(\bigcup_{k=0}^{n} f_{\theta}^{-k}(\mathbf{D})\right)=F_{\theta}^{-n}\left(\mathbf{S}^{1}\right)$ and $J_{f_{\theta}}^{\text {skeleton }}$ is dense in $J_{f_{\theta}}$.

Lemma 0.1. Let $n \geqslant 0$ and let $\omega$ be a connected component of $F_{\theta}^{-n}\left(\bar{U}_{0}\right)$. Then the restriction $F_{\theta}^{n}=f_{\theta}^{n}: \omega \rightarrow \bar{U}_{0}$ is a diffeomorphism.

Proof. The map $f_{\theta}$ is a branched covering map. Moreover, the set $\bar{U}_{0}$ is simplyconnected and does not intersect the forward orbits of critical points.

For $\omega$ and $n$ as in the lemma we shall say that $\omega$ is a (closed) $n$-drop or just a drop, if $n$ is understood. We shall say that the interior of $\omega$ is an (open) $n$-drop. Moreover, we define the root $z$ of $\omega$ to be the boundary point given by $\{z\}=f_{\theta}^{-n}(1) \cap \partial \omega$. Then the relation root of drop defines a bijection between $F_{\theta}^{-n}(1)$ and the set of $n$-drops, $n \geqslant 0$.

Lemma 0.2. Let $\omega$ be an $n$-drop for some $n \geqslant 0$ and let $z$ be the root of $\omega$. Then either $z \in \mathbf{S}^{1}$ or $z$ belongs to the boundary of an $n^{\prime}$-drop $\omega^{\prime}$ with $0 \leqslant n^{\prime}<n$.

Proof. Let $0 \leqslant k \leqslant n$ be minimal with the property $F_{\theta}^{k}(z) \in \mathbf{S}^{1}$. If $z \notin \mathbf{S}^{1}$ then $k>0$ and $F_{\theta}^{k-1}(z) \in \partial U_{0}-\{1\}$, because $F_{\theta}^{-1}\left(\mathbf{S}^{1}\right)=\mathbf{S}^{1} \cup \partial U_{0}$. Let $n^{\prime}=k-1<n$ and let $\omega^{\prime}$ be the closed $n^{\prime}$-drop containing $z$. Then $n^{\prime}$ and $\omega^{\prime}$ satisfies the conclusion of the lemma.

For $\omega$ as in the lemma above we say that $\mathbf{S}^{1}$ and $\omega^{\prime}$ respectively is the parent (drop) of $\omega$. More generally we shall define generations as follows: The two discs $\overline{\mathbf{D}}$ and $\bar{U}_{0}$


Fig. 6. Addresses of some drops and their roots
form generation zero. The drops of first generation are the drops $\omega$ with root $z \in \mathbf{S}^{\mathbf{1}} \cup \partial U_{0}$. A drop $\omega$ and its root are of generation $m \geqslant 2$ precisely if the root belongs to the boundary of a drop $\omega^{\prime} \neq \omega$ of the ( $m-1$ )st generation.

Let $x_{j}=f_{\theta}^{-j}(1) \cap \mathbf{S}^{1}, \forall j \in \mathbf{Z}$, and let $y_{j} \in \partial U_{0}$ be given by $f_{\theta}\left(x_{j}\right)=f_{\theta}\left(y_{j}\right)$ for $n \geqslant 1$. For $s \geqslant 1$ let $U_{s}, V_{s}$ be the open $s$-drops with roots $x_{s}$ and $y_{s}$ respectively. The first generation drops and their respective roots are precisely the drops $U_{s}, V_{s}$ and roots $x_{s}, y_{s}$, for $s \geqslant 1$. See Figure 5.

More generally we shall label the drops and roots of all generations by finite but arbitrarily long tuples with positive integers as entries. See Figure 6. A label should be thought of as an address: Suppose that $\omega$ is a drop of generation $m \geqslant 1$ with root $x$. They will be labelled by a common $m$-tuple $\left(s_{1}, \ldots, s_{m}\right) \in \mathbf{N}^{m}$, where $\left(s_{1}, \ldots, s_{m-1}\right) \in \mathbf{N}^{m-1}$ is the address of the parent and the sum $n=\sum_{i=1}^{m} s_{i}$ is the number of iterates it takes to $\operatorname{map} \omega$ onto $U_{0}$ and $x$ onto 1 . Another way to view the last entry is to apply $F_{\theta}^{s_{1}+\ldots+s_{m-1}}$ to $\omega$ and its parent, thus mapping the parent onto $U_{0}$ and $\omega$ onto $V_{s_{m}}$. It turns out to be convenient to denote drops descending directly to $\mathbf{S}^{1}$ by $U_{s_{1}, \ldots, s_{m}}$ and drops descending to $U_{0}$ by $V_{s_{1}, \ldots, s_{m}}$. Moreover, we let $x_{s_{1}, \ldots, s_{m}}$ and $y_{s_{1}, \ldots, s_{m}}$ denote the respective roots. To complete the picture we let $\varepsilon$ denote the empty sequence and define $U_{\varepsilon}=\mathbf{D}$ and $V_{\varepsilon}=U_{0}$. In this way there is a natural bijection between drops and labels. Finally let us note that $F_{\theta}^{s_{1}}\left(x_{s_{1}, \ldots, s_{m+1}}\right)=F_{\theta}^{s_{1}}\left(y_{s_{1}, \ldots, s_{m+1}}\right)=y_{s_{2}, \ldots, s_{m+1}}$.

We define limbs and sublimbs $X_{s_{1}, \ldots, s_{m}}^{\text {skeleton }}, Y_{s_{1}, \ldots, s_{m}}^{\text {skeleton }}$ of $J_{\theta}^{\text {skeleton }},\left(s_{1}, \ldots, s_{m}\right) \in \mathbf{N}^{m}, m \geqslant 0$, as the union of $U_{s_{1}, \ldots, s_{m}}$ with all its descendents, and $V_{s_{1}, \ldots, s_{m}}$ with all its descendents
respectively:

$$
\begin{aligned}
& X_{s_{1}, \ldots, s_{m}}^{\text {skeleton }}=\partial U_{s_{1}, \ldots, s_{m}} \cup \bigcup_{m^{\prime} \geqslant 1} \bigcup_{\left(t_{1}, \ldots, t_{m}\right) \in \mathbf{N}^{m^{\prime}}} \partial U_{s_{1}, \ldots, s_{m}, t_{1}, \ldots, t_{m^{\prime}}} \\
& Y_{s_{1}, \ldots, s_{m}}^{\text {skeleton }}=\partial V_{s_{1}, \ldots, s_{m}} \cup \bigcup_{m^{\prime} \geqslant 1} \bigcup_{\left(t_{1}, \ldots, t_{m}\right) \in \mathbf{N}^{m^{\prime}}} \partial V_{s_{1}, \ldots, s_{m}, t_{1}, \ldots, t_{m^{\prime}}}
\end{aligned}
$$

Then

$$
\begin{equation*}
X_{\varepsilon}^{\text {skeleton }} \cap Y_{\varepsilon}^{\text {skeleton }}=\{1\} \quad \text { and } \quad J_{\theta}^{\text {skeleton }}=X_{\varepsilon}^{\text {skeleton }} \cup Y_{\varepsilon}^{\text {skeleton }} \tag{1}
\end{equation*}
$$

Moreover, apart from this, any two limbs of $J_{\theta}^{\text {skeleton }}$ are either disjoint or contained one in the other.

The model set $J_{\theta}^{\mathrm{abs}}$ and the model dynamics $F_{\theta}^{\mathrm{abs}}: J_{\theta}^{\mathrm{abs}} \rightarrow J_{\theta}^{\mathrm{abs}}$ are defined as follows: Let $\left\{\hat{x}_{\underline{s}}, \hat{y}_{\underline{s}}\right\}, \underline{s} \in \mathbf{N}^{\mathbf{N}}$, be a family of distinct ideal points, i.e. points not already in $J_{\theta}^{\text {skeleton }}$. Define $J_{\theta}^{\text {abs }}=J_{\theta}^{\text {skeleton }} \cup \bigcup_{\underline{s} \in \mathbf{N}^{\mathbf{N}}}\left\{\hat{x}_{\underline{s}}, \hat{y}_{\underline{s}}\right\}$. Moreover, define $F_{\theta}^{\text {abs }}=F_{\theta}$ on $J_{\theta}^{\text {skeleton }}$ and for $\underline{s}=\left(s_{1}, s_{2}, \ldots, s_{n}, \ldots\right) \in \mathbf{N}^{\mathbf{N}}$,

$$
F_{\theta}^{\mathrm{abs}}\left(\hat{x}_{\underline{s}}\right)=F_{\theta}^{\mathrm{abs}}\left(\hat{y}_{\underline{s}}\right)= \begin{cases}\hat{x}_{s_{1}-1, s_{2}}, \ldots, s_{n}, \ldots & \text { if and only if } s_{1}>1 \\ \hat{y}_{s_{2}, s_{3}, \ldots, s_{n}, \ldots} & \text { if and only if } s_{1}=1\end{cases}
$$

Define abstract limbs

$$
\begin{aligned}
& X_{s_{1}, \ldots, s_{m}}^{\text {abs }}=X_{s_{1}, \ldots, s_{m}}^{\text {skeleton }} \cup \bigcup_{\underline{t} \in \mathbf{N}^{N}} \hat{x}_{s_{1}, \ldots, s_{m}, \underline{t}}, \\
& Y_{s_{1}, \ldots, s_{m}}^{\mathrm{abs}}=Y_{s_{1}, \ldots, s_{m}}^{\text {skeleton }} \cup \bigcup_{\underline{t} \in \mathbf{N}^{N}} \hat{y}_{s_{1}, \ldots, s_{m}, \underline{t}},
\end{aligned}
$$

for all $m \geqslant 0$ and for all $\left(s_{1}, \ldots, s_{m}\right) \in \mathbf{N}^{m}$. We shall say that $x_{s_{1}, \ldots, s_{m}}, y_{s_{1}, \ldots, s_{m}}$ are the roots of the respective (abstract) limbs.

We topologize the set $J_{\theta}^{\text {abs }}$ as follows: Define the nested sequences $\left\{X_{s_{1}, \ldots, s_{m}}^{\text {abs }}\right\}_{m \geqslant 1}$ and $\left\{Y_{s_{1}, \ldots, s_{m}}^{\mathrm{abs}}\right\}_{m \geqslant 1}$ to be neighbourhood bases of $\hat{x}_{\underline{s}}$ and $\hat{y}_{\underline{s}}$ respectively. In order to define a neighbourhood basis for any point in $J_{\theta}^{\text {skeleton }} \subset J_{\theta}^{\text {abs }}$ also, we first do so by defining a neighbourhood basis for any point in $\mathbf{S}^{1}-\{1\}$ and then pull these back by $F_{\theta}^{\text {abs }}$, thus making this map automatically continuous. Given $z \in \mathbf{S}^{1}-\{1\}$, take as element of a neighbourhood basis at $z$ any arc $I \in \mathbf{S}^{1}$ containing $z$ as an interior point together with all limbs $X_{s}^{\text {abs }}$ with root $x_{s} \in I, s>0$. Our proof of local connectivity of $J_{\theta}$ implies that $J_{\theta}$ is homeomorphic to $J_{\theta}^{\mathrm{abss}}$ by a homeomorphism which conjugates dynamics.

Recall Theorem 1.3 and let $R_{+}, R_{-}$be the external rays of $J_{\theta}$ (and $J_{f_{\theta}}$ ) landing on the critical point 1 . Let $\Pi_{\theta} \subset \mathbf{C}$ denote the closed subset containing $U_{0}$ and bounded by the $\operatorname{arc} R_{+} \cup\{1\} \cup R_{-}$. For $n \geqslant 0$ and $\Omega$ a connected component of $F_{\theta}^{-n}\left(\Pi_{\theta}\right)$, the restriction

$$
\begin{equation*}
F_{\theta}^{n}=f_{\theta}^{n}: \Omega \rightarrow \Pi_{\theta} \tag{2}
\end{equation*}
$$



Fig. 7. Sketch of some wakes
is a diffeomorphism. The proof is identical with that of Lemma 0.1 . For $\Omega$ and $n$ as above we say that $\Omega$ is a (closed) $n$-wake or just a wake if $n$ is understood. See Figure 7 . We shall say that the interior of $\Omega$ is an (open) $n$-wake. Any two $n$-wakes are trivially disjoint, being preimages of the same set by a covering map. If $\Omega$ is an $n$-wake and $\Omega^{\prime}$ is an $n^{\prime}$-wake with $0 \leqslant n^{\prime}<n$. Then either $\Omega^{\prime} \cap \Omega=\varnothing$ or $\Omega \subset \Omega^{\prime}$, because external rays do not cross.

An $n$-wake contains a central $n$-drop, whose root $x$ is also called the root of $\Omega$. It is the meeting point of the two external rays bounding the wake. This defines a one-to-one correspondence between roots of $n$-drops and $n$-wakes. The notions of generation and address is naturally carried over to wakes. To distinguish wakes descending to $\mathbf{S}^{1}$ and $\partial U_{0}$ we shall denote by $\Omega_{x, s_{1}, \ldots, s_{m}}$ and $\Omega_{y, s_{1}, \ldots, s_{m}}$ the wakes with central drops $U_{s_{1}, \ldots, s_{m}}$ and $V_{s_{1}, \ldots, s_{m}}$ respectively. Define $X_{s_{1}, \ldots, s_{m}}=\overline{X_{s_{1}, \ldots, s_{m}}^{\text {skeleton }}}=J_{\theta} \cap \Omega_{x, s_{1}, \ldots, s_{m}}$ and $Y_{s_{1}, \ldots, s_{m}}=\overline{Y_{s_{1}, \ldots, s_{m}}^{\text {skeleto }}}=J_{\theta} \cap \Omega_{y, s_{1}, \ldots, s_{m}}$, where the later equalities follows from $J_{\theta}^{\text {skeleton }}$ being dense in $J_{\theta}$. Each limb is mapped diffeomorphically onto $Y_{\varepsilon}$ by (2) (and $Y_{\varepsilon}$ is mapped homeomorphically onto $J_{\theta}$ by $F_{\theta}$ ).

Theorem 3.7. For each $\theta \in] 0,1[-\mathbf{Q}$ the Euclidean diameter of the principal limbs $X_{s}$ and $Y_{s}$ tends to 0 as $s \rightarrow \infty$.

The proof of this theorem shall be given in $\S 3$. We obtain immediately some corol-
laries. We shall use $X_{0}$ as a synonym for $Y_{\varepsilon}$.
Corollary 0.3. For each $\theta \in] 0,1\left[-\mathbf{Q}\right.$ there are no ghost limbs of $\mathbf{D}$ and $U_{0}$ in $J_{\theta}$. That is,

$$
J_{\theta}=\mathbf{S}^{1} \cup \bigcup_{j \geqslant 0} X_{j}=\mathbf{S}^{1} \cup \partial U_{0} \cup \bigcup_{j \geqslant 1}\left(X_{j} \cup Y_{j}\right) .
$$

Proof. It suffices to prove the first equality sign, as $Y_{\varepsilon}=X_{0}$ maps homeomorphically onto $J_{\theta}$. Any point in $z \in J_{\theta}=\overline{J_{\theta}^{\text {skeleton }}}$ is accumulated by the proper limbs. This is possible only if $z$ is already in $\mathbf{S}^{1}$ or in one of the proper limbs $X_{j}$, because the size of the limb $X_{j}$ tends to 0 as $j$ tends to $\infty$, and each limb touches $\mathbf{S}^{\mathbf{1}}$.

Theorem 0.4. Let $\theta \in] 0,1\left[-\mathbf{Q}\right.$ be arbitrary. Any point of $\mathbf{S}^{1}$, and more generally any point of $J_{\theta}^{\text {skeleton }}=\bigcup_{n \geqslant 0} F_{\theta}^{-n}\left(\mathbf{S}^{1}\right)$, has a fundamental system of open connected neighbourhoods in $J_{\theta}$.

Proof. Let us first prove the corollary for any $z \in \mathbf{S}^{1}-\left\{x_{s}\right\}_{s} \geqslant 0$. Let $\varepsilon>0$ be given. We shall find an open connected neighbourhood $\varpi$ of $z$ in $J_{\theta}$ with $\varpi \subset \mathbf{D}_{\varepsilon}(z)$, where $\mathbf{D}_{\varepsilon}(z)$ is the Euclidean disc of center $z$ and radius $\varepsilon$. Let $s_{0} \geqslant 0$ be such that the Euclidean diameters $\operatorname{diam}_{E}\left(X_{s}\right) \leqslant \frac{1}{2} \varepsilon$ for all $s \geqslant s_{0}$. Let $z_{1}, z_{2} \in \mathbf{S}^{1}-\left\{x_{s}\right\}_{s \geqslant 0} \cap \mathbf{D}_{\varepsilon / 2}(z)$ be points bounding an open subarc $\rceil z_{1}, z_{2}\left\lceil\right.$ of $\mathbf{S}^{1}-\bigcup_{j=0}^{s_{0}} x_{j}$ with $\left.z \in\right\rceil z_{1}, z_{2}\left\lceil\subset \mathbf{D}_{\varepsilon / 2}(z)\right.$. Define

$$
\varpi=7 z_{1}, z_{2}\left\lceil\cup \bigcup_{s, x_{s} \in \mid z_{1}, z_{2}\lceil } X_{s}\right.
$$

Then $\varpi$ is the required neighbourhood. The above works in particular for the critical value $v$. Thus we can construct a fundamental system of connected neighbourhoods of 1 in $J_{\theta}$ from the system around $v$, as $J_{\theta}$ is invariant $F_{\theta}$. By the same argument we prove local connectivity for the remaining points, first of $\mathbf{S}^{1}$ and secondly of $J_{\theta}^{\text {skeleton }}$.
 the repelling periodic points for $F_{\theta}$. In order to complete the proof of local connectivity of $J_{\theta}$ we need to produce a fundamental system of connected neighbourhoods for each point in $E_{\theta}$. We shall introduce some notation in order to facilitate this discussion. This notation is inspired by the "puzzle" notation of Branner and Hubbard $[\mathrm{BH}]$.

Definition 0.5. For each $\underline{s}=\left(s_{1}, s_{2}, \ldots, s_{m}, \ldots\right) \in \mathbf{N}^{\mathbf{N}}$ define $\mathcal{X}_{\underline{s}}, \mathcal{Y}_{\underline{s}}$ to be the nested sequences of compact connected sets, $\mathcal{X}_{\underline{s}}=\left\{X_{s_{1}, \ldots, s_{m}}\right\}_{m \geqslant 1}$ and $\mathcal{Y}_{\underline{s}}=\left\{Y_{s_{1}, \ldots, s_{m}}\right\}_{m \geqslant 1}$. We call each such sequence a Nest.

We have $X_{s_{1}, \ldots, s_{m}} \subset X_{s_{1}, \ldots, s_{m-1}}, Y_{s_{1}, \ldots, s_{m}} \subset Y_{s_{1}, \ldots, s_{m-1}}$ for any $m$-tuple $\left(s_{1}, \ldots, s_{m}\right)$, because it holds already for the corresponding wakes.

Definition 0.6. We define the Core of a Nest $\mathcal{Y}_{\underline{s}}$ to be the set

$$
\operatorname{Core}\left(\mathcal{Y}_{\underline{s}}\right)=\bigcap_{m \geqslant 1} Y_{s_{1}, \ldots, s_{m}} \subset E_{\theta}
$$

And likewise for Core $\left(\mathcal{X}_{\underline{s}}\right)$. The Core of a Nest is always non-empty, as it is the intersection of a nested sequence of compact non-empty sets. We shall say that the Core of the Nest $\mathcal{Y}_{\underline{s}}\left(\mathcal{X}_{\underline{s}}\right)$ is trivial if and only if it is a one-point set.

If $\operatorname{Core}\left(\mathcal{Y}_{\underline{s}}\right)=\{z\}$, for some $z \in E_{\theta}$, then $\mathcal{Y}_{\underline{s}}$ is a neighbourhood basis of compact connected neighbourhoods of $z$ in $J_{\theta}$ and likewise for Core $\left(\mathcal{X}_{\underline{s}}\right)$.

Let us recall that for any $s \geqslant 1$ the map $F_{\theta}^{s}$ maps the $\operatorname{limb} X_{s, s_{1}, \ldots, s_{m}}$ homeomorphically (even diffeomorphically) onto the limb $Y_{s_{1}, \ldots, s_{m}}$ for every $\left(s_{1}, \ldots, s_{m}\right) \in \mathbf{N}^{m}, m \geqslant 0$. Thus for the question of triviality of Cores, it suffices to consider only Nest $\mathcal{Y}_{\underline{s}}, \underline{s} \in \mathbf{N}^{\mathbf{N}}$.

Proposition 0.7. For each $\theta \in] 0,1[-\mathbf{Q}$ the following two statements are equivalent:
(1) The set $J_{\theta}$ is locally connected.
(2) For all $\underline{s} \in \mathbf{N}^{\mathbf{N}}$, $\operatorname{Core}\left(\mathcal{Y}_{\theta, \underline{s}}\right)$ is trivial.

Proof. Let $\theta \in] 0,1[-\mathrm{Q}$ be given.
$(2) \Rightarrow(1)$. It suffices to show that any $z \in E_{\theta}$ has a fundamental system of connected neighbourhoods in $J_{\theta}$, because of Theorem 0.4 . Thus (2) $\Rightarrow(1)$ follows from the two remarks preceeding this proposition.
$(1) \Rightarrow(2)$. We shall actually prove the equivalent, non-(2) implies non-(1). Suppose that $\operatorname{Core}\left(\mathcal{Y}_{\underline{s}}\right)$ is non-trivial for some $\underline{s} \in \mathbf{N}^{\mathbf{N}}$ (this case suffices by the remark preceding this proposition). For each $m \geqslant 1$ let $\Omega_{m}$ be the ( $s_{1}+\ldots+s_{m}$ )-wake with root $y_{s_{1}, \ldots, s_{m}}$ and let $\eta_{m}^{+}, \eta_{m}^{-} \in \mathbf{T}=\mathbf{R} / \mathbf{Z}$ be the arguments of the two external rays bounding $\Omega_{m}$. Moreover, let $\delta_{m} \subset \mathbf{T}$ be the interval of arguments of external rays in $\Omega_{m}$. Then an external ray of argument $\eta \in \mathbf{T}$ accumulates $Y_{s_{1}, \ldots, s_{m}}$ if and only if $\eta \in \delta_{m}$. Moreover, $2^{s_{0}+\ldots+s_{m}} \cdot l\left(\delta_{m}\right)=\frac{1}{2}$. We deduce that exactly 1 external ray accumulates $\operatorname{Core}\left(\mathcal{Y}_{\underline{s}}\right)$. On the other hand if $J_{\theta}=\partial \Lambda_{0}(\infty)$ is locally connected, then any point $z \in J_{\theta}$ is the landing point of at least one ray and any external ray lands. Thus non-(2) and (1) (logical and) lead to a contradiction. This completes the proof.

Theorem 3.25. For each $\theta \in] 0,1\left[-\mathbf{Q}\right.$ the $\operatorname{Core}\left(\mathcal{Y}_{\underline{s}}\right)$ is trivial for any $\underline{s} \in \mathbf{N}^{\mathbf{N}}$. In particular, $J_{\theta}$ is locally connected for each irrational $\theta$.

Before we open the final discussion leading to local connectivity of the Julia sets $J_{f_{\theta}}$, let us discuss a side result, which is interesting in its own right. It identifies for instance large compact hyperbolic subsets of $J_{\theta}$.

Definition 0.8. Define a map of first return from the collection of first generation sublimbs of $Y_{\varepsilon}$ onto $Y_{\varepsilon}$,

$$
\mathcal{F}_{\theta}: \bigcup_{s \geqslant 1} Y_{s} \rightarrow Y_{\varepsilon} \quad \text { by } \quad \mathcal{F}_{\theta \mid Y_{s}}=F_{\theta}^{s}=f_{\theta}^{s}
$$

The map is infinite-to-1, but for each $s \geqslant 1$ it is the restriction of a univalent map from a neighbourhood of $Y_{s}$ to a neighbourhood of $Y_{\varepsilon}$. Moreover, $\mathcal{F}_{\theta}$ leaves the set $E Y_{\theta}:=E_{\theta} \cap Y_{\varepsilon}$ invariant and carries all the essential dynamics of $F_{\theta}$ on $E Y_{\theta}$. Let $\lambda$ denote both the hyperbolic metric on $\mathbf{C}-\overline{\mathbf{D}}$ and its coefficient function.

Theorem 3.26. For all $\theta \in] 0,1\left[-\mathbf{Q}\right.$ and for all $z \in E Y_{\theta}$ we have

$$
\begin{equation*}
\left\|D_{z} \mathcal{F}_{\theta}\right\|_{\lambda}=\frac{\lambda\left(\mathcal{F}_{\theta}(z)\right)}{\lambda(z)}\left|\mathcal{F}_{\theta}^{\prime}(z)\right|>1 \quad \text { and } \quad\left\|D_{z} \mathcal{F}_{\theta}^{m}\right\|_{\lambda} \underset{m \rightarrow \infty}{\longrightarrow} \infty \tag{1}
\end{equation*}
$$

Moreover, if $\theta$ is of constant type there exists $M>1$ such that

$$
\begin{equation*}
\left\|D_{z} \mathcal{F}_{\theta}\right\|_{\lambda} \geqslant M \quad \text { for all } z \in E Y_{\theta} \tag{2}
\end{equation*}
$$

The shift $\sigma: \mathbf{N}^{\mathbf{N}} \rightarrow \mathbf{N}^{\mathbf{N}}$ is the map which forgets the first entry and shifts all other entries one to the left, that is, $\sigma\left(\left(s_{1}, s_{2}, \ldots, s_{m}, \ldots\right)\right) \mapsto\left(s_{2}, \ldots, s_{m-1}, \ldots\right)$. We define for any $\underline{s}=\left(s_{1}, \ldots, s_{m}, \ldots\right) \in \mathbf{N}^{\mathbf{N}}$,

$$
\mathcal{F}_{\theta}\left(\mathcal{Y}_{\underline{s}}\right):=\left\{\mathcal{F}_{\theta}\left(Y_{s_{1}, \ldots, s_{m}}\right)\right\}_{m \geqslant 2}=\left\{F_{\theta}^{s_{1}}\left(Y_{s_{1}, \ldots, s_{m}}\right)\right\}_{m \geqslant 2}=\mathcal{Y}_{\sigma(\underline{s})}
$$

We see immediately that

$$
\operatorname{Core}\left(\mathcal{Y}_{\sigma(\underline{s})}\right)=f_{\theta}^{s_{1}}\left(\operatorname{Core}\left(\mathcal{Y}_{\underline{s}}\right)\right)=\mathcal{F}_{\theta}\left(\operatorname{Core}\left(\mathcal{Y}_{\underline{s}}\right)\right)
$$

as $f_{\theta}^{s_{1}}$ is holomorphic. In particular, the property of having trivial Core is invariant under $\sigma$.

The map $\operatorname{dist}(\cdot, \cdot): \mathbf{N}^{\mathbf{N}} \rightarrow \mathbf{N}^{\mathbf{N}}$ given by $\operatorname{dist}(\underline{s}, \underline{t})=\sum_{m \geqslant 1} \delta\left(s_{j}, t_{j}\right) / 2^{j}$ is a metric on $\mathbf{N}^{\mathbf{N}}$, which makes the space complete but not compact. For $\left\{s_{1}<s_{2}<\ldots<s_{m}\right\} \subset \mathbf{N}$ let $\Sigma_{s_{1}, \ldots, s_{m}}=\left\{s_{1}, \ldots, s_{m}\right\}^{\mathbf{N}}$. Then $\Sigma_{s_{1}, \ldots, s_{m}}$ is a shift-invariant Cantor subset of $\mathbf{N}^{\mathbf{N}}$.

The following corollary of Proposition 0.7 shows that we can use symbolic dynamics to try to understand the dynamics of $\mathcal{F}_{\theta}$ on $E Y_{\theta}$, and thus the dynamics of $F_{\theta}$ on $E_{\theta}$.

Corollary 0.9 (of Proposition 0.7 and Theorem 3.25). Define a map $\Psi_{\theta}: \mathbf{N}^{\mathbf{N}} \rightarrow E Y_{\theta}$ $b y$

$$
\operatorname{Core}\left(\mathcal{Y}_{\underline{s}}\right)=\left\{\Psi_{\theta}(\underline{s})\right\}
$$

The map $\Psi_{\theta}$ is a homeomorphism which conjugates the shift map $\sigma: \mathbf{N}^{\mathbf{N}} \rightarrow \mathbf{N}^{\mathbf{N}}$ to the $\operatorname{map} \mathcal{F}_{\theta}: E Y_{\theta} \rightarrow E Y_{\theta}$.

Corollary 0.10. For any $\theta \in] 0,1\left[-\mathbf{Q}\right.$ and for any $\Sigma_{s_{1}, \ldots, s_{m}}$ the set $\Psi_{\theta}\left(\Sigma_{s_{1}}, \ldots, s_{m}\right)$ is an $\mathcal{F}_{\theta}$-invariant hyperbolic Cantor set on which the dynamics of $\mathcal{F}_{\theta}$ is conjugate to the one-sided shift on $m$ symbols.

