LOCAL CONVERGENCE THEOREMS FOR QUASI-NEWTON METHODS

By

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ABSTRACT

This paper presents generalizations of two results which have been useful for analyzing methods of the form $x_{k+1} = x_k - B_k^{-1}F(x_k)$. The bounded deterioration theorem of Broyden-Dennis-Moré is generalized to show that if $\{B_k\}$ or $\{B_k^{-1}\}$ is of bounded deterioration as a sequence of approximants to some B_* or B_*^{-1} then the iteration above has the same local convergence properties and arbitrarily nearly the same linear rate as would be achieved by the stationary iteration function which uses $B_k = B_*$. The characterization theorem for superlinear convergence given by Dennis-More is then generalized to give conditions under which the rates are the same. In the case when $B_* = F'(x_*)$, these results reduce to those already known.

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1. Introduction.

The results of this paper were discovered in the course of a continuing investigation of convergence properties of least-change secant updates (see Dennis-Schnabel [3]). Specific results of that investigation will appear elsewhere [4] but this paper was written because we feel that the results it contains are sufficiently interesting and attractive to merit separate attention. To support this position we apply these results to the convergence for the Jacobi-secant method in section 4 and to the rate of convergence of the general class of Newton-iterative methods studied by Ortega-Rheinboldt [5] and Sherman [7] in section 5.

Our interest is in <u>quasi-Newton</u> or <u>Newton-like</u> methods

$$x_{k+1} = x_k - B_k^{-1}F(x_k) = x_k - K_kF(x_k),\ B_k \in R^{n\times n},\ K_k = B_k^{-1} \qquad (1.1)$$

for solving $F(x) = 0$, where $F\colon\ \Omega \to R^n$ and Ω is an open convex set in R^n. We assume that $F(x_*) = 0$ for some $x_* \in \Omega$ and that for some $\gamma \geq 0$ and $p \in (0,1]$

$$|F'(x) - F'(x_*)| \leq \gamma|x - x_*|^p \quad \text{for every} \quad x \in \Omega, \qquad (1.2)$$

where $|\cdot|$ denotes a vector norm and subordinate operator norm of interest.

A familiar procedure in analyzing least change secant methods for this problem (c.f. Broyden-Dennis-Moré [1], Powell [6], Sorensen [8]) is first

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to establish the existence and q-linear convergence of the iteration sequence $\{x_k\}$ and then to show q-superlinear convergence by the use of the characterization theorem given in Dennis-Moré [2]. The technique used for proving q-linear convergence is generally based on some variant of the principle of bounded deterioration, which states that while the sequence $\{B_k\}$ of approximate partial derivative matrices need not get nearer $F'(x_*)$, they should only get worse in a controlled way as the iteration proceeds.

Our purpose here is to show how to extend this method of analysis to the case when the iteration (1.1) uses a sequence $\{B_k\}$ which is taken to have some desirable property not necessarily shared by $F'(x)$ at any x. A familiar example to illustrate this is the nonlinear Jacobi iteration where $B_k = \text{diag}(F'(x_k))$ although $F'(x_k)$ is not a diagonal matrix. Certainly we know for this iteration that no matter how near x_0 is to x_*, we can't reasonably expect convergence unless there is some diagonal matrix B_* (perhaps $B_* = \text{diag}(F'(x_*))$) for which $B_k \equiv B_*$ would yield a convergent iteration (1.1).

The theorems in section 2 assume the existence of such a B_* and give a bounded deterioration condition on $\{B_k\}$ or $\{B_k^{-1}\}$ as a sequence of approximations to B_* or B_*^{-1} sufficient for the sequence $\{x_k\}$ defined by (1.1) to converge locally and q-linearly to x_*. In section 3 we give a necessary and sufficient condition on $\{B_k\}$ for $\{x_k\}$ to have the same q-linear rate constant as the idealized stationary iteration (1.1) which uses $B_k \equiv B_*$ for every k. In case the convergence of the stationary sequence is q-superlinear, the theorem reduces to the Dennis-Moré [2] result.

The last two sections illustrate the utility of the theorems of sections 2 and 3. In section 4 we show that the Jacobi-secant method([5],[9])with a modification which extends its usefulness can be readily analyzed by the techniques of section 4. In section 5 we show that the results of section 3 give a simple proof characterizing the Q_1 factors for iterates generated by a general class of Newton-iterative methods. A special case gives a proof of q-superlinearity mentioned in NR 11.2-9 of Ortega-Rheinboldt [5, p. 366].

2. Convergence Results.

The results of this section are direct generalizations along the lines mentioned above of the results of section 3 of Broyden-Dennis-Moré [1]. We also use their notion of an updating function here, and any reader who desires more of a discussion of this useful abstraction is referred to the original paper.

