New Journal of Physics
An Institute of Physics and Deutsche Physikalische Gesellschaft Journal

Local correlations in a strongly interacting one-dimensional Bose gas

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New Journal of Physics **5** (2003) 79.1–79.11 (http://www.njp.org/)
Received 3 April 2003
Published 18 June 2003

Abstract. We develop an analytical method for calculating local correlations in strongly interacting one-dimensional (1D) Bose gases, based on the exactly solvable Lieb–Liniger model. The results are obtained at zero and finite temperatures. They describe the interaction-induced reduction of local many-body correlation functions and can be used for achieving and identifying the strong-coupling Tonks–Girardeau regime in experiments with cold Bose gases in the 1D regime.

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PII: S1367-2630(03)61681-2

1. Introduction

Recently, one-dimensional (1D) Bose gases have been created in long cylindrical traps by tightly confining the transverse motion of particles to zero-point oscillations [1]–[3]. At the present stage, one of the main goals is to achieve the strong-coupling Tonks–Girardeau (TG) regime [4, 5] where, due to repulsion between particles, the 1D Bose gas acquires fermionic properties. These studies have revived an interest in dynamical and correlation properties of 1D gases. In most cases, correlations and dynamics of trapped atomic gases in the 1D regime can be investigated [6]–[12] by using the Lieb–Liniger model [5], which assumes that particles interact via a delta-function repulsive potential. Of particular importance are local three-body and two-body correlations [11, 12] as they govern the rates of inelastic processes, such as three-body recombination and photoassociation in pair collisions. The measurement of these rates provides a way to identify the strong-coupling Tonks–Girardeau regime, and the intrinsic decay process of recombination in three-body collisions is crucial for the stability of the gas.

The Lieb–Liniger model for a uniform system of N bosons belongs to the class of integrable models of statistical physics [13]. The ground state and spectrum of elementary excitations for this model were found by Lieb and Liniger [5] and a theory for finding thermodynamic functions at finite temperatures was constructed by Yang and Yang [14]. These quantities are determined by relatively simple integral equations for the distribution of quantum numbers (quasi-momenta or rapidities). On the other hand, the problem of correlation properties is far from being completely resolved, except for some correlation functions in the limiting cases of weak and strong interactions. For example, the case of infinitely strong interactions is, to a certain extent, equivalent to that of free fermions and the interactions play the role of the Pauli principle [4]. In this limit, any correlation function of the density is given by the corresponding expression for fermions, and the one-particle density matrix is the determinant of a certain integral operator [15, 16]. The expressions for the one-body and two-body correlations for an arbitrary interaction strength were obtained by using the inverse scattering method [17]. However, closed analytical results can be found only as perturbative expansions in the limiting cases of weak and strong interactions [18]–[20].

In this paper, we calculate local many-body correlations, that is the expectation value of a product of $2m$ field operators ($m = 2, 3, \ldots$) at zero space and time separation. The locality allows us to obtain closed analytical results at zero and finite temperatures in the limiting cases of strong and weak interactions. The paper is organized as follows. In section 2 we present the Hamiltonian and introduce the relevant parameters of the problem. Section 3 contains the main result of the paper, the derivation of local correlation functions at zero and finite temperature in the strong-coupling regime. In section 4 we use this result for the particular case of two-body and three-body correlations. For the sake of completeness, local correlations for the weak-coupling regime, following from the Bogoliubov approach, are presented in section 5. We conclude in section 6.

2. Interacting Bose gas in 1D

We consider the system of N bosons on a 1D ring of length L in the thermodynamic limit $N, L \to \infty$, with a fixed density $n = N/L$. The particles interact via a repulsive delta-function

interaction potential and are described by the Hamiltonian

$$H = \frac{\hbar^2}{2M}\left[\sum_{j=1}^{N} -\partial_{x_j}^2 + 2c\sum_{i<j}\delta(x_i - x_j)\right]. \tag{1}$$

Here M is the particle mass, x_j is the coordinate of the jth particle and $c = mg/\hbar^2$, with $g > 0$ being the coupling constant. The quantity c is the inverse interaction length for the two-body problem with the delta-function potential $g\delta(x)$. In other words, the wavefunction of two particles decreases on a distance scale $1/c$ as they approach each other.