Corollary 0.11. For any $\theta \in] 0,1[-\mathbf{Q}$ of constant type there exists a constant $L=L(\theta)>1$ such that $L<|\mu|$ for $\mu$ the multiplier of any repelling periodic orbit for $P_{\theta}$.

Proof. Given $\theta \in] 0,1[-\mathbf{Q}$ of constant type let $M$ be as in Theorem 3.26 (2). Moreover, let $\phi_{\theta}: \overline{\mathbf{C}} \rightarrow \overline{\mathbf{C}}$ be a quasi-conformal homeomorphism conjugating $F_{\theta}$ to $P_{\theta}$, and let $K>1$ be the constant of quasi-conformality of $\phi_{\theta}$. Then the constant $L=M^{1 / K}$ works, because of the following two remarks.
(1) Any repelling periodic orbit for $F_{\theta}$ intersects $E Y_{\theta}$.
(2) The homeomorphism $\phi_{\theta}$ preserves repelling periodic points, and moreover, if $\mu_{F}$ and $\mu_{P}$ are multipliers of corresponding repelling orbits then $d_{\lambda}\left(\mu_{F}, \mu_{P}\right) \leqslant \log K$, where $d_{\lambda}(\cdot, \cdot)$ denotes distance with respect to the hyperbolic metric on $\mathbf{C}-\overline{\mathbf{D}}$.

For $K \subset \mathbf{C}$ a compact connected subset define $\operatorname{Hull}(K)$ to be the set $K$ union the bounded connected components of $\mathbf{C}-K$.

Lemma 0.12. Let $\Omega \subset \mathbf{C}$ be any wake. Then

$$
\Omega \cap J_{\theta} \subset \Omega \cap J_{f_{\theta}} \subset \Omega-\Lambda_{0}(\infty)=\operatorname{Hull}\left(\Omega \cap J_{\theta}\right)
$$

In particular, $\operatorname{diam}_{e}\left(\Omega \cap J_{\theta}\right)=\operatorname{diam}_{e}\left(\Omega \cap J_{f_{\theta}}\right)$. Moreover, the "limb" of $J_{f_{\theta}}, \Omega \cap J_{f_{\theta}}$, is connected.

Proof. The only non-trivial verification is the equal sign: $\Omega-\Lambda_{0}(\infty)=\operatorname{Hull}\left(\Omega \cap J_{\theta}\right)$. However, this follows from $\partial \Lambda_{0}(\infty) \subseteq J_{\theta}$ and the definition of wakes.

Corollary 0.13 (of Proposition 0.7 and Theorem 3.25). Any point in $E_{\theta} \cup \tau\left(E_{\theta}\right)$ has a fundamental system of connected neighbourhoods in $J_{f_{\theta}}$, and thus so has also any point in

$$
\bigcup_{n \geqslant 0} f_{\theta}^{-n}\left(E_{\theta} \cup \tau\left(E_{\theta}\right)\right)
$$

For $s \geqslant 0$ let $\Omega_{s}$ be the $s$-wake with root $x_{s}$. Define for $s \geqslant 0$ limbs of $S^{1}$ in $J_{f_{\theta}}$ by $X_{+s}=J_{f_{\theta}} \cap \Omega_{s}$ and $X_{-s}=\tau\left(X_{+s}\right)$ (the indices should be read plus $s$ and minus $s$ ).

Corollary 0.14 (of Theorem 3.7). The Euclidean diameters of the limbs $X_{+s}$ and $X_{-s}$ tend to 0 as $n \rightarrow \infty$. Moreover, $\mathbf{S}^{1}$ has no ghost limbs, i.e. $J_{f_{\theta}}=\mathbf{S}^{1} \cup \bigcup_{s \geqslant 0}\left(X_{+s} \cup X_{-s}\right)$, and any point of $\bigcup_{n \geqslant 0} f_{\theta}^{-n}\left(\mathbf{S}^{1}\right)$ has a fundamental system of connected neighbourhoods in $J_{f_{\theta}}$.

Let $Z_{\theta}=J_{f_{\theta}}-\bigcup_{n \geqslant 0} f^{-n}\left(J_{\theta} \cup \tau\left(J_{\theta}\right)\right)$. Rename $U_{0}$ to $U_{+}$and define $U_{-}=\tau\left(U_{+}\right)$. Another caracterization of $Z_{\theta}$ is that it is the set consisting of those points $z \in J_{f_{\theta}}$ whose forward orbit passes infinitely often through alternately $U_{0}$ and $\tau\left(U_{0}\right)$.

The two corollaries above prove that $J_{f_{\theta}}$ is locally connected at any of its points except those in $Z_{\theta}$. As a last theorem on local connectivity we present

Theorem 4.1. For all $\theta \in] 0,1\left[-\mathbf{Q}\right.$ any point of $Z_{\theta}$ has a fundamental system of connected neighbourhoods in $J_{f_{\theta}}$, and thus $J_{f_{\theta}}$ is locally connected.

## 1. Local connectivity at the critical point 1

A family of Jordan curves. Recall that $F_{\theta}=f_{\theta}$ on $\overline{\mathbf{C}}-\mathbf{D}$. They shall thus be used synonymously on this domain. Let $\beta_{\theta}$ be the unique repelling fixed point for $F_{\theta}$ in $\mathbf{C}-\overline{\mathbf{D}}$. Recall that $x_{j}=F_{\theta}^{-j}(1) \cap \mathbf{S}^{1}$ for each $j \in \mathbf{Z}$, that $y_{j} \in \partial U_{0}$ is given by $F_{\theta}\left(x_{j}\right)=F_{\theta}\left(y_{j}\right)$ for each $j \geqslant 1$, and moreover that $J_{\theta}^{\text {skeleton }}=\bigcup_{n \geqslant 0} F_{\theta}^{-n}\left(\mathbf{S}^{1}\right)$.

For $z_{1}, z_{2}$ both in $\mathbf{S}^{1}$ the symbols $\left\lceil z_{1}, z_{2}\right\rceil$ and $\rceil z_{1}, z_{2}\lceil$ denote the shorter, closed and open subarc respectively of $\mathbf{S}^{1}$ bounded by $z_{1}$ and $z_{2}$, if not stated explicitly otherwise. We shall furthermore use the same notation for subarcs of $\partial U_{0}$.

We shall construct a family of Jordan curves with nice properties. Each Jordan curve $\Gamma$ in the family shall possess the following five fundamental properties:
(1) $\Gamma \cap J_{\theta}=\Gamma \cap J_{f_{\theta}}$ is a connected subset of $J_{\theta}^{\text {skeleton }} \cup \bigcup_{n \geqslant 0} F_{\theta}^{-n}\left(\beta_{\theta}\right)$.
(2) $\Gamma=\left(\Gamma \cap J_{\theta}\right) \cup\left(\Gamma \cap \Lambda_{0}(\infty)\right) \subset \mathbf{C}-\left(\mathbf{D} \cup U_{0}\right)$.
(3) $\Gamma \cap S^{1}$ and $\Gamma \cap \partial U_{0}$ are non-trivial arcs of the form $\left\lceil 1, x_{m}\right\rceil$ and $\left\lceil 1, y_{l}\right\rceil$ respectively for some $m, l \geqslant 1$.
(4) $l_{e}(\Gamma)<\infty$, where $l_{e}(\cdot)$ denotes the Euclidean curve length.
(5) $\operatorname{Ind}_{\Gamma}(0)=0$.
(See also Figure 8 and the subsection "An initial curve".)
Theorem 1.1. There exists a family of Jordan curves, $\left\{\Gamma_{k}\right\}_{k \geqslant 0}$, such that each curve has the five fundamental properties stated above and, moreover,

$$
l_{e}\left(\Gamma_{k}\right) \rightarrow 0 \quad \text { as } k \rightarrow \infty .
$$

For any Jordan curve $\gamma \in \mathbf{C}$ let $D(\gamma)$ denote the closure of the bounded connected component of $\mathbf{C}-\gamma$.

Corollary 1.2. For each $\theta \in] 0,1[-\mathbf{Q}$ there exists a fundamental system of connected neighbourhoods of 1 in both $J_{\theta}$ and $J_{f_{\theta}}$.

Proof. We construct, using the family $\left\{\Gamma_{k}\right\}_{k \geqslant 0}$, a neighbourhood basis of connected neighbourhoods of 1 in $J_{\theta}$ and in $J_{f_{\theta}}$ as follows. For each $k$ let $\Xi_{k}$ be the union of


Fig. 8. The first three curves of the family $\left\{\Gamma_{k}\right\}_{k \geqslant 0}$
$f_{\theta}\left(D\left(\Gamma_{k}\right)\right)$ with its reflection in $\mathbf{S}^{1}$. Then $\Xi_{k} \cap J_{\theta}$ and $\Xi_{k} \cap J_{f_{\theta}}$ are connected neighbourhoods of $v=f_{\theta}(1)$ in $J_{\theta}$ and $J_{f_{\theta}}$ respectively, because of properties (1) and (3) and because both $J_{\theta}$ and $J_{f_{\theta}}$ are connected. The diameter of $\Xi_{k}$ tends to 0 as $k \rightarrow \infty$, because $f_{\theta}$ is continuous and $\operatorname{diam}\left(D\left(\Gamma_{k}\right)\right) \rightarrow 0$. Hence the sequences $\left\{\Xi_{k} \cap J_{\theta}\right\}_{k \geqslant 0}$ and $\left\{\Xi_{k} \cap J_{f_{\theta}}\right\}_{k \geqslant 0}$ form neighbourhood bases of $v$ in $J_{\theta}$ and $J_{f_{\theta}}$ respectively. Consequently the sequences of preimages $\left\{f_{\theta}^{-1}\left(\Xi_{k} \cap J_{\theta}\right)\right\}_{k \geqslant 0}$ and $\left\{f_{\theta}^{-1}\left(\Xi_{k} \cap J_{f_{\theta}}\right)\right\}_{k \geqslant 0}$ form neighbourhood bases with connected neighbourhoods of 1 in $J_{\theta}$ and $J_{f_{\theta}}$ respectively.

We obtain as an immediate corollary
ThEOREM 1.3. For each $\theta \in] 0,1\left[-\mathbf{Q}\right.$ the critical point $1 \in J_{f_{\theta}}$ is in the impression of precisely two prime ends of the immediate basin of $\infty$ for $f_{\theta}$. The impressions of these two prime ends equal $\{1\}$. In particular, there are precisely two external rays landing on 1 .

An initial curve. Denote by $v$ the critical value $x_{-1}=f_{\theta}(1) \in \mathbf{S}^{1}$ and recall that $y_{1}$ is the preimage of 1 in $\partial U_{0}$. We shall suppose that $0<\theta<\frac{1}{2}$, so that also $0<t(\theta)<\frac{1}{2}$. Then $v$ is in the upper half-plane and $x_{1}$ is in the lower half-plane. The other cases, $\frac{1}{2}<\theta<1$, can be obtained using, for instance, the symmetry under conjugation by complex conjugation.

Let $\varkappa_{0}$ be the closed subarc of $\partial U_{0}$ mapping homeomorphically to the subarc $\lceil 1, v\rceil \subset S^{1}$ in the upper half-plane and let $\gamma_{0}$ be the closure of the complementary subarc of $\partial U_{0}$. The arcs $\varkappa_{0}, \gamma_{0}$ are thought of as starting at 1 and ending at $y_{1}$. Define $\varkappa_{n}$ and $\gamma_{n}$ inductively as the arcs which start at the common endpoint of $\varkappa_{n-1}$ and $\gamma_{n-1}$ and


Fig. 9. The arcs $\gamma$ and $\varkappa$ with some of their constituents
which map homeomorphically to $\varkappa_{n-1}$ and $\gamma_{n-1}$ respectively by $f_{\theta}$ (see Figure 9 ). Let $\gamma$ denote the arc $\gamma_{0} \gamma_{1} \ldots \gamma_{n} \ldots$, i.e. $\gamma_{0}$ followed by $\gamma_{1}$ etc., and let $\varkappa$ denote the arc which is the preimage of $\varkappa_{0} \varkappa_{1} \ldots \varkappa_{n} \ldots$ starting at $x_{1}=F_{\theta}^{-1}(1)$.

Theorem (Sullivan, Douady, Hubbard, Yin). Let $R$ be a rational map and let $C_{R}$ denote the closure of the post-critical set union possible rotation domains for $R$. Suppose that $\gamma:]-\infty, 0] \rightarrow \overline{\mathbf{C}}-C_{R}$ is a curve with $R^{n}(\gamma(t))=\gamma(t+1)$ for all $t \leqslant-1$. Then $\lim _{t \rightarrow-\infty} \gamma(t)$ exists and is a repelling or parabolic $n$-periodic point $\beta$ for $R$. Moreover, if $\beta$ is parabolic then its multiplier is an $n$-th root of unity.

Proof. See [TY].
We make the arcs $\gamma$ and $\varkappa$ closed by adding the points $\beta_{\theta}$ and $\beta_{\theta}^{\prime}$ respectively, where $\beta_{\theta}^{\prime}$ at the end of $\varkappa$ is a preimage of the repelling fixed point $\beta_{\theta}$. Join the two arcs by the lower subarc of $\mathbf{S}^{1}$ between the two root points 1 and $x_{1}$. Also join the two arcs by following $\gamma$ by the segment of the external ray of external argument 0 from $\beta_{\theta}$ to equipotential level 1, say. Next follow the equipotential curve at level 1 in the clockwise direction to the external ray of external argument $\frac{1}{2}$. Finally follow the later external ray into the endpoint $\beta_{\theta}^{\prime}$ of $\varkappa$. We call the Jordan curve just constructed $\Gamma_{0}$. Evidently


Fig. 10. Iterating $\Gamma$ backwards until it hits the critical value $v$
$\Gamma_{0} \cap J_{\theta}=\Gamma_{0} \cap J_{f_{\theta}}$ is connected.
Lemma 1.4. The arc $\Gamma_{0}$ has the five fundamental properties (1) through (5).
Proof. The only non-trivial verification is (4). However, this follows from the fact that the point $\beta_{\theta}$ is repelling.

A binary tree $\mathcal{T}_{\theta}$ of Jordan curves. Let $\left.\theta \in\right] 0,1[-\mathbf{Q}$. We shall construct a binary tree $\mathcal{T}_{\theta}$ of Jordan curves $\Gamma$ possessing the five fundamental properties (1) to (5) above. The root of the tree $\mathcal{T}_{\theta}$ is the curve $\Gamma_{0}$ constructed above. The two children of any $\Gamma \in \mathcal{T}_{\theta}$ shall be lifts of $\Gamma$ to some appropriate iterate of $f_{\theta}$. The motivation for creating the tree $\mathcal{T}_{\theta}$ is that we shall find the sequence $\left\{\Gamma_{k}\right\}_{k \geqslant 0}$ of Theorem 1.1 as a descending path in $\mathcal{T}_{\theta}$.

Moving from one Jordan curve to the next. Let $\Gamma$ be a Jordan curve satisfying the five fundamental properties (1) through (5) above and with $I:=\Gamma \cap \mathbf{S}^{1}=\left\lceil 1, x_{m}\right\rceil$. We move from $\Gamma$ to anyone of its two children $\Gamma^{\prime} \in \mathcal{T}_{\theta}$ as follows. If $I$ does not contain the critical value $v$, then there is a unique inverse branch of $f_{\theta}$ defined on $D(\Gamma)$ and mapping $I$ to some subarc of $\mathbf{S}^{1}$. If also the inverse image of $I$ does not contain $v$ we may continue to find a unique branch of $f_{\theta}^{-2}$ on $D(\Gamma)$ mapping $I$ to some subarc of $\mathbf{S}^{1}$, and so on. We may continue this until we have obtained a branch $h$ of $f_{\theta}^{-(j-1)}$ for some $j \geqslant 1$ defined on $D(\Gamma)$ and mapping $I$ to some subarc of $\mathbf{S}^{1}$ containing $v$ in the interior ( $\theta$ is irrational).

Here we have to make a choice. The preimage of $h(\Gamma)$ by $F_{\theta}$ can be viewed as two Jordan curves with 1 as a common point. Each of the two choices for $\Gamma^{\prime}$ satisfies the fundamental properties (1) through (5) above, because $\Gamma$ does so and they are lifts of $\Gamma$ to $F_{\theta}^{j}$ (and to $f_{\theta}^{j}$ ). See Figure 10.

Let $g$ denote the composition of the final choice of inverse branch of $F_{\theta}$ with $h$. We will call $g$ a move. The map $g: D(\Gamma) \rightarrow D\left(\Gamma^{\prime}\right)$ is a homeomorphism with $f_{\theta}^{j} \circ g=\mathrm{Id}$


Fig. 11. The colouring of the Jordan curves $\Gamma$
on $D(\Gamma)$. It is easily checked that $I^{\prime}=\Gamma^{\prime} \cap \mathbf{S}^{1}=\left\lceil 1, x_{m^{\prime}}\right\rceil$ and $O^{\prime}=\Gamma^{\prime} \cap \partial U_{0}=\left\lceil 1, y l^{\prime}\right\rceil$ with $\left\{m^{\prime}, l^{\prime}\right\}=\{j, m+j\}$. The long composition $h$ of inverse branches of $f_{\theta}$ is univalent on a domain containing $D(\Gamma)$ in its interior and $g$ is a local diffeomorphism at each point of $D(\Gamma)$ except at the point $x_{-j} \in I=\mathbf{S}^{1} \cap \Gamma$, which is mapped to 1 . Even though $g$ is defined on all of $D(\Gamma)$ we shall often just write $g: \Gamma \rightarrow \Gamma^{\prime}$. We note also that $x_{-j}$ is the first return of 1 into $I$.

One of the two choices for $\Gamma^{\prime}$ has $I^{\prime}$ above 1 , the other choice has $I^{\prime}$ below 1. This leads us to distinguish the following two types of moves $g: \Gamma \rightarrow \Gamma^{\prime}$.
(1) The move $g$ is called a Gain if $I$ and $I^{\prime}$ are on the same side of 1, i.e. either both above or both below 1 .
(2) The move $g$ is called a $S w a p$ if $I$ and $I^{\prime}$ are on different sides of 1 .

The binary tree $\mathcal{T}_{\theta}$ of Jordan curves is constructed inductively with $\Gamma_{0}$ as root and using the above two moves. Moreover, for $k \geqslant 1$ we let $\mathcal{T}_{\theta, k}$ denote the union of the $2^{k}$ subtrees of $\mathcal{T}_{\theta}$ for which the root points are the curves $k$ moves down from $\Gamma_{0}$.


Fig. 12. The dynamics of colours under the two kinds of moves
Colouring the Jordan curves $\Gamma$. Recall that $U_{j}$ and $V_{j}$ are the unique connected components of $F_{\theta}^{-j}\left(U_{0}\right)$ (open $j$-drops) with roots $x_{j}$ and $y_{j}$ respectively, $j \geqslant 1$.

Any curve $\Gamma \in \mathcal{T}_{\theta}$ naturally falls into five subarcs, two of which, $I=\Gamma \cap \mathbf{S}^{1}=\left\lceil 1, x_{m}\right\rceil$ and $O=\Gamma \cap \partial U_{0}=\left\lceil 1, y_{l}\right\rceil$, have already been introduced. Let $B=\Gamma \cap \partial U_{m}, R=\Gamma \cap \partial V_{l}$, and let $G$ be the closure of the complementary subarc of $\Gamma$ left out by the others. When making drawings the reader is invited to colour the different subarcs of $\Gamma, B$ (lue), $G$ (reen), $\mathrm{R}(\mathrm{ed})$ and O (range) and invent a colour for $I$. The careful reader will have observed that the arc $G$ actually consists of three parts, one part at either end contained in the Julia set and the middle part contained entirely in the basin of infinity: To emphasize the colouring we shall also at times write $\Gamma(I, B, G, R, O)$ for $\Gamma$ (see also Figure 11).

Moving the colours. Let $g: \Gamma(I, B, G, R, O) \rightarrow \Gamma^{\prime}\left(I^{\prime}, B^{\prime}, G^{\prime}, R^{\prime}, O^{\prime}\right)$ be a move between Jordan arcs satisfying (1) through (3). Then always $g(I)=I^{\prime} \cup O^{\prime}$ and $g(R \cup G)=G^{\prime}$; whereas $g(O)=B^{\prime}, g(B)=R^{\prime}$ if $g$ is a Swap; and $g(O)=R^{\prime}, g(B)=B^{\prime}$ if $g$ is a Gain. See Figure 12.

A good subtree. As mentioned above the family $\left\{\Gamma_{k}\right\}_{k \geqslant 0}$ in Theorem 1.1 shall be found as a descending path in $\mathcal{T}_{\theta}$. To facilitate the choice of a good path we shall consider especially certain branches of $\mathcal{T}_{\theta}$. More precisely we shall consider such paths in $\mathcal{T}_{\theta}$ for which the sequence of moves does not contain consecutive Gains. We may illustrate this by the flowchart Figure 13.

We call a sequence of moves admissible if it complies with the flowchart. We let $\mathcal{G}_{\theta}^{*}$ denote the subtree of $\mathcal{T}_{\theta}$ consisting of those Jordan curves $\Gamma$ obtained from $\Gamma_{0}$ by an admissible sequence of moves. Furthermore, we let $\mathcal{G}_{\theta, k}^{*}=\mathcal{G}_{\theta}^{*} \cap \mathcal{T}_{\theta, k}$ for $k \geqslant 0$.

We shall at the end of this section (Bounding $G$ (reen)) describe how to choose the sequence $\left\{\Gamma_{k}\right\}_{k \geqslant 0}$ which satisfies the statement of Theorem 1.1. We are however already in position to control all but the length of the $G$ part of each $\Gamma \in \mathcal{G}_{\theta}^{*}$. Let $\lambda$ denote the hyperbolic metric on $\mathbf{C}-\overline{\mathbf{D}}$.


Fig. 13. Flowchart for defining $\mathcal{G}_{\theta}^{*}$
Proposition 1.5. For each $\theta \in] 0,1\left[-\mathbf{Q}\right.$ there exist constants $K_{O, \theta}, K_{B, \theta}, K_{R, \theta}>0$ and $L_{R, \theta}>0$ such that any Jordan curve $\Gamma(I, B, G, R, O) \in \mathcal{G}_{\theta}^{*}$ satisfies:
(1) $l_{e}(O) \leqslant K_{O, \theta} \cdot l_{e}(I)$,
(2) $l_{e}(B) \leqslant K_{B, \theta} \cdot l_{e}(I)$,
(3) $l_{e}(R) \leqslant K_{R, \boldsymbol{\theta}} \cdot l_{e}(I)$,
(4) $l_{\lambda}(R) \leqslant L_{R, \theta}$.

Here $l_{e}(\cdot)$ denotes Euclidean length and $l_{\lambda}(\cdot)$ denotes length with respect to the hyperbolic metric $\lambda$ on $\mathbf{C}-\overline{\mathbf{D}}$. We shall postpone the proof of Proposition 1.5 to the next chapter. That chapter is devoted to proving a universal version of this proposition. The proof essentially consists in obtaining complex bounds from the Siagtec a priori real bounds.

We shall study the endpoints $x_{m}$ of $I$ and $y_{l}$ of $O$. This leads us to discuss first and closest return.

Moments and points of closest return. Let $\theta \in] 0,1[-\mathbf{Q}$. The $n$th convergent of $\theta$ is the rational number $p_{n} / q_{n}$ obtained by truncating the continued fraction expansion of $\theta$ at level $n-1$, i.e.

$$
\frac{p_{n}}{q_{n}}=\frac{1}{a_{1}+\frac{1}{a_{2}+\frac{1}{\ddots+\frac{1}{a_{n-1}}}} .}
$$

Defining $q_{0}=p_{1}=0$ and $q_{1}=p_{0}=1$ gives the recurrence formulas $p_{n+1}=a_{n} p_{n}+p_{n-1}$ and
$q_{n+1}=a_{n} q_{n}+q_{n-1}$. We are however not interested in the $p_{n}$.
The integers $q_{n}$ are called moments of closest return and the integers $\alpha q_{n}+q_{n-1}$, $0<\alpha \leqslant a_{n}$, are called moments of first return for orbits of $f_{\theta}: \mathbf{S}^{1} \rightarrow \mathbf{S}^{1}$ and $f_{\theta}^{-1}: \mathbf{S}^{1} \rightarrow \mathbf{S}^{1}$ ( $\alpha=0, a_{n}$ are also closest returns). The corresponding points $x_{ \pm q_{n}}$ and $x_{\mp\left(\alpha q_{n}+q_{n-1}\right)}$ are called points of closest and first return respectively of 1 to itself under $f_{\theta}^{ \pm 1}$. Note that as usual in this paper the backward iterates of 1 in $\mathbf{S}^{1}$ have positive indices, whereas the forward iterates have negative indices.

Let $\tilde{v}$ denote the logarithm of the critical value $v$ in $] 0, i 2 \pi\left[\right.$. For $j \in \mathbf{Z}-\{-1\}$ let $\tilde{x}_{j}$ be the logarithm of $x_{j}$ in $] \tilde{v}-2 \pi i, \tilde{v}[$.

Lemma 1.6. Let $<$ denote the natural ordering on $i \mathbf{R}$. Suppose that $n$ is even and $0<\theta<\frac{1}{2}$ (the case $n$ odd or $\frac{1}{2}<\theta<1$ is analogous, but with all inequalities reversed). Then, if $a_{n} \neq 1$ :

$$
\begin{aligned}
\tilde{x}_{-\left(q_{n}-q_{n-1}\right)} & <\tilde{x}_{q_{n-1}}<\tilde{x}_{q_{n}+q_{n-1}}<\ldots<\tilde{x}_{q_{n+1}-q_{n}}<\tilde{x}_{-q_{n}} \\
& <\tilde{x}_{a_{n} q_{n}+q_{n-1}}=\tilde{x}_{q_{n+1}}<0<\tilde{x}_{q_{n}}<\tilde{x}_{-q_{n-1}}
\end{aligned}
$$

and if $a_{n}=1$ :

$$
\tilde{x}_{-\left(q_{n}-q_{n-1}\right)}<\tilde{x}_{q_{n-1}}<\tilde{x}_{-q_{n}}<\tilde{x}_{q_{n}+q_{n-1}}=\tilde{x}_{q_{n+1}}<0<\tilde{x}_{q_{n}}<\tilde{x}_{-q_{n-1}} .
$$

Proof. For the rigid rotations, $R_{\theta}(z)=z \cdot e^{i 2 \pi \theta}$, the above is a standard result, thus it follows from the Poincaré semiconjugation theorem for circle homeomorphisms.

Lemma 1.7. The largest subarc of $\mathbf{S}^{1}$ around 1 which is mapped diffeomorphically into $\mathbf{S}^{1}$ by $f_{\theta}^{-\left(q_{n}-1\right)}$ is the arc bounded by $x_{-\left(q_{n}-q_{n-1}\right)}$ and $x_{-q_{n-1}}$.

Proof. Follows from the previous lemma, because $q_{n}=a_{n-1} q_{n-1}+q_{n-2}$ and the first return of 1 under $f_{\theta}$ into the subarc of the lemma is the point $x_{-q_{n}}$.

Tracing the endpoints of $I$ and $O$. In the sequel we shall focus on the combinatorics of the end points of the subarcs $I$ and $O$ of the Jordan curve $\Gamma(I, B, G, R, O)$. For this reason it will be convenient to introduce $\Gamma\left(x_{m}, y_{l}\right)$ as a synonym for $\Gamma$, where $x_{m}$ and $y_{l}$ are given by $I=\left\lceil 1, x_{m}\right\rceil$ and $O=\left\lceil 1, y_{l}\right\rceil$. (Note that the points $x_{m}$ and $y_{l}$ alone do not specify the curve $\Gamma\left(x_{m}, y_{l}\right)$ uniquely.)

Lemma 1.8. For each $\Gamma\left(x_{m}, y_{l}\right) \in \mathcal{T}_{\theta}$ there exist $n \geqslant 1$ and $0 \leqslant \alpha \leqslant a_{n}$ such that

$$
\{m, l\}=\left\{q_{n}, \alpha q_{n}+q_{n-1}\right\} \quad(\text { equal as sets }) .
$$

Note that $a_{n} q_{n}+q_{n-1}=0 q_{n+2}+q_{n+1}$, and hence the numbers $n$ and $\alpha$ are not unique when $\alpha=0$ or $\alpha=a_{n}$.