In our theorems we find it convenient to use two norms. As in section 1 it is useful to denote a vector norm by $|v|$ for $v \in R^n$ and the subordinate matrix operator norm by $|A|$ for $A \in R^{n\times n}$. The notation $||A||$ for $A \in R^{n\times n}$ stands for any arbitrary but fixed norm on $R^{n\times n}$ which may not be subordinate to a vector norm. We make strong use of the equivalence of all norms on $R^{n\times n}$; in particular for $|\cdot|$ and $||\cdot||$ we assume for some $\mu,\eta > 0$ and any $A \in R^{n\times n}$, that

$$\mu||A|| \leq |A| \leq \eta||A||.$$

Theorem 2.1. Let F satisfy conditions (1.2) and let $B_* \in R^{n\times n}$ have the property that B_*^{-1} exists and for some operator norm

$$|I - B_*^{-1}F'(x_*)| \leq r_* < 1.$$

Let $U: R^n \times R^{n\times n} \to 2^{R^{n\times n}}$ be defined in a neighborhood $N = N_1 \times N_2$ of (x_*,B_*) where $N_1 \subset \Omega$ and N_2 contains only nonsingular matrices. Assume that there are nonnegative constants α_1 and α_2 such that for each $(x,B) \in N$, and for $x_+ = x - B^{-1}F(x)$, every $B_+ \in U(x,B)$ satisfies

$$||B_+-B_*|| \leq [1 +\alpha_1\sigma(x,x_+)^p]\cdot||B-B_*|| + \alpha_2\sigma(x,x_+)^p$$

for $\sigma(x,x_+) = \max\{|x - x_*|,|x_+ - x_*|\}$.

Under these hypotheses, for any $r \in (r_*,1)$, there exist constants ϵ_r, δ_r such that if $|x_0 - x_*| < \epsilon_r$ and $|B_0 - B_*| < \delta_r$, then any iteration sequence $\{x_k\}$ defined by

$$x_{k+1} = x_k - B_k^{-1}F(x_k),\ B_{k+1} \in U(x_k,B_k)$$

$k = 0,1,\ldots$, exists, converges q-linearly to x_* with

$$|x_{k+1}-x_*| \leq r\cdot|x_k - x_*|,$$

and $\{|B_k|\},\{|B_k^{-1}|\}$ are uniformly bounded.

<u>Proof:</u> Let $r \in (r_*,1)$ and choose δ,ϵ_r so small that for $\beta \geq |B_*^{-1}|$ and $\psi \geq |F'(x_*)|$, one has $2\beta\eta\delta < 1$,

$$r \geq r_* + \frac{\beta}{1-2\beta\eta\delta}(\gamma\epsilon_r^p + 2\beta\eta\psi\delta),\ \text{and}\ (2\alpha_1\delta + \alpha_2)\frac{\epsilon_r^p}{1-r^p} \leq \delta.$$

Now select δ_r small enough so that $||B - B_*|| < \delta$ whenever $|B - B_*| < \delta_r$. If necessary further restrict ϵ_r,δ_r so that $(x,B) \in N$ whenever $|B - B_*| \leq 2\eta\delta$ and $|x - x_*| < \epsilon_r$. Let $|B_0 - B_*| < \delta_r$ and $|x_0 - x_*| < \epsilon_r$.

It follows from the Banach Perturbation Lemma [5], since $|B_*^{-1}|\cdot|B_0-B_*| \leq \beta\eta||B_0-B_*|| < \beta\eta\delta < 2\beta\eta\delta < 1$, that B_0^{-1} exists and $|B_0^{-1}| \leq \frac{\beta}{1-2\beta\eta\delta}$. Thus, from standard arguments,

$$|x_1 - x_*| \leq |B_0^{-1}|\ |F(x_0)-F(x_*)-F'(x_*)(x_0-x_*)| + |I - B_0^{-1}F'(x_*)|\ |x_0-x_*|$$

$$\leq [|B_0^{-1}|\gamma\epsilon_r^p + |I-B_*^{-1}F'(x_*)| + |B_*^{-1}-B_0^{-1}|\cdot|F'(x_*)|]\ |x_0-x_*|$$

$$\leq [|B_0^{-1}|(\gamma\epsilon_r^p + |B_*^{-1}|\cdot|B_0-B_*|\cdot\psi) + r_*]\cdot|x_0-x_*|$$

$$\leq [\frac{\beta}{1-2\beta\eta\delta}(\gamma\epsilon_r^p + 2\beta\eta\delta\psi) + r_*]\cdot|x_0-x_*| \leq r|x_0-x_*|.$$

Assume by way of induction that for $k = 0,1,\ldots,m-1$, $||B_k-B_*|| \leq 2\delta$ and $|x_{k+1}-x_*| \leq r|x_k-x_*|$. Then, $||B_{k+1}-B_*||-||B_k-B_*|| \leq 2\alpha_1\delta\epsilon_r^p r^{pk} + \alpha_2\epsilon_r^p r^{pk}$. We can sum both sides from $k = 0$ to $m-1$ to obtain

$$||B_m-B_*|| \leq ||B_0-B_*|| + (2\alpha_1\delta + \alpha_2)\frac{\epsilon_r^p}{1-r^p} \leq 2\delta, \text{ so } |B_m' - B_*| \leq 2\eta\delta \text{ and}$$

again by the Banach Lemma, B_m^{-1} exists and $|B_m^{-1}| \leq \frac{\beta}{1-2\beta\eta\delta}$. To complete the induction we proceed as for $m = 0$:

$$|x_{m+1}-x_*| \leq [|B_m^{-1}|\ (\gamma\epsilon_r^p + |B_*^{-1}|\cdot|B_m-B_*|\cdot\psi) + r_*]\cdot|x_m-x_*|$$

$$\leq [\frac{\beta}{1-2\beta\eta\delta}(\gamma\epsilon_r^p + 2\beta\eta\delta\psi) + r_*]\cdot|x_m-x_*| \leq r\cdot|x_m - x_*|.$$