The Hamiltonian (1) is diagonalized by means of the Bethe ansatz [5]. The many-body wavefunction is symmetric with respect to the permutation of particle coordinates, and in the domain $0 < x_1 < \cdots < x_N < L$ the wavefunction is

$$\Phi(x_1, x_2, \ldots, x_N) = \frac{1}{\mathcal{N}(c)}\sum_{P} a(P)\,\mathrm{e}^{\mathrm{i}\sum k_{P_j}x_j}. \tag{2}$$

In equation (2) the sum runs over $N!$ permutations P acting on N indices of quantum numbers (quasi-momenta) k_j. These quantum numbers are chosen such that $k_1 < k_2 < \cdots < k_N$. They determine the amplitudes

$$a(P) = \prod_{i<j}\left(\frac{\mathrm{i}c + k_{P_i} - k_{P_j}}{\mathrm{i}c - k_{P_i} + k_{P_j}}\right)^{1/2} \tag{3}$$

of each term in equation (2) and their quantization in a finite system follows from the requirement of periodicity of the wavefunction (2). The corresponding eigenenergy is $E = \sum k_j^2$, and $\mathcal{N}(c)$ in equation (2) is the normalization constant. The latter has been calculated in [22, 23].

The key parameter of the system is the ratio of the mean interparticle separation $1/n$ to the interaction length $1/c$:

$$\gamma = \frac{c}{n} = \frac{mg}{\hbar^2 n}. \tag{4}$$

At sufficiently low temperatures, for $\gamma \gg 1$ the gas is in the strong-coupling regime: the many-body wavefunction strongly decreases at interparticle distances much smaller than $1/n$. In the extreme case $\gamma = \infty$ the amplitudes (3) are determined by the sign of the permutation:

$$a(P) = (-1)^P. \tag{5}$$

So, the wavefunction (2) vanishes when at least two particle coordinates coincide. In this limit any correlation function of the density operators can be calculated by using the theory of free fermions, since in the domain $0 < x_1 < x_2 < \cdots < x_N < L$ the wavefunction (2) is the Slater determinant constructed out of the plane waves with momenta k_j. In the case of large but finite γ, the calculations can be performed perturbatively as expounded in the next section for the local density correlations.

In the opposite limit, $\gamma \ll 1$, the gas is in the weak-coupling regime. In this case the ground state of the system is well described by the Gross–Pitaevskii mean-field theory, and correlation functions at $T = 0$ can be found on the basis of the Bogoliubov–Popov approach (see [21]). This approach also covers a significant part of the weak-coupling regime at finite temperatures.

For finite temperatures, aside from $1/c$ and $1/n$, one has another length scale, the thermal de Broglie wavelength $\Lambda = (2\pi\hbar^2/mT)^{1/2}$. The latter is especially important at temperatures $T > T_d$, where $T_d = \hbar^2 n^2/2m$ is the temperature of quantum degeneracy. The relation between the three length scales determines various finite-temperature regimes of the 1D Bose gas, which we discuss below in the limits of strong and weak interactions.

3. Local correlations in the strong-coupling limit

We now consider the strong-coupling limit $\gamma \gg 1$ and calculate the local m-particle correlation function:

$$g_m(\gamma) = \langle (\Psi^\dagger(0))^m (\Psi(0))^m \rangle, \tag{6}$$

where $\Psi^\dagger(x), \Psi(x)$ are the creation and annihilation operators of bosons. At $T = 0$ the expectation value $\langle \cdots \rangle$ is taken with respect to the ground state of the system, and at finite temperatures we assume the average in the grand-canonical ensemble. We first present the derivation at zero temperature and then generalize the method to the case of finite temperatures.