Complement to Lemma 1.8. Let $g: \Gamma\left(x_{m}, y_{l}\right) \rightarrow \Gamma^{\prime}\left(x_{m^{\prime}}, y_{l^{\prime}}\right)$ be a move and suppose that $m=\alpha q_{n}+q_{n-1}, n \geqslant 1$ and $0 \leqslant \alpha<a_{n}$. Then $f_{\theta}^{q_{n}} \circ g=\mathrm{Id}$. Moreover,
(1) $\left(m^{\prime}, l^{\prime}\right)=\left((\alpha+1) q_{n}+q_{n-1}, q_{n}\right)$ if $g$ is a Gain,
(2) $\left(m^{\prime}, l^{\prime}\right)=\left(q_{n},(\alpha+1) q_{n}+q_{n-1}\right)$ if $g$ is a Swap.
(Note that here we have equality of ordered pairs.)
Proof. We prove the lemma by induction on the number of moves it takes to produce $\Gamma\left(x_{m}, y_{l}\right)$ from $\Gamma_{0}\left(x_{1}, y_{1}\right)$. In doing so we shall simultaneously prove the complement. As $\Gamma_{0}\left(x_{m}, y_{l}\right)$ has $m=l=1=q_{1}=q_{1}+q_{0}$ the induction basis is okay. Assume next that any Jordan curve $\Gamma \in \mathcal{T}_{\theta}$ which is at most $k \geqslant 0$ moves down from $\Gamma_{0}$ satisfies the statement of the lemma, and let $\Gamma\left(x_{m}, y_{l}\right) \in \mathcal{T}_{\theta}$ be any such curve. Write $m=\alpha q_{n}+q_{n-1}$ with $n \geqslant 1,0 \leqslant \alpha<a_{n}$, and let $g: \Gamma\left(x_{m}, y_{l}\right) \rightarrow \Gamma^{\prime}\left(x_{m^{\prime}}, y_{l^{\prime}}\right)$ be any of the two moves from $\Gamma$. Then $f_{\theta}^{q_{n}} \circ g=\mathrm{Id}$ by Lemma 1.6 and the definition of moves. It follows that $\left\{m^{\prime}, l^{\prime}\right\}=\left\{q_{n},(\alpha+1) q_{n}+q_{n-1}\right\}$. This proves the lemma and moreover the complement, because $g\left(x_{m}\right)=x_{m^{\prime}}$ if and only if $g$ is a Gain.

Lemma 1.9. Suppose that $\Gamma, \Gamma^{\prime} \in \mathcal{T}_{\theta}$ and that $g: \Gamma\left(x_{m}, y_{l}\right) \rightarrow \Gamma^{\prime}\left(x_{m^{\prime}}, y_{l^{\prime}}\right)$ is a move. Then there exists $n \geqslant 2$ such that $f_{\theta}^{q_{n}} \circ g=\mathrm{Id}$, and moreover,

$$
\begin{aligned}
m & =\alpha q_{n}+q_{n-1}, \quad 0 \leqslant \alpha<a_{n} \\
l & =\beta q_{n-1}+q_{n-2}, \quad 0 \leqslant \beta \leqslant a_{n-1} .
\end{aligned}
$$

Proof. Let $n \geqslant 2$ and $0 \leqslant \alpha<a_{n}$ be given by $m=\alpha q_{n}+q_{n-1}$. It follows from Lemma 1.8 and its complement that $f_{\theta}^{q_{n}} \circ g=$ Id. If $0<\alpha$, then $l=q_{n}=a_{n-1} q_{n-1}+q_{n-2}$ by Lemma 1.8. And if $0=\alpha$, then $m=q_{n-1}$, and Lemma 1.8 implies $l=\beta q_{n-1}+q_{n-2}$, with $0 \leqslant \beta \leqslant a_{n-1}$. Thus in either case the lemma follows.

Proposition 1.10. Let $\Gamma \in \mathcal{T}_{\theta}$ be arbitrary and let $\Gamma_{1}$ and $\Gamma_{2}$ be the arcs obtained by the two moves from $\Gamma$. Then

$$
D\left(f_{\theta}\left(\Gamma_{1}\right)\right)=D\left(f_{\theta}\left(\Gamma_{2}\right)\right) \subset D\left(f_{\theta}(\Gamma)\right)
$$

Proof. The first equality sign follows from the definition of moves. For each $\Gamma \in \mathcal{T}_{\theta}$ let $h_{\Gamma}: D(\Gamma) \rightarrow \mathbf{C}$ be the long composition of inverse branches of $f_{\theta}$ with $v \in h_{\Gamma}(I) \subset \mathbf{S}^{1}$, where $I$ equals $\mathbf{S}^{1} \cap \Gamma$. We shall prove the following equivalent formulation of the proposition: $\forall \Gamma \in \mathcal{T}_{\theta}$,

$$
\begin{equation*}
h_{\Gamma}(\Gamma) \subset D\left(f_{\theta}(\Gamma)\right) \tag{1}
\end{equation*}
$$

Moreover, (1) is equivalent to $J_{\theta} \cap h_{\Gamma}(\Gamma)=h_{\Gamma}\left(\Gamma \cap J_{\theta}\right) \subset D\left(f_{\theta}(\Gamma)\right)$, because external rays do not cross and $f_{\theta}$ maps the equipotential curve at level $p>0$ in $\Lambda_{0}(\infty)$ to the equi-
potential curve at level $2 p$. We divide the curves $\Gamma \in \mathcal{T}_{\theta}$ into two complementary classes. The first class consists of all curves $\Gamma$ of the form $\Gamma=\Gamma\left(x_{q_{n}}, y_{l}\right)$ for some $n \geqslant 1$ and some $l=\beta q_{n}+q_{n-1}, 0 \leqslant \beta \leqslant a_{n}$. The second class consists of all curves $\Gamma$ of the form $\Gamma=\Gamma\left(x_{m}, y_{q_{n}}\right)$ for some $n \geqslant 1$ and some $m=\alpha q_{n}+q_{n-1}, 0<\alpha<a_{n}$ (recall Lemma 1.8).

Suppose first that $\Gamma=\Gamma(I, B, G, R, O)$ belongs to the first class. If $0 \leqslant \beta<a_{n}$, then

$$
h_{\Gamma}(I)=f_{\theta}\left(\left\lceil x_{q_{n+1}+q_{n}}, x_{q_{n+1}}\right\rceil\right) \subset f_{\theta}( \rceil x_{q_{n}}, x_{\beta q_{n}+q_{n-1}}\lceil ) \subset f_{\theta}(I \cup O)
$$

The fundamental curve properties (1) through (3) then implies that $h_{\Gamma}\left(\Gamma \cap J_{\theta}\right) \subset D\left(f_{\theta}(\Gamma)\right)$, and thus (1) holds.

For $\beta=a_{n}$ we have to look a little bit further.
Claim. For any $\Gamma\left(x_{m}, y_{l}\right)=\Gamma(I, B, G, R, O) \in \mathcal{T}_{\theta}$, with $m=\alpha q_{n}+q_{n-1}$, for some $n \geqslant 1,0 \leqslant \alpha<a_{n}$, we have $\left\lceil x_{m}, x_{m, q_{n-1}}\right\rceil \subset B$, where

$$
\left\lceil x_{m}, x_{m, q_{n-1}}\right\rceil=f_{\theta}^{-m}\left(\left\lceil 1, y_{q_{n-1}}\right\rceil\right) \cap \partial U_{m}
$$

Proof of the claim. Note at first that it suffices to observe that the claim holds for $\Gamma_{0}$ and to prove that the claim holds for those $\Gamma$ obtained by a Swap from their predecessor, because a Gain preserves $B$ (see the subsection "Moving the colours"). Thus we can suppose that $m=q_{n-1}$ and that the last move $g^{\prime}: \Gamma^{\prime}\left(x_{m^{\prime}}, y_{l^{\prime}}\right) \rightarrow \Gamma\left(x_{q_{n-1}}, y_{l}\right)$ is a Swap. Then $l^{\prime}=\beta q_{n-2}+q_{n-3}$ for some $0 \leqslant \beta \leqslant a_{n-2}$. The claim then follows because $g^{\prime}$ maps $O^{\prime}$ to $B$ and $\left\lceil 1, y_{q_{n-1}}\right\rceil \subseteq O^{\prime}$ by Lemma 1.9.

Suppose that $\Gamma$ is in the first class with $\beta=a_{n}$, so that $\Gamma=\Gamma\left(x_{q_{n}}, y_{q_{n+1}}\right)$. Let the last move in obtaining $\Gamma$ be $g^{\prime}: \Gamma^{\prime}\left(x_{m^{\prime}}, y_{l^{\prime}}\right) \rightarrow \Gamma\left(x_{q_{n}}, y_{q_{n+1}}\right)$. It follows from Lemma 1.8 with its complement that $g^{\prime}$ is a Swap and $m^{\prime}=\left(a_{n}-1\right) q_{n}+q_{n-1}=q_{n+1}-q_{n}$. Let

$$
B^{\prime}=\left\lceil x_{q_{n+1}-q_{n}}, x_{q_{n+1}-q_{n}, k}\right\rceil=\Gamma^{\prime} \cap \partial U_{q_{n+1}-q_{n}}
$$

Then

$$
\left\lceil x_{q_{n+1}-q_{n}}, x_{q_{n+1}-q_{n}, q_{n-1}}\right\rceil \subset B^{\prime} \quad \text { and } \quad\left\lceil y_{q_{n+1}}, y_{q_{n+1}, q_{n-1}}\right\rceil \subset\left\lceil y_{q_{n+1}}, y_{q_{n+1}, k}\right\rceil=g^{\prime}\left(B^{\prime}\right)=R
$$

by the claim, and hence

$$
\left.\left.h_{\Gamma}(I)=f_{\theta}\left(\left\lceil x_{q_{n+1}+q_{n}}, x_{q_{n+1}}\right\rceil\right) \subset f_{\theta}( \rceil x_{q_{n}}, x_{q_{n+1}}\right\rceil\right) \subset f_{\theta}(I \cup O)
$$

and

$$
h_{\Gamma}(O)=f_{\theta}\left(\left\lceil y_{q_{n+1}}, y_{q_{n+1}, q_{n+1}}\right\rceil\right) \subset f_{\theta}\left(\left\lceily_{q_{n+1}}, y_{q_{n+1}, k}\lceil )=f_{\theta}(R) .\right.\right.
$$



Fig. 14. The "worst" case $\beta=a_{n}$
The fundamental curve properties (1) through (3) then imply that $h_{\Gamma}\left(\Gamma \cap J_{\theta}\right) \subset D\left(f_{\theta}(\Gamma)\right)$ (see also Figure 14), and thus (1) holds.

This proves that (1) holds for any $\Gamma$ in the first class. Suppose next that $\Gamma$ belongs to the second class, i.e. $\Gamma\left(x_{\alpha q_{n}+q_{n-1}}, y_{q_{n}}\right)$ for some $0<\alpha<a_{n}$. It follows from the complement to Lemma 1.8 that there exists a $\Gamma^{\prime}=\Gamma^{\prime}\left(x_{q_{n-1}}, y_{l}\right) \in \mathcal{T}_{\theta}$ such that $\Gamma$ is obtained from $\Gamma^{\prime}$ by $\alpha$ consecutive Gains. Moreover, each Gain is a local inverse of $f_{\theta}^{q_{n}}$ mapping 1 to $y_{q_{n}}$. Let $g^{\prime}: \Gamma^{\prime}\left(x_{q_{n-1}}, y_{l^{\prime}}\right) \rightarrow \Gamma^{\prime \prime}\left(x_{q_{n}+q_{n-1}}, y_{q_{n}}\right)$ be the Gain of $\Gamma^{\prime}$. Then $h_{\Gamma^{\prime}}=f_{\theta} \circ g^{\prime}$ and $D\left(g^{\prime}\left(\Gamma^{\prime}\right)\right) \subset D\left(\Gamma^{\prime}\right)$, because $\Gamma^{\prime}$ belongs to the first class and hence satisfies (1) by the above. But then the Gain of $\Gamma^{\prime \prime}$ coincides with the restriction of $g^{\prime}$ to $D\left(\Gamma^{\prime \prime}\right)$. It then follows by induction that $D(\Gamma) \subset D\left(\Gamma^{\prime}\right)$ and that the Gain $g$ of $\Gamma$ coincides with the restriction of $g^{\prime}$ to $D(\Gamma)$, because $\alpha<a_{n}$ implies that $g$ is also an inverse branch of $f_{\theta}^{q_{n}}$ mapping 1 to $y_{q_{n}}$. But then $\Gamma$ satisfies (1). This completes the proof.

Bounding $G($ reen $)$. We shall bound $G$ and complete the proof of Theorem 1.1 (assuming Proposition 1.5) before we end this section. Let $W_{1}=F_{\theta}^{-1}(\mathbf{C}-\overline{\mathbf{D}}) \subset \mathbf{C}-\overline{\mathbf{D}}$ so that $f_{\theta}=F_{\theta}: W_{1} \rightarrow \mathbf{C}-\overline{\mathbf{D}}$ is a degree-two covering. In particular, it is infinitesimally expanding with respect to the hyperbolic metric $\lambda$ on $\mathbf{C}-\overline{\mathbf{D}}$. Fix $\theta \in] 0,1[-\mathbf{Q}$ and let $h_{0}: \Gamma_{0} \rightarrow \mathbf{C}$ be the long composition of inverse branches of $f_{\theta}$ in the definition of moves from $\Gamma_{0}$. Recall that $h_{0}$ extends to a diffeomorphism from a neighbourhood of $D\left(\Gamma_{0}\right)$ onto a neighbourhood of $D\left(h_{0}\left(\Gamma_{0}\right)\right)$. Consider the half-line $l$ from 0 to $\infty$ through the critical value $v$. Let $\alpha$ be the first intersection outside $\overline{\mathbf{D}}$ of $l$ with $h_{0}\left(\Gamma_{0}\right)$. Let $[v, \alpha]$ denote the closed line segment from $v$ to $\alpha$. The segment $[v, \alpha]$ cuts $D\left(h_{0}\left(\Gamma_{0}\right)\right)$ into two


Fig. 15. The strongly contracting inverse branches
closed pieces (we include $[v, \alpha]$ in both pieces and orient $[v, \alpha]$ outwards). Let $\widehat{\omega}_{\theta,+}$ be the piece containing the points in $h_{0}\left(D\left(\Gamma_{0}\right)\right)$ immediately to the right of $[v, \alpha]$ and let $\widehat{\omega}_{\theta,-}$ be the piece containing the points in $D\left(h_{0}\left(\Gamma_{0}\right)\right)$ immediately to the left of $[v, \alpha]$. Finally let $\omega_{\theta, \pm}=\widehat{\omega}_{\theta, \pm}-\mathbf{S}^{1}$. Let $\Omega_{\theta,+}, \Omega_{\theta,-}$ be the connected components of $f_{\theta}^{-1}\left(\omega_{\theta,+}\right)$ and $f_{\theta}^{-1}\left(\omega_{\theta,-}\right)$ respectively, having a non-trivial boundary arc in common with $U_{0}$. See Figure 15. Let $f_{\theta_{+}+}^{-1}, f_{\theta_{1}-}^{-1}: D\left(h_{0}\left(\Gamma_{0}\right)\right) \rightarrow \mathbf{C}$ denote the inverse branches of $f_{\theta}$ with $f_{\theta_{,+}}^{-1}\left(\omega_{\theta,+}\right)=\Omega_{\theta,+}$ and $f_{\theta,-}^{-1}\left(\omega_{\theta,-}\right)=\Omega_{\theta,-}$ respectively.

Lemma 1.11. Let $\theta \in] 0,1\left[-\mathbf{Q}\right.$. On $\Omega_{\theta, \pm}$ the map $f_{\theta}$ is strongly infinitesimally expanding with respect to the hyperbolic metric $\lambda$ on $\mathbf{C}-\overline{\mathbf{D}}$. That is, there exists $0<2 \varepsilon(\theta)<1$ such that

$$
\left\|D_{z} f_{\theta}\right\|_{\lambda}=\frac{\lambda\left(f_{\theta}(z)\right)}{\lambda(z)}\left|f_{\theta}^{\prime}(z)\right| \geqslant \frac{1}{1-2 \varepsilon} \quad \forall z \in \Omega_{\theta, \pm},
$$

and consequently the corresponding local inverse branches are strongly contracting,

$$
\begin{equation*}
\left\|D_{z} f_{\theta, \pm}^{-1}\right\|_{\lambda} \leqslant(1-2 \varepsilon) \quad \forall z \in \omega_{\theta, \pm} . \tag{1}
\end{equation*}
$$

Moreover, $\liminf \left\|D_{z} f_{\theta}\right\|_{\lambda}=\frac{3}{2}$, where inf is over $\mathbf{D}_{r}(1) \cap \Omega_{\theta, \pm}$ and $\lim$ is for $r \rightarrow 0$.
Proof. On $W_{1}$ the map $f_{\theta}$ is expanding with respect to $\lambda$ and moreover $\left\|D_{z} f_{\theta}\right\|_{\lambda} \rightarrow \frac{3}{2}$, when $z \rightarrow 1$ in $f_{\theta}^{-1}([v, \alpha]) \cap W_{1}$.

Let $\Gamma_{1}$ and $\Gamma_{2}$ be the curves obtained by one and two Swaps respectively from $\Gamma_{0}$. (This choice is not essential but convenient.) Let $\Gamma(I, B, G, R, O)$ be any curve in the subtree of $\tau_{\theta}$ with root $\Gamma_{2}$. Let $h$ be the long composition of inverse branches of $f_{\theta}$ on
$\Gamma$ with $v \in h(I)$. It is easy to check that $h(I) \subset h_{2}\left(I_{2}\right) \subset\left(h_{0}\left(I_{0}\right)\right)^{\circ}$. It then follows that $D(h(\Gamma)) \subset D\left(h_{0}\left(\Gamma_{0}\right)\right)$. In particular we have $h(R \cup G) \subset \omega_{\theta,+} \cup \omega_{\theta,-}$. Alternatively we can appeal to Proposition 1.10, whose proof however requires a little more work.

We are now ready to choose the sequence $\left\{\Gamma_{k}\right\}_{k \geqslant 0} \subset \mathcal{G}_{\theta}^{*}$ for Theorem 1.1. The first three curves, $\Gamma_{0}, \Gamma_{1}$ and $\Gamma_{2}$, have already been chosen above. We shall choose the sequence as a descending path in $\mathcal{G}_{\theta}^{*}$. Thus we need only specify how the decision between a Swap and a Gain is taken at the top of the flowchart (Figure 13). Suppose that $\Gamma_{k}\left(I_{k}, B_{k}, G_{k}, R_{k}, O_{k}\right), k \geqslant 2$, has already been chosen. Let $h_{k}$ be the long composition of univalent inverse branches of $f_{\theta}$ defined on some neighbourhood of $D\left(\Gamma_{k}\right)$ and with $v \in h_{k}\left(I_{k}\right)$. Now at least half of the $\lambda$-length of $h_{k}\left(R_{k} \cup G_{k}\right)$ is in either $\omega_{\theta,+}$ or $\omega_{\theta, \ldots}$. If in $\omega_{\theta,+}$ we choose $f_{\theta,+}^{-1}$ as final inverse branch of $f_{\theta}$ on $D\left(h_{k}\left(\Gamma_{k}\right)\right)$, and if in $\omega_{\theta,-}$ we choose the other branch $f_{\theta,-}^{-1}$. If the obtained move is a Swap, then we have chosen the Swap, and if it is a Gain we have chosen the Gain. If both $\omega_{\theta,+}$ and $\omega_{\theta,-}$ contain at least half the $\lambda$-length, we choose the Swap.

Definition 1.12. Define $\left\{\Gamma_{k}\right\}_{k \geqslant 0}$ to be the descending sequence in $\mathcal{G}_{\theta}^{*}$ chosen above.
Lemma 1.13. There exist constants $L_{G, \theta}, K_{G, \theta}>0$ such that

$$
l_{\lambda}\left(G_{k}\right) \leqslant L_{G, \theta} \quad \text { and } \quad l_{e}\left(G_{k}\right) \leqslant K_{G, \theta} \cdot l_{e}\left(I_{k}\right)
$$

for all $k \geqslant 0$.
Proof. We have $g_{k}\left(R_{k} \cup G_{k}\right)=G_{k+1}$ and thus $l_{\lambda}\left(G_{k+1}\right) \leqslant l_{\lambda}\left(G_{k}\right)+l_{\lambda}\left(R_{k}\right)$ for all $k$, as $f_{\theta}$ is expanding with respect to $\lambda$. Moreover, $l_{\lambda}\left(G_{0}\right)<\infty$ and by Proposition 1.5 (4) there exists a constant $L_{R, \theta}$ such that $l_{\lambda}\left(R_{k}\right) \leqslant L_{R, \theta}$ for all $k$. By construction and Lemma 1.11 (1) we thus have

$$
\begin{aligned}
l_{\lambda}\left(G_{k+1}\right) & \leqslant \frac{1}{2}\left(l_{\lambda}\left(G_{k}\right)+l_{\lambda}\left(R_{k}\right)\right)+(1-2 \varepsilon) \frac{1}{2}\left(l_{\lambda}\left(G_{k}\right)+l_{\lambda}\left(R_{k}\right)\right) \\
& \leqslant(1-\varepsilon)\left(l_{\lambda}\left(G_{k}\right)+L_{R, \theta}\right)
\end{aligned}
$$

for at least every second $k$. Let $L^{\prime}=2 L_{\theta, R} / \varepsilon$. If $l_{\lambda}\left(G_{k}\right) \geqslant L^{\prime}$, then

$$
l_{\lambda}\left(G_{k+2}\right) \leqslant(1-\varepsilon)\left(l_{\lambda}\left(G_{k}\right)+L_{R, \theta}\right)+L_{R, \theta} \leqslant l_{\lambda}\left(G_{k}\right)-\varepsilon \cdot L_{R, \theta} .
$$

Thus $\lim \sup l_{\lambda}\left(G_{k}\right) \leqslant L^{\prime}+L_{R}$. This proves the existence of an upper bound $L_{G, \theta}$ for $l_{\lambda}\left(G_{k}\right)$.

To prove the existence of $K_{G, \theta}$ note that $G_{k}$ and $B_{k}$ have a common endpoint and that $B_{k}$ touches $\mathbf{S}^{1}$. Moreover, $l_{e}(B) \leqslant K_{B, \theta} \cdot l_{e}(I)$ by Proposition $1.5(2)$. The weight function $\lambda(z)$ of the hyperbolic metric on $\mathbf{C}-\overline{\mathbf{D}}$ is asymptotic to $1 /(|z|-1)$ when $z$ approaches $\mathbf{S}^{1}$. Hence we also get the existence of $K_{G, \theta}$.

Proof of Theorem 1.1. Let $\left\{\Gamma_{k}\right\}_{k \geqslant 0} \in \mathcal{G}_{\theta}^{*}$ be the sequence defined in Definition 1.12. Set $K_{\Gamma, \theta}=1+K_{B, \theta}+K_{G, \theta}+K_{R, \theta}+K_{O, \theta}$. Then $l_{e}\left(\Gamma_{k}\right) \leqslant K_{\Gamma, \theta} \cdot l_{e}\left(I_{k}\right)$. It follows from the definition of $\mathcal{G}_{\theta}^{*}$ and the fact that $f_{\theta}$ is conjugate to the rigid rotation $R_{\theta}$ on $\mathbf{S}^{1}$, that $l_{e}\left(I_{k}\right) \rightarrow 0$ when $k \rightarrow \infty$. This proves Theorem 1.1. Appealing to the Świagtec a priori real bounds (see $\S 2$ ) we even obtain exponential convergence to zero.

## 2. Complex bounds from real bounds

The Świgtec a priori real bounds. The following theorem is often referred to as the Świa̧tec a priori real bounds:

Theorem (Świạtec, Herman). There exists a constant $0<a<1$ such that for all $\theta \in] 0,1\left[-\mathbf{Q}\right.$ the points of closest return under $f_{\theta}^{-1}: \mathbf{S}^{1} \rightarrow \mathbf{S}^{1}, x_{q_{n}}, n \geqslant 1$, satisfy

$$
a \leqslant \frac{\left|x_{q_{n+1}}-1\right|}{\left|x_{q_{n}}-1\right|} \leqslant \frac{1}{a} .
$$

Proof. [Sw], [Yo2, §3, proposition, p. 6].
An initial version of this result in the case of rational $\theta$ and for $n$ up to $\theta=p_{n} / q_{n}$ appeared in [Sw]. M. Herman [He] observed that Świątec's inequalities hold for all $n$, when $\theta$ is irrational. He then used the inequalitites to prove the Herman-Świątec conjugation theorem. The Światec a priori real bounds are actually better than stated above. The constant $a$ depends only on $f_{\theta}^{\prime}$ and hence only on $f_{0}$. More precisely, it depends on macroscopic properties of $f_{\theta}$, such as the order of the critical point and the total variation of $\log \left|f_{\theta}^{\prime}\right|$ on $\mathbf{S}^{1}-J$, where $J$ is an interval around the critical point $c$, such that $f_{\theta}$ has negative Schwarzian derivative on $J-\{c\}$. For more precise statements see the manuscript [Yo2].

Terminology 2.1. Let $\Gamma\left(x_{m}, y_{l}\right) \in \mathcal{T}_{\theta}$. We say that $\Gamma$ has a Fresh $B(l u e)$ if $\Gamma=\Gamma_{0}$ or if the last move in obtaining $\Gamma$ was a Swap. If the last move in obtaining $\Gamma$ was a Gain, we distinguish two cases. First we note that $(m, l)=\left(\alpha q_{n}+q_{n-1}, q_{n}\right), 0<\alpha \leqslant a_{n}$, for some $n$, by the complement to Lemma 1.8. Moreover, an easy induction argument on that same complement shows that $\Gamma$ is obtained from a $\Gamma^{\prime}\left(x_{q_{n}}, y_{l}\right)$ by $\alpha$ consecutive Gains. If $\Gamma^{\prime}$ has a Fresh $B$, then $\Gamma$ is said to have a Young $B$ (lue). If not, then $\Gamma$ is said to have an old $B$ (lue).

We shall use a similar notion for $R(e d)$. Let $\Gamma\left(x_{m}, y_{l}\right) \in \mathcal{T}_{\theta}$. We say that $\Gamma$ has a Fresh $R(e d)$ if $\Gamma=\Gamma_{0}$ or if the last move in obtaining $\Gamma$ was a Gain. If the last move in obtaining $\Gamma$ was a Swap, then we distinguish three cases. It follows from Lemma 1.8 and its complement that $(m, l)=\left(q_{n},(\alpha+1) q_{n}+q_{n-1}\right)$ for some $n$ and $0 \leqslant \alpha<a_{n}$, and that the


Fig. 16. A branch of $\mathcal{G}_{\theta}$ with $\Gamma$ 's up to old $B(l u e)$ and old $R(e d)(G=$ Gain, $S=$ Swap $)$
predecessor $\Gamma^{\prime}\left(x_{m^{\prime}}, y_{l^{\prime}}\right)$ of $\Gamma$ has $m=\alpha q_{n}+q_{n-1}$. We say that $\Gamma$ has a Young $R(e d)$ if $\Gamma^{\prime}$ has a Fresh $B$ or if $\Gamma^{\prime}$ has a Young $B$ and $0<\alpha$. Moreover, we say that $\Gamma$ has a Juvenile $R(e d)$ if $\Gamma^{\prime}$ has a Young $B$, but $\alpha=0$. If none of the above, then $\Gamma$ is said to have an old $R(e d)$ (see also Figure 16).

The philosophy or motivation for the above terminology is that the $O$ part of a curve $\Gamma \in \mathcal{T}_{\theta}$ is always newborn (and well controlled). A Fresh $B$ or $R$ is only one move away from the $O$ stage (recall Figure 12). Moreover, a Fresh $B$ belongs to a $\Gamma\left(x_{q_{n}}, y_{l}\right)$ for some $n$. An arc $B$ or $R$ is considered Young, if it is at most $a_{n}$ moves away from the Fresh $B$ stage. Finally the borderline Juvenile $R$ is $a_{n}+1$ moves away from the Fresh $B$ stage. In Propositions 2.10 and 2.11 as well as in the proof of Theorem 2.2 we shall see how we can carry over the initial control of $O$ to its close descendents.

Define $\mathcal{G}_{\theta}$ to be the subtree of $\mathcal{T}_{\theta}$ consisting of the set of $\Gamma$ for which the descending path from $\Gamma_{0}$ to $\Gamma$ does not pass any $\Gamma^{\prime}$ with an old $B$. We note that trivially $\mathcal{G}_{\theta}^{*} \subseteq \mathcal{G}_{\theta}$. Moreover, we note that any $\Gamma \in \mathcal{G}_{\theta}$ has either a Fresh, a Young or a Juvenile $R$. Define $\mathcal{G}_{\theta, k}=\mathcal{G}_{\theta} \cap \mathcal{T}_{\theta, k}$.

THEOREM 2.2. There exist universal constants $K_{O}, K_{B}, K_{R}>0$ and $L_{R}>0$, i.e. not depending on $\theta \in] 0,1[-\mathbf{Q}$, such that

$$
\begin{align*}
& \limsup _{\mathcal{T}_{\theta, k}} \frac{l_{e}(O)}{l_{e}(I)} \leqslant K_{O},  \tag{1}\\
& \limsup _{\mathcal{G}_{\theta, k}} \frac{l_{e}(B)}{l_{e}(I)} \leqslant K_{B},  \tag{2}\\
& \limsup \frac{l_{e}(R)}{\mathcal{G}_{\theta, k}(I)} \leqslant K_{R},  \tag{3}\\
& \underset{\mathcal{G}_{\theta, k}}{\limsup _{\lambda}} l_{\lambda}(R) \leqslant L_{R} . \tag{4}
\end{align*}
$$

We note that Proposition 1.5 is an immediate corollary of the above Theorem 2.2, because $\mathcal{G}_{\theta}^{*} \subset \mathcal{G}_{\theta}$. In the language of Sullivan [Su], the $O, B$ and $R$ of a $\Gamma \in \mathcal{G}_{\theta}$ are "beau",
i.e. bounded (here in terms of $I$ ), and after a finite number of moves the bounds are universal. We shall express this by saying that the bounds are asymptotically universal. We remark also that it now easily follows that the bounds $L_{G, \theta}$ and $K_{G, \theta}$ are asymptotically universal. This chapter is devoted to proving the above theorem. We shall immediately prove the first statement.