Note that we have easily that $|B_k^{-1}| \leq \frac{\beta}{1-2\beta\eta\delta}$ and that $|B_k| \leq 2\eta\delta + |B_*|$, and this completes the proof.

This theorem is applied to the Jacobi-secant iteration in section 4. Sometimes it is useful to have conditions directly on $\{B_k^{-1}\}$ rather than on $\{B_k\}$.

Theorem 2.2. Let F satisfy conditions (1.2) and let K_* be an invertible matrix with $|I - K_*F'(x_*)| \leq r_* < 1$.

Let $U: R^n \times R^{n\times n} \to 2^{R^{n\times n}}$ be defined in a neighborhood $N = N_1 \times N_2$ of (x_*,K_*) where $N_1 \subset \Omega$. Assume that there are nonnegative constants α_1,α_2 such that for each (x,K) in N, and for $x_+ = x - KF(x)$, the function U satisfies

$$||K_+ - K_*|| \leq [1 + \alpha_1\sigma(x,x_+)^p]\ ||K - K_*|| + \alpha_2\sigma(x,x_+)^p$$

for each $K_+ \in U(x,K)$. Then for each $r \in (r_*,1)$ there exist positive constants ϵ_r,δ_r such that for $|x_0-x_*| < \epsilon_r$ and $|K_0-K_*| < \delta_r$, any sequence $\{x_k\}$ defined by

$$x_{k+1} = x_k - K_kF(x_k), K_{k+1} \in U(x_k,K_k)$$

$k = 0,1,\ldots,$ exists, converges q-linearly to x_* with

$$|x_{k+1} - x_*| \le r \; |x_k - x_*|$$

and $\{|K_k|\}$, $\{|K_k^{-1}|\}$ are uniformly bounded.

Proof: Let $r \in (r_*, 1)$ and choose ϵ_r, δ so that

$$(2\alpha_1\delta + \alpha_2)\frac{\epsilon_r^p}{1-r^p} \le \delta \text{ and } |F'(x_*)|\cdot 2\eta\delta + r_* + (|K_*| + 2\eta\delta)\gamma\epsilon_r^p \le r.$$

Now select δ_r small enough so that $|K - K_*| < \delta_r$ implies that $||K - K_*|| < \delta$. If necessary, further restrict ϵ_r, δ_r so that $(x,K) \in N$ whenever $|K - K_*| < 2$ and $|x - x_*| < \epsilon_r$.

Let x_0, K_0 be chosen to satisfy $|x_0 - x_*| < \epsilon_r$ and $|K_0 - K_*| < \delta_r$. Then

$$|x_1 - x_*| \le |K_0| \; |F(x_0) - F(x_*) - F'(x_*)(x_0 - x_*)|$$

$$+ |I - K_0F'(x_*)||x_0 - x_*| \le [(|K_*| + |K_0 - K_*|)\gamma\epsilon_r^p$$

$$+ |I - K_*F'(x_*)| + |K_* - K_0|\cdot|F'(x_*)|]\cdot|x_0 - x_*|$$

$$\le [(|K_*| + \eta\delta)\gamma\epsilon_r^p + r_* + \eta\delta\cdot|F'(x_*)|]\cdot|x_0 - x_*|$$

$$\le r\;|x_0 - x_*|.$$

Now assume by way of induction that

$$||K_k - K_*|| \le 2\delta \text{ and } |x_{k+1} - x_*| \le r\cdot|x_k - x_*|$$

for $k = 0,1,\ldots,m-1$. It follows that

$$||K_{k+1} - K_*|| - ||K_k - K_*|| \le 2\alpha_1\delta\epsilon_r^p\, r^{pk} + \alpha_2\epsilon_r^p r^{pk}$$

and, again as in Theorem 2.1, by summing from $k = 0$ to $m-1$, we obtain

$$||K_m - K_*|| \le ||K_0 - K_*|| + (2\alpha_1\delta + \alpha_2)\frac{\epsilon_r^p}{1-r^p} \le 2\delta. \quad \text{Thus}$$

$$\begin{aligned} |x_{m+1} - x_*| &\le [|K_m|\gamma\epsilon_r^p + |I - K_m F'(x_*)|]\cdot|x_m - x_*| \\ &\le [(|K_*| + 2\eta\delta)\gamma\epsilon_r^p + r_* + 2\eta\delta\cdot|F'(x_*)|]\cdot|x_m - x_*| \\ &\le r\cdot|x_m - x_*| \end{aligned}$$

and the induction is complete.