For $T = 0$, in first quantization the correlation function g_m (6) is

$$g_m(\gamma) = \frac{N!}{m!(N-m)!} \int \mathrm{d}x_{m+1} \ldots \mathrm{d}x_N |\Phi_0^{(\gamma)}(0, \ldots, 0, x_{m+1}, \ldots, x_N)|^2, \tag{7}$$

where the ground state wavefunction $\Phi_0^{(\gamma)}$ is given by the Bethe ansatz expression (2). In the strong-coupling limit the amplitudes $a(P)$ (3) can be expanded in inverse powers of the interaction strength

$$a(P) = \prod_{i<j} \left(\frac{\mathrm{i}c + k_{P_i} - k_{P_j}}{\mathrm{i}c - k_{P_i} + k_{P_j}} \right)^{1/2} = (-1)^P \prod_{i<j} \left(1 + \frac{k_{P_i} - k_{P_j}}{\mathrm{i}c} + \cdots \right). \tag{8}$$

The quasi-momenta are close to their ground state values at $\gamma = \infty$, given by the uniform Fermi–Dirac distribution in the interval $[-k_F, k_F]$, with the Fermi momentum $k_F = \pi n$. One then sees that, in fact, equation (8) gives the expansion in inverse powers of γ. For $\gamma \gg 1$, we extract the leading behaviour of the wavefunction Φ_0^γ at m coinciding points by appropriately symmetrizing the amplitudes $a(P)$ for each permutation:

$$\frac{1}{m!} \sum_p a(P_{p_1}, P_{p_2}, \ldots, P_{p_m}, P_{m+1}, \ldots P_N) \simeq \frac{(-1)^P}{(\mathrm{i}c)^{m(m-1)/2}} \Delta_m(k_{P_1}, \ldots, k_{P_m}), \tag{9}$$

where the sum runs over $m!$ permutations p of numbers $1, 2, \ldots, m$, and

$$\Delta_m(k_1, \ldots, k_m) = \prod_{i<j} (k_i - k_j) \tag{10}$$

is the Vandermonde determinant. Up to the leading term in $1/c$, the ground state wavefunction at m coinciding points is therefore given by

$$\Phi_0^{(\gamma)}(0, \ldots, 0, x_{m+1}, \ldots, x_N) = \frac{1}{(\mathrm{i}c)^{m(m-1)/2}} \frac{1}{\mathcal{N}(\infty)} \sum_P (-1)^P \Delta_m(k_{P_1}, \ldots, k_{P_m}) \times \exp\left(\mathrm{i} \sum_{j=m+1}^N k_{P_j} x_j \right). \tag{11}$$

This allows us to express $\Phi_0^{(\gamma)}(0, \ldots, 0, x_{m+1}, \ldots, x_N)$ through spatial derivatives of the wavefunction of non-interacting fermions:

$$\Phi_0^{(\infty)}(x_1, \ldots, x_N) = \frac{1}{\mathcal{N}(\infty)} \sum_P (-1)^P \exp\left(\mathrm{i} \sum_{j=1}^N k_{P_j} x_j \right).$$

From equation (11) we find

$$\Phi_0^{(\gamma)}(0,\ldots,0,x_{m+1},\ldots,x_N) = \left(\frac{-1}{c}\right)^{m(m-1)/2} \Delta_m(\partial_{x_1},\ldots,\partial_{x_m})\Phi_0^{(\infty)}(x_1,\ldots,x_m,x_{m+1},\ldots,x_N)|_{x_1=\cdots=x_m=0}. \quad (12)$$

Returning to equation (7), one sees that the local correlation function is given by derivatives of the m-particle correlation function of free fermions at $x_1 = \cdots = x_m = y_1 = \cdots = y_m = 0$:

$$g_m = \frac{1}{c^{m(m-1)}}\Delta_m(\partial_{x_1},\ldots,\partial_{x_m})\Delta_m(\partial_{y_1},\ldots,\partial_{y_m})\langle\psi^\dagger(x_1)\cdots\psi^\dagger(x_m)\psi(y_m)\cdots\psi(y_1)\rangle. \quad (13)$$

The operators $\psi^\dagger$, ψ are now fermionic field operators with canonical anticommutation relations. Using Wick's theorem we express the expectation value of these operator as a product of one-particle Green functions:

$$\langle\psi^\dagger(x_1)\cdots\psi^\dagger(x_m)\psi(y_m)\cdots\psi(y_1)\rangle = \sum_p (-1)^p\, G(x_1,y_{p_1})G(x_2,y_{p_2})\cdots G(x_m,y_{p_m}). \quad (14)$$