Proof of Theorem $2.2(1)$. Let $\Gamma\left(x_{m}, y_{l}\right) \in \mathcal{T}_{\theta, 1}$ so that $\{m, l\}=\left\{q_{n}, \alpha q_{n}+q_{n-1}\right\}$ for some $n \geqslant 1$ and $0 \leqslant \alpha \leqslant a_{n}$, by Lemma 1.8. Thus $\left\lceil 0, q_{n}\right\rceil \subset I$ and $O \subset\left\lceil 0, q_{n-1}\right\rceil$ for some $n \geqslant 1$. As $f_{\theta}$ has a double critical point at 1 we obtain

$$
\frac{l_{e}(O)}{l_{e}(I)} \leqslant \frac{l_{e}\left(\left\lceil 1, y_{q_{n-1}}\right\rceil\right)}{l_{e}\left(\left\lceil 1, x_{q_{n}}\right\rceil\right)} \quad \text { and } \quad \limsup _{\Gamma \in \tau_{\theta, k}} \frac{l_{e}(O)}{l_{e}(I)} \leqslant \frac{1}{a}
$$

where the latter comes from the Sisiatec a priori real bounds.
Some hyperbolic geometry. Besides Terminology 2.3 the rest of this subsection are elementary facts about the hyperbolic geometry of some particular sets. The reader making a first reading of the paper is recommended to read the terminology and continue reading in the following subsection, "The complex bounds", thus using this subsection as a reference when needed.

Terminology 2.3. For $K \subsetneq \mathbf{S}^{1}$ an open subarc define

$$
A_{K}=\overline{\mathbf{C}}-\left(\mathbf{S}^{1}-K\right) \quad \text { and } \quad A_{K}^{*}=A_{K}-\{0, \infty\}
$$

Let $\lambda_{K}$ denote both the hyperbolic metric and the coefficient function of the hyperbolic metric on $A_{K}^{*}$. Moreover, let $d_{K}(\cdot, \cdot)$ and $l_{K}(\cdot)$ denote the $\lambda_{K}$-distance and curve length functions respectively.

For $J \subsetneq \mathbf{S}^{1}$ an open subarc let $K=f_{\theta}(J)$. Define $W_{J}^{*}=f_{\theta}^{-1}\left(A_{K}^{*}\right) \subsetneq A_{J}^{*}$ so that the restriction $f_{\theta}: W_{J}^{*} \rightarrow A_{K}^{*}$ is a branched covering map of degree three, with 1 as only possible branch point. Let $\varrho_{J}$ denote both the hyperbolic metric and the coefficient function of the hyperbolic metric on $W_{J}^{*}$.

Define $\mathbf{H}_{ \pm}=\{z \mid \operatorname{Re}(z) \gtrless 0\}$. For $a \neq b \in \mathbf{C}$ let $[a, b]$ and $] a, b[$ denote the closed and open line segments from $a$ to $b$ respectively. For $\widetilde{K} \subsetneq i \mathbf{R}$ an open interval let $\mathbf{C}_{\tilde{K}}$ denote the set $\mathbf{H}_{-} \cup \mathbf{H}_{+} \cup \widetilde{K}$ and let $\delta_{\tilde{K}}$ denote the hyperbolic metric on $\mathbf{C}_{\tilde{K}}$. We shall often refer to $\mathbf{C}_{\tilde{K}}$ as a doubly slit plane (with gab $\widetilde{K}$ ) when $\widetilde{K}$ is relatively compact in $i \mathbf{R}$.

Lemma 2.4. Suppose that $V \subset U \subset \mathbf{C}$ are hyperbolic subsets of $\mathbf{C}$, i.e. each carries a hyperbolic metric. Let $\lambda_{V}$ and $\lambda_{U}$ denote the coefficient functions of the respective hyperbolic metrics and let $d_{U}(\cdot, \cdot)$ denote the hyperbolic distance in $U$. Then

$$
\tanh \left(\frac{1}{2} d_{U}(z, \partial V)\right) \leqslant \frac{\lambda_{U}(z)}{\lambda_{V}(z)}<1 \quad \forall z \in V
$$



Fig. 17. The arc $\gamma_{K}$ and its two lifts to $f_{\theta}$ outside $\mathbf{D}$
Proof. Let $z \in V$ be arbitrary. Lifting to a universal cover of $U$ if necessary, we can suppose that $U=\mathbf{D}$ and $z=0$. Then the largest Euclidean disc centered at 0 and entirely contained in $V$ has Euclidean radius $\tanh \left(\frac{1}{2} d_{U}(z, \partial V)\right)$, so the lemma follows.

Given $K \subsetneq \mathbf{S}^{1}$ an open subarc let $e^{i \eta}$ denote the midpoint of $K$. Let $\gamma_{K}:[0,1] \rightarrow A_{K}^{*}$ denote the arc $\gamma_{K}(t)=\exp \left(i \pi\left(1-e^{i \pi t}\right)+i \eta\right)$. Then $\gamma_{K}$ is a closed arc beginning and ending at $e^{i \eta} \in K$ and with index 1 around 0 . Let $[\gamma]_{K}$ be the homotopy class of $\gamma_{K}$ in $A_{K}^{*}$ through curves $\gamma^{\prime}$ with endpoints in $K$. See Figure 17. Define

$$
E_{K}:=\inf _{\gamma^{\prime} \in[\gamma]_{K}} l_{K}\left(\gamma^{\prime}\right) \quad \text { and } \quad F_{K}:=d_{\sigma_{K}}(K, \infty)
$$

where $\sigma_{K}$ denotes the hyperbolic metric on $A_{K}$ (see also Figure 17).
There exist (continuous) decreasing functions $E$ and $F$ with $E_{K}=E\left(l_{E}(K)\right)$ and $F_{K}=F\left(l_{e}(K)\right)$, because multiplication by a constant $\eta$ of norm 1 is a hyperbolic isometry between $A_{K}, A_{K}^{*}$ and $A_{\eta K}, A_{\eta K}^{*}$ respectively. Our sole interest here is however that $E(l) \rightarrow \infty$ as $l \rightarrow 0$. The latter can be seen as follows: An elementary calculation shows that $F(l)=\log \left(\cot \left(\frac{1}{8} l\right)\right)$. Moreover, $E(l)>2 F(l)$ for all $0<l \leqslant 2 \pi$ as $A_{K}^{*} \subset A_{K}$. We shall use the function $E$ through the following lemma.

Lemma 2.5. Let $K, J \subset \mathbf{S}^{1}$ and $J^{\prime} \subset \partial U_{0}$ be open arcs such that $f_{\theta}$ maps $J$ and $J^{\prime}$ diffeomorphically to $K$. Then

$$
d_{\varrho_{J}}\left(\mathbf{S}^{1}, J^{\prime}\right)=d_{\varrho_{J}}\left(J, J^{\prime}\right)=E\left(l_{e}(K)\right)
$$

In particular, the distance $d_{\varrho_{J}}\left(\mathbf{S}^{1}, J^{\prime}\right)$ depends only on $l_{e}(K)$ and tends to $\infty$ as $l_{e}(K) \rightarrow 0$.
Proof. The first equality is because $J=\mathbf{S}^{1} \cap W_{J}^{*}$. The restriction $f_{\theta}: W_{J}^{*} \rightarrow A_{K}^{*}$ is a covering map and an isometry with respect to the hyperbolic metrics $\varrho_{J}$ and $\lambda_{K}$, because $1 \notin J$. Moreover, any curve in $[\gamma]_{K}$ has two lifts to $f_{\theta}$ joining $J$ and $J^{\prime}$ in $W_{J}^{*}$, and any simple curve which joins $J$ and $J^{\prime}$ in $W_{J}^{*}$ maps by $f_{\theta}$ to a curve in the equivalence class $[\gamma]_{K}$ (see also Figure 17).

Lemma 2.6. For $K \subsetneq \mathbf{S}^{1}$ an open arc and $t>0$ define

$$
\Omega_{K, t}=\left\{\gamma \subset A_{K}^{*} \mid \gamma \text { a curve with } \gamma \cap K \neq \varnothing \text { and } l_{K}(\gamma) \leqslant t\right\} .
$$

The number $\sup \left\{l_{e}(\gamma) \mid \gamma \in \Omega_{K, t}\right\}$ depends only on $l=l_{e}(K)$. We denote it $S_{l, t}$. It satisfies

$$
\begin{equation*}
\lim _{l \rightarrow 0} \frac{S_{l, t}}{l}=\frac{1}{2} \sinh t \tag{1}
\end{equation*}
$$

Proof. We shall first prove an analogous statement for doubly slit planes.
SUBLEMMA. Let $\widetilde{K} \subsetneq i \mathbf{R}$ be an open relatively compact interval and let $\delta_{\widetilde{K}}$ denote the hyperbolic metric on $\mathbf{C}_{\tilde{K}}$, the doubly slit plane with gab $\widetilde{K}$. Any curve $\widetilde{\gamma} \in \mathbf{C}_{\tilde{K}}$ with $\widetilde{\gamma} \cap \widetilde{K} \neq \varnothing$ satisfies

$$
\begin{equation*}
l_{e}(\widetilde{\gamma}) \leqslant \frac{1}{2} \cdot l_{e}(\widetilde{K}) \cdot \sinh \left(l_{\delta_{\widetilde{K}}}(\widetilde{\gamma})\right) \tag{2}
\end{equation*}
$$

with equality if and only if $\widetilde{\gamma}$ is a horizontal line segment emanating from the midpoint of $\widetilde{K}$.

Proof of the sublemma. It suffices to consider $\left.\widetilde{K}_{2}=\right]-i, i[$, as the affine map $z \mapsto s+r z$ is both an Euclidean congruence and a hyperbolic isometry between $\mathbf{C}_{2}=\mathbf{C}_{\tilde{K}_{2}}$ and $\mathbf{C}_{s+r \tilde{K}_{2}}$ for all $r>0$ and $s \in i \mathbf{R}$. We let $\delta_{2}$ denote the hyperbolic metric on $\mathbf{C}_{2}$.

Let $\pi: \mathbf{H}_{+} \rightarrow \mathbf{C}_{2}$ be the univalent map $\pi(z)=\frac{1}{2}(z-1 / z)$. Then $\pi\left(\left[1, e^{t}\right]\right)=[0, \sinh t]$, which shows that the horizontal linesegment $\widetilde{\gamma}_{t}=[0, \sinh t]$ has $\delta_{2}$-length $t$. This proves the optimality of (2). We shall prove that for any $x \in \mathbf{R}$,

$$
\begin{equation*}
\delta_{2}(x) \leqslant \delta_{2}(z) \text { for all } z \text { with } d_{\delta_{2}}\left(z, \widetilde{K}_{2}\right) \leqslant d_{\delta_{2}}\left(x, \widetilde{K}_{2}\right), \tag{3}
\end{equation*}
$$

with equality if and only if $z= \pm x$. Let us first prove that $\left|x_{1}\right|<\left|x_{2}\right|$ implies $\delta_{2}\left(x_{1}\right)>\delta_{2}\left(x_{2}\right)$. This follows from the computation

$$
\delta_{2}(\pi(s))=\frac{1}{s\left|\pi^{\prime}(s)\right|}=\frac{2}{|s+1 / s|} \quad \forall s>0
$$

because $\pi$ is an increasing homeomorphism of $\mathbf{R}_{+}$onto $\mathbf{R}$ mapping 1 to 0 , and $1 / x$ is the coefficient of the hyperbolic metric on $\mathbf{H}_{+}$at the point $x+i y$. Next we consider the automorphisms of $\mathbf{C}_{2}$,

$$
H_{r}(z)=\frac{z+i r}{1-i r z}, \quad-1<r<1
$$

Each $H_{r}$, being a Möbius transformation, preserves the circles through its fixed points $\pm i$. We deduce that the arcs of circle between $\pm i$ are lines of equidistance to $\tilde{K}_{2}$, which is itself such an arc. Let $x \in \mathbf{R}$ be arbitrary. We compute

$$
\delta_{2}(x)=\delta_{2}\left(H_{r}(x)\right) \cdot\left|H_{r}^{\prime}(x)\right|=\delta_{2}\left(H_{r}(x)\right) \cdot \frac{1-r^{2}}{1+r^{2} x^{2}}<\delta_{2}\left(H_{r}(x)\right)
$$

The computation shows that $\delta_{2}$, when restricted to the arc of circle between $\pm i$ through $x$, attains its infimum only at $x$. This completes the proof of (3).

We shall prove the sublemma only for piecewise differentiable curves and leave the generalization to rectifiable curves to the reader, as we only need the piecewise differentiable case. Let $\widetilde{\gamma} \subset \mathbf{C}_{2}$ be a piecewise differentiable curve with $\widetilde{\gamma} \cap \widetilde{K}_{2} \neq \varnothing$ and $l_{\delta_{2}}(\tilde{\gamma})=t \leqslant \infty$. We can suppose that $t<\infty$, as (2) is void if $t=\infty$. It suffices to consider curves with one end point on $\widetilde{K}_{2}$, because $\sinh t_{1}+\sinh t_{2}<\sinh \left(t_{1}+t_{2}\right)$. Reparametrizing if necessary we can suppose that $\widetilde{\gamma}:[0, t] \rightarrow \mathbf{C}_{2}$ is parametrized by hyperbolic curve length and starts on $\widetilde{K}_{2}$. Then $\left|\widetilde{\gamma}^{\prime}(s)\right|=1 / \delta_{2}(\widetilde{\gamma}(s))$ for all $0 \leqslant s \leqslant t$ and

$$
l_{e}(\widetilde{\gamma})=\int_{0}^{t}\left|\widetilde{\gamma}^{\prime}(s)\right| d s=\int_{0}^{t} \frac{1}{\delta_{2}(\widetilde{\gamma}(s))} d s \leqslant \int_{0}^{t} \frac{1}{\delta_{2}(s)} d s=l_{e}\left(\widetilde{\gamma}_{0}\right)=\sinh t
$$

where the inequality comes from (3), and equality applies if and only if $\gamma=\widetilde{\gamma}_{t}$. This proves the sublemma.

To justify the definition of $S_{l, t}$ we note that the rigid rotations $z \mapsto \lambda z$, with $|\lambda|=1$, are both Euclidean isometries and hyperbolic isometries between $A_{K}^{*}$ and $A_{\lambda K}^{*}$. We shall hence only consider the arcs $K_{l} \subsetneq \mathbf{S}^{1}$ of Euclidean length $0<l \leqslant 2 \pi$ and with mid point 1. The lemma then follows from the sublemma by considering the Möbius transformation $H(z)=(1+z) /(1-z)$, which maps 1 to 0 and each $A_{K_{l}}^{*}$ univalently into a doubly slit plane.

The complex bounds.
Definition 2.7. Let $\theta \in] 0,1\left[-\mathbf{Q}\right.$. Define $Q_{0}=\left\lceil 1, y_{q_{1}}\right\rceil, Q_{1}=\partial U_{0}-\stackrel{\circ}{Q}_{0}$ and, for $n \geqslant 2$, $Q_{n}=\left\lceil 1, y_{q_{n}}\right\rceil$. Moreover, define $\left.K_{m}=\right\rceil x_{-q_{m}}, x_{-q_{m+1}}\left\lceil\subset \mathbf{S}^{1}\right.$ for $m \geqslant 1$. (For typographical reasons we shall not add an index $\theta$ to $Q_{n}$ and $K_{m}$.)

Proposition 2.8. For each $\theta \in] 0,1\left[-\mathbf{Q}\right.$ there exist positive constants $L_{d, \theta}, d=$ $1,2,3, \ldots$, such that for all $n$,

$$
l_{K_{n+d}}\left(Q_{n}\right) \leqslant L_{d, \theta}
$$

Complement to Proposition 2.8. There exist (explicit) universal constants $L, M>0$, i.e. not depending on $\theta$, such that

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} l_{K_{n+d}}\left(Q_{n}\right) \leqslant L+(d+2) M \tag{1}
\end{equation*}
$$

For $d \geqslant 1$ we define $L_{d}=L+(d+2) M$.


Fig. 18. The (bounded) geometry of $Q_{n}$ relative to $K_{n+d}$

Sketch of proof. When $n$ tends to $\infty$, the set $A_{K_{n+d}}^{*}$ looks more and more like a doubly slit plane with gab $K_{n+d}$. Moreover, the $\operatorname{arc} Q_{n}$ looks more and more like a line segment sticking out of $K_{n+d}$ making an angle $\pm 30^{\circ}$ with the horizontal, because $f_{\theta}$ has a double critical point at 1 and preserves the unit circle. Finally the Świạtec a priori real bounds imply that the described configuration has bounded geometry independent of $\theta$. The proof we shall give renders the above sketch into a proof, essentially by changing coordinates so that we get actual slit planes and actual line segments. See Figure 18.

Proof. Let $\tilde{v}$ be a logarithm of $v$ and define $F_{\tilde{v}}(z)=\tilde{v}-\pi z^{3}$. Define a univalent parameter $\Phi: \mathbf{D} \rightarrow V \subset \mathbf{C}$ with $\Phi(0)=1$ and $\Phi^{\prime}(0)>0$ such that the following diagram is commutative:


Note that $\exp$ is univalent on $\mathbf{D}_{\pi}(\tilde{v})$ and let $V^{\prime}=\exp \left(\mathbf{D}_{\pi}(\tilde{v})\right)$. Define $V$ to be the preimage of $V^{\prime}$ under $f_{\theta}$. Then $V$ is simply-connected and the restriction $f_{\theta}: V \rightarrow V^{\prime}$ is a branched triple cover, branched only above $\tilde{v}$. Define $\Phi: \mathbf{D} \rightarrow V$ as the Riemann map with $\Phi(0)=1$ and $\Phi^{\prime}(0)>0$. It is easy to verify that $\Phi$ satisfies (2).

Define $\hat{x}_{j}=\Phi^{-1}\left(x_{j}\right)$ for $j \in \mathbf{Z}-\{-1\}$ and $\hat{y}_{j}=\Phi^{-1}\left(y_{j}\right)$ for $j \geqslant 1$. The quotients of $\hat{x}_{j}$ over $\hat{y}_{j}$ for $j \geqslant 1$ are all some third root of unity, as they have the same image under $F_{\bar{v}}$.

For all $n$ and $d$ we have $l_{K_{n+d}}\left(Q_{n}\right)<\infty$ and thus it suffices to prove the complement
of the lemma. Fix $\theta \in] 0,1[-\mathbf{Q}$ and $d \geqslant 1$. For $n \geqslant 2$ let

$$
\left.\widehat{Q}_{n}=\Phi^{-1}\left(Q_{n}\right)=\left[0, \hat{y}_{q_{n}}\right] \quad \text { and } \quad \widehat{K}_{n+d}=\Phi^{-1}\left(K_{n+d}\right)=\right] \hat{x}_{-q_{n+d}}, \hat{x}_{-q_{n+d+1}}[
$$

We shall first find explicit universal constants $L, M>0$ such that

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} l_{\delta_{\widehat{K}_{n+d}}}\left(\widehat{Q}_{n}\right) \leqslant L+(d+2) M \tag{3}
\end{equation*}
$$

where $\delta_{\widehat{K}_{n+d}}$ is the hyperbolic metric on the doubly slit plane with gab $\widehat{K}_{n+d}$. Second and finally we shall prove that (3) implies (1) with the same constants.

Let $r_{n+d}=\min \left\{\left|\hat{x}_{-q_{n+d}}\right|,\left|\hat{x}_{-q_{n+d+1}}\right|\right\}$. Let $z_{n, d} \in\left[0, \hat{y}_{q_{n}}\right]$ be given by $\left|z_{n, d}\right|=r_{n+d}$. The hyperbolic length $L$ of the segment $\left[0, z_{n, d}\right]$ measured in the doubly slit plane with gab $]-i r_{n+d}, i r_{n+d}$ [ does not depend on $n$ or $d$, because the geometry is fixed so that the only thing which changes is the scale. As $]-i r_{n+d}, i r_{n+d}\left[\subsetneq \widehat{K}_{n+d}\right.$ we also have $l_{\delta_{\widehat{K}_{n+d}}}\left(\left[0, z_{n, d}\right]\right)<L$. Moreover, $l_{\delta_{\widehat{K}_{n+d}}}\left(\left[z_{n, d}, \hat{y}_{q_{n}}\right]\right) \leqslant l_{\mathbf{H}_{+}}\left(\left[z_{n, d}, \hat{y}_{q_{n}}\right]\right)$ because $\mathbf{H}_{+} \subset \mathbf{C}_{\widehat{K}_{n+d}}$. Combining the Síatec a priori real bounds with the univalence of $\Phi$ we obtain

$$
\begin{equation*}
\liminf \frac{\left|z_{n, d}\right|}{\left|\hat{y}_{q_{n}}\right|}=\liminf \frac{r_{n+d}}{\left|\hat{y}_{q_{n}}\right|} \geqslant \liminf \frac{\min \left\{\left|\hat{x}_{q_{n+d+1}}\right|,\left|\hat{x}_{q_{n+d+2}}\right|\right\}}{\left|\hat{y}_{q_{n}}\right|} \geqslant a^{d+2} \tag{4}
\end{equation*}
$$

Hence we obtain by direct computation $\lim \sup l_{\mathbf{H}_{+}}\left(\left[z_{n, d}, \hat{y}_{q_{n}}\right]\right) \leqslant(2 / \sqrt{3})(d+2) \log 1 / a$. This proves (3) with $M=(2 / \sqrt{3}) \log 1 / a$. Moreover, the bound does not depend on $\theta$, because the Świątec a priori real bounds which imply (4) do not depend on $\theta$, and $L$ is universal.

It follows from (3) that the hyperbolic distance in the slit planes $d_{\delta_{\widehat{K}_{n+d}}}\left(\widehat{Q}_{n}, \partial \mathbf{D}\right)$ diverges to $\infty$ as $n \rightarrow \infty$ and $d$ is fixed. It then follows from Lemma 2.4 that we may replace $\delta_{\widehat{K}_{n+d}}$ in (3) with the hyperbolic metrics on the slit disc $\mathbf{D}_{\widetilde{K}_{n+d}}=\mathbf{D} \cap \mathbf{C}_{\widehat{K}_{n+d}}$. Finally this remark proves that (3) implies (1) and thus the complement, because $\Phi$ maps $\mathbf{D}_{\widetilde{K}_{n+d}}$ univalently into $A_{K_{n+d}}^{*}$ and in particular is a hyperbolic contraction.

Lemma 2.9. Suppose that $N \geqslant 1$ and $J, K \subset \mathbf{S}^{1}$ are any pair of subarcs such that the restriction $f_{\theta}^{N}: J \rightarrow K$ is a diffeomorphism. Then any univalent branch $G: U \rightarrow W_{J}^{*}$ of $f_{\theta}^{-N}$ defined on a subset $U \subset A_{K}^{*}$ satisfies

$$
\left\|D_{z} G\right\|_{\lambda_{J}, \lambda_{K}}<\left\|D_{z} G\right\|_{\varrho_{J}, \lambda_{K}}=\frac{\varrho_{J}(G(z))}{\lambda_{K}(z)}\left|G^{\prime}(z)\right| \leqslant 1 \quad \forall z \in U
$$

with equality if and only if $N=1$. That is, $G$ is infinitesimally contracting or possibly a local isometry with respect to the involved metrics.

Proof. We have $\varrho_{J}(z)>\lambda_{J}(z)$ for all $z \in W_{J}^{*}$. This proves the first inequality. Suppose first that $N=1$. Then the restriction $f_{\theta}: W_{J}^{*} \rightarrow A_{K}^{*}$ is a covering map of degree three,
because the assumption that $f_{\theta}$ maps $J$ diffeomorphically to $K$ implies that $v \notin K$. In particular, the restriction is a local isometry for the respective hyperbolic metrics $\varrho_{J}$ and $\lambda_{K}$. This proves the case $N=1$. We shall prove the general case by induction. So suppose that (1) holds for $N-1 \geqslant 1$, that $f_{\theta}^{N}$ maps $J$ diffeomorphically to $K$ and that $G: U \rightarrow W_{J}^{*}$ is a univalent branch of $f_{\theta}^{-N}$ defined on a subset $U \subset A_{K}^{*}$. Define $J_{1}=K_{1}=f_{\theta}(J), G_{1}=f_{\theta} \circ G$ and $U_{1}=G_{1}(U)$. Then $G_{1}: U \rightarrow U_{1}$ is a univalent branch of $f_{\theta}^{N-1}$ which by the induction hypotheses satisfies

$$
\left\|D_{z} G_{1}\right\|_{\lambda_{J_{1}}, \lambda_{K}}<\left\|D_{z} G_{1}\right\|_{\varrho_{J_{1}}, \lambda_{K}} \leqslant 1 \quad \forall z \in U .
$$

Let $G_{2}: U_{1} \rightarrow W_{J}^{*}$ be the inverse branch of $f_{\theta}$ on $U_{1}$ with $G=G_{2} \circ G_{1}$. Then we proved above that

$$
\left\|D_{z} G_{2}\right\|_{e_{J}, \lambda_{K_{1}}}=1 \quad \forall z \in U_{1}
$$

Since $J_{1}=K_{1}$ we can conclude by the chain rule.
Recall Terminology 2.1 on Fresh and Young $B$.
Proposition 2.10. Suppose that $\Gamma\left(x_{q_{n}}, y_{l}\right) \in \mathcal{T}_{\theta, 1}$ has a Fresh $B$. Let $J$ be the open arc $J=\rceil 1, x_{q_{n}-q_{n-1}}\left\lceil\subset \mathbf{S}^{1}\right.$. Then

$$
l_{J}(B) \leqslant l_{K_{n-1}}\left(Q_{n-2}\right) \leqslant L_{1, \theta}
$$

where $K_{n-1}$ and $Q_{n-2}$ are as in Definition 2.7 and $L_{1, \theta}$ is as in Proposition 2.8.
Proof. The arc $J$ is mapped diffeomorphically onto $\rceil x_{-q_{n}}, x_{-q_{n-1}}\left\lceil=K_{n-1}\right.$ by $f_{\theta}^{q_{n}}$. Let $g^{\prime}: \Gamma^{\prime} \rightarrow \Gamma$ be the final Swap in obtaining $\Gamma$ ( $\Gamma$ has a Fresh $B$ ). Then $f^{q_{n}} \circ g^{\prime}=I d$ and

$$
\begin{gathered}
\Gamma^{\prime}\left(x_{m^{\prime}}, y_{\alpha^{\prime} q_{n-1}+q_{n-2}}\right) \xrightarrow[\text { Swap }]{g^{\prime}} \Gamma\left(x_{q_{n}}, y_{l}\right), \\
\left(Q_{n-2} \supseteq\right) O^{\prime} \xrightarrow[\text { Swap }]{g^{\prime}} B
\end{gathered}
$$

for some $0 \leqslant \alpha^{\prime} \leqslant a_{n-1}$, by Lemma 1.9. Moreover, $g^{\prime}$ can be defined univalently in a neighbourhood $U \subset A_{K}^{*}$ of $O^{\prime}$. Thus by Lemma 2.9 and Proposition 2.8,

$$
l_{J}(B) \leqslant l_{K_{n-1}}\left(O^{\prime}\right) \leqslant l_{K_{n-1}}\left(Q_{n-2}\right) \leqslant L_{1, \theta}
$$

Proposition 2.11. Suppose that $\Gamma\left(x_{\alpha q_{n}+q_{n-1}}, y_{l}\right) \in \mathcal{T}_{\theta}, 0<\alpha \leqslant a_{n}$, has a Young $B$. Let $J$ be the open subarc $J=\mid 1, x_{(\alpha-1) q_{n}+q_{n-1}}\left\lceil\subset \mathbf{S}^{1}\right.$. Then

$$
l_{J}(B) \leqslant l_{K_{n}}\left(Q_{n-3}\right) \leqslant L_{3, \theta},
$$

where $K_{n}$ and $Q_{n-3}$ are as in Definition 2.7 and $L_{3, \theta}$ is as in Proposition 2.8.
Proof. The open arc $J \subset \mathbf{S}^{1}$ is mapped diffeomorphically onto

$$
\left.K=\rceil x_{-\left(\alpha q_{n}+q_{n-1}\right)}, x_{-q_{n}}\right\rceil
$$

by $f_{\theta}^{\left(\alpha q_{n}+q_{n-1}\right)}$. Let $G: \Gamma^{\prime \prime} \rightarrow \Gamma$ be the long composition of the final $\alpha$ consecutive Gains in constructing $\Gamma$ and let $g^{\prime}: \Gamma^{\prime} \rightarrow \Gamma^{\prime \prime}$ be the final Swap in constructing $\Gamma^{\prime \prime}$ ( $\Gamma$ has a Young $B$ ). We assume here that $\Gamma^{\prime \prime} \neq \Gamma_{0}$. In case $\Gamma^{\prime \prime}=\Gamma_{0}$, one should replace $g^{\prime}$ by the branch of $f_{\theta}^{-1}$ mapping $Q_{0}$ to $B_{0}$ and $Q_{n-3}$ by $Q_{0}$. The details are left to the reader. Then $f^{q_{n-1}} \circ g^{\prime}=\mathrm{Id}$ and

$$
\begin{gathered}
\Gamma^{\prime}\left(x_{m^{\prime}}, y_{\alpha^{\prime} q_{n-2}+q_{n-3}}\right) \xrightarrow[\text { Swap }]{g^{\prime}} \Gamma^{\prime \prime}\left(x_{q_{n-1}}, y_{l^{\prime \prime}}\right) \xrightarrow[\alpha \text { Gains }]{G} \Gamma\left(x_{\alpha q_{n}+q_{n-1}}, y_{l}\right) \\
\left(Q_{n-3} \supseteq\right) O^{\prime} \xrightarrow[\text { Swap }]{g^{\prime}} B^{\prime \prime} \xrightarrow[\alpha \text { Gains }]{G} B
\end{gathered}
$$

where $0 \leqslant \alpha^{\prime} \leqslant a_{n-2}$, by Lemma 1.9. Moreover, $f_{\theta}^{\left(\alpha q_{n}+q_{n-1}\right)} \circ G \circ g^{\prime}=\mathrm{Id}, G \circ g^{\prime}$ can be defined univalently in a neighbourhood $U \subset A_{K}^{*}$ of $O^{\prime}$ and $\left.K \supseteq K_{n}=\right\rceil x_{-q_{n+1}}, x_{-q_{n}}\lceil$. Thus

$$
l_{J}(B) \leqslant l_{K}\left(O^{\prime}\right) \leqslant l_{K_{n}}\left(Q_{n-3}\right) \leqslant L_{3, \theta}
$$

where the first inequality comes from Lemma 2.9 and the last two inequalities come from Proposition 2.8.