In order to finish the proof we need to derive the bounds for $\{|K_k|\},\{|K_k^{-1}|\}$. These follow readily from the induction relations; in fact we already have $|K_k| \le |K_*| + \eta||K_k - K_*|| \le |K_*| + 2\eta\delta$. Furthermore, $|I - K_*F'(x_*)| \le r_* < 1$ implies that $|F'(x_*)^{-1}|$ exists and $|F'(x_*)^{-1}| \le \frac{|K_*|}{1-r_*}$. Thus we have $|I - K_kF'(x_*)| < r_* + 2\eta\delta\cdot|F'(x_*)| \le r < 1$, and so K_k^{-1} exists and $|K_k^{-1}| \le \frac{|F'(x_*)|}{1-r}$

3. Speed of convergence.

In this section we will present theorems which give necessary and sufficient conditions for the iteration (1.1) to have the same q-linear rate of convergence as some idealized stationary iteration

$$x_{k+1} = x_k - B_*^{-1}F(x_k).$$

In the case when this stationary iteration is q-superlinear our theorems reduce to the Dennis-Moré results [2].

Some remarks are in order about q-linear convergence and the use of different norms. Given any vector norm, Ortega and Rheinboldt [5, p. 281] define the linear q-factor of $\{x_k\}$ as

$$Q_1\{x_k\} = \begin{cases} 0, & \text{if } x_k = x^*,\ k \geq \text{some } k_0 \\ \overline{\lim_{k\to\infty}} \dfrac{|x_{k+1}-x_*|}{|x_k-x_*|}, & \text{if } x_k \neq x_*,\ k \geq k_0 \\ +\infty, & \text{otherwise} \end{cases}$$

For a given norm, the statement that $\{x_k\}$ converges q-linearly to x_* means that $Q_1\{x_k\} < 1$, and q-superlinear convergence means that $Q_1\{x_k\} = 0$. Since all norms are equivalent, the condition that $Q_1\{x_k\} = 0$ will clearly be norm-independent. On the other hand $Q_1\{x_k\} < 1$ in one norm only ensures that $Q_1\{x_k\} < +\infty$ in any other norm.

In this terminology, the following theorems give norm-invariant necessary and sufficient conditions for any sequence $\{x_k\}$ generated by (1.1) which is q-linearly convergent to x_* in some norm to have

$$Q_1\{x_k\} = \overline{\lim_{k\to\infty}} \left|(I - B_*^{-1}F'(x_*)) \frac{x_k-x_*}{|x_k-x_*|}\right|$$

in every norm. With these remarks in mind, perhaps the reader will be patient with the attention given to norm independence in the statements of the following theorems.

Theorem 3.1. Suppose that F satisfies condition (1.2) and that $\{x_k\}$ is a sequence generated by (1.1) which converges to x_* with $x_k \neq x_*$ for all but finitely many k and that for some norm $|\cdot|_1$, and some $r \in (0,1)$,

$$|x_{k+1}-x_*|_1 \leq r|x_k-x_*|_1, \; k = 0,1,2,\ldots \tag{3.1}$$

If $s_k = x_{k+1} - x_k$ and $B_* \in R^{n\times n}$ is any invertible matrix, then the norm-independent condition

$$\lim_{k\to\infty} \frac{|(B_k-B_*)s_k|}{|s_k|} = 0 \tag{3.2}$$

holds if and only if the norm-independent condition

$$\lim_{k\to\infty} |[I-B_*^{-1}F'(x_*)]\frac{(x_k-x_*)}{|x_k-x_*|} - \frac{(x_{k+1}-x_*)}{|x_k-x_*|}| = 0 \tag{3.3}$$

holds. In particular, if (3.2) holds in some norm, then for any vector norm $|\cdot|$,

$$Q_1\{x_k\} = \overline{\lim_{k\to\infty}} \frac{|x_{k+1}-x_*|}{|x_k-x_*|} = \overline{\lim_{k\to\infty}} |[I-B_*^{-1}F'(x_*)]\frac{(x_k-x_*)}{|x_k-x_*|}| \leq |I-B_*^{-1}F'(x_*)| \tag{3.4}$$

and $\{x_k\}$ converges q-superlinearly to x_* if and only if

$$\lim_{k\to\infty} |[I-B_*^{-1}F'(x_*)]\frac{(x_k-x_*)}{|x_k-x_*|}| = 0 \tag{3.5}$$

Proof: Notice that (3.2), (3.3), and (3.5) hold in any norm if and only if they hold in every norm. Since

$$(B_k-B_*)s_k = [B_*-F'(x_*)](x_k-x_*)-B_*(x_{k+1}-x_*)+F'(x_*)(x_k-x_*)-F(x_k)$$

and, since (1.2) can be shown to hold in $|\cdot|_1$ for a constant γ_1, one has

$$\lim_{k\to\infty} \frac{|F'(x_*)(x_k-x_*)-F(x_k)|_1}{|s_k|_1} \le \lim_{k\to\infty} \frac{\gamma_1 \sigma_k^p}{1-r} = 0,$$

