Local Covariant Operator Formalism of Non-Abelian Gauge Theories and Quark Confinement Problem

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A manifestly covariant and local canonical operator formalism of non-Abelian gauge theories is presented in its full detail. This formalism, applicable to Yang-Mills theories as well as to gravity, not only provides us a transparent understanding in the scattering theoretical aspects, but also makes it possible to discuss other important problems directly related to the (Heisenberg) operators and the state vectors: As for the former, the physical S-matrix unitarity is proved quite generally on the basis of the representation of the algebra of the BRS charge, and asymptotic field analysis is explicitly performed for some examples. As for the latter, the problems of observables and the well-definedness of charge operators are discussed and clear results are obtained, where the locality and covariance of the formalism are indispensable. Observables are shown to be invariant under the BRS transformation as well as the unbroken global gauge groups. By analyzing the structure of "Maxwell" equations in YM theories, the converse of the Higgs theorem is found to hold. This turns out to lead to a remarkably simple criterion of quark confinement in QCD. The present formalism is found useful also for the $U(1)$ problem and the charge universality proof in the Weinberg-Salam model. General theory of indefinite metric quantum fields is developed to some extent.

Contents

Chapter I

Introduction

§ 1.1. Motive and Outline

A view, which may be interpreted, in a sense, as a revived and revised version of Einstein's "geometrization" programme of physics, becomes recently prevailing in elementary particle physics, with the increasing experimental data supporting the Weinberg-Salam (W.S.) model in the weak-electromagnetic interaction as well as the quantum chromodynamics (QCD) in the strong interaction:[1] Namely, all of the four types of the interactions ruling the nature, not only electromagnetic and gravitational but also strong and weak interactions, are intermediated by *gauge fields* universally. Apart from the gravitational interaction described by the metric tensor $g_{\mu\nu}(x)$ or the vierbein field $e_\mu{}^a(x)$, which has a *non-compact* gauge group, all other gauge fields are, in the geometrical language, the *connection* fields[2] of compact *non-Abelian* gauge groups of internal symmetries [$SU(2)\times U(1)$ for W.S. model and $SU(3)$ for QCD, etc.], which are called, usually in physics, *Yang-Mills* (*YM*) *fields.*

So the present physics requires a consistent and powerful formalism of non-Abelian gauge theories on a sound basis. We will present, in this paper, a manifestly *covariant* and *local* canonical operator-formalism of non-Abelian gauge theories in its full detail. In this formalism initiated in Ref. 3), we can directly deal with the *Heisenberg operators* and the *state vector space* as well as its subspace of physical states. Accordingly, this formalism not only gives us a satisfactory and transparent understanding in such *scattering theoretical aspects* as the physical S-matrix unitarity, but also enables us to discuss such problems as the physical *observables,* the well-definedness of *charge operators* and so on. The latter point is very important. The analysis of the general structure of non-Abelian gauge theories is first made possible in this formalism where we can utilize with slight modifications the useful consequences derived from locality and covariance in the general quantum field theory with positive definite metric. On the basis of such apparatus, we can attack the outstanding problem of quark confinement, and will, in fact, find a simple and clear criterion of confinement of colored particles.

The non-Abelian character of YM field has been shown to be crucial for the peculiar property of *asymptotic freedom*[4] (which explains the Bjorken-scaling and its violation[5] in the deep-inelastic scattering of leptons off nucleons), and further, is supposed to hold the essential key for the solution of the *quark-confinement* problem.[6] This very non-Abelianness, however, has simultaneously caused many difficulties in quantizing such gauge theories as YM field and gravity. In fact, they have long been so far treated only in the path-integral formalism for lack of consistent (Heisenberg-) operator formalisms except for those in such non-covariant and non-local gauges as Coulomb, axial and timelike-axial gauges. To be sure, the path integral method has yielded many such successful results as the correct Feynman rules including the famous Faddeev-Popov (FP) ghosts,[7] the Ward-Takahashi (W.T.) identities and the proof of renormalizability of the theory,[8] etc., and it enables us to calculate some Green's functions and transition amplitudes, perturbatively. But, the absence of the notions of the *state vector space* and the *Heisenberg operators* in this formalism obstructs us to get an insight into the general and fundamental aspects of the logical structure of the theory in a non-perturbative fashion. The understanding of the aspects of this sort seems quite necessary not only for the explanation of the Higgs mechanism in a consistent way with the Goldstone theorem, as already done in the Abelian case,[9] but also for the theoretical investigation of the quark confinement, in both of which the problem 'what are *physical* and *observable*?' should be clarified.