Controlling Fresh and Young B(lue).
Proof of Theorem $2.2(2)$. Let $\Gamma(I, B, G, R, O) \in \mathcal{G}_{\theta}$ be arbitrary. The Jordan curve $\Gamma$ has either a Fresh or Young $B$ by definition of $\mathcal{G}_{\theta}$. In particular, $\Gamma$ satisfies the hypotheses of either Lemma 2.10 or Lemma 2.11. We shall treat the two cases separately.

Suppose that $\Gamma=\Gamma\left(x_{q_{n}}, y_{l}\right)$ has a Fresh $B$ so that $l_{J_{n}}(B) \leqslant l_{K_{n-1}}\left(Q_{n-2}\right)$, where $\left.J_{n}=\right\rceil 1, x_{q_{n}-q_{n-1}}\left\lceil\subset \mathbf{S}^{1}\right.$. As $l_{e}\left(J_{n}\right) \rightarrow 0$, when $n \rightarrow \infty$, and $\limsup l_{K_{n-1}}\left(Q_{n-2}\right) \leqslant L_{1}$ we obtain from Lemma 2.6 that

$$
\begin{equation*}
\limsup \frac{l_{e}(B)}{l_{e}\left(J_{n}\right)} \leqslant \frac{1}{2} \sinh \left(L_{1}\right) \tag{1}
\end{equation*}
$$

where sup is over curves $\Gamma\left(x_{q_{n}}, y_{l}\right)$ with a Fresh $B$ and $\lim$ is for $n \rightarrow \infty$. Moreover,

$$
\left\lceil 1, x_{q_{n}}\right\rceil=I \subset \bar{J}_{n}=\left\lceil 1, x_{q_{n}-q_{n-1}}\right\rceil \subset\left\lceil 1, x_{q_{n-2}}\right\rceil
$$

so that

$$
\begin{equation*}
\limsup \frac{l_{e}\left(J_{n}\right)}{l_{e}(I)} \leqslant a^{-2} \tag{2}
\end{equation*}
$$

by the Świa̧tec a priori real bounds, and where limsup has the same sense as above. Combining (1) and (2) we obtain

$$
\begin{equation*}
\limsup _{\text {Fresh }} \frac{l_{e}(B)}{l_{e}(I)} \leqslant \frac{1}{2 a^{2}} \sinh \left(L_{1}\right) . \tag{3}
\end{equation*}
$$

This takes care of the case of $\Gamma$ with a Fresh $B$. For $\Gamma$ with a Young $B$ one obtains

$$
\begin{equation*}
\limsup _{\text {Young }} \frac{l_{e}(B)}{l_{e}(I)} \leqslant \frac{1}{2 a^{2}} \sinh \left(L_{3}\right) \tag{4}
\end{equation*}
$$

by copying the above arguments. The details are left to the reader as an exercise. This completes the proof of Proposition 2.2 (2).

Uniform bounds for Fresh, Young and Juvenile $R$.
Proof of Theorem 2.2 (4). We shall prove the following more precise estimates:

$$
\begin{align*}
& \underset{\text { Fresh }}{\limsup } l_{\lambda}(R) \leqslant L_{1},  \tag{1}\\
& \limsup l_{\lambda}(R) \leqslant L_{3},  \tag{2}\\
& \text { Young }  \tag{3}\\
& \limsup l_{\lambda}(R) \leqslant L_{5} .
\end{align*}
$$

Here sup is over $\Gamma \in \mathcal{T}_{\theta, k}$ with respectively a Fresh, Young or Juvenile $R$, and lim is for $k$ tending to $\infty$.

Let $\Gamma\left(x_{m}, y_{l}\right)=\Gamma(I, B, G, R, O) \in \mathcal{T}_{\theta}$. We shall consider separately the three different cases (see Terminology 2.1 for the definitions of Fresh, Young and Juvenile). The relevant final moves in the three cases are for Fresh $R$,

$$
\begin{align*}
& \Gamma^{\prime}\left(x_{\alpha q_{n}+q_{n-1}}, y_{l^{\prime}}\right) \xrightarrow[\text { Gain }]{g^{\prime}} \Gamma\left(x_{(\alpha+1) q_{n}+q_{n-1}}, y_{q_{n}}\right) \\
& \quad\left(Q_{n-2} \supseteq\right) O^{\prime} \xrightarrow[\text { Gain }]{g^{\prime}} R
\end{align*}
$$

where $0 \leqslant \alpha<a_{n}$ and $f_{\theta}^{q_{n}} \circ g^{\prime}=\mathrm{Id}$, so that $l^{\prime}=\beta q_{n-1}+q_{n-2}$ with $0 \leqslant \beta \leqslant a_{n-1}$, by Lemma 1.9.
For Young $R$ we have $(m, l)=\left(q_{n},(\alpha+1) q_{n}+q_{n-1}\right)$ for some $n$ and $0 \leqslant \alpha<a_{n}$, and

$$
\begin{align*}
& \Gamma^{\prime}\left(x_{m^{\prime}}, y_{l^{\prime}}\right) \xrightarrow[\text { Swap }]{g^{\prime}} \Gamma^{\prime \prime}\left(x_{q_{n-1}}, y_{l^{\prime \prime}}\right) \xrightarrow[\alpha \text { Gains }]{G} \Gamma^{\prime \prime \prime}\left(x_{\alpha q_{n}+q_{n-1}}, y_{l^{\prime \prime \prime}}\right) \xrightarrow[\text { Swap }]{g^{\prime \prime \prime}} \Gamma\left(x_{m}, y_{l}\right) \\
& \left(Q_{n-3} \supseteq\right) O^{\prime} \xrightarrow[\text { Swap }]{g^{\prime}} B^{\prime \prime} \xrightarrow[\alpha \text { Gains }]{G} B^{\prime \prime \prime} \xrightarrow[\text { Swap }]{g^{\prime \prime \prime}} R,
\end{align*}
$$

where $f_{\theta}^{q_{n-1}} \circ g^{\prime}=\mathrm{Id}$, so that $l^{\prime}=\beta q_{n-2}+q_{n-3}$ for some $0 \leqslant \beta \leqslant a_{n-2}$. Here the indices of the marked points $\left(x_{m^{\prime}}, y_{l^{\prime}}\right)$ of $\Gamma^{\prime}$ follow from Lemma 1.9 and from Lemma 1.8 with its complement for the others.

Finally for Juvenile $R$ we have ( $m, l$ ) $=\left(q_{n}, q_{n}+q_{n-1}\right)$ for some $n$, and

$$
\begin{align*}
& \Gamma^{\prime}\left(x_{m^{\prime}}, y_{l^{\prime}}\right) \xrightarrow[\text { Swap }]{g^{\prime}} \Gamma^{\prime \prime}\left(x_{q_{n-3}}, y_{l^{\prime \prime}}\right) \xrightarrow[a_{n-2} \text { Gains }]{G} \Gamma^{\prime \prime \prime}\left(x_{q_{n-1}}, y_{q_{n-2}}\right) \xrightarrow[\text { Swap }]{g^{\prime \prime \prime}} \Gamma\left(x_{m}, y_{l}\right) \\
& \left(Q_{n-5} \supseteq\right) O^{\prime} \xrightarrow[\text { Swap }]{g^{\prime}} B^{\prime \prime} \xrightarrow[\alpha \text { Gains }]{G} B^{\prime \prime \prime} \xrightarrow[\text { Swap }]{g^{\prime \prime \prime}} R
\end{align*}
$$

where $f_{\theta}^{q_{n-3}} \circ g^{\prime}=\mathrm{Id}$, so that $l^{\prime}=\beta q_{n-4}+q_{n-5}$ for some $0 \leqslant \beta \leqslant a_{n-4}$. As above the indices of the marked points for $\Gamma^{\prime}$ follow from Lemma 1.9 and from Lemma 1.8 with its complement for the others.

Passing from ( $\mathrm{i}^{\prime}$ ) to (i) is essentially the same for $i=1,2$ and 3 . We shall hence only go through the details for $i=3$, the Juvenile (and worst) case. Let $\left.J_{n}=\right\rceil 1, x_{q_{n-1}}\lceil$ and $\left.J_{n}^{\prime}=\right\rceil 1, y_{q_{n-1}}\left\lceil\right.$. Then $J_{n}$ is mapped onto the arc $\left.K_{n}^{\prime}=\right\rceil x_{-\left(q_{n}+q_{n-1}\right)}, x_{-q_{n}}\left\lceil\supseteq K_{n}\right.$ diffeomorphically by $f_{\theta}^{\left(q_{n}+q_{n-1}\right)}$ and $f_{\theta}^{\left(q_{n}+q_{n-1}\right)} \circ\left(g^{\prime \prime \prime} \circ G \circ g^{\prime}\right)=$ Id. Let $Z_{n}$ be given by $f_{\theta}\left(J_{n}\right)=f_{\theta}\left(J_{n}^{\prime}\right)=Z_{n}$, and let $\varrho_{n}=\varrho_{J_{n}}$ denote the hyperbolic metric on $W_{J_{n}}^{*}$. Then we get from Lemma 2.9,

$$
\begin{equation*}
l_{Q_{n}}(R)=l_{\varrho_{n}}\left(g^{\prime \prime \prime} \circ G \circ g^{\prime}\left(O^{\prime}\right)\right) \leqslant l_{K_{n}^{\prime}}\left(O^{\prime}\right) \leqslant l_{K_{n}}\left(Q_{n-5}\right) \leqslant L_{5, \theta} \tag{4}
\end{equation*}
$$

Let $\eta_{n}$ denote the hyperbolic metric on $W_{J_{n}}^{*}-\overline{\mathbf{D}}$. Then

$$
\begin{align*}
l_{\lambda}(R)<l_{\eta_{n}}(R) & \leqslant l_{\varrho_{n}}(R) \cdot \operatorname{coth}\left(\frac{1}{2} d_{\varrho_{n}}\left(R, J_{n}\right)\right) \\
& \leqslant l_{K_{n}}\left(Q_{n-5}\right) \cdot \operatorname{coth}\left(\frac{1}{2} d_{\varrho_{n}}\left(R, J_{n}\right)\right) \tag{5}
\end{align*}
$$

by Lemma 2.4. Moreover, from Lemma 2.5 and (4) we find that

$$
\begin{equation*}
d_{\varrho_{n}}\left(R, J_{n}\right) \geqslant E\left(l_{e}\left(Z_{n}\right)\right)-L_{\theta, 5} \underset{n \rightarrow \infty}{\longrightarrow} \infty \tag{6}
\end{equation*}
$$

Finally (3) follows from (5) combined with the complement of Proposition 2.8 and (6). The reader is encouraged to fill in the details for (1) and (2) to complete the proof of Theorem 2.2 (4).

Proof of Theorem 2.2 (3). Combine Theorem 2.2, (4) and (1) with the fact that the hyperbolic metric $\lambda$ on $\mathbf{C}-\overline{\mathbf{D}}$ is asymptotic to $1 /(|z|-1)$ as $|z| \rightarrow 1$.

As above the careful reader will have noticed that we only used that $\Gamma$ had a Fresh, a Young or a Juvenile $R$ in the above proof. Hence we might replace $\mathcal{G}_{\theta}$ in Theorem 2.2, (3) and (4) by the set of $\Gamma \in \mathcal{T}_{\theta}$ with a Fresh, a Young or a Juvenile $R$.

## 3. Local connectivity of $J_{\theta}$

We introduce the notation $t_{0}=1$ and $t_{n}=q_{n}+q_{n-1}$ for $n \geqslant 1$. Moreover, we shall use the extended notation $\Gamma\left(x_{m}, y_{l}, x_{m, k}\right)$ for the Jordan curve $\Gamma(I, B, G, R, O) \in \mathcal{T}_{\theta}$, where $x_{m}$ and $y_{l}$ are the "free" endpoints of $I$ and $O$ as before and $x_{m, k}$ is the "free" endpoint of $B$.

Estimating the sizes of limbs. We let $\mathcal{F}_{\theta}=\left\{\Gamma_{k}\left(x_{m_{k}}, y_{l_{k}}\right)\right\}_{k \geqslant 0} \subset \mathcal{G}_{\theta}^{*}$ be the descending sequence of Jordan curves defined in Definition 1.12.

Proposition and Definition 3.1. For each $\theta \in] 0,1[-\mathbf{Q}$ there exists a sequence of Jordan curves $\left\{\Sigma_{n}\left(x_{q_{n}}, y_{t_{n}}, x_{q_{n}, t_{n-1}}\right)\right\}_{n \geqslant 1} \subset \mathcal{G}_{\theta}^{*}$ such that each $\Sigma_{n}$ either belongs to $\mathcal{F}_{\theta}$ or is obtained from some $\Gamma_{k} \in \mathcal{F}_{\theta}$ by one, two or three Swaps.

Proof. We are looking for curves of the form $\Sigma_{n}=\Sigma_{n}\left(x_{q_{n}}, y_{t_{n}}, x_{q_{n}, t_{n-1}}\right) \in \mathcal{G}_{\theta}^{*}$. Such curves are obtained by two consecutive Swaps of a curve of the form $\Gamma^{\prime}\left(x_{q_{n-2}, y_{l^{\prime}}}\right) \in \mathcal{G}_{\theta}^{*}$, i.e.

$$
\Gamma^{\prime}\left(x_{q_{n-2}}, y_{l^{\prime}}\right) \xrightarrow[\text { Swap }]{g^{\prime}} \Gamma^{\prime \prime}\left(x_{q_{n-1}}, y_{t_{n-1}}\right) \xrightarrow[\text { Swap }]{g^{\prime \prime}} \Gamma\left(x_{q_{n}}, y_{t_{n}}, x_{q_{n}, t_{n-1}}\right)
$$

Recall that each $\Gamma_{k+1} \in \mathcal{F}_{\theta}$ is obtained from $\Gamma_{k}$ by a move $g_{k}: \Gamma_{k} \rightarrow \Gamma_{k+1}$. It follows that each number $q_{n}, n \geqslant 1$, appears at most once in the sequence $\left\{m_{k}\right\}_{k \geqslant 0}$.

Define

$$
\Sigma_{1}\left(x_{q_{1}}, y_{q_{1}+q_{0}}, x_{q_{1}, t_{0}}\right)=\Gamma_{0}\left(x_{q_{1}}, y_{q_{1}}\right) \quad \text { and } \quad \Sigma_{2}\left(x_{q_{2}}, y_{t_{2}}, x_{q_{2}, t_{1}}\right)=\Gamma_{1}\left(x_{q_{2}}, y_{q_{2}+q_{1}}\right)
$$

For $n \geqslant 3$ we define $\Sigma_{n}=\Sigma_{n}\left(x_{q_{n}}, y_{t_{n}}, x_{q_{n}, t_{n-1}}\right)$ as follows: If there exists $k \geqslant 0$ with $\Gamma_{k}=$ $\Gamma_{k}\left(x_{q_{n}}, y_{t_{n}}, x_{q_{n}, t_{n-1}}\right)$, then we define $\Sigma_{n}=\Gamma_{k}$. If not, we look for a $k \geqslant 0$ with $\Gamma_{k}=$ $\Gamma_{k}\left(x_{q_{n-1}}, y_{t_{n-1}}\right)$. If such a $k$ exists define $\Sigma_{n}$ to be the Swap of $\Gamma_{k}$. If such a $k$ does not exist either, we look for a $k \geqslant 0$ with $\Gamma_{k}=\Gamma_{k}\left(x_{q_{n-2}}, y_{l}\right)$ for some $l$. If such a $k$ exists we define $\Sigma_{n}$ as the curve obtained by two consecutive Swaps of $\Gamma_{k}$. Finally if such a $k$ does not exist either, then there exists a $k \geqslant 0$ with $\Gamma_{k}=\Gamma_{k}\left(x_{q_{n-3}}, y_{l}\right)$ for some $l$. Because the sequence $\left\{m_{k}\right\}_{k \geqslant 0}$ can jump over a $q_{n}$ only if there exists $k \geqslant 0$ with $m_{k}=q_{n-1}$, the move $g_{k}$ is a Gain and $a_{n}=1$, so that $m_{k+1}=q_{n+1}$. In the final case we let $\Sigma_{n}$ be the curve obtained by three consecutive Swaps from $\Gamma_{k}$. The reader shall easily verify that with the above definition $\Sigma_{3}$ either equals $\Gamma_{2}$ or is obtained by a Swap from $\Gamma_{1}$, so that the final case in the definition occurs no earlier than for $n=4$. This completes the definition and the proposition.

For each $n \geqslant 1$ and $0 \leqslant j<q_{n+1}$ let

$$
\begin{aligned}
& \left.J_{n, j}=\right\rceil x_{-q_{n}+j}, x_{-q_{n+1}+q_{n}+j}\left\lceil\subset \mathbf{S}^{1},\right. \\
& \left.J_{n, j}^{\prime}=\right\rceil y_{-q_{n}+j}, y_{-q_{n+1}+q_{n}+j}\left\lceil\subset \partial U_{0},\right.
\end{aligned}
$$

so that $f_{\theta}\left(J_{n, j}\right)=f_{\theta}\left(J_{n, j}^{\prime}\right)=J_{n, j-1}$ for $j>0$. Note that the arcs $J_{n, j}$ and $J_{n, j}^{\prime}$ depend on $\theta$.
Lemma 3.2. For each $\theta \in] 0,1\left[-\mathbf{Q}\right.$ the arc $J_{n, j}$ is mapped diffeomorphically onto $J_{n, 0}$ by $f_{\theta}^{j}$ for every $n \geqslant 1$ and $0<j<q_{n+1}$.

Proof. Evidently, $J_{n, 0}=f_{\theta}^{j}\left(J_{n, j}\right)$. Next we note that $x_{-q_{n+1}}$ is the first return of 1 into $\left.J_{n, 0}=\right\rceil x_{-q_{n}}, x_{-q_{n+1}+q_{n}}\left\lceil\subset \mathbf{S}^{1}\right.$ under $f_{\theta}$. This proves that the restrictions $f_{\theta}^{j}$ : $J_{n, j} \rightarrow J_{n, 0}$ are diffeomorphisms for each $0 \leqslant j<q_{n+1}$, and so completes the proof.


Fig. 19. The Jordan arc $\Sigma_{n}$ relative to the gab $J_{n, 0}$
Lemma 3.3. Given $\theta \in] 0,1[-\mathbf{Q}$ write

$$
\Sigma_{n}\left(x_{q_{n}}, y_{t_{n}}, x_{q_{n}, t_{n-1}}\right)=\Sigma_{n}\left(I_{n}, B_{n}, G_{n}, R_{n}, O_{n}\right)
$$

for each $n \geqslant 1$. Then

$$
l_{J_{n, 0}}\left(O_{n}\right) \leqslant L_{1, \theta}, \quad l_{J_{n, 0}}\left(B_{n}\right) \leqslant L_{2, \theta} \quad \text { and } \quad l_{\lambda}\left(G_{n}\right) \leqslant L_{G, \theta}+3 L_{R, \theta}
$$

Moreover, the bounds are asymptotically universal.
Proof. See Figure 19. To obtain the estimate on $O_{n}$ we note that $J_{n, 0}$ contains $\rceil x_{-q_{n}}, x_{-q_{n+1}}\left\lceil=K_{n}\right.$ and that $O_{n} \subset\left\lceil 1, y_{q_{n-1}}\right\rceil=Q_{n-1}$. Thus the first estimate is immediate from Lemma 2.8. To obtain the estimate on $B_{n}$ we note that $f_{\theta}^{q_{n}} \operatorname{maps} K^{\prime}=$ $\rceil 1, x_{-q_{n+1}+q_{n}}\left\lceil\subset J_{n, 0}\right.$ diffeomorphically onto $\rceil x_{-q_{n}}, x_{-q_{n+1}}\left\lceil=K_{n}\right.$ of Lemma 2.8. Let $g: \Gamma(I, B, G, R, O) \rightarrow \Sigma_{n}$ be the last move in obtaining $\Sigma_{n}$. Then $B_{n}=g(O), f^{q_{n}} \circ g=\mathrm{Id}$ and $g$ can be defined univalently in a neighbourhood $U \subset A_{K_{n}}^{*}$ of $O$. Moreover, $O$ is a subarc of $\left\lceil 1, y_{q_{n-2}}\right\rceil=Q_{n-2}$, because the free endpoint of $B$ is $x_{q_{n}, t_{n-1}}$. Thus by combining Lemma 2.9 and Lemma 2.8 we obtain

$$
l_{J_{n, 0}}\left(B_{n}\right)<l_{K^{\prime}}\left(B_{n}\right) \leqslant l_{K_{n}}(O) \leqslant l_{K_{n}}\left(Q_{n-2}\right) \leqslant L_{2, \theta}
$$

The estimate for $l_{\lambda}(G)$ comes from $\Sigma_{n}$ being at most three admissible moves (Swaps) away from a $\Gamma_{k}$, for which we have $l_{\lambda}(G) \leqslant L_{G, \theta}$. Finally the bounds are easily seen to be asymptotically universal.

Lemma 3.4. For each $\theta \in] 0,1\left[-\mathbf{Q}\right.$ the Euclidean length $l_{e}\left(J_{n, j}\right)$ tends to 0 uniformly in $j$ as $n \rightarrow \infty$.

Proof. The restriction $f_{\theta}: \mathbf{S}^{1} \rightarrow \mathbf{S}^{1}$ is conjugate to the rigid rotation $R_{\theta}$ according to Yoccoz [Yo1; there are no analytic Denjoy counterexamples]. Let $h: \mathbf{S}^{1} \rightarrow \mathbf{S}^{1}$ be a conjugating homeomorphism, i.e. $h \circ R_{\theta}=f_{\theta} \circ h$. Then $l_{e}\left(h^{-1}\left(J_{n, j}\right)\right)=l_{e}\left(h^{-1}\left(J_{n, 0}\right)\right)$ for all $n \geqslant 1$ and all $0 \leqslant j<q_{n+1}$. Moreover, $h$ is uniformly continuous as $\mathbf{S}^{1}$ is compact and $l_{e}\left(h^{-1}\left(J_{n, 0}\right)\right) \rightarrow 0$ as $n \rightarrow \infty$. Thus the lemma follows.

Definition 3.5. For each $n \geqslant 1$ rename $\Sigma_{n}$ to $\Sigma_{n, 0}$. Furthermore, for each $0<j<q_{n+1}$ let $\Sigma_{n, j}$ and $\Delta_{n, j}$ be the unique lifts of $\Sigma_{n, 0}$ to $f_{\theta}^{j}$ intersecting $\mathbf{S}^{1}$ and $\partial U_{0}$ respectively. Let $I_{n, 0}$ be the $I$ of $\Sigma_{n, 0}$. Moreover, for $0<j<q_{n+1}$ let $I_{n, j}=\Sigma_{n, j} \cap f_{\theta}^{-j}\left(I_{n, 0}\right)$.

Lemma 3.6. For each $\theta \in] 0,1\left[-\mathbf{Q}\right.$ there exist constants $L_{\Sigma, \theta}, L_{\Delta, \theta}>0$ such that

$$
\begin{align*}
l_{J_{n, j}}\left(\Sigma_{n, j}-I_{n, j}\right) \leqslant L_{\Sigma, \theta} & \forall n \geqslant 1,0 \leqslant j<q_{n+1}  \tag{1}\\
l_{\lambda}\left(\Delta_{n, j}\right) \leqslant L_{\Delta, \theta} & \forall n \geqslant 1,0<j<q_{n+1} \tag{2}
\end{align*}
$$

Moreover, the constants $L_{\Sigma, \theta}$ and $L_{\Delta, \theta}$ are asymptotically universal.
Proof. Let $L_{\Sigma, \theta}=L_{2, \theta}+L_{G, \theta}+4 L_{R, \theta}+L_{1, \theta}$. Then $l_{J_{n, 0}}\left(\Sigma_{n, 0}-I_{n, 0}\right) \leqslant L_{\Sigma, \theta}$ as it follows from Lemma 3.3. Let $\varrho_{n, j}=\varrho_{J_{n, j}}$ denote the hyperbolic metric on $W_{J_{n, j}}^{*} \subset A_{J_{n, j}}^{*}$. Let $I_{n, j}^{\prime}=\Delta_{n, j} \cap f_{\theta}^{-j}\left(I_{n, 0}\right)$ for $0<j<q_{n+1}$. Then $I_{n, j}^{\prime} \subset J_{n, j}^{\prime}$ is a $\varrho_{n, j}$-geodesic, because $J_{n, j-1}=f_{\theta}\left(J_{n, j}^{\prime}\right)$ is a $\lambda_{J_{n, j-1}}$-geodesic. In particular, $l_{\varrho_{n, j}}\left(\Delta_{n, j}\right) \leqslant 2 l_{\varrho_{n, j}}\left(\Delta_{n, j}-I_{n, j}^{\prime}\right)$. From Lemma 3.2 and Lemma 2.9 we obtain

$$
\begin{align*}
l_{J_{n, j}}\left(\Sigma_{n, j}-I_{n, j}\right) & <l_{\varrho_{n, j}}\left(\Sigma_{n, j}-I_{n, j}\right) \leqslant l_{J_{n, 0}}\left(\Sigma_{n, 0}-I_{n, 0}\right) \leqslant L_{\Sigma, \theta} \\
l_{\varrho_{n, j}}\left(\Delta_{n, j}\right) & \leqslant 2 l_{\varrho_{n, j}}\left(\Delta_{n, j}-I_{n, j}^{\prime}\right) \leqslant 2 l_{J_{n, 0}}\left(\Sigma_{n, 0}-I_{n, 0}\right) \leqslant 2 L_{\Sigma, \theta} \tag{3}
\end{align*}
$$

The first line is identical with (1). Lemma 2.5 implies $d_{\varrho_{n, j}}\left(J_{n, j}, J_{n, j}^{\prime}\right)=E\left(l_{e}\left(J_{n, j-1}\right)\right)$, so that

$$
d_{\varrho_{n, j}}\left(J_{n, j}, \Delta_{n, j}\right) \geqslant\left(E\left(l_{e}\left(J_{n, j-1}\right)\right)-L_{\Sigma, \theta}\right) \rightarrow \infty \quad \text { as } n \rightarrow \infty
$$

by (3) and Lemma 3.4. Hence by Lemma 2.4, $\lim \sup _{n \rightarrow \infty} l_{\lambda}\left(\Delta_{n, j}\right) \leqslant 2 L_{\Sigma, \theta}$, from which the existence of $L_{\Delta, \theta}$ as in (2) easily follows. Moreover, as $L_{1, \theta}, L_{2, \theta}, L_{R, \theta}$ and $L_{G, \theta}$ are asymptotically universal, so are $L_{\Sigma, \theta}$ and $L_{\Delta, \theta}$.

Theorem 3.7. For each $\theta \in] 0,1[-\mathbf{Q}$ the Euclidean diameter of the principal limbs $X_{s}$ and $Y_{s}, s \geqslant 1$, tends to 0 as $s \rightarrow \infty$.