(3.2) holds if and only if

$$\lim_{k\to\infty} \frac{|[B_*-F'(x_*)](x_k-x_*)-B_*(x_{k+1}-x_*)|_1}{|s_k|_1} = 0.$$

Since $(1-r)|x_k-x_*|_1 \le |s_k|_1 \le (1+r)|x_k-x_*|_1$, one has

$$\frac{|[B_*-F'(x_*)](x_k-x_*)-B_*(x_{k+1}-x_*)|_1}{|s_k|_1} \le \frac{|B_*|_1}{(1-r)}\left|[I-B_*^{-1}F'(x_*)]\frac{(x_k-x_*)}{|x_k-x_*|_1} - \frac{(x_{k+1}-x_*)}{|x_k-x_*|_1}\right|_1$$

and

$$\frac{|[B_*-F'(x_*)](x_k-x_*)-B_*(x_{k+1}-x_*)|_1}{|s_k|_1} \ge \frac{1}{|B_*^{-1}|_1(1+r)}\left|[I-B_*^{-1}F'(x_*)]\frac{(x_k-x_*)}{|x_k-x_*|_1} - \frac{(x_{k+1}-x_*)}{|x_k-x_*|_1}\right|_1$$

Thus (3.2) holds if and only if (3.3) holds, and the proof is complete since only norm-independent "zero" limits have been used.

It is easy to see from Theorem 3.1 that condition (3.2) and hence (3.3) and (3.4) follow from $\lim_{k\to\infty} B_k = B_*$. This is not so easy to see from the following theorem because the sequence $\{y_k\}$ in condition (3.6) is unfamiliar. In fact if $\lim_{k\to\infty} K_k = K$ then $\{y_k\} = \{B_* s_k\}$ will certainly do since with $\alpha_k = 0$ and any norm, (3.6) holds. We hope these remarks will make the statement of Theorem 3.2 easier to understand.

<u>Theorem 3.2</u>. Suppose that the hypotheses of Theorem 3.1 hold. Suppose further that $\{||K_k||\}$ and $\{||K_k^{-1}||\}$ are bounded and that $\{y_k\}$ is a sequence satisfying the norm-independent condition

$$|K_* y_k - s_k| \le \alpha_k |s_k|, \tag{3.6}$$

where $s_k = x_{k+1} - x_k$, K_* is some invertible matrix, and $\lim_{k\to\infty} \alpha_k = 0$. Then the norm-independent condition

$$\lim_{k\to\infty} \frac{|(K_k - K_*)y_k|}{|y_k|} = 0 \tag{3.7}$$

holds if and only if the norm-independent condition

$$\lim_{k\to\infty} \left| [I - K_* F'(x_*)] \frac{(x_k - x_*)}{|x_k - x_*|} - \frac{(x_{k+1} - x_*)}{|x_k - x_*|} \right| = 0$$

holds. In particular, if (3.7) holds in some norm, then for any norm,

$$\overline{\lim_{k\to\infty}} \frac{|x_{k+1} - x_*|}{|x_k - x_*|} = \overline{\lim_{k\to\infty}} \left| [I - K_* F'(x_*)] \frac{(x_k - x_*)}{|x_k - x_*|} \right| \le |I - K_* F'(x_*)|$$

and $\{x_k\}$ converges q-superlinearly to x_* if and only if the norm-independent condition

$$\lim_{k\to\infty} \left| [I - K_* F'(x_*)] \frac{(x_k - x_*)}{|x_k - x_*|} \right| = 0$$

holds.

<u>Proof:</u> Set $B_* = K_*^{-1}$ and $B_k = K_k^{-1}$ for $k = 0,1,2,\ldots$, and note that

$$(B_k - B_*)s_k = (B_* - B_k)(K_* y_k - s_k) - B_k(K_k - K_*)y_k.$$

It follows that

$$\frac{|(B_k - B_*)s_k|}{|s_k|} \le |B_* - B_k| \frac{|K_* y_k - s_k|}{|s_k|} + \frac{|B_k||y_k|}{|s_k|} \frac{|(K_k - K_*)y_k|}{|y_k|} \tag{3.8}$$

and, since $|B_k v| \geq \frac{1}{|K_k|}|v|$ for $v \in R^n$,

$$\frac{|y_k|}{|K_k||s_k|} \quad \frac{|(K_k-K_*)y_k|}{|y_k|} - |B_*-B_k|\frac{|K_*y_k-s_k|}{|s_k|} \leq \frac{|(B_k-B_*)s_k|}{|s_k|} \tag{3.9}$$

Now (3.6) yields

$$\begin{aligned} |y_k| &\leq |B_*||K_*y_k| \leq |B_*|(|K_*y_k-s_k| + |s_k|) \\ &\leq |B_*|(1+\alpha_k)|s_k| \end{aligned} \tag{3.10}$$

and

$$|y_k| \geq \frac{|K_*y_k|}{|K_*|} \geq \frac{1}{|K_*|}(|s_k| - |K_*y_k-s_k|) \geq \frac{1}{|K_*|}(1-\alpha_k)|s_k|, \tag{3.11}$$

and since $\{||K_k||\}$ and $\{||B_k||\}$ are bounded and $\lim_{k\to\infty} \alpha_k = 0$, one sees from (3.8), (3.9), (3.10) and (3.11) that (3.7) holds if and only if

$$\lim_{k\to\infty} \frac{|(B_k-B_*)s_k|}{|s_k|} = 0.$$

The theorem then follows from Theorem 3.1.