The one-particle Green function is given by

$$G(x,y) = G(x-y) = \langle\psi^\dagger(x)\psi(y)\rangle = \frac{1}{L}\sum_k N^0(k)\,\mathrm{e}^{\mathrm{i}k(x-y)}, \quad (15)$$

where $N^0(k)$ is the ground state Fermi–Dirac distribution. Then, from equations (15), (14) and (13) we obtain

$$g_m = \frac{1}{c^{m(m-1)}}\sum_p \frac{(-1)^p}{L^m}\Delta_m(\partial_x)\Delta_m(\partial_y)\sum_{k_1\ldots k_m} N^0_{k_1}\cdots N^0_{k_m}\mathrm{e}^{\mathrm{i}k_1(x_1-y_{p_1})}\cdots\mathrm{e}^{\mathrm{i}k_m(x_m-y_{p_m})}\Big|_{\substack{x_1=\cdots=x_m=0\\ y_1=\cdots=y_m=0}}, \quad (16)$$

where $\Delta_m(\partial_x)$ stands for $\Delta_m(\partial_{x_1},\ldots,\partial_{x_m})$. This expression can be further simplified as the Vandermonde determinant is a totally antisymmetric function of its variables. Therefore, reordering y_{p_j} arguments cancels the factor $(-1)^p$. Using this fact, acting with derivatives on the corresponding arguments in the exponents, and noting that all permutations give the same result, we arrive at the equation

$$g_m = \frac{m!}{c^{m(m-1)}}\frac{1}{L^m}\sum_{k_1\ldots k_m} N^0(k_1)\cdots N^0(k_m)\Delta_m^2(k_1,\ldots,k_m). \quad (17)$$

The method developed was briefly outlined in [11] for three-particle local correlations. It is readily generalized to the case of finite temperatures. In the grand canonical ensemble the expression for the m-particle local correlation function g_m (6) is

$$g_m = \frac{1}{Z}\mathrm{Tr}\{(\Psi^\dagger(0))^m(\Psi(0))^m\exp\{-(H-\mu N)/T\}\} = \sum_{N=1}^{\infty}\frac{z^N}{N!}\sum_\alpha \mathrm{e}^{-E_\alpha/T}\frac{N!}{m!(N-m)!} \times \int \mathrm{d}x_{m+1}\cdots\mathrm{d}x_N|\Phi_\alpha^{(\gamma)}(0,\ldots,0,x_{m+1},\ldots,x_N)|^2, \quad (18)$$

where Z is the partition function, $z = \exp(\mu/T)$ is the fugacity, μ is the chemical potential and the index α labels eigenstates of the system. Expanding the eigenfunctions $\Phi_\alpha^{(\gamma)}$ at coinciding points exactly as above in the case of the ground state wavefunction, we express the leading

contribution to g_m through derivatives of the finite-temperature correlation function of free fermions at $x_1 = x_2 = \cdots = x_m = y_1 = \cdots = y_m = 0$:

$$g_m = \frac{1}{c^{m(m-1)}} \Delta_m(\partial_x)\Delta_m(\partial_y)\langle \psi^\dagger(x_1)\cdots\psi^\dagger(x_m)\psi(y_m)\cdots\psi(y_1)\rangle_T. \tag{19}$$

The finite-temperature m-fermion correlation function is again calculated with the use of Wick's theorem. This gives equation (14) in which $G(x_i, y_{P_i})$ are now finite-temperature Green functions for free fermions. We then proceed in the same way as at $T = 0$ and arrive at equation (17), with $N^0(k)$ replaced by the Fermi–Dirac distribution at finite temperatures:

$$N(k) = \frac{z\exp(-\Lambda^2 k^2/4\pi)}{1 + z\exp(-\Lambda^2 k^2/4\pi)}. \tag{20}$$