Next, as for the *covariance* and the *locality*, we should recall that without these principles even the renormalizations could not be carried out in a complete form for lack of simple kinematics to determine the forms of counterterms. As is seen from this example, the *relativistic kinematics following from the covariance and the locality* furnishes us with some prospects to the unsolved problem restricting the forms of possible solutions. From a more pragmatic viewpoint also, non-covariant gauges are not convenient for practical calculations and have appreciably worse ultraviolet properties than the local covariant gauges. Furthermore, the locality combined with the covariance leads to the validity of the dispersion relations,[10] which means the *analyticity* in p-space, while the spectrum condition with covariance and locality concludes the *analyticity* in x-space, namely, analyticity of the Wightman functions.[11),12] From this analyticity, such far-reaching results are derived, as Reeh-Schlieder theorem,[13),12] PCT theorem[14),12] and Borchers classes,[15),12] the relation between cluster property and uniqueness of the vacuum,[16] and some general theorems about the symmetry[17)~20] and so on.[21),22]

Thus, in order for us to approach such difficult dynamical problem as quark confinement in QCD, as well as to consolidate the foundation of Weinberg-Salam model, it is desirable and even crucial to formulate first the canonical formalism of YM fields in the framework of relativistic covariant local quantum field theory, which enables us to utilize such general and useful apparatus as the above.

Several attempts to construct such canonical operator formulations of YM theory in covariant gauges have been made so far. All those attempts[23)~26] made before Ref. 3), however, have failed in giving satisfactory formulations especially by the following crucial two defects: First, the *hermiticity of the Hamiltonian and Lagrangian* is violated, and hence, the *(pseudo-) unitarity of (full) S-matrix* also breaks down. Second, the consistent *subsidiary condition(s)* to specify the physical subspace invariant under time-evolution is not given. The former defect comes from the incorrect *hermiticity assignment to the FP ghosts*. With such non-hermitian Hamiltonian, one could not obtain a consistent Heisenberg operator formalism from the beginning. The latter one is related to the complicated non-Abelian character which hinders the straightforward application of Gupta-Bleuler subsidiary condition to the YM case. The key for the solution of this problem is given by the *charge of BRS transformation*[24] providing, essentially, a global version of the local gauge transformation. These points will be explained in detail in § 1.2.

The contents of this paper are organized as follows: In the first half (Chaps. II~IV), we present basic framework of the present formalism and deal mainly with the scattering theoretical aspects of theory. On the basis of these, in the latter part (Chaps. V~VII), various interesting contents of gauge theories are revealed in full use of the general consequences of covariance and locality.

In § 1.2, we explain *physicality criteria* which state indispensable conditions for a theory to be consistent and physically meaningful. In connection with it, we present the correct hermiticity assignment to FP ghosts and the consistent and concise subsidiary condition specifying the physical subspace.

Chapter II is devoted to the presentation of basic ingredients of our formalism. The Lagrangian density of the system to be discussed, its BRS symmetry and other symmetries are presented. Some of the consequences of the BRS symmetry, for example, "Maxwell" equation of motion, the W.T. identities, etc., are derived there.

Through the unitarity proof of the physical S-matrix in Chap. III, the role of the BRS invariance as a gauge invariance is made clear, which operates to "confine" the unphysical particles into the physically invisible unphysical world by the mechanism which we call "quartet mechanism". In Chap. IV, the essence of the general discussions made in Chap. III is elucidated through various example models including the gravitation theory.

In Chaps. V and VI, where the physical contents described in $H_{\rm phys}$ of the YM theory remaining after the "confinement" of the unphysical particles are examined, another aspect of the gauge (BRS) invariance shows itself in the notion of the observable (Chap. V) and the dynamical consequences of the "Maxwell" equation are discussed (Chap. VI). Here, the local covariant formalism exhibits its significance by allowing the general techniques developed in the Appendix to function effectively. These analyses reveal the interesting features of the global gauge symmetry in the YM theory concerning the Higgs phenomenon, the color confinement and their relations with the mass spectrum of the theory.

In Chap. VII, some other applications of the formalism are discussed: the $U(1)$ problem from the consideration of 'local' gauge invariance, the universality of the electric charges in the Weinberg-Salam model and other topics.

Some discussions are given in the final chapter.

In the Appendix, some of the useful consequences of the general theory of relativistic quantum fields obtained in the positive definite metric cases, are extended to the cases with indefinite metric.

§ 1.2. Indefinite Metric and Physicality Criteria

As is well-known in QED, one should inevitably bring in an *indefinite metric* into the theory in order to quantize gauge fields in a Lorentz covariant manner.[27] This means the presence of *negative* probabilities which might damage the probabilistic interpretation of the quantum theory. Moreover, the quantization of the YM theory (and also of the gravitation theory) requires the introduction of the unphysical fields called Faddeev-Popov (FP) ghosts[7] with wrong spin-statistics relation. Thus, the main problem in the covariant quantization of gauge fields consists in how these unphysical negative norm states as well as such unphysical particles as FP ghosts can be "confined" so as not to come out in the physical world.