4. The Jacobi-secant iteration.

Our purpose in this section is to give a non-trivial application of Theorem 2.1 to something other than a least change secant update method. The iteration we consider is a modification of a method suggested by Wegge [9].

Let $F = (f_1, \ldots, f_n)^T$ and let u_i denote the i^{th} column of the identity matrix. Remember that F is defined on the convex set Ω.

The Jacobi-secant iteration:

Given $x_{k-1}, x_k \in R^n$ and $B_{k-1} = \text{diag}(b^1_{k-1}, \ldots, b^n_{k-1})$ with $b^i_{k-1} \neq 0$ for $1 \leq i \leq n$, set $s_{k-1} = x_k - x_{k-1}$ and for $i = 1, \ldots, n$ set

$$x^i_{k+1} = x^i_k - \frac{f_i(x_k)}{b^i_k}$$

where b^i_k is either b^i_{k-1} or it may be chosen to be $[f_i(x_k - s^i_{k-1}u_i) - f_i(x_k)]/-s^i_{k-1}$, provided $x_k - s^i_{k-1}u_i \in \Omega$ and this quotient is defined and nonzero.

This is very close to the Steffensen iteration of section 5 of Wegge [9] but there is one very important computational difference. When $f_i(x_k)$, and hence s^i_k, is zero, Wegge's proof requires that $b^i_k = \partial_i f_i(x_k)$ and so he points out the necessity for coding $\partial_i f_i$ in order to handle these cases by his methods. Our techniques allow the much more reasonable choice of $b^i_k = b^i_{k-1}$ because we don't need b^i_k to be a single-valued, much less a continuous, function of s^i_k. We hasten to point out that our pride is not in the rather obvious modification but in the natural way that our analysis handles the more useful form of the algorithm while Wegge's analysis, because it is based on Ostrowski's theorem, requires very severe restrictions on the iteration function. We feel that our analysis successfully resolves the difficulty in NR 11.2-6 of Ortega and Rheinboldt [5, p. 365].

This may or may not be a practical numerical method but it certainly won't always be locally convergent. In particular, we wouldn't expect to be successful unless the standard nonlinear Jacobi iteration, $b_k^i = \partial_i f_i(x_k)$ can be shown to be locally convergent. In the theorem below we make the standard assumption on $F'(x_*)$ to give Jacobi-iteration convergence; the remainder of the hypotheses are restrictions on the starting (x_0, B_0) to get local convergence of the Jacobi-secant method.

Because the Jacobi-secant iteration is not simple to analyze, it is worth pointing out that the Jacobi iteration requires only a fairly trivial application of Theorem 2.1. This is easily seen since for $B(x) = \mathrm{diag}(\partial_1 f_1(x), \ldots, \partial_n f_n(x))$ and

$$B_* = \mathrm{diag}(\partial_1 f_1(x_*), \ldots, \partial_n f_n(x_*)) \tag{4.1}$$

we need only Lipschitz continuity on the diagonal of $F'(x)$ to have, for any $x_+ \in \Omega$,

$$||B(x_+) - B_*|| \le \gamma ||x_+ - x_*||^p \le \alpha_2 \sigma(x, x_+)^p.$$

We define the ℓ_∞ norm $|v|_\infty = \max_{1 \le i \le n} |v^i|$, and for $A \in R^{n \times n}$, $|A|_\infty$ is the induced ℓ_∞ operator norm.

<u>Theorem 4.1.</u> Let F satisfy conditions (1.2). Assume also that B_* defined by (4.1) is invertible and that $\rho(I - B_*^{-1}F'(x_*)) \le \bar{r} < 1$, where ρ denotes the spectral radius. Under these hypotheses, for any $r > \bar{r}$, there exists an $\epsilon_r, \delta_r > 0$ such that if $|B_0 - B_*| < \delta_r$ for a diagonal B_0 and $|x_0 - x_*| < \epsilon_r$ then the sequence of Jacobi-secant iterates

$$x_{k+1} = x_k - B_k^{-1}F(x_k) \quad k = 0, 1, \ldots$$

is well-defined and $\{x_k\}$ converges to x_* with

$$\overline{\lim_{k\to\infty}} \frac{|x_{k+1}-x_*|_r}{|x_k-x_*|_r} < r,$$

for some vector norm $|\cdot|_r$.

Proof: Let $r > \bar{r}$. Set $r_* = \bar{r} + \frac{(r-\bar{r})}{2} < r$, and from [5, p. 44], we know that for some vector norm $|\cdot|_r$, the induced operator norm satisfies

$$|I - B_*^{-1}F'(x_*)|_r \leq r_*.$$

Furthermore for some $\alpha > 0$, $|v|_\infty \leq \alpha|v|_r$. The operator norm subordinate to $|\cdot|_r$ is the one we use in satisfying the hypotheses of Theorem 2.1.