At temperatures $T \gg T_d$ the characteristic momentum of particles is the thermal momentum $k_T \sim 1/\Lambda$. Therefore, the small parameter for the expansion of the amplitudes $a(P)$ in equation (8) is $1/\Lambda c$. Thus, one must satisfy the inequality $\Lambda c \gg 1$, which requires temperatures

$$T \ll \gamma^2 T_d. \tag{21}$$

From this point on we consider together the cases of zero and finite temperature. In the thermodynamic limit, expressing the momenta in terms of the size of the Fermi zone, $k = 2k_F x = 2\pi n x$, we have

$$\frac{g_m}{n^m} = \left(\frac{2\pi}{\gamma}\right)^{m(m-1)} m!\, I_m(\Lambda n, z), \tag{22}$$

where the m-fold integral $I_m(\Lambda n, z)$ depends on the ratio of the de Broglie wavelength to the mean interparticle separation and on the fugacity z (or chemical potential), and is given by

$$I_m(\Lambda n, z) = \int_{-\infty}^{+\infty} \mathrm{d}x_1 \cdots \mathrm{d}x_m N(x_1)\cdots N(x_m)\Delta_m^2(x_1, \ldots, x_m), \tag{23}$$

where $N(x)$ is given by equation (20) with $k = 2\pi n x$. In order to keep the density constant one must fix z as a function of Λn by the normalization condition

$$\int_{-\infty}^{+\infty} N(x)\,\mathrm{d}x = 1. \tag{24}$$

Under this condition the integral in equation (23) becomes a function of Λn only. For calculating this function we employ the method of orthogonal polynomials used in random matrix theory [24]. Consider a set of polynomials $P_j(x) = x^j + \cdots$ for $j = 0, 1, 2, \ldots$, orthogonal with the weight $N(x)$ on an infinite interval:

$$\int_{-\infty}^{+\infty} N(x) P_i(x) P_j(x)\,\mathrm{d}x = h_i \delta_{ij}. \tag{25}$$

Then the value of the integral (23) is expressed through the normalization coefficients h_j as follows:

$$I_m(\Lambda n, z) = m! h_0 \cdots h_{m-1}. \tag{26}$$

At $T = 0$ we have $\Lambda n = \infty$. In this limit the distribution function $N(x)$ is uniform in the interval $-1/2 < x < 1/2$, and is zero otherwise. Polynomials $P_j(x)$ can be expressed in terms of Jacobi polynomials [25] and we obtain

$$I_m(\Lambda n) = m! \prod_{j=0}^{m-1} \frac{[\Gamma(j+1)]^4}{\Gamma(2j+1)\Gamma(2j+2)}, \qquad \Lambda n = \infty. \tag{27}$$

From equation (22) we then obtain the local correlation function

$$\frac{g_m}{n^m} = A_m \left(\frac{\pi}{\gamma}\right)^{m(m-1)}, \tag{28}$$

where the coefficients A_m satisfy the recurrence relation

$$\frac{A_{m+1}}{A_m} = \pi \frac{2m+1}{2^{2m+2}} \left[\frac{\Gamma(m+2)}{\Gamma(m+3/2)}\right]^2, \qquad A_1 = 1. \tag{29}$$

Although general expressions for the normalization coefficients h_j in equation (25) are not known, for sufficiently small m these coefficients can be calculated straightforwardly as they are related to the moments of the distribution function $N(x)$. For example, one finds

$$h_0 = 1, \quad h_1 = < x^2 >, \quad h_2 = < x^4 > - < x^2 >^2, \qquad < x^{2j} > = \int_{-\infty}^{+\infty} x^{2j} N(x)\, \mathrm{d}x. \tag{30}$$

We finally consider high temperatures $T \gg T_d$ that still satisfy equation (21). In this case the fugacity is $z = \Lambda n$ and the distribution function is Gaussian: $N(x) = z \exp(-\pi \Lambda^2 n^2 x^2)$. The polynomials $P_j(x)$ are Hermite polynomials, and we find

$$I_m(\Lambda n) = \frac{m!}{(\sqrt{2\pi}\,\Lambda n)^{m(m-1)}} \prod_{j=0}^{m-1} \Gamma(1+j), \qquad 1/c \ll \Lambda \ll 1/n. \tag{31}$$

Then equation (22) for the local correlation function takes the form

$$\frac{g_m}{n^m} = B_m \left(\frac{\sqrt{2\pi}}{\Lambda c}\right)^{m(m-1)}, \qquad 1/c \ll \Lambda \ll 1/n, \tag{32}$$

where the recurrence relation for the coefficients B_m is $B_{m+1} = (m+1)\Gamma(m+2)\, B_m$, and $B_1 = 1$.