Proof. It suffices to consider the limbs $X_{s}, s \geqslant 1$, as $f_{\theta}\left(Y_{s}\right)=f_{\theta}\left(X_{s}\right)$ for all $s \geqslant 1$. Let $\theta \in] 0,1[-\mathbf{Q}$ be given. We shall first prove that

$$
\begin{equation*}
l_{e}\left(\Sigma_{n, j}\right) \rightarrow 0 \quad \text { as } n \rightarrow \infty \tag{1}
\end{equation*}
$$

uniformly in $0 \leqslant j<q_{n+1}$.
Combining Lemma 3.6 (1) with Lemma 3.4 and Lemma 2.6 we obtain

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} \frac{l_{e}\left(\Sigma_{n, j}-I_{n, j}\right)}{l_{e}\left(J_{n, j}\right)} \leqslant 2 \sinh \left(\frac{1}{2} L_{\Sigma, \theta}\right) \tag{3}
\end{equation*}
$$

It follows that there exists a constant $K_{\Sigma, \theta}$ such that

$$
\begin{equation*}
l_{e}\left(\Sigma_{n, j}\right) \leqslant K_{\Sigma, \theta} \cdot l_{e}\left(J_{n, j}\right) \quad \forall n \geqslant 1,0 \leqslant j<q_{n+1} \tag{4}
\end{equation*}
$$

as $l_{e}\left(\Sigma_{n, j}\right)<\infty$ for all $n, j$ and $I_{n, j} \subset J_{n, j}$. Combining (4) and Lemma 3.4 we obtain (1). Finally the theorem follows from (1), because $X_{\alpha q_{n+1}+q_{n}+j} \subset D\left(\Sigma_{n, j}\right)$ for all $n \geqslant 1,0 \leqslant$ $j<q_{n+1}$ and $0<\alpha \leqslant a_{n+1}$.
"The bridge across Lille Baelt". Suppose that $\eta_{1}, \eta_{2}$ are two curves with one or two common endpoint(s). We shall write $\eta_{1} \cdot \eta_{2}$ for the curve obtained by gluing the two arcs at their common endpoint(s).

Proposition and Definition 3.8. Let $\theta \in 10,1[-\mathbf{Q}$. There exist positive constants $L_{d, \theta}^{\prime}, d=0,1,2,3, \ldots$, and for each pair $\left(l, l^{\prime}\right)=\left(\alpha q_{n}+q_{n-1}, \beta q_{n+d}+q_{n+d-1}\right)$ with $n \geqslant 2$ and $0 \leqslant \alpha<a_{n}, 0<\beta \leqslant a_{n+d}$, there exists an are $\gamma_{l, l^{\prime}} \in \bar{U}_{0}$ joining $y_{l}$ to $y_{l^{\prime}}$ and with

$$
l_{\lambda}\left(\gamma_{l, l^{\prime}}\right) \leqslant L_{d, \theta}^{\prime}
$$

where $\lambda$ denotes the hyperbolic metric on $\mathbf{C}-\overline{\mathbf{D}}$. See Figure 20.
Complement to Proposition 3.8. There exist explicit and universal constants $L^{\prime}, M$, i.e. not depending on $\theta$ such that the same curves satisfy

$$
\limsup l_{\lambda}\left(\gamma_{l, l^{\prime}}\right) \leqslant \begin{cases}L^{\prime}+(d+2) M & \text { for } d \text { odd } \\ (d+2) M & \text { for } d \text { even }\end{cases}
$$

Proof. We shall define the curves $\gamma_{l, l^{\prime}}$ so that they all have finite $\lambda$-length and then prove the complement, from which the proposition follows. Let $\Phi: \mathbf{D} \rightarrow V \subset \mathbf{C}$ be the univalent parameter with $\Phi(0)=1, \Phi^{\prime}(0)>0$ and $\exp \circ\left(z \mapsto \tilde{v}-\pi z^{3}\right)=f_{\theta} \circ \Phi$ defined in the


Fig. 20. The curves $\gamma_{l, l^{\prime}}$ in $\bar{U}_{0}$
proof of Proposition 2.8 (here $\exp (\tilde{v})=v)$. Define $\hat{y}_{j}=\Phi^{-1}\left(y_{j}\right)$ for $j \geqslant 1, z_{j}=\left|\hat{y}_{j}\right| \in \mathbf{R}_{+}$and let $C_{j}$ be the cirle with center 0 and radius $z_{j}$. Moreover, define

$$
l_{ \pm}=\left\{z \left\lvert\, \operatorname{Arg}(z)= \pm \frac{1}{6} i \pi\right.\right\} \cap \mathbf{D}
$$

so that

$$
\Phi\left(l_{ \pm}\right)=\partial U_{0} \cap\{z \mid \operatorname{Im}(z) \gtrless 0\}
$$

For $\left(l, l^{\prime}\right)=\left(\alpha q_{n}+q_{n-1}, \beta q_{n+d}+q_{n+d-1}\right)$ with $n \geqslant 2,0 \leqslant \alpha<a_{n}, 0<\beta \leqslant a_{n+d}$, define an $\operatorname{arc} \widehat{\gamma}_{l, l^{\prime}}$ as follows (see Figure 20): If $d$ is even then $y_{l}$ and $y_{l^{\prime}}$ are in the same line segment $l_{+}$or $l_{-}$. We define $\widehat{\gamma}_{l, l^{\prime}}=\left[\hat{y}_{l}, \hat{y}_{l^{\prime}}\right]$. For $d$ odd $y_{l}$ and $y_{l^{\prime}}$ are in opposite line segments. Let $\left\lceil\hat{y}_{l}, z_{l}\right\rceil$ and $\left\lceil\hat{z}_{l^{\prime}}, \hat{y}_{l^{\prime}}\right\rceil$ be the smaller subarcs of $C_{l}$ and $C_{l^{\prime}}$ respectively, between the respective points. Define $\widehat{\gamma}_{l, l^{\prime}}=\left\lceil\hat{y}_{l}, z_{l}\right\rceil \cdot\left[z_{l}, z_{l^{\prime}}\right] \cdot\left\lceil\hat{z}_{l^{\prime}}, \hat{y}_{l^{\prime}}\right\rceil$. Let $\tilde{\lambda}$ denote the hyperbolic metric on $\mathbf{H}_{+}$and let $L^{\prime}$ be the $\tilde{\lambda}$-distance between $l_{+}$and $l_{-}$. Then the $\tilde{\lambda}$-length of any of the arcs $\left\lceil\hat{y}_{j}, z_{j}\right\rceil$ equals $\frac{1}{2} L^{\prime}$. Moreover, $\left[\hat{y}_{l}, \hat{y}_{l^{\prime}}\right] \subset\left[\hat{y}_{q_{n}-1}, \hat{y}_{q_{n+d+1}}\right]$ if $d$ is even and $\left[z_{l}, z_{l^{\prime}}\right] \subset\left[z_{q_{n-1}}, z_{q_{n+d+1}}\right]$. Hence we get by direct calculation

$$
\begin{aligned}
& l_{\tilde{\lambda}}\left(\left[\hat{y}_{l}, \hat{y}_{l^{\prime}}\right]\right) \leqslant l_{\tilde{\lambda}}\left(\left[\hat{y}_{q_{n}-1}, \hat{y}_{q_{n+d+1}}\right]\right)=\frac{2}{\sqrt{3}} \log \frac{\left|\hat{y}_{q_{n-1}}\right|}{\left|\hat{y}_{q_{n+d+1}}\right|} \quad \text { for } d \text { even } \\
& l_{\tilde{\lambda}}\left(\left[z_{l}, z_{l^{\prime}}\right]\right) \leqslant l_{\tilde{\lambda}}\left(\left[z_{q_{n}-1}, z_{q_{n+d+1}}\right]\right)=\log \frac{z_{q_{n-1}}}{z_{q_{n+d+1}}}
\end{aligned}
$$

Combining the univalence of $\Phi$ with the Świa̧tec a priori real bounds we obtain

$$
\limsup \log \frac{z_{q_{n-1}}}{z_{q_{n+d+1}}}=\lim \sup \log \frac{\left|\hat{y}_{q_{n-1}}\right|}{\left|\hat{y}_{q_{n+d+1}}\right|} \leqslant(d+2) \log \frac{1}{a}
$$

This proves asymptotic bounds as in the complement, but for the curves $\widehat{\gamma}_{l, l^{\prime}}$ in the metric $\tilde{\lambda}$. We define $\gamma_{l, l^{\prime}}=\Phi\left(\widehat{\gamma}_{l, l^{\prime}}\right)$ and then appeal to Lemma 2.4 to carry over the
asymptotic bounds for the curves $\widehat{\gamma}_{l, l^{\prime}}$ to the asymptotic bounds of the complement for the curves $\gamma_{l, l^{\prime}}$, as in the proof of Proposition 2.8.

For $0<\eta<\frac{1}{2} \pi$ define

$$
\widetilde{S}(\eta)=\{z| | \operatorname{Arg}(z) \mid \leqslant \eta\} \cup\{0\} \quad \text { and } \quad \tilde{l}( \pm \eta)=\{z| | \operatorname{Arg}(z) \mid= \pm \eta\}
$$

$\widetilde{S}_{r}(\eta)=\widetilde{S}(\eta) \cap \mathbf{D}_{r}$, where $\mathbf{D}_{r}$ is the Euclidean disc of center 0 and radius $r>0$. Moreover, let $S(\eta)=\exp (\widetilde{S}(\eta)), S_{r}(\eta)=\exp \left(\widetilde{S}_{r}(\eta)\right)$ and $l( \pm \eta)=\exp (\tilde{l}( \pm \eta))$. Recall that $\Pi_{\theta}$ is the principal wake containing the limb $X_{0}=Y_{\varepsilon}$.

Theorem 3.9. Let $\theta \in] 0,1\left[-\mathbf{Q}\right.$ be of constant type. There exists an angle $0<\eta<\frac{1}{2} \pi$ and $r>0$ such that

$$
X_{0} \cap \exp \left(\mathbf{D}_{r}\right) \subset \Pi_{\theta} \cap \exp \left(\mathbf{D}_{r}\right) \subset S_{r}(\eta)
$$

Moreover, the angle $\eta$ depends only on $N=\lim \sup a_{n}$, whereas $r$ depends on the number $n_{0}$ for which $a_{n} \leqslant N$ for all $n \geqslant n_{0}$.

Complement to Theorem 3.9. For $\theta$ as in the theorem there exists a constant $L_{\theta}>0$ such that

$$
\operatorname{diam}_{\lambda}\left(Y_{s}\right) \leqslant L_{\theta} \quad \forall s \geqslant 1 .
$$

Moreover, $\limsup \operatorname{diam}_{\lambda}\left(Y_{s}\right) \leqslant L\left(\limsup a_{n}\right)$, where the constants $L(n)>0, n \geqslant 1$, are independent of $\theta$.

Proof. We shall construct a "suspension bridge", which will prevent $\Pi_{\theta}$ and $X_{0}$ from coming too close to $\mathbf{S}^{1}$, and which will imply both the theorem and its complement. See Figure 21.

For each $n \geqslant 1$ write $\Sigma_{n}=\Sigma_{n}\left(I_{n}, B_{n}, G_{n}, R_{n}, O_{n}\right)$ and let $P_{n}=R_{n} \cdot G_{n}$. For each $n \geqslant 2$ define arcs as follows:
(1) $P_{n, \alpha}=f_{\theta}^{-\left(\alpha q_{n}+q_{n-1}\right)}\left(P_{n}\right) \cap \Sigma_{n, \alpha q_{n}+q_{n-1}}$ for $0 \leqslant \alpha<a_{n}$,
(2) $\gamma_{t_{n-1}, t_{n}, \alpha}=f_{\theta}^{-\left(\alpha q_{n}+q_{n-1}\right)}\left(\gamma_{t_{n-1}, t_{n}}\right) \cap \bar{U}_{\alpha q_{n}+q_{n-1}}$ for $0<\alpha \leqslant a_{n}$,
(3) $\gamma_{t_{n-2}, t_{n}}^{\prime}=f_{\theta}^{-q_{n-1}}\left(\gamma_{t_{n-2}, t_{n}}\right) \cap \bar{U}_{q_{n-1}}$,
(4) $\mathcal{P}_{n}=\gamma_{t_{n-2}, t_{n}}^{\prime} \cdot P_{n, 0} \cdot \gamma_{t_{n-1}, t_{n}, 1} \cdot P_{n, 1} \cdot \ldots \cdot \gamma_{t_{n-1}, t_{n}, a_{n}-1} \cdot P_{n, a_{n}-1} \cdot \gamma_{t_{n-1}, t_{n}, a_{n}}$,
(5) $\Xi_{n}=P_{n-1} \cdot \mathcal{P}_{n} \cdot\left(-P_{n+1}\right) \cdot\left(-\gamma_{t_{n-1}, t_{n+1}}\right)$,
(6) $D_{n}=D\left(\Xi_{n}\right)$.

Finally define long arcs
(7) $\Delta_{\text {even }}=\mathcal{P}_{2} \cdot \mathcal{P}_{4} \cdot \ldots \cdot \mathcal{P}_{2 n} \cdot \ldots$,
(8) $\Delta_{\text {odd }}=\mathcal{P}_{3} \cdot \mathcal{P}_{5} \cdot \ldots \cdot \mathcal{P}_{2 n+1} \cdot \ldots$.

Then $\Delta_{\text {even }}$ and $\Delta_{\text {odd }}$ converges to 1 and can be made into closed arcs by adding the point 1 to each. Let $R_{+}$and $R_{-}$be the two external rays landing on 1 . Then

$$
P_{n, \alpha} \cap R_{+}=P_{n, \alpha} \cap R_{-}=P_{n, \alpha} \cap X_{0}=\varnothing,
$$



Fig. 21. The barriers $\Delta_{\text {even }}$ and $\Delta_{\text {odd }}$ separating the principal wake $\Pi_{\theta}$ from $\mathbf{D}$ because

$$
J_{\theta} \cap \Sigma_{n, \alpha q_{n}+q_{n-1}} \subset X_{\alpha q_{n}+q_{n-1}} \cup X_{(\alpha+1) q_{n}+q_{n-1}} \cup\left\lceil x_{\alpha q_{n}+q_{n-1}}, x_{(\alpha+1) q_{n}+q_{n-1}}\right\rceil
$$

$P_{n, \alpha} \subset \Sigma_{n, \alpha q_{n}+q_{n-1}}$ and external rays do not cross (see also Figure 21).
As $f_{\theta}^{-1}$ is contracting with respect to $\lambda$ we obtain from Lemma 3.3 and Proposition 3.8 that

$$
\begin{aligned}
l_{\lambda}\left(\Xi_{n}\right) & \leqslant l_{\lambda}\left(P_{n-1}\right)+l_{\lambda}\left(\mathcal{P}_{n}\right)+l_{\lambda}\left(P_{n+1}\right)+l_{\lambda}\left(\gamma_{t_{n-1}, t_{n+1}}\right) \\
& \leqslant 2\left(L_{2}^{\prime}+L_{G, \theta}+4 L_{R, \theta}\right)+a_{n}\left(L_{1}^{\prime}+L_{G, \theta}+4 L_{R, \theta}\right) \\
& =K_{2, \theta}+a_{n} \cdot K_{1, \theta},
\end{aligned}
$$

where the constants $K_{1, \theta}, K_{2, \theta}$ are defined by the equality sign. The above curves can be constructed and the estimate on $\Xi_{n}$ holds for all $\left.\theta \in\right] 0,1[-\mathbf{Q}$. We note however that the estimate depends on $a_{n}$.

Suppose that $\theta$ is of constant type. Let $N=\limsup a_{n}$ and let $n_{0} \geqslant 2$ be minimal with $a_{n} \leqslant N$ for all $n \geqslant n_{0}$. Let us prove that there exists $0<\eta<\frac{1}{2} \pi$ such that

$$
\begin{equation*}
\mathcal{P}_{n} \subset S(\eta) \text { for all } n \geqslant n_{0} \tag{1}
\end{equation*}
$$

Let $\frac{1}{6} \pi \leqslant \eta_{1}<\frac{1}{2} \pi$ be minimal with $U_{0} \subset S\left(\eta_{1}\right)$. Next let $\eta_{1}<\eta<\frac{1}{2} \pi$ be given by

$$
\operatorname{dist}_{\tilde{\lambda}}\left(\tilde{l}\left(\eta_{1}\right), \tilde{l}(\eta)\right)=\frac{1}{2}\left(K_{2, \theta}+N \cdot K_{1, \theta}\right)
$$

where $\tilde{\lambda}$ denotes the hyperbolic metric on $\mathbf{H}_{+}$. Then $\eta$ satisfies (1).
Let $0<r<\pi$ be small enough that

$$
\exp \left(\mathbf{D}_{r}\right) \subset\left(D\left(\Sigma_{n_{0}, 0}\right) \cup D\left(\Sigma_{n_{0}+1,0}\right) \cup U_{0} \cup \mathbf{D}\right)
$$

Then $X_{0} \cap \exp \left(\mathbf{D}_{r}\right) \subset S_{r}(\eta)$, where $\eta$ is as in (1). This proves the theorem.
To prove the complement we note first that for any $n \geqslant 2,0<\alpha \leqslant a_{n}$ and $0<j<q_{n}$ we have $Y_{\alpha q_{n}+q_{n-1}+j} \subset D\left(\Delta_{n, j}\right)$. Moreover,

$$
\operatorname{diam}_{\lambda}\left(D\left(\Delta_{n, j}\right)\right)<\frac{1}{2} l_{\lambda}\left(\Delta_{n, j}\right) \leqslant L_{\Delta, \theta}
$$

by Lemma 3.3. Thus we need only bound the $\lambda$-diameters of the limbs $Y_{\alpha q_{n}+q_{n-1}}$ for $n \geqslant 1$ and $0<\alpha \leqslant a_{n}$. Each limb $Y_{s}$ is a compact subset of $\mathbf{C}-\overline{\mathbf{D}}$. Thus we need only give a bound for the cases $n \geqslant n_{0}+2$, say. For any $n \geqslant n_{0}+2$ we have $Y_{\alpha q_{n}+q_{n-1}} \subset D_{n} \cup D_{n-2}$ and $\operatorname{diam}_{\lambda}\left(D_{n} \cup D_{n-2}\right) \leqslant K_{2, \theta}+N \cdot K_{1, \theta}$. This proves the existence of a bound in the remaining case. Finally the statement of asymptotic universality follows from the asymptotic universality of the bounding constants.

We obtain as immediate corollary a proof of the last half of Theorem 3.26. (Recall Definition 0.8.)

Theorem 3.26 (2). Let $\theta \in] 0,1[$ be of constant type. Then there exists a constant $M=M(\theta)>1$ such that

$$
\left\|D_{z} F_{\theta}\right\|_{\lambda} \geqslant M \quad \text { for all } z \in Y_{\theta} .
$$

Proof. Given $\theta \in] 0,1\left[-\mathbf{Q}\right.$ of bounded type let $\frac{1}{6} \pi<\eta<\frac{1}{2} \pi$ and $r>0$ be as in Theorem 3.9. Then $Y_{\theta} \subset\left(S_{r}(\eta)-\{0\}\right) \cup K$, where $K \subset \mathbf{C}-\overline{\mathbf{D}}$ is a compact subset. Thus there exists $M>1$ such that $\varrho(z) / \lambda(z) \geqslant M$ for all $z \in Y_{\theta}$. As

$$
\left\|D_{z} f_{\theta}\right\|_{\lambda}=\frac{\varrho(z)}{\lambda(z)} \cdot\left\|D_{z} f_{\theta}\right\|_{\lambda, \varrho}=\frac{\varrho(z)}{\lambda(z)}>1 \quad \forall z \in W_{1}
$$

the theorem follows from the definition of $F_{\theta}$.
Lifting to the exponential. This subsection except Theorem 3.12 shall be used both in the subsequent subsection and in the subsection "Controlling the core of nests for all irrational $\theta^{\prime \prime}$. Let $\left.J_{0}=\right] \tilde{v}-i 2 \pi, \tilde{v}[$, where $\tilde{v}$ is the logarithm in $] 0, i 2 \pi[$ of the critical
value $v$. Moreover, let $\tilde{U}_{0}$ be the connected component of $\exp ^{-1}\left(U_{0}\right)$ with 0 on the boundary.

Recall that $f_{\theta}: W_{1} \rightarrow \mathbf{C}-\overline{\mathbf{D}}$ is a degree-two covering map, where $W_{1}$ is the unbounded connected component of $f_{\theta}^{-1}(\mathbf{C}-\overline{\mathbf{D}})$. Let $\widetilde{W}_{1}=\exp ^{-1}\left(W_{1}\right)$ and let $G_{\theta}: \mathbf{H}_{+} \rightarrow \widetilde{W}_{1}$ be a lift of exp to $f_{\theta} \circ$ exp. The map $G_{\theta}$ extends to a continuous map of $\overline{\mathbf{H}}_{+}$onto $\widetilde{W}_{1}$. The extended map $G_{\theta}$ is injective except on the set $\{\tilde{v}-i 2 \pi, \tilde{v}\}+i 4 \pi \mathbf{Z}$, which is mapped two-to-one onto $i 2 \pi \mathbf{Z}$. Choosing another lift, if necessary, we shall suppose that $G_{\theta}$ maps $J_{0}$ homeomorphically onto $\partial \widetilde{U}_{0}-\{0\}$. The map $G_{\theta}$ is an isometry with respect to the hyperbolic metrics $\tilde{\lambda}$ on $\mathbf{H}_{+}$and $\tilde{\varrho}$ on $\widetilde{W}_{1}$. Let $\tilde{f_{\theta}}: \widetilde{W}_{1} \rightarrow \mathbf{H}_{+}$denote the inverse of $G_{\theta}$ and extend $\tilde{f}_{\theta}$ continuously to $\overline{\widetilde{W}}_{1}-i 2 \pi \mathbf{Z}$. We note that $\widetilde{\widetilde{W}}_{1}$ does not depend on $\theta$ and that $\tilde{f}_{\theta_{1}}, \tilde{f}_{\theta_{2}}$ for $\left.\theta_{1}, \theta_{2} \in\right] 0,1\left[-\mathbf{Q}\right.$ differ only by an additive, imaginary constant, because $f_{\theta_{1}}$ and $f_{\theta_{2}}$ differ only by a multiplicative constant of modulus 1 .

We shall write $\mathbf{C}_{0}$ for $\mathbf{C}_{J_{0}}$, the doubly slit plane with gab $J_{0}$. Moreover, we let $\delta_{0}$ denote the hyperbolic metric on $\mathbf{C}_{0}$. For $0<T<\infty$ we define (see Figure 22)

$$
\omega(T)=\left\{z \in \mathbf{H}_{+} \mid 0<d_{\delta_{0}}\left(J_{0}, z\right) \leqslant T\right\} \quad \text { and } \quad \Omega(T)=G_{\theta}(\omega(T))
$$

By the above remark the set $\Omega(T)$ does not depend on $\theta$ and the sets $\omega(T)=\omega_{\theta}(T)$ differ only by a purely imaginary translation. We remind the reader that for each $T>0$, the set $\omega(T)$ is bounded in $\mathbf{H}_{+}$by an arc of circle through the endpoints of $J_{0}$. We call this arc of circle $C_{T}$. The angle between $J_{0}$ and $C_{T}$ at any one of their common endpoints is in one-to-one correspondence with $T$. An elementary calculation shows that the angle $\frac{1}{2} \pi$ corresponds to the distance $T_{C}=\log (1+\sqrt{2})$. The arc $C_{T_{C}}$ is a $\tilde{\lambda}$-geodesic. Moreover, each $C_{T}$ is an arc of $\tilde{\lambda}$-equidistance from $C_{T_{C}}$. Thus the set $\omega(T)$ is a $\tilde{\lambda}$-convex subset of $\mathbf{H}_{+}$if and only if $T \geqslant T_{C}$. Then also $\Omega(T)$ is a $\tilde{\varrho}$-convex subset of $\widetilde{W}_{1}$ if and only if $T \geqslant T_{C}$, as $G_{\theta}$ is an isometry.

Lemma 3.10. There exists an increasing function $\left.M: \mathbf{R}_{+} \rightarrow\right] 0,1[$ such that

$$
\frac{\tilde{\lambda}(z)}{\tilde{\varrho}(z)} \leqslant M(T) \quad \forall z \in \Omega(T),
$$

where $\tilde{\lambda}$ is the coefficient of the hyperbolic metric on $\mathbf{H}_{+}$and $\tilde{\varrho}$ is the coefficient of the hyperbolic metric on $\widetilde{W}_{1}$. In particular, if $T \geqslant T_{C}$, then

$$
\operatorname{diam}_{\tilde{\lambda}}(K) \leqslant M(T) \cdot \operatorname{diam}_{\tilde{e}}(K)
$$

for all compact subsets $K \subset \Omega(T)$.
Proof. The boundary of $\Omega(T)$ different from $\partial \widetilde{U}_{0}$ makes a non-zero angle with the imaginary axis at 0 .


Fig. 22. The set $\widetilde{W}$ and $\Omega(T)$ with $T=T_{C}$
Let $\widetilde{X}_{j}, j \geqslant 0$, be the connected component of $\exp ^{-1}\left(X_{j}\right)$ intersecting the segment $J_{0}$ and let $\widetilde{X}_{0}^{*}=\widetilde{X}_{0}-\{0\}$. Moreover, define $\widetilde{Y}_{s_{1}, \ldots, s_{m}}=\exp ^{-1}\left(Y_{s_{1}, \ldots, s_{m}}\right) \cap \widetilde{X}_{0}$ for each $\left(s_{1}, \ldots, s_{m}\right) \in \mathbf{N}^{m}, m \geqslant 1$, and $\tilde{y}_{s}=\tilde{X}_{0} \cap \exp ^{-1}\left(y_{s}\right)$ for $s \geqslant 1$. Define $W_{0}=\mathbf{C}-\overline{\mathbf{D}}$ and for $n \geqslant 2$ let $W_{n}$ be the unbounded connected component of $f_{\theta}^{-1}\left(W_{n-1}\right)$. Then each restriction $f_{\theta}: W_{n} \rightarrow W_{n-1}$ is a degree-two covering map. Define $\widetilde{W}_{n}=\exp ^{-1}\left(W_{n}\right)$ for all $n \geqslant 1$ (the case $n=1$ has previously been taken care of).

Definition 3.11. Let $\theta \in] 0,1\left[-\mathbf{Q}\right.$. For each $s \geqslant 1$ let $G_{s, \theta}: \overline{\mathbf{H}}_{+} \rightarrow \widetilde{\widetilde{W}}_{s} \subseteq \overline{\widetilde{W}}_{1}$ be the lift of exp: $\overline{\mathbf{H}}_{+} \rightarrow \mathbf{C}-\mathbf{D}$ to $f_{\theta}^{s}{ }^{\circ} \exp : \widetilde{\widetilde{W}}_{s} \rightarrow \mathbf{C}-\mathbf{D}$ with $G_{s, \theta}(0)=\tilde{y}_{s}$. We shall usually omit the index $\theta$ however. Moreover, for $\left(s_{1}, \ldots, s_{m}\right) \in \mathbf{N}^{m}$ we shall use the shorthand notation

$$
G_{s_{1}, \ldots, s_{m}}=G_{s_{1} \circ \ldots \circ} G_{s_{m}} .
$$

Each $G_{s}$ is Lipschitz with constant 1 with respect to the hyperbolic metrics $\tilde{\lambda}$ and $\tilde{\varrho}$. Moreover, for all $m \geqslant 1$ and for all $\left(s_{1}, \ldots, s_{m}\right) \in \mathbf{N}^{m}$,

$$
Y_{s_{1}, \ldots, s_{m}}=G_{s_{1}, \ldots, s_{m}}\left(X_{0}\right)=G_{s_{1}, \ldots, s_{m-1}}\left(Y_{s_{m}}\right)=G_{s_{1}, \ldots, s_{m-2}}\left(Y_{s_{m-1}, s_{m}}\right)
$$

and $G_{s_{1}, \ldots, s_{m}}: \mathbf{H}_{+} \rightarrow \widetilde{W}_{s_{1}+\ldots+s_{m}}$ is biholomorphic, because it is a lift of the universal covering exp: $\mathbf{H}_{+} \rightarrow \mathbf{C}-\overline{\mathbf{D}}$ to the universal covering $f_{\theta}^{s_{1}+\ldots+s_{m}}{ }_{\circ} \exp : \widetilde{W}_{s_{1}+\ldots+s_{m}} \rightarrow \mathbf{C}-\overline{\mathbf{D}}$.

Theorem 3.12. For any $\theta \in] 0,1\left[-\mathbf{Q}\right.$ of constant type the Core $\operatorname{Core}\left(\mathcal{Y}_{\underline{s}, \theta}\right)$ is trivial for each $\underline{s} \in \mathbf{N}^{\mathbf{N}}$.