The most convenient norm in which to prove bounded deterioration is the ℓ_∞ operator norm, since $|D|_\infty = \max_{1\leq i\leq n} |D^{ii}|$ for a diagonal matrix D. We were vague about the norm in (1.2) because it is convenient not to be continually redefining the constant γ; we assume (1.2) holds for the ℓ_∞ norm. This allows us to conclude that for $x \in \Omega$ and each i, $|\partial_i f_i(x)-\partial_i f_i(x_*)| \leq |[F'(x)-F'(x_*)]u_i|_\infty \leq |F'(x)-F'(x_*)|_\infty \leq \gamma|x-x_*|_\infty^p$.

In order to define the update function, we abstract the definition of b_k^{ii} given earlier. Let $N_2 = \{A \in R^{n\times n}: A^{-1}$ exists and $A^{ii} \neq 0$ for every $i, 1 \leq i \leq n\}$ and let $B \in N_2$. Then for $x \in \Omega$, set $x_+ = x - B^{-1}F(x)$ and define $U(x,B) = N_2 \cap \{\bar{B} \in R^{n\times n}: \bar{B}$ is a diagonal matrix and for each i, $\bar{B}^{ii}$ is either B^{ii} or else if $x_+ + (x-x_+)^i u_i \in \Omega$, then $\bar{B}^{ii} = \dfrac{f_i(x_+ + (x-x_+)^i u_i) - f_i(x_+)}{(x-x_+)^i}$ may be chosen$\}$. Note that for any $B \in N_2$, and any x, $\mathrm{diag}(B) \in U(x,B) \neq \{\ \}$. N_2 is obviously a neighborhood of B_*.

We are now ready to show that $U(x,B)$ is of bounded deterioration when $(x,B) \in N = \Omega \times N_2$. Let $\alpha_1 = 0$ and $\alpha_2 = \gamma\alpha^p$. If $B_+^{ii} \neq B^{ii}$ then by the mean value theorem, since $x_+ + (x-x_+)^i u_i \in \Omega$,

$B_+^{ii} = \dfrac{f_i(x_+ + (x-x_+)^i u_i) - f_i(x_+)}{(x-x_+)^i} = \partial_i f_i(t_i x_+ + (1-t_i)(x_+ + (x-x_+)^i u_i))$ for some $t_i \in (0,1)$. Thus

$$|B_+ - B_*|_\infty = \max_{1 \le i \le n} |B_+^{ii} - B_*^{ii}|$$

$$\le \max_{1 \le i \le n} \{|B^{ii} - B_*^{ii}|, |\partial_i f_i(t_i x_+ + (1-t_i)(x_+ + (x-x_+)^i u_i)) - \partial_i f(x_*)|\}$$

$$\le \max_{1 \le i \le n} \{|B^{ii} - B_*^{ii}|, \gamma \ |t_i x_+ + (1-t_i)(x_+ + (x-x_+)^i u_i) - x_*|_\infty^p\}$$

$$\le \max_{1 \le i \le n} \{|B^{ii} - B_*^{ii}|, \gamma \max(|x_+ - x_*|_\infty^p, |x - x_*|_\infty^p)\}$$

$$\le |B - B_*|_\infty + \gamma_* \alpha^p \sigma(x_+, x)^p.$$

The result now follows from Theorem 2.1.

This completes our analysis of the Jacobi-secant iteration because Theorem 4.1 already contains all of the most interesting speed of convergence information, but in the next section we offer an interesting application of the results of section 3.

5. Newton-iterative methods.

The basic idea behind the methods considered in this section is that if $F'(x)$ is large and sparse, then the main difficulty in carrying out Newton's method might not lie in the computation of $F'(x)$ but rather in solving the linear system

$$F'(x)s^N = -F(x) \tag{5.1}$$

for the Newton step s^N. It may be desirable to use a linear iterative method like SOR to approximately solve (5.1). While it is easy to believe that this should be equivalent to solving

$$B(x)s^{q-N} = -F(x) \tag{5.2}$$

for a quasi-Newton step, where B comes from the iteration somehow, it turns out [5] that B^{-1} reveals itself more naturally than B.

We refer the reader to the paper by Sherman [7] or to Ortega and Rheinboldt [5] for a clear presentation of all the details; we restrict our discussion to an abstraction of the essential identities and assumptions from those references.

If $H(x)$ is the iteration matrix defined by the (interior) linear iteration applied to (5.1), then the quasi-Newton step (5.2) resulting from m steps of the linear iteration applied to (5.1) is

$$x_+ - x = s(x)^{q-N} = -[I - H(x)^m]F'(x)^{-1}F(x).$$

In fact, Sherman seems to restrict all his tests and specific references, though not his main convergence result, to the case when the number of inner iterations is either constant or else increasing as the outer (Newton) iteration proceeds.