4. Three-body and two-body correlations

In this section we use the general results of section 3 for the particular case of three-body local correlations in the strong-coupling limit. Two-body correlations have been discussed in [12] and are expounded here for completeness.

At temperatures $T \ll T_d$ we have $\Lambda n \gg 1$ and the local correlation functions g_2 and g_3 are close to their zero-temperature values. Thus, the system remains in the Tonks–Girardeau regime. For calculating g_2 and g_3 we use equations (22) and (26), with normalization coefficients h_1 and h_2 following from equation (30). The quantities $\langle x^2 \rangle$ and $\langle x^4 \rangle$ at finite T are obtained on the basis of the Sommerfeld expansion:

$$< x^{2j} >_T - < x^{2j} >_0 \simeq \frac{2}{3} \frac{2j}{2^{2j}} \frac{1}{(\Lambda n)^4} = \frac{1}{24\pi^2} \frac{2j}{2^{2j}} \left(\frac{T}{T_d}\right)^2. \tag{33}$$

This gives the following expressions:

$$\frac{g_2}{n^2} = \frac{4}{3}\left(\frac{\pi}{\gamma}\right)^2 \left[1 + \frac{4}{(\Lambda n)^4}\right] = \frac{4}{3}\left(\frac{\pi}{\gamma}\right)^2 \left[1 + \frac{1}{4\pi^2}\left(\frac{T}{T_d}\right)^2\right] \tag{34}$$

$$\frac{g_3}{n^3} = \frac{16}{15}\left(\frac{\pi}{\gamma}\right)^6 \left[1 + \frac{28}{(\Lambda n)^4}\right] = \frac{16}{15}\left(\frac{\pi}{\gamma}\right)^6 \left[1 + \frac{7}{4\pi^2}\left(\frac{T}{T_d}\right)^2\right]. \tag{35}$$

The results of equations (34) and (35) have a clear physical explanation. For $T \ll T_d$ the characteristic momentum of particles is of the order of $k_F = \pi n$. Fermionic correlations are

present at interparticle distances $x \gtrsim 1/c$, since for smaller distances the correlations do not change. Therefore, g_2 is nothing other than the pair correlation function for free fermions at a distance $x \sim 1/c$ [12]. This function is of the order of $(k_F/c)^2 \sim (\pi/\gamma)^2$, which agrees with equation (34). With the same arguments, one finds that the three-body correlation function is $g_3 \sim (k_F/c)^2 \sim (\pi/\gamma)^6$, which coincides with the result of equation (35).

For temperatures in the interval $T_d \ll T \ll \gamma^2 T_d$, where $1/c \ll \Lambda \ll 1/n$, we use equation (32) directly and obtain

$$\frac{g_2}{n^2} = \frac{8\pi}{(\Lambda c)^2} = \frac{2}{\gamma^2}\frac{T}{T_d} \tag{36}$$

$$\frac{g_3}{n^3} = \frac{576\pi^3}{(\Lambda c)^6} = \frac{9}{\gamma^6}\left(\frac{T}{T_d}\right)^3. \tag{37}$$

This regime can be called the regime of 'high-temperature fermionization' [12]. The gas is no longer degenerate, but due to the strong interaction between particles the correlations still have a fermionic character. With regard to local correlations, the main difference from the low-temperature case is related to the value of the characteristic momentum of particles. It is now of the order of $1/\Lambda$, instead of k_F at $T \ll T_d$. Accordingly, the two-body and three-body local correlation functions are $g_2 \sim (1/\Lambda c)^2$ and $g_3 \sim (1/\Lambda c)^6$, as described by equations (36) and (37).