Proof. We shall prove the following equivalent statement of the theorem: For every $\underline{s}=\left(s_{1}, \ldots, s_{m}, \ldots\right) \in \mathbf{N}^{\mathbf{N}}$ there exists $\tilde{z}_{\underline{s}} \in \mathbf{H}_{+}$such that

$$
\bigcap_{m \geqslant 1} \widetilde{Y}_{s_{1}, \ldots, s_{m 2}}=\left\{\tilde{z}_{\underline{s}}\right\} .
$$

Let $\theta$ of constant type be given. It follows from Theorem 3.9 that there exists a constant $T \geqslant T_{C}$ such that $\left(\widetilde{X}_{0}-\partial \widetilde{U}_{0}\right) \subset \Omega(T)$. This is because the angle between $\partial \Omega(T)$ at 0 and $i \mathbf{R}$ is determined by $T$ and tends to 0 as $T$ tends to $\infty$. Let $M=M(T)$ and let $L$ be an upper bound for $\operatorname{diam}_{\tilde{\lambda}}\left(\widetilde{Y}_{s}\right)=\operatorname{diam}_{\lambda}\left(Y_{s}\right)$ for $s \geqslant 1$ as given by the complement to Theorem 3.9. Then we have for any $\underline{s} \in \mathbf{N}^{\mathbf{N}}$,

$$
\operatorname{diam}_{\tilde{\lambda}}\left(\widetilde{Y}_{s_{1}, \ldots, s_{m}}\right)=\operatorname{diam}_{\tilde{\lambda}}\left(G_{s_{1}} \circ \ldots \circ G_{s_{m-1}}\left(Y_{s_{m}}\right)\right) \leqslant L \cdot M^{(m-1)} \underset{m \rightarrow \infty}{\longrightarrow} 0
$$

Here we have applied iteratively the second statement of Lemma 3.10 and the fact that each $G_{s}$ is Lipschitz with constant 1 with respect to the hyperbolic metrics $\tilde{\lambda}$ and $\tilde{\varrho}$.

On the Lebesgue measure of $J_{\theta}$ for $\theta$ of constant type. Let $\tilde{J}_{\theta}=\exp ^{-1}\left(J_{\theta}\right)$. Moreover, for $\delta$ a continuous conformal metric on a connected open subset $\mathcal{W} \subset \mathbf{C}$ and a point $z \in \mathcal{W}$, let $B_{\delta, R}(z)=\left\{w \in \mathcal{W} \mid d_{\delta}(w, z) \leqslant R\right\}$. If no metric is specified, then $W=\mathbf{C}$ and $\delta$ is the Euclidean metric. Finally for $\omega \subset \mathcal{W}$ a Borel-measurable subset let Area $(\omega, \delta)$ denote the Area (infinite or not) of $\omega$ with respect to $\delta$. We write however mes $(\omega)$ for the Lebesgue measure of $\omega$.

Proposition 3.13. Let $\theta \in] 0,1\left[-\mathbf{Q}\right.$ be of constant type. There exist $R=R_{\theta}>0$ and $0<\alpha=\alpha_{\theta} \leqslant 1$ such that for all $z \in \tilde{X}_{0}^{*}$,

$$
\frac{\operatorname{Area}\left(\tilde{J}_{\theta} \cap B_{\tilde{\lambda}, R}(z), \tilde{\lambda}\right)}{\operatorname{Area}\left(B_{\tilde{\lambda}, R}(z), \tilde{\lambda}\right)} \leqslant 1-\alpha
$$

Proof. Given $\theta \in] 0,1\left[-\mathbf{Q}\right.$ of constant type let $0<\eta<\frac{1}{2} \pi$ and $r>0$ be as in Proposition 3.9 so that $\widetilde{X}_{0} \cap \overline{\mathbf{D}}_{r} \subset \widetilde{S}_{r}(\eta)$. Moreover, let $0<\eta_{1} \leqslant \frac{1}{6} \pi$ and $r_{1}>0$ be such that $S_{r_{1}}\left(\eta_{1}\right) \subset \widetilde{U}_{0} \cup\{0\}$. Define $R=\operatorname{dist}_{\tilde{\lambda}}\left(\tilde{l}(-\eta), \tilde{l}\left(\eta_{1}\right)\right)$. The function

$$
\operatorname{relA}(z)=\frac{\operatorname{Area}\left(\tilde{J}_{\theta} \cap B_{\tilde{\lambda}, R}(z), \tilde{\lambda}\right)}{\operatorname{Area}\left(B_{\bar{\lambda}, R}(z), \tilde{\lambda}\right)}
$$

is a continuous function of $z \in \mathbf{H}_{+}$. Moreover, $\operatorname{relA}(z)<1$ for all $z \in \widetilde{X}_{0}^{*}$, because $J_{\theta}$ and hence $\tilde{J}_{\theta}$ has empty interior. Thus it suffices to prove that

$$
\begin{equation*}
\limsup _{|z| \rightarrow 0} \operatorname{relA}(z)<1, \quad z \in \widetilde{X}_{0}^{*} \tag{1}
\end{equation*}
$$

Let $R_{1}=\frac{1}{2} \operatorname{dist}_{\tilde{\lambda}}\left(\tilde{l}\left(-\eta_{1}\right), \tilde{l}\left(\eta_{1}\right)\right)$. Moreover, let $r_{2}=\exp (-R) \cdot \min \left\{r, r_{1}\right\}$. Then for any $z \in \widetilde{S}_{r_{2}}(\eta)$ we have

$$
\begin{equation*}
B_{R_{1}, \tilde{\lambda}}(|z|) \subset B_{R, \tilde{\lambda}}(z) \cap \widetilde{U}_{0} \subset \mathbf{H}_{+}-\tilde{J}_{\theta} \tag{2}
\end{equation*}
$$

The number

$$
\alpha=\frac{\operatorname{Area}\left(B_{R_{1}, \tilde{\lambda}}(|z|), \tilde{\lambda}\right)}{B_{R, \tilde{\lambda}}(z)}>0
$$

does not depend on $z \in \widetilde{S}_{r_{2}}(\eta)$. Hence we have proved that the limsup of (1) is bounded by $1-\alpha$.

Lemma 3.14. Let $U, V \subsetneq \mathbf{C}$ be open hyperbolic subsets, i.e. carrying hyperbolic metrics $\lambda_{U}$ and $\lambda_{V}$ respectively, and let $d_{U}(\cdot, \cdot)$ denote the corresponding distance function on $U$. Suppose that $U$ is simply-connected and let $f: U \rightarrow V$ be a univalent map. For $z, w \in U$ arbitrary let $T=\left\|D_{z} f\right\|_{\lambda_{V}, \lambda_{U}}$ and $R=d_{U}(z, w)$. Then

$$
\frac{\sinh R}{\exp R} \cdot \frac{T}{\lambda_{V}(f(z))} \leqslant|f(w)-f(z)| \leqslant \frac{T}{\lambda_{V}(f(z))} \cdot \frac{\sinh R}{\exp (-R)}
$$

Proof. Let $z \in U$ be arbitrary and let $\phi: \mathbf{D} \rightarrow U$ be biholomorphic with $\phi(0)=z$. Then $\phi$ is a hyperbolic isometry, so that we can suppose that $U=\mathbf{D}$. The lemma then follows from the estimates

$$
\frac{|w|}{(1+|w|)^{2}}\left|g^{\prime}(0)\right| \leqslant|g(w)-g(0)| \leqslant\left|g^{\prime}(0)\right| \frac{|w|}{(1-|w|)^{2}}
$$

for univalent maps $g: \mathbf{D} \rightarrow \mathbf{C}$.
Lemma 3.15. Let $U \subsetneq \mathbf{C}$ be an open simply-connected subset. Let $\lambda_{U}$ be the coefficient function of the hyperbolic metric and let $d_{U}(\cdot, \cdot)$ denote the hyperbolic distance function. For $z, w \in U$ arbitrary let $R=d_{U}(z, w)$. Then

$$
\exp (-2 R) \leqslant \frac{\lambda_{U}(w)}{\lambda_{U}(z)} \leqslant \exp (2 R)
$$

Proof. Easy consequence of the distortion theorem for univalent mappings of the disc.

ThEOREM 3.16. Let $\theta \in] 0,1[-\mathbf{Q}$ be of constant type, let $R>0$ and $0<\alpha \leqslant 1$ be as in Proposition 3.13 and let $\alpha^{\prime}=\min \left\{\frac{1}{2}, \alpha \cdot e^{(-12 R)}\right\}>0$. Then for all $z \in \widetilde{X}_{0}^{*}$,

$$
\liminf _{r \rightarrow 0} \frac{\operatorname{mes}\left(B_{r}(z) \cap \tilde{J}_{\theta}\right)}{\operatorname{mes}\left(B_{r}(z)\right)} \leqslant 1-\alpha^{\prime}<1
$$

Proof. Let $\theta \in] 0,1\left[-\mathbf{Q}\right.$ of constant type be given. Define Cdens: $\mathbf{H}_{+} \times \mathbf{R}_{+} \rightarrow[0,1]$ by

$$
\operatorname{Cdens}(z, r)=\frac{\operatorname{mes}\left(B_{r}(z)-\tilde{J}_{\theta}\right)}{\operatorname{mes}\left(B_{r}(z)\right)}
$$

The statement of the theorem is that $\lim \sup _{r \rightarrow 0} \operatorname{Cdens}(z, r) \geqslant \alpha^{\prime}$ for any $z \in \widetilde{X}_{0}^{*}$. For any $z \in \exp ^{-1}\left(J_{\theta}^{\text {skeleton }}\right) \cap \widetilde{X}_{0}^{*}$ we have limsup ${ }_{r \rightarrow 0} \operatorname{Cdens}(z, r) \geqslant \frac{1}{2} \geqslant \alpha^{\prime}$, because any inverse image of $\partial U_{0}-\{1\}$ is an analytic arc with $\mathbf{C}-J_{\theta}$ on one side. Thus we need only consider points $z \in \widetilde{X}_{0}^{*}$ with $\exp (z)$ of infinite address.

Let $z \in \widetilde{X}_{0}^{*}$ be a point with $\exp (z)$ of infinite address (itinerary) $s \in \mathbf{N}^{\mathbf{N}}$. Let $z_{m} \in \tilde{X}_{0}^{*}$ be given by $\exp \left(z_{m}\right)=f_{\theta}^{\left(s_{1}+\ldots+s_{m}\right)}(\exp (z))=F_{\theta}^{m}(\exp (z))$ so that $z=G_{s_{1}, \ldots, s_{m}}\left(z_{m}\right)$ for all $m \geqslant 1$. Let $\tilde{\delta}_{m}$ denote the hyperbolic metric on $\widetilde{W}_{s_{1}+\ldots+s_{m}}$ so that $G_{s_{1}, \ldots, s_{m}}$ is an isometry with respect to $\tilde{\lambda}$ and $\tilde{\delta}_{m}$. Define

$$
\varpi_{m}(z)=G_{s_{1}, \ldots, s_{m}}\left(B_{R, \tilde{\lambda}}\left(z_{m}\right)\right)=B_{R, \delta_{m}}(z) .
$$

Let $0<M<1$ be as in the proof of Theorem 3.12. Then $\left\|D_{z_{m}} G_{s_{1}, \ldots, s_{m}}\right\|_{\tilde{\lambda}} \leqslant M^{m} \rightarrow 0$ when $m \rightarrow \infty$. Thus the Euclidean diameter of $\varpi_{m}(z)$ tends to 0 when $m \rightarrow \infty$ by Lemma 3.14. Moreover,

$$
\frac{\operatorname{Area}\left(\varpi_{m}(z)-\tilde{J}_{\theta}, \delta_{m}\right)}{\operatorname{Area}\left(\varpi_{m}(z), \delta_{m}\right)}=\frac{\operatorname{Area}\left(B_{\tilde{\lambda}, R}\left(z_{m}\right)-\tilde{J}_{\theta}, \tilde{\lambda}\right)}{\operatorname{Area}\left(B_{\tilde{\lambda}, R}\left(z_{m}\right), \tilde{\lambda}\right)} \geqslant \alpha
$$

because $G_{s_{1}, \ldots, s_{m}}$ is an isometry. Next Lemma 3.15 implies that

$$
\frac{\operatorname{mes}\left(\varpi_{m}(z)-\tilde{J}_{\theta}\right)}{\operatorname{mes}\left(\varpi_{m}(z)\right)} \geqslant \alpha \exp (-8 R)
$$

Finally Lemma 3.14 implies that there exists $r_{m}>0$ such that

$$
\overline{\mathbf{D}}_{r_{m}}(z) \subset \varpi_{m} \subset \mathbf{D}_{r_{m} \exp (2 R)}(z)
$$

Hence $\operatorname{Cdens}\left(z, e^{2 R} r_{m}\right) \geqslant \alpha \exp (-12 R)$. This completes the proof.
Corollary 3.17. For $\theta \in] 0,1\left[-\mathbf{Q}\right.$ of constant type, the set $J_{\theta}$ has zero Lebesgue measure and so has $J_{P_{\theta}}$, the Julia set of the quadratic polynomial $P_{\theta}$.

Proof. Let $K \subset \mathbf{R}^{2}$ be a compact set. A point $z \in K$ is called a density point for $K$ if

$$
\lim _{r \rightarrow 0} \frac{\operatorname{mes}\left(K \cap B_{r}(z)\right)}{\operatorname{mes}\left(B_{r}(z)\right)}=1
$$

The Lebesgue density theorem states that: For any compact set $K \subset \mathbf{R}^{2}$ almost all points of $K$ are density points for $K$.

Theorem 3.16 implies that $X_{0}^{*}=X_{0}-\{1\}$ does not contain any density points for $J_{\theta}$, because $\widetilde{X}_{0}^{*}$ does not contain any density points for $\tilde{J}_{\theta}$ and exp is locally biholomorphic. Thus $X_{0}^{*}$ is a null set by the Lebesgue density theorem. Finally $J_{\theta}=f_{\theta}\left(X_{0}^{*}\right) \cup\{v\}$ is a null set because $f_{\theta}$ is holomorphic and injective on $X_{0}^{*}$.

Controlling the Core of Nests for all irrational $\theta$. This subsection is devoted to spreading local connectivity to all points $z$ with infinite address. The special case $\theta$ of constant type was handled in Theorem 3.12. Here we give a slightly different proof independent of the combinatorics of $\theta$. We shall use frequently all of the subsection "Lifting to the exponential" except Theorem 3.12.

For each irrational $\theta$ we single out the following set of exceptional addresses:

$$
\mathcal{E}_{\theta}=\left\{j=k q_{n}+q_{n-1} \mid n \geqslant 1,0<k \leqslant a_{n}\right\} .
$$

We divide $\mathbf{N}^{\mathbf{N}}$ into the following three classes:

$$
\begin{aligned}
\mathrm{It}_{E B}(\theta) & =\left\{\left(s_{1}, \ldots, s_{m}, \ldots\right) \mid \liminf s_{m}<\infty\right\} \\
\mathrm{It}_{R B}(\theta) & =\left\{\left(s_{1}, \ldots, s_{m}, \ldots\right) \notin \mathrm{It}_{E B}(\theta) \mid \sup \left\{m \mid s_{m} \notin \mathcal{E}\right\}=\infty\right\} \\
\mathrm{It}_{S R}(\theta) & =\left\{\left(s_{1}, \ldots, s_{m}, \ldots\right) \notin\left(\operatorname{It}_{E B}(\theta) \cup \mathrm{It}_{R B}(\theta)\right)\right\} \\
& =\left\{\left(s_{1}, \ldots, s_{m}, \ldots\right) \mid s_{m} \underset{m \rightarrow \infty}{ } \infty \text { and } \exists m_{0}: \forall m \geqslant m_{0}, s_{m} \in \mathcal{E}\right\}
\end{aligned}
$$

Addresses in the first two classes can be handled with essentially the information at hand, but in order to handle also the more difficult addresses in $\mathrm{It}_{S R}(\theta)$ we shall obtain a new family of curves by cutting and pasting iterated preimages of the $\Sigma_{n}$.

Proposition and Definition 3.18. There exists a family of Jordan curves $\left\{\Upsilon_{n, 0}\right\}_{n \geqslant 3}, \Upsilon_{n, 0}=\Upsilon_{n}^{\prime} \cup\left\lceil 1, x_{q_{n}}\right\rceil$, and a constant $L_{\Upsilon, \theta}^{\prime}>0$ such that

$$
\begin{gather*}
l_{J_{n, 0}}\left(\Upsilon_{n}^{\prime}\right) \leqslant L_{\Upsilon, \theta}^{\prime} \quad \forall n \geqslant 3  \tag{1}\\
Y_{\alpha q_{n}+q_{n-1}} \subset D\left(\Upsilon_{n, 0}\right) \quad \forall n \geqslant 3 \text { and for each } 0<\alpha \leqslant a_{n} \tag{2}
\end{gather*}
$$

Proof. For $n \geqslant 1$ let $g_{n}: D\left(\Sigma_{n}\right) \rightarrow D\left(\Gamma_{n}\right)$ be the Gain of $\Sigma_{n}$. Then $g_{n}$ is a local branch of $f_{\theta}^{-q_{n+1}}$ mapping 1 to $y_{q_{n+1}}$ and $D\left(\Gamma_{n}\right) \subset D\left(\Sigma_{n}\right)$ by Proposition 1.10. If $a_{n+1}>1$ then the Gain of $\Gamma_{n}$ coincides with the restriction of $g_{n}$ to $D\left(\Gamma_{n}\right)$, because it is a local inverse of $f_{\theta}^{q_{n+1}}$ mapping 1 to $y_{q_{n+1}}$. Let $H_{n}=g_{n}^{a_{n+1}}$,

$$
\Sigma_{n}\left(x_{q_{n}}, y_{t_{n}}, x_{q_{n}, t_{n-1}}\right) \xrightarrow[a_{n+1} \text { Gains }]{H_{n}} \Xi_{n+2}\left(x_{q_{n+2}}, y_{q_{n+1}}, x_{q_{n+2}, t_{n-1}}\right)
$$

be the long composition of the $a_{n+1}$ consecutive Gains starting from $\Sigma_{n}$. Essentially the curve $\Xi_{n+2}$ is the curve we want, except that its $\lambda_{J_{n, 0}}$-length grows linearly with $a_{n+1}$. Thus if $a_{n+1} \leqslant 2$ we define $\Upsilon_{n+2,0}=\Xi_{n+2}$ and $\left.\Upsilon_{n+2}^{\prime}=\Upsilon_{n+2,0}-\right\rceil 1, x_{q_{n+2}}$. If $a_{n+1} \geqslant 3$ we shall replace the part of $\Xi_{n+2}$ whose $\lambda_{J_{n, 0}}$-length is proportional to $a_{n+1}$ by a shortcut of bounded length.


Fig. 23. Constructing the new Jordan curves $\Upsilon_{n, 0}$ cutting and pasting bits and pieces
Define $P_{n+2}=H_{n}\left(B_{n} \cdot G_{n} \cdot R_{n} \cdot \gamma_{t_{n}, q_{n+1}}\right)$ if $a_{n} \neq 1$ and $P_{n+2}=H_{n}\left(B_{n} \cdot G_{n} \cdot R_{n}\right)$ if $a_{n}=1$. Thus

$$
P_{n+2}=H_{n}\left(\Sigma_{n}-\left\lceil 1, x_{q_{n}}\left\lceil-\left\lceil 1, y_{q_{n+1}}\lceil )\right.\right.\right.\right.
$$

and it starts at $x_{q_{n+2}}$, leaves $\partial U_{q_{n+2}}$ at $x_{q_{n+2}, t_{n-1}}$ and ends at

$$
y_{\underbrace{q_{n+1}, \ldots, q_{n+1}}_{1+a_{n+1} \text { times }}}
$$

(see also Figure 23). Suppose that $a_{n+1} \geqslant 3$, and define

$$
\begin{aligned}
\Upsilon_{n+2}^{\prime}=P_{n+2} & \cdot g_{n}^{1+a_{n+1}}\left(\left\lceil 1, y_{q_{n+1}}\right\rceil\right) \cdot g_{n}^{2+a_{n+1}}\left(\left\lceil 1, y_{q_{n+1}}\right\rceil\right) \\
& \cdot\left(g_{n}^{2}\left(-P_{n+2}\right)\right) \cdot g_{n}\left(\left\lceil y_{t_{n+2}}, 1\right\rceil\right) \cdot\left\lceil y_{q_{n+1}}, 1\right\rceil
\end{aligned}
$$

and $\Upsilon_{n+2,0}=\Upsilon_{n}^{\prime} \cup\left\lceil 1, x_{q_{n+2}}\right\rceil$. We have $\Upsilon_{n+2,0} \cap J_{f_{\theta}} \subset Y_{q_{n+1}} \cup \partial U_{0} \cup \mathbf{S}^{1} \cup X_{q_{n+2}}$ by construction. Thus $Y_{\alpha q_{n+2}+q_{n+1}} \subset D\left(\Upsilon_{n+2,0}\right)$ for each $0<\alpha \leqslant a_{n+2}$. This proves property (2).

To prove (1) and thus the existence of $L_{\Upsilon, \theta}^{\prime}$ we prove that there exists a universal constant $L_{\Upsilon}^{\prime}$, i.e. independent of $\theta$ such that

$$
\begin{equation*}
\lim \sup l_{J_{n+2,0}}\left(\Upsilon_{n+2,0}^{\prime}\right) \leqslant L_{\Upsilon}^{\prime} \tag{3}
\end{equation*}
$$

We shall prove (3) by proving that the $\lambda_{J_{n+2,0}}$-lengths of the 6 constituent subarcs of $\Upsilon_{n+2,0}^{\prime}$ are asymptotically universally bounded. Whenever a constituent is compactly contained in $\mathbf{C}-\overline{\mathbf{D}}$ it suffices to give a bound for its $\lambda$-length, because $\mathbf{C}-\overline{\mathbf{D}} \subset A_{J_{n, 0}}$. Moreover, $g_{n}$ is infinitesimally contracting with respect to $\lambda$, so it suffices to give bounds for the following four lengths:

$$
\begin{align*}
& l_{J_{n+2,0}}\left(P_{n+2}\right)  \tag{4}\\
& l_{\lambda}\left(g_{n}\left(\left\lceil 1, y_{q_{n+1}}\right\rceil\right)\right) \quad\left(\left\lceil 1, y_{t_{n+2}}\right\rceil \subset\left\lceil 1, y_{q_{n+1}}\right\rceil\right)  \tag{5}\\
& l_{\lambda}\left(g_{n}\left(P_{n+2}\right)\right)  \tag{6}\\
& l_{J_{n+2,0}}\left(\left\lceil 1, y_{q_{n+1}}\right\rceil\right) \tag{7}
\end{align*}
$$

The careful reader may easily verify that this also suffices to cover the cases $a_{n+1} \leqslant 2$. We shall concentrate on (4) and (6), as the two others essentially are treated in Lemma 3.3 and the proof of Theorem $2.2(4)$. Furthermore, $P_{n+2}=H_{n}\left(B_{n}\right) \cdot H_{n}\left(G_{n} \cdot R_{n} \cdot \gamma_{t_{n}, q_{n+1}}\right)$ with $\gamma_{t_{n}, q_{n+1}}$ possibly being a point. As we have asymptotically universal bounds for $l_{\lambda}\left(G_{n} \cdot R_{n} \cdot \gamma_{t_{n}, q_{n+1}}\right)$ (for $\gamma_{t_{n}, q_{n+1}}$ see Proposition and Definition 3.8) we are left with only $l_{J_{n+2,0}}\left(H_{n}\left(B_{n}\right)\right)$ and $l_{\lambda}\left(g_{n}\left(H_{n}\left(B_{n}\right)\right)\right)$.

Note that $f_{\theta}^{q_{n+2}}$ maps $H_{n}\left(B_{n}\right)$ diffeomorphically onto $\left\lceil 1, y_{t_{n-1}}\right\rceil \subset Q_{n-2}$, because $B_{n}=\left\lceil x_{q_{n}}, x_{q_{n}, t_{n-1}}\right\rceil \subset \partial U_{q_{n}}$ by construction of $\Sigma_{n}$ (recall Definition 2.7). Moreover, $f_{\theta}^{q_{n+2}}$ maps the arc $K_{n+2}^{\prime}=\left\lceil 1, x_{-q_{n+3}+q_{n+2}}\right\rceil \subset J_{n+2,0}$ diffeomorphically onto $K_{n+2}$. Let $\varrho_{n+2}$ denote the hyperbolic metric on $W_{J_{n+2, q_{n+1}}}$ and note that $f_{\theta}^{q_{n+1}} \circ g_{n}=$ Id. Combining Lemma 3.2 with Proposition 2.8 and Lemma 2.9 we obtain

$$
\begin{align*}
l_{Q_{n+2}}\left(g_{n}\left(H_{n}\left(B_{n}\right)\right)\right) & \leqslant l_{J_{n+2,0}}\left(H_{n}\left(B_{n}\right)\right)  \tag{8}\\
& <l_{K_{n+2}^{\prime}}\left(H_{n}\left(B_{n}\right)\right) \leqslant l_{K_{n+2}}\left(Q_{n-2}\right) \leqslant L_{4, \theta} .
\end{align*}
$$

This takes care of $l_{J_{n+2,0}}\left(H_{n}\left(B_{n}\right)\right)$. To obtain a bound for $l_{\lambda}\left(g_{n}\left(H_{n}\left(B_{n}\right)\right)\right)$ from (8) we combine Lemma 2.4 with Lemma 2.5 as in the proof of Lemma 3.6. This completes the proof.

Definition 3.19. For $1 \leqslant n$ and $0<j<q_{n+1}$ let $\Upsilon_{n, j}$ be the unique lift of $\Upsilon_{n, 0}$ to $f_{\theta}^{j}$ intersecting $\partial U_{0}$. Note that the previously defined arc $I_{n, 0}$ equals both the $I$ of $\Sigma_{n, 0}$ and the intersection $\Upsilon_{n, 0} \cap \mathbf{S}^{1}$ for $n \geqslant 3$. Let $I_{n, j}^{\prime}=\Upsilon_{n, j} \cap f_{\theta}^{-j}\left(I_{n, 0}\right)$ for each $n \geqslant 3$ and $0<j<q_{n+1}$.

LEMMA 3.20. For each $\theta \in] 0,1\left[-\mathbf{Q}\right.$ there exist constants $L_{\Upsilon, \theta}^{\prime}, L_{\Upsilon, \theta}>0$ ( $L_{\Upsilon, \theta}^{\prime}$ equals the constant of Proposition and Definition 3.18) such that $\forall n \geqslant 3$ and $0<j<q_{n+1}$,

$$
\begin{array}{r}
l_{J_{n, j-1}}\left(f_{\theta}\left(\Upsilon_{n, j}-I_{n, j}^{\prime}\right)\right) \leqslant L_{\Upsilon, \theta}^{\prime}, \\
l_{\lambda}\left(\Upsilon_{n, j}\right) \tag{2}
\end{array} \leqslant L_{\Upsilon, \theta} .
$$

Proof. The proof is a simple copy of the proof of Lemma 3.6 and is left to the reader.

For $n \geqslant 1$ and $0<j<q_{n+1}$ let $\tilde{\Delta}_{n, j}$ be the connected component of $\exp ^{-1}\left(\Delta_{n, j}\right)$ intersecting $\partial \widetilde{U}_{0}$. Moreover, for $n \geqslant 3$ and $0<j<q_{n+1}$ let $\widetilde{\Upsilon}_{n, j}$ be the connected component of $\exp ^{-1}\left(\Upsilon_{n, j}\right)$ intersecting $\partial \widetilde{U}_{0}$. Recall the definition of $\Omega(T), T>0$, from the subsection "Lifting to the exponential".