Thus, we can view these methods as

$$x_+ = x - K(x)F(x)$$
$$K(x_+) \in U(x,K(x)) = \{[I - H(x_+)^{m_+}]F'(x_+)^{-1}: m_+ \text{ is a positive integer}\}. \quad (5.3)$$

The question now arises as to what assumptions we can make on the matrix-valued function $H(x)$. Since $H(x)$ is the iteration matrix for the linear iteration applied to (5.1), it certainly seems reasonable to assume that the spectral radius satisfies

$$\rho(H(x_*)) < \lambda < 1. \quad (5.4)$$

Hence, as in section 4, there is some vector norm $|\cdot|$ for which $|H(x_*)| < \lambda < 1$ in the induced operator norm. It is also reasonable to assume that if (1.2) holds with $p = 1$, then for $\gamma_1 \geq 0$ and $x \in \Omega' \subset \Omega$, Ω' a convex neighborhood of x_*,

$$\left.\begin{array}{l} |H(x)| < \lambda \\ \text{and for any } m \geq 1, \\ |H(x)^m - H(x_*)^m| \leq \gamma_1 |x - x_*|. \end{array}\right\} \quad (5.5)$$

We stress that γ_1 is independent of x and m and also that these are not direct assumptions but rather that they follow from (1.2) and reasonable assumptions on the inner iteration. [5] [7].

Ortega and Rheinboldt [5, p. 349-351] show that the iteration is q-linear in the norm of (5.5) and R-superlinear if $\lim_{k\to\infty} m_k = \infty$. They later point out that this condition is sufficient for q-superlinear convergence because it forces $\lim_{k\to\infty} K(x_k) = F'(x_*)^{-1}$, a condition known as consistency. We generalize this result to the case where $\{m_k\}$ is required to have some positive limit, possibly finite, and obtain an expression for the q-linear factor in any given norm.

Sherman [7] refines the R-order part of the Ortega-Rheinboldt result. In particular he shows that if $m_k = 2^k$ then convergence is q-superlinear and R-quadratic. He also makes a restriction on $\{m_k\}$

$$\infty > \max[\{1,m_0\} \cup \{(m_k - \sum_{\ell=0}^{k-1} m_\ell): k = 1,2,3,\ldots\}]$$

which seems to be necessitated by the somewhat artificial nature of R-convergence.

<u>Theorem 5.1</u>. Let F satisfy conditions (1.2) with $p = 1$ and let H satisfy conditions (5.5). If $\{m_k\}$ is any sequence of positive integers then there is an $\epsilon > 0$ such that, for $|x_0 - x_*| < \epsilon$, the iteration

$$\begin{aligned} K(x_k) &= [I - H(x_k)^{m_k}]F'(x_k)^{-1} \\ x_{k+1} &= x_k - K(x_k)F(x_k), \end{aligned} \tag{5.6}$$

exists and converges q-linearly to x_*. Furthermore, if $\bar{H} = \lim_{k\to\infty} H(x_*)^{m_k}$ exists then, for any norm,

$$\overline{\lim_{k\to\infty}} \frac{|x_{k+1} - x_*|}{|x_k - x_*|} = \overline{\lim_{k\to\infty}} \left|\bar{H} \frac{x_k - x_*}{|x_k - x_*|}\right| \leq |\bar{H}| \tag{5.7}$$

so that if $\lim_{k\to\infty} m_k = \infty$ then $\{x_k\}$ converges q-superlinearily to x_*.

<u>Proof</u>: The local and q-linear convergence follows from Ortega and Rheinboldt [5, pp. 349-352] although it is in the proof but not the statement. We now use Theorem 3.2 to derive the new expression (5.7).

Let $K_* = (I - \overline{H})F'(x_*)^{-1}$ and expand judiciously to obtain

$$\begin{aligned} K(x_k)-K_* &= (I - H(x_k)^{m_k})F'(x_k)^{-1}-(I-\overline{H})F'(x_*)^{-1} \\ &= F'(x_k)^{-1}-F'(x_*)^{-1} + [\overline{H} - H(x_*)^{m_k}]F'(x_*)^{-1} \\ &+ [H(x_*)^{m_k} - H(x_k)^{m_k}]F'(x_*)^{-1} \\ &+ H(x_k)^{m_k}[F'(x_*)^{-1}-F'(x_k)^{-1}]. \end{aligned}$$

The continuity of $F'(x)^{-1}$ at x_* follows from (1.2) and from (5.5) we have that $H(x)^m$ is uniformly continuous in m and at x_*. The convergence of $K(x_k)$ to K_* thus follows from the definition and existence of $\overline{H} = \lim_{k\to\infty} H(x_*)^{m_k}$. The result now follows directly from Theorem 3.2 with $y_k = K_*^{-1}s_k$ and $\alpha_k = 0$ for each $k = 0,1,\ldots$.

Notice that we were able to use the results of section 3 easily to study the order of convergence of an iteration whose q-linear convergence had not been shown by the results of section 2. We think this is an important point to make.

Although the full power of Theorem 3.2 was not called upon here, since $\{K_k\}$ satisfied the generalized consistency condition, $\lim_{k\to\infty} K_k = K_*$, $|I - K_*F'(x_*)| < 1$, we will need the full power of all these results in [4].

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