5. Weak-coupling limit

In the weak-coupling limit, where $\gamma \ll 1$, one can rely on the mean-field approach. The mean-field interaction energy per particle is proportional to ng and it is reasonable to introduce the correlation length $l_c = \hbar/\sqrt{mng}$. At temperatures $T \ll T_d$, over a wide range of parameters the correlation length $l_c \ll l_\phi$, where l_ϕ is the phase coherence length. Then the equilibrium state is a quasicondensate, that is the state in which density fluctuations are suppressed but the phase still fluctuates [26]. A review of earlier studies of 1D Bose gases in the weak-coupling limit may be found in [21]. The zero-temperature phase coherence length is always larger than l_c, and at finite T one has $l_\phi = \hbar^2 n/mT$. Therefore, the condition $l_c \ll l_\phi$ can be rewritten as

$$\frac{T}{T_d} \ll \sqrt{\gamma}. \tag{38}$$

One then sees that the quasicondensate regime requires sufficiently low temperatures or relatively large interactions between particles.

Due to the presence of phase coherence on a long distance scale, local correlation functions in the quasicondensate regime are close to n^2. Deviations from this value can be calculated straightforwardly by using the Bogoliubov approximation [27]. For g_2 the results are known (see [12, 28] and references therein), and g_3 at $T = 0$ is given in [11]. Here we complete the picture and present general results for g_m. They are obtained in the same way as g_2 in [12].

We represent the bosonic field operator as a sum of a macroscopic component Ψ_0 and a small component Ψ' describing finite-momentum excitations. To be more precise, the component Ψ_0 contains the contribution of excitations with momenta $k \lesssim k_0 \ll l_c^{-1}$, whereas Ψ' includes the contribution of larger k. At the same time, the momentum k_0 is chosen such that most of the particles are contained in the part Ψ_0. This picture is along the lines of [21] and the momentum k_0 drops out of the answer as the main contribution of the excitation part Ψ' to local correlation

functions is provided by excitations with $k \sim l_c^{-1}$. Confining ourselves to the terms quadratic in Ψ' and taking then into account that $|\Psi_0|^{2m} = n^m - mn^{m-1}\langle\Psi'^{\dagger}\Psi'\rangle$, the m-particle local correlation function is given by

$$g_m = \langle(\Psi_0^* + \Psi'^{\dagger})^m(\Psi_0 + \Psi')^m\rangle = n^m\left[1 - \frac{m(m-1)}{n}(\langle\Psi'^{\dagger}\Psi'\rangle + \langle\Psi'\Psi'\rangle)\right]. \tag{39}$$

The normal and anomalous averages, $\langle\Psi'^{\dagger}\Psi'\rangle$ and $\langle\Psi'\Psi'\rangle$, can be calculated by using the same Bogoliubov transformation for Ψ' as in the 3D case:

$$\Psi' = \sum_k (u_k a_k e^{ikx} - v_k a_k^{\dagger} e^{-ikx}), \tag{40}$$

where a_k and $a_k^{\dagger}$ are annihilation and creation operators of elementary excitations, $u_k, v_k = (\varepsilon_k \pm E_k)/2\sqrt{\varepsilon_k E_k}$, $\varepsilon_k = \sqrt{E_k^2 + 2ngE_k}$ is the Bogoliubov excitation energy and $E_k = \hbar^2k^2/2m$. This immediately gives

$$\frac{g_m}{n^m} = 1 + m(m-1)\int_{-\infty}^{\infty}\frac{dk}{2\pi n}\left[\frac{E_k}{\varepsilon_k}(1+\tilde{N}_k) - 1\right], \tag{41}$$

with $\tilde{N}_k = [\exp(\varepsilon_k/T) - 1]^{-1}$ being the equilibrium occupation numbers for the Bogoliubov excitations.

The integral term in equation (41) contains the contribution of both vacuum and thermal fluctuations. The former is determined by excitations with $k \sim l_c^{-1}$, and at $T = 0$ we find

$$\frac{g_m}{n^m} = 1 - \frac{m(m-1)}{\pi}\sqrt{\gamma}. \tag{42}$$

At temperatures $T \ll ng \sim \gamma T_d$, thermal fluctuations provide only a small correction $(\pi\sqrt{\gamma}/24)(T/\gamma T_d)^2$ to the result of equation (42).