Lemma 3.21. For each $\theta \in] 0,1[-\mathbf{Q}$ there exist constants

$$
T_{\Delta, \theta}, T_{\Upsilon, \theta} \geqslant T_{C}=\log (1+\sqrt{2})
$$

such that

$$
\begin{array}{cl}
\stackrel{\circ}{D}\left(\tilde{\Delta}_{n, j}\right) \subset \Omega\left(T_{\Delta, \theta}\right) & \forall n \geqslant 1,0<j<q_{n+1}, \\
\stackrel{\circ}{D}\left(\widetilde{\Upsilon}_{n, j}\right) \subset \Omega\left(T_{\Upsilon, \theta}\right) & \forall n \geqslant 3,0<j<q_{n+1} . \tag{2}
\end{array}
$$

Proof. We prove (1) and leave the similar proof of (2) to the reader. Let $\tilde{J}_{n, j-1}$ be the connected component of $\exp ^{-1}\left(J_{n, j-1}\right)$ contained in $J_{0}$ and let $\tilde{I}_{n, j}^{\prime}=\exp ^{-1}\left(I_{n, j}^{\prime}\right) \cap \tilde{\Delta}_{n, j}$. We shall prove the following slightly better statement, from which (1) follows. Let $\delta_{n, j-1}$ denote the hyperbolic metric on the doubly slit plane $\mathbf{C}_{\tilde{j}_{n, j-1}}$. There exists $T_{\Delta, \theta}>0$ such that

$$
\begin{equation*}
l_{\delta_{n, j-1}}\left(\tilde{f}_{\theta}\left(\tilde{\Delta}_{n, j}-\tilde{I}_{n, j}^{\prime}\right)\right) \leqslant 2 T_{\Delta, \theta}, \quad 1 \leqslant n, 0<j<q_{n+1} . \tag{3}
\end{equation*}
$$

To prove (3) it suffices to prove that

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} l_{\delta_{n, j-1}}\left(\tilde{f}_{\theta}\left(\tilde{\Delta}_{n, j}-\tilde{I}_{n, j}^{\prime}\right)\right) \leqslant \limsup _{n \rightarrow \infty} l_{J_{n, j-1}}\left(\Sigma_{n, j-1}-I_{n, j-1}\right) \leqslant L_{\Sigma, \theta} \tag{4}
\end{equation*}
$$

because the length $l_{\delta_{n, j-1}}\left(\tilde{f}_{\theta}\left(\tilde{\Delta}_{n, j}\right)\right)$ is always finite. Let $\tilde{A}_{n, j-1}=\exp ^{-1}\left(A_{J_{n, j-1}}\right)$ and let $\tilde{\lambda}_{n, j-1}$ denote the hyperbolic metric on $\tilde{A}_{n, j-1}$. Then

$$
\begin{equation*}
l_{\tilde{\lambda}_{n, j-1}}\left(\tilde{f}_{\theta}\left(\tilde{\Delta}_{n, j}-I_{n, j}^{\prime}\right)\right)=l_{J_{n, j-1}}\left(\Sigma_{n, j-1}-I_{n, j-1}\right) \leqslant L_{\Sigma, \theta}, \tag{5}
\end{equation*}
$$

by Lemma 3.6 and because the restriction exp: $\tilde{A}_{n, j-1} \rightarrow A_{J_{n, j-1}}$ is a local hyperbolic isometry. Moreover, by the same argument

$$
\begin{equation*}
d_{\tilde{\lambda}_{n, j-1}}\left(\tilde{f}_{\theta}\left(\tilde{\Delta}_{n, j}\right), \partial \mathbf{C}_{\tilde{J}_{n, j-1}}\right) \geqslant E\left(l_{e}\left(J_{n, j-1}\right)\right)-L_{\Sigma, \theta} \tag{6}
\end{equation*}
$$

The right hand side of (6) diverges to $\infty$ as $n$ diverges to $\infty$, by Lemma 3.4. We obtain (4) by combining this fact with Lemma 2.4 and (5). Recall Lemma 3.10.

Lemma 3.22. For each $\theta \in] 0,1\left[-\mathbf{Q}\right.$ there exist constants $L_{N, \theta}>0$ and $T_{N, \theta} \geqslant T_{C}$, $1 \leqslant N$, such that for each $0<s \leqslant N$,

$$
\begin{equation*}
\operatorname{diam}_{\tilde{\lambda}}\left(\tilde{Y}_{s}\right) \leqslant L_{N, \theta} \tag{1}
\end{equation*}
$$

and moreover, for all compact subsets $K \subset \tilde{X}_{0}^{*}$,

$$
\begin{equation*}
\operatorname{diam}_{\tilde{\lambda}}\left(G_{s}(K)\right) \leqslant M\left(T_{N, \theta}\right) \cdot \operatorname{diam}_{\tilde{\lambda}}(K) \tag{2}
\end{equation*}
$$

Proof. The sets $Y_{s}, 0<s \leqslant N$, are compact subsets of $\mathbf{H}_{+}$. This proves the existence of $L_{N}$ as in (1). We have $\bigcup_{s=1}^{N} \widetilde{X}_{s-1} \subset \omega\left(T_{N, \theta}\right)$ for $T_{n, \theta}$ sufficiently big, because the former is a compact subset of $\mathbf{C}_{0}$. Moreover, $\widetilde{Y}_{s}=G_{s}\left(\widetilde{X}_{0}\right)=G_{\theta}\left(\widetilde{X}_{s-1}\right)$ for each $s=1, \ldots, N$, and in particular $Y_{s}^{*} \subset \Omega\left(T_{N, \theta}\right)$. Increasing $T_{N, \theta}$ if necessary we can suppose that $T_{C} \leqslant T_{N, \theta}$. Since $G_{s}$ is Lipschitz with constant 1 for the hyperbolic metrics $\tilde{\lambda}$ and $\tilde{\varrho}$ we have

$$
\operatorname{diam}_{\tilde{e}}\left(G_{s}(K)\right) \leqslant \operatorname{diam}_{\tilde{\lambda}}(K)
$$

for any compact subset $K \subset X_{0}^{*}$. Furthermore, Lemma 3.10 implies

$$
\operatorname{diam}_{\tilde{\lambda}}\left(G_{s}(K)\right) \leqslant M\left(T_{N, \theta}\right) \cdot \operatorname{diam}_{\tilde{\varrho}}\left(G_{s}(K)\right)
$$

Recall that $\mathcal{E}=\left\{j=k q_{n}+q_{n-1} \mid n \geqslant 1,0<k \leqslant a_{n}\right\}$.
Lemma 3.23. Suppose that $s \notin \mathcal{E}$ and $s>q_{2}+1$. Then

$$
\begin{equation*}
\operatorname{diam}_{\tilde{\lambda}}\left(\widetilde{Y}_{s}\right) \leqslant L_{\Delta, \theta} \tag{1}
\end{equation*}
$$

and for all compact subsets $K \subset \widetilde{X}_{0}^{*}$,

$$
\begin{equation*}
\operatorname{diam}_{\tilde{\lambda}}\left(G_{s}(K)\right) \leqslant M\left(T_{\Delta, \theta}\right) \cdot \operatorname{diam}_{\tilde{\lambda}}(K) \tag{2}
\end{equation*}
$$

Proof. Let $q_{2}+1<s \notin \mathcal{E}$ be arbitrary, so that $s$ is of the form $s=\alpha q_{n+1}+q_{n}+j$ for some $n \geqslant 1,0<\alpha \leqslant a_{n+1}$ and $0<j<q_{n+1}$. As $Y_{\alpha q_{n+1}+q_{n}+j} \subset D\left(\Delta_{n, j}\right)$ and the covering map exp: $\mathbf{H}_{+} \rightarrow \mathbf{C}-\overline{\mathbf{D}}$ is a hyperbolic isometry, we obtain (1) from Lemma 3.6 and we obtain

$$
\widetilde{Y}_{\alpha q_{n+1}+q_{n}+j}^{*}=G_{\alpha q_{n+1}+q_{n}+j}\left(\tilde{X}_{0}^{*}\right) \subset \Omega\left(T_{\Delta, \theta}\right)
$$

by Lemma 3.21 (1). Applying Lemma 3.10 as in the above proof of Lemma 3.22 we obtain (2).

LEMMA 3.24. For $s_{2}=\alpha q_{n}+q_{n-1} \in \mathcal{E}, n \geqslant 3,0<\alpha \leqslant a_{n}$ and $s_{1}<s_{2}$, we have

$$
\begin{equation*}
\operatorname{diam}_{\tilde{\lambda}}\left(\tilde{Y}_{s_{1}, s_{2}}\right) \leqslant L_{\Upsilon, \theta} \tag{1}
\end{equation*}
$$

and for all compact subsets $K \subset \widetilde{Y}_{s_{2}}$,

$$
\begin{equation*}
\operatorname{diam}_{\tilde{\lambda}}\left(G_{s_{1}}(K)\right) \leqslant M\left(T_{\Upsilon, \theta}\right) \cdot \operatorname{diam}_{\tilde{\lambda}}(K) \tag{2}
\end{equation*}
$$

Proof. We have $s_{1}<s_{2}=\alpha q_{n}+q_{n-1} \leqslant q_{n+1}, 0<\alpha \leqslant a_{n}$ and $Y_{s_{2}} \subset D\left(\Upsilon_{n, 0}\right)$, which implies that $Y_{s_{1}, s_{2}} \subset D\left(\Upsilon_{n, s_{1}}\right)$. From here and onwards the proof goes as in the previous lemma, except that we use Lemma 3.20 to obtain (1) and Lemma 3.21 (2) to obtain (2).

Theorem 3.25. For each $\theta \in] 0,1\left[-\mathbf{Q}\right.$, $\operatorname{Core}\left(\mathcal{Y}_{\theta, \underline{s}}\right)$ is trivial for any $\underline{s} \in \mathbf{N}^{\mathbf{N}}$. In particular, $J_{\theta}$ is locally connected for each irrational $\theta$.

Proof. Let $\theta \in] 0,1[-\mathbf{Q}$ be given. We shall prove the following equivalent statement of the theorem: Let $\underline{s}=\left(s_{1}, \ldots, s_{m}, \ldots\right)$ be arbitrary. There exists $\tilde{z}_{\underline{s}} \in \mathbf{H}_{+}$such that

$$
\bigcap_{m \geqslant 1} \tilde{Y}_{s_{1}, \ldots, s_{m}}=\left\{\tilde{z}_{\underline{s}}\right\}
$$

We shall treat the three types of addresses $\mathrm{It}_{E B}(\theta), \mathrm{It}_{R R}(\theta)$ and $\mathrm{It}_{S R}(\theta)$ separately. Suppose first that $\underline{s} \in \operatorname{It}_{E B}(\theta)$ and let $N=\liminf s_{m}$. Let $L_{N, \theta}$ and $T_{N, \theta}$ be constants as in Lemma 3.22. Define $M_{N}=M\left(T_{N, \theta}\right)$, where $M(\cdot)$ is the function of Lemma 3.10. Moreover, define $\chi(m)=\#\left\{l<m \mid s_{l} \leqslant N\right\}$, where $\#(\cdot)$ denotes the cardinality. Then for any $m$ with $s_{m} \leqslant N$,

$$
\operatorname{diam}_{\tilde{\lambda}}\left(Y_{s_{1}, \ldots, s_{m}}\right)=\operatorname{diam}_{\tilde{\lambda}}\left(G_{s_{1}} \circ \ldots \circ G_{s_{m-1}}\left(Y_{s_{m}}\right)\right) \leqslant L_{N, \theta} \cdot\left(M_{N}\right)^{\chi(m)} \underset{m \rightarrow \infty}{\longrightarrow} 0
$$

This proves the case $\underline{s} \in \mathrm{It}_{E B}(\theta)$.
Suppose next that $\underline{s} \in \operatorname{It}_{R R}(\theta)$. Shifting $\underline{s}$ some number of times if necessary, we can suppose that $s_{m}>q_{2}+1$ for all $m$, so that Lemma 3.23 applies whenever $s_{m} \notin \mathcal{E}$. Define $M_{\Delta}=M\left(T_{\Delta, \theta}\right)$ and $\chi(m)=\#\left\{l<m \mid s_{l} \notin \mathcal{E}\right\}$. Then for any $m$ with $s_{m} \notin \mathcal{E}$,

$$
\operatorname{diam}_{\tilde{\lambda}}\left(Y_{s_{1}, \ldots, s_{m}}\right)=\operatorname{diam}_{\tilde{\lambda}}\left(G_{\left.s_{1} \circ \ldots \circ G_{s_{m-1}}\left(Y_{s_{m}}\right)\right) \leqslant L_{\Delta, \theta} \cdot\left(M_{\Delta}\right)^{\chi(m)} \underset{m \rightarrow \infty}{\longrightarrow} 0 . . . . .}\right.
$$

This proves the case $\underline{s} \in \operatorname{It}_{R R}(\theta)$.
Finally suppose that $\underline{s} \in \mathrm{It}_{S R}(\theta)$. Shifting $\underline{s}$ some number of times if necessary, we can suppose that $s_{m} \in \mathcal{E}$ and $s_{m}>q_{3}$ for all $m$.

Let $L=L_{\Upsilon, \theta}$ and $M=M\left(T_{\Upsilon, \theta}\right)$. Define $\chi(m)=\#\left\{l<m-1 \mid s_{l}>s_{l-1}\right\}$. Then for any $m$ with $s_{m}>s_{m-1}$,

$$
\operatorname{diam}_{\tilde{\lambda}}\left(Y_{s_{1}, \ldots, s_{m}}\right)=\operatorname{diam}_{\tilde{\lambda}}\left(G_{s_{1}} \circ \ldots \circ G_{s_{m-2}}\left(Y_{s_{m-1}, s_{m}}\right)\right) \leqslant L \cdot M^{\chi(m)} \underset{m \rightarrow \infty}{\longrightarrow} 0
$$

This proves the case $\underline{s} \in \mathrm{It}_{S R}(\theta)$ and completes the proof.

ThEOREM $3.26(1)$. For all $\theta \in] 0,1\left[-\mathbf{Q}\right.$ and for all $z \in Y_{\theta}$ we have

$$
\left\|D_{z} F_{\theta}\right\|_{\lambda}=\frac{\lambda\left(F_{\theta}(z)\right)}{\lambda(z)}\left|F_{\theta}^{\prime}(z)\right|>1 \quad \text { and } \quad\left\|D_{z} F_{\theta}^{m}\right\|_{\lambda} \underset{m \rightarrow \infty}{\longrightarrow} \infty
$$

Proof. Let $\theta \in] 0,1[-\mathbf{Q}$ be given. The first inequality follows from

$$
\left\|D_{z} f_{\theta}\right\|_{\lambda}=\left\|D_{z} f_{\theta}\right\|_{\lambda, \varrho} \cdot \frac{\varrho(z)}{\lambda(z)}=\frac{\varrho(z)}{\lambda(z)}>1 \quad \forall z \in W_{1}
$$

and the definition of $F_{\theta}$. For each $T>0$ we have $\varrho(z) / \lambda(z) \geqslant 1 / M(T)$ for all $z \in \exp (\Omega(T))$ (recall Lemma 3.10), because $\exp$ is a local isometry for both the pair of metrics $\tilde{\lambda}, \lambda$ and $\tilde{\varrho}, \varrho$.

We proceed to prove the second part of the theorem. Let $z \in Y_{\theta}$ with itinerary $\left(s_{1}, \ldots, s_{m}, \ldots\right)=\underline{s} \in \mathbf{N}^{\mathbf{N}}$ be arbitrary. We shall consider separately the three cases of $\underline{s}$ belonging to the three different classes of addresses $\mathrm{It}_{E B}, \mathrm{It}_{R R}, \mathrm{It}_{S R}$.

Suppose first that $\underline{s} \in \mathrm{It}_{E B}$. Let $N=\liminf s_{m}$ and let $T_{N, \theta}$ be as in the proof of Lemma 3.22, so that $Y_{s} \in \exp \left(\Omega\left(T_{N}\right)\right)$ for all $0<s \leqslant N$. Let $M=M\left(T_{N, \theta}\right)$, where $M(\cdot)$ is the function of Lemma 3.10. Moreover, define $\chi(m)=\#\left\{l \leqslant m \mid s_{l} \leqslant N\right\}$, where \#( $)$ denotes the cardinality. Then

$$
\begin{equation*}
\left\|D_{z} F_{\theta}^{m}\right\|_{\lambda} \geqslant M^{-\chi(m)} \underset{m \rightarrow \infty}{\longrightarrow} \infty \tag{1}
\end{equation*}
$$

This proves the case $s \in \mathrm{It}_{E B}(\theta)$.
Secondly suppose that $\underline{s} \in \operatorname{It}_{R R}(\theta)$. Shifting $\underline{s}$ some number of times if necessary, we can suppose that $s_{m}>q_{2}+1$ for all $m$. Recall from the proof of Lemma 3.23 that $Y_{s} \subset \exp \left(\Omega\left(T_{\Delta, \theta}\right)\right)$ for all $s \notin \mathcal{E}$. Define $M=M\left(T_{\Delta, \theta}\right)$ and $\chi(m)=\#\left\{l \leqslant m \mid s_{l} \notin \mathcal{E}\right\}$. Then (1) holds again. This proves the case $\underline{s} \in \mathrm{It}_{R R}(\theta)$.

Finally suppose that $\underline{s} \in \mathrm{It}_{S R}(\theta)$. Shifting $\underline{s}$ some number of times if necessary, we can suppose that $s_{m} \in \mathcal{E}$ and $s_{m}>q_{3}$ for all $m$. Recall from the proof of Lemma 3.24 that $Y_{s_{1}, s_{2}} \subset \exp \left(\Omega\left(T_{\Upsilon, \theta}\right)\right)$ whenever $s_{1}<s_{2} \in \mathcal{E}$ and $s_{2}>q_{3}$.

Let $M=M\left(T_{\Upsilon, \theta}\right)$. Define $\chi(m)=\#\left\{l \leqslant m \mid s_{l}<s_{l+1}\right\}$. Then (1) holds again. This proves the case $\underline{s} \in \mathrm{It}_{S R}(\theta)$ and completes the proof.

## 4. Local connectivity of $\boldsymbol{J}_{\boldsymbol{f}}$

Let $Z_{\theta}$ be the subset of $J_{f_{\theta}}$ consisting of those points which pass infinitely often through $U_{+}=U_{0}$ and $U_{-}=\tau\left(U_{+}\right)$. In this section we shall prove the following theorem.

Theorem 4.1. For all $\theta \in] 0,1\left[-\mathbf{Q}\right.$ any point of $Z_{\theta}$ has a fundamental system of connected neighbourhoods in $J_{f_{\theta}}$.

For $s \geqslant 1$ we define $Z_{+s}=f_{\theta}^{-1}\left(X_{-(s-1)}\right) \cap \bar{U}_{0}$ and $Z_{-s}=\tau\left(Z_{+s}\right)$. The sets $Z_{-s}$ and $Z_{+s}$ are connected and $y_{s} \in Z_{+s}$. We shall rename $y_{s}$ to $y_{+s}$ and define $y_{-s}=\tau\left(y_{+s}\right)$. We shall say that $Z_{+s}$ is the internal limb of $U_{+}$with root $y_{+s}$ and that $Z_{-s}$ is the internal limb of $U_{-}$with root $y_{-s}$.

Rename the hyperbolic metric $\lambda$ on $\mathbf{C - D}$ to $\lambda_{+}$and let $\lambda_{-}$denote the hyperbolic metric on $\mathbf{D}^{*}=\mathbf{D}-\{0\}$. Moreover, let $\delta_{+}$denote the hyperbolic metric on $U_{+}^{*}=$ $U_{+}-f_{\theta}^{-1}(0)$ and let $\delta_{-}$denote the hyperbolic metric on $U_{-}^{*}=U_{-}-f_{\theta}^{-1}(\infty)$. We note immediately that $\tau$ is an isometry with respect to both of the pairs of metrics $\lambda_{ \pm}$and $\delta_{ \pm}$.

Lemma 4.2. The local inverse branches $f_{\theta}^{-1}: \mathbf{D}^{*} \rightarrow U_{+0}^{*}$ and $f_{\theta}^{-1}: \mathbf{C}-\overline{\mathbf{D}} \rightarrow U_{-0}^{*}$ are strong contractions with respect to the pair of hyperbolic metrics $\lambda_{+}$and $\lambda_{-}$. More precisely, there exists a constant $0<C<1$ such that

$$
\left\|D_{z} f_{\theta}^{-1}\right\|_{\lambda_{+}, \lambda_{-}} \leqslant C \cdot\left\|D_{z} f_{\theta}^{-1}\right\|_{\delta_{+}, \lambda_{-}}=C \quad \forall z \in \mathbf{D}^{*}
$$

and

$$
\left\|D_{z} f_{\theta}^{-1}\right\|_{\lambda_{-}, \lambda_{+}} \leqslant C\left\|D_{z} f_{\theta}^{-1}\right\|_{\delta_{-}, \lambda_{+}}=C \quad \forall z \in \mathbf{C}-\overline{\mathbf{D}}
$$

Proof. Let us prove that there exists a constant $0<C<1$ such that $\lambda_{+}(z) / \delta_{+}(z) \leqslant C$ for all $z \in U_{+}$and $\lambda_{-}(z) / \delta_{-}(z) \leqslant C$ for all $z \in U_{-}$. These two inequalities are equivalent because $\tau$ is a hyperbolic isometry.

The first inequality, say, follows by observing that $U_{+}^{*} \subset \mathbf{C}-\overline{\mathbf{D}}$ and observing that the boundary of $U_{+}$makes an angle of $\frac{1}{3} \pi$ with $\mathbf{S}^{1}$ at 1 , their unique point of intersection.

Lemma 4.3. For each $\theta \in] 0,1\left[-\mathbf{Q}\right.$ there exists a constant $L_{\theta}>0$ such that for all $s \geqslant 1$,

$$
\operatorname{diam}_{\lambda_{+}}\left(Z_{+s}\right)=\operatorname{diam}_{\lambda_{-}}\left(Z_{-s}\right) \leqslant L_{\theta}
$$

Proof. The equality sign follows from $\tau$ being an isometry with respect to the pair of metrics $\lambda_{+}$and $\lambda_{-}$and $\tau\left(Z_{+s}\right)=Z_{-s}$.

We shall thus concentrate on giving an absolute upper bound for $\operatorname{diam}_{\lambda_{-}}\left(Z_{-s}\right)$. For $n \geqslant 2$ write $\Sigma_{n}=\Sigma_{n}\left(I_{n}, B_{n}, G_{n}, R_{n}, O_{n}\right)$ where $\Sigma_{n}=\Sigma_{n, 0}$ is the arc defined in $\S 3$. Define an arc $\Xi_{n} \subset \bar{U}_{-}$by $f_{\theta}\left(\Xi_{n}\right)=f_{\theta}\left(B_{n} \cup G_{n} \cup R_{n}\right)$, so that $\Xi_{n}$ connects $y_{-q_{n}}$ and $y_{-t_{n}}$ in $U_{-}$. Essentially repeating the arguments of Lemma 3.3 we find that

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} l_{\lambda_{-}}\left(\Xi_{n}\right) \leqslant L_{1}+L_{2}+C\left(4 L_{R, \theta}+L_{G, \theta}\right) \tag{1}
\end{equation*}
$$

(Let $g: \Gamma\left(x_{q_{n-1}}, y_{l}\right) \rightarrow \Sigma_{n}\left(x_{q_{n}}, y_{t_{n}}\right)$ be the last move in obtaining $\Sigma_{n}$. Let $h$ be the long composition of inverse branches of $f_{\theta}$ in the construction of $g$. Consider instead of $g$, the composition of $h$ with the previously unused inverse branch of $f_{\theta}$, which maps $\overline{\mathbf{C}}-\mathbf{D}$ into $\bar{U}_{-}$. The details are left to the reader.)

We extend the notion $\lceil\cdot, \cdot\rceil$ to include also subarcs of $U_{-}$as well (as $\mathbf{S}^{1}$ and $U_{+}$ where it has previously been defined).

Let $D_{n} \subset \bar{U}_{-}, n \geqslant 1$, be topological discs defined as follows. For $n=1$ we take $D_{1}$ to be bounded by $\Xi_{4}$ and the subarc $\left\lceil y_{-q_{4}}, y_{-t_{4}}\right\rceil \subset \partial U_{-}$not containing 1. For $n \geqslant 2$ take $D_{n}$ to be bounded by $\Xi_{n}, \tau\left(\gamma_{t_{n}, q_{n+3}}\right), \Xi_{n+3}$ and $\left(-\tau\left(\gamma_{q_{n}, t_{n+3}}\right)\right)$, where $\gamma_{l, l^{\prime}} \subseteq \partial U_{-}$are the curves of Proposition and Definition 3.8. Then it follows from (1) that

$$
\limsup _{n \rightarrow \infty} \operatorname{diam}_{\lambda_{-}}\left(D_{n}\right) \leqslant L_{2}^{\prime}+L_{1}+L_{2}+C\left(4 L_{R, \theta}+L_{G, \theta}\right)
$$

In particular, there exists a constant $L_{\theta}$ such that $\operatorname{diam}_{\lambda_{-}}\left(D_{n}\right) \leqslant L_{\theta}$ for all $n$, as each $D_{n}$ is a compact subset of $\mathbf{D}^{*}$ and so has finite $\lambda_{\text {_ }}$-diameter. Next we easily check that any $\operatorname{limb} Z_{-s}, s \geqslant 1$, is contained in at least one $D_{n}$, so that its $\lambda_{-}$-diameter is at most $L_{\theta}$.

Proof of Theorem 4.1. Let $z \in Z_{\theta}$ be arbitrary. Let $n_{1} \geqslant 0$ be minimal with the property $z_{1}:=f_{\theta}^{n_{1}}(z) \in U_{+} \cup U_{-}$. Replacing $z$ by $\tau(z)$ we can suppose that $z_{1} \in \bar{U}_{+}$. Hence $z_{1} \in Z_{+s_{1}}$ for some $s_{1} \geqslant 1$. Let $n_{2} \geqslant 1$ be minimal with $z_{2}:=f_{\theta}^{n_{2}}\left(z_{1}\right) \in U_{-}$and let $s_{2}$ be given by $z_{2} \in Z_{-s_{2}}$. Define inductively $n_{j} \geqslant 1, z_{j} \in U_{+} \cup U_{-}$and $s_{j} \geqslant 1$ by $n_{j}$ being minimal with

$$
z_{j}:=f_{\theta}^{n_{j}}\left(z_{j-1}\right) \in \begin{cases}Z_{+s_{j}} \subset \bar{U}_{+} & \text {if and only if } j \text { is odd } \\ Z_{-s_{j}} \subset \bar{U}_{-} & \text {if and only if } j \text { is even. }\end{cases}
$$

Next let $\varpi_{j}$ denote the connected component, containing $z$, of $f_{\theta}^{-\left(n_{1}+n_{2}+\ldots+n_{j}\right)}\left(Z_{+s_{j}}\right)$ if $j$ is odd and of $f_{\theta}^{-\left(n_{1}+n_{2}+\ldots+n_{j}\right)}\left(Z_{-s_{j}}\right)$ if $j$ is even. Then each $\varpi_{j}$ is a connected neighbourhood of $z$ in $J_{f_{\theta}}$. Moreover, applying first Lemma 4.3 and then Lemma 4.2 we obtain

$$
\operatorname{diam}_{\lambda_{+}}\left(\varpi_{j}\right) \leqslant L_{\theta} \cdot C^{j} \underset{j \rightarrow \infty}{\longrightarrow} 0
$$

which implies that the sequence $\left\{\varpi_{j}\right\}_{j \geqslant 1}$ forms a fundamental system of connected neighbourhoods of $z$ in $J_{f_{\theta}}$. As $z \in Z_{\theta}$ was arbitrary we have proved the theorem.

## References

[Be] Beardon, A. F., Iteration of Rational Functions. Graduate Texts in Math., 132. SpringerVerlag, New York-Berlin, 1991.
[BH] Branner, B. \& Hubbard, J. H., The iteration of cubic polynomials, Part II: Patterns and parapatterns. Acta Math., 169 (1992), 229-325.
[CG] Carleson, L. \& Gamelin, T. W., Complex Dynamics. Universitext: Tracts in Mathematics. Springer-Verlag, New York, 1993.
[Do] Douady, A., Disques de Siegel et anneaux de Herman. Sém. Bourbaki, 39ème année, 1986/87, $\mathrm{n}^{\circ} 677$.
[He] Herman, M. R., Conjugaison quasi symmetrique des homéomorphismes analytiques du cercle a des rotations. Preliminary manuscript.
[Hu] Hubbard, J. H., Local connectivity of Julia sets and bifurcation loci: three theorems by Yoccoz, in Topological Methods in Modern Mathematics (Stony Brook, NY, 1991), pp. 467-511. Publish or Perish, Houston, TX, 1993.
[Ke] Keller, K., Symbolic dynamics for angle-doubling on the circle III. Sturmian sequences and the the quadratic map. Ergodic Theory Dynamical Systems, 14 (1994), 787-805.
[LV] Lehto, O. \& Virtanen, K. I., Quasiconformal Mappings in the Plane, 2nd edition. Grundlehren Math. Wiss., 126. Springer-Verlag, New York-Berlin, 1973.
[Mc] McMullen, C. T., Self-similarity of Siegel disks and Hausdorff dimension of Julia sets. Manuscript, Univ. of California, Berkeley, CA, October 1995.
[Si] Siegel, L., Iteration of analytic functions. Ann. of Math. (2), 43 (1942), 607-612.
[St] Steinmetz, N., Rational Iteration. Complex Analytic Dynamical Systems. de Gruyter Stud. Math., 16. de Gruyter, Berlin, 1993.
[Su] Sullivan, D., Bounds, quadratic differentials, and renormalization conjectures, in American Mathematical Society Centennial Publications, Vol. II (Providence, RI, 1988), pp. 417-466. Amer. Math. Soc., Providence, RI, 1992.
[Sw] Świstec, G., Rational rotation numbers for maps of the circle. Comm. Math. Phys., 119 (1988), 109-128.
[TY] TAN, L. \& Yin, Y., Local connectivity of the Julia set for geometrically finite rational maps. Preprint, École Normale Supérieure de Lyon, UMPA-94-nº 121, 1994. To appear in Acta Math. Sinica.
[Ya] Yampolsky, M., Complex bounds for critical circle maps. Preprint, SUNY, StonyBrook, Institute for Mathematical Sciences, \#1995/12.
[Yol] Yoccoz, J.-C., Il n'y a pas de contre-exemple de Denjoy analytique. C. R. Acad. Sci. Paris Sér. I Math., 298 (1984), 141-144.
[Yo2] - Structure des orbites des homéomorphismes analytiques possedant un point critique. Manuscript.

## Carsten Lunde Petersen <br> IMFUFA

Roskilde University
Postbox 260
DK-4000 Roskilde
Denmark
lunde@mmf.ruc.dk
Received June 27, 1994
Received in revised form May 22, 1996