For temperatures $T \gg ng \sim \gamma T_d$ (but still $T \ll \sqrt{\gamma}T_d$), the contribution of thermal fluctuations to the integral term in equation (41) is the most important. It comes from excitation energies $\varepsilon_k \sim ng$ and, hence, one may put $\tilde{N}(k) = T/\varepsilon_k$ in the integrand. This yields

$$\frac{g_m}{n^m} = 1 + \frac{m(m-1)}{4}\frac{T}{\sqrt{\gamma}T_d}. \tag{43}$$

One clearly sees from equation (43) that g_m increases with decreasing interaction strength or increasing temperature. The ratio $T/T_d\sqrt{\gamma}$ is simply l_c/l_ϕ. For $l_c \sim l_\phi$ or $T \sim T_d\sqrt{\gamma}$, the gas leaves the quasicondensate regime and equation (43) is no longer valid. At a temperature and interaction strength satisfying the condition $T \gg T_d\sqrt{\gamma}$ the gas is in the decoherent regime where the interaction between particles is much less important. Note that, for a very small interaction strength, this regime is present even at temperatures far below T_d. The cross-over from the quasicondensate to the decoherent regime was discussed in [12, 28] on the basis of calculations for g_2. In the decoherent regime the local correlation function is close to the result for an ideal Bose gas following from Wick's theorem: $g_m/n^m = m!$. The corrections to this value can be calculated perturbatively [12]. The lower T is, the smaller is the γ at which the gas enters the decoherent regime. For $T = 0$ this transition is discontinuous and occurs at zero interaction strength.

6. Concluding remarks

From a general point of view, local correlations of the 1D system are less universal than the long-wave behaviour. Local correlation functions depend on a particular model used for calculations. Our approach leads to physically transparent analytical results for g_m of the 1D Bose gas in the Lieb–Liniger model.

The use of this model for trapped atomic gases in the 1D regime is justified by the short-range character of interatomic interaction: the characteristic radius of interactions is much smaller than any length scale of the Lieb–Liniger model. For an atomic gas in an infinitely long cylindrical trap, harmonically confined with frequency ω_0 in the transverse direction to zero-point oscillations, the coupling constant g is expressed through the 3D scattering length a and is [29] $g = 2\hbar^2 a/Ml_0^2$, where $l_0 = \sqrt{\hbar/M\omega_0}$ is the amplitude of zero-point oscillations. Then, to describe the system by the 1D Hamiltonian (1), it is sufficient to satisfy the inequalities $l_0 \ll 1/n$, Λ_T [11, 12].

The reduction of g_3 in the strong-coupling limit, which follows from our results, is important in two respects. First, it indicates a possibility of achieving this limit at a high gas density (large number of particles), since the rate of decay due to three-body recombination is proportional to g_3 and will be strongly suppressed. Second, the measurement of three-body losses of particles can be used for identifying the strong-coupling Tonks–Girardeau regime and the regime of high-temperature fermionization. The identification of these regimes can also be provided by the measurement of photoassociation in pair atomic collisions, as the rate of this process is proportional to the two-particle local correlation function g_2. In this respect, it is worth mentioning that recent Monte Carlo calculations of g_2 at $T = 0$ for the number of particles as low as 100 [30] agree with our results obtained in the thermodynamic limit.

Acknowledgments

We are grateful to Lincoln Carr for useful comments and we acknowledge the hospitality of the European Centre for Theoretical Studies in Nuclear Physics and Related Areas (ECT) during the 2002 BEC Summer Programme, where part of the present work was done. This work was financially supported by the French Ministère des Affaires Etrangères, by the Dutch Foundations NWO and FOM, by INTAS, and by the Russian Foundation for Fundamental Research. Laboratoire Kastler–Brossel is a research unit of Université Pierre et Marie Curie and Ecole Normale Supérieure, associated with CNRS (UMR 8562).

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