For this purpose, we recall a former example of the Abelian gauge theory, especially the Nakanishi-Lautrup (N.L.) formalism[28] as a prototype of satisfactory formulation in the Abelian case. The N.L. formalism is an elegant canonical formulation in covariant gauges which provides an extension of the Gupta-Bleuler (G.B.) formalism[29] in Feynman gauge. In the N.L. formalism, this problem of unphysical negative-norm states is solved in the following way. First, since gauge must be fixed before quantization, we add, to the original local-gauge-invariant Lagrangian density $\mathcal{L}_s$ of the system, the following gauge fixing term:

$$\mathcal{L}_{\mathrm{GF}} = B\partial_\mu A^\mu + \alpha B^2/2 . \tag{1·1a}$$

Then, the total Lagrangian to be quantized is

$$\mathcal{L} = \mathcal{L}_s + \mathcal{L}_{\mathrm{GF}} . \tag{1·1b}$$

The auxiliary gauge-fixing multiplier field B, which becomes a canonical momentum variable conjugate to A_0, satisfies the following equations:

$$\partial^\mu A_\mu + \alpha B = 0 , \tag{1·2}$$

$$\Box B = 0 , \tag{1·3}$$

and obeys the commutation relations:

$$[B(x), B(y)] = 0, \tag{1·4a}$$

$$[A_\mu(x), B(y)] = -i\partial_\mu D(x-y). \tag{1·4b}$$

By virtue of (1·3), the positive and negative frequency parts (in other words, the annihilation and creation parts), $B^{(+)}(x)$ and $B^{(-)}(x)$, of the B field can be defined without any inconsistency with the time evolution of the system. Now, we can select the *physical states* $|\text{phys}\rangle$ from the total state vector space $\mathcal{V}$ with an indefinite metric $\langle|\rangle$ by a *subsidiary* condition:

$$B^{(+)}(x)|\text{phys}\rangle = 0, \tag{1·5}$$

which is equivalent, by (1·2), to a more familiar one

$$(\partial^\mu A_\mu)^{(+)}(x)|\text{phys}\rangle = 0, \tag{1·5′}$$

as long as $\alpha \neq 0$.*) Then, the *physical subspace* $\mathcal{V}_{\text{phys}} \equiv \{|\text{phys}\rangle\}$ consisting of all the physical states is shown to satisfy the following two conditions:

(i) The physical subspace $\mathcal{V}_{\text{phys}}$ is invariant under the time evolution of the system, namely,

$$H\mathcal{V}_{\text{phys}} \subset \mathcal{V}_{\text{phys}} \tag{1·6}$$

holds for the Hamiltonian $H(=P_0$: generator of time translation).

(ii) The inner product $\langle|\rangle$ is *positive semi-definite in* $\mathcal{V}_{\text{phys}}$:

$$|\Psi\rangle \in \mathcal{V}_{\text{phys}} \Rightarrow \langle\Psi|\Psi\rangle \geq 0. \tag{1·7}$$

Under the usual hermiticity assignment to field operators, the above Hamiltonian H is duly hermitian:

(0) $$H^\dagger = H, \tag{1·8}$$

which implies (on the assumption of asymptotic completeness), the (pseudo-) unitarity of the total S-matrix S with respect to the indefinite metric $\langle|\rangle$:

(0′) $$S^\dagger S = SS^\dagger = 1. \tag{1·8′}$$

In this case, the condition (1·6) can be rewritten as

(i′) $$S\mathcal{V}_{\text{phys}} = S^{-1}\mathcal{V}_{\text{phys}} = \mathcal{V}_{\text{phys}} \tag{1·7′}$$

or equivalently

(i″) $$\mathcal{V}^{\text{in}}_{\text{phys}} = \mathcal{V}^{\text{out}}_{\text{phys}}, \tag{1·7″}$$

where $\mathcal{V}^{\text{in,out}}_{\text{phys}}$ is the physical subspace of the Fock space of in- and out-states.

*) The α^{-1} factor in $B = -\alpha^{-1}\partial^\mu A_\mu$ clearly indicates the reason why the G.B. subsidiary condition (1·5′) does not work well for Landau gauge ($\alpha = 0$).

Since, as shown in the following theorem, these three conditions (0)~(ii) are sufficient to guarantee the consistent physical interpretation of theory (at least, in its scattering theoretical aspects), we call them the *physicality criteria* of theory.

Theorem 1.1. If the theory satisfies the following three *physicality criteria*, for the Hamiltonian H and the physical subspace $\mathcal{V}_{\rm phys}$ in the total state vector space $\mathcal{V}$ having indefinite inner product $\langle | \rangle$,

$$\left.\begin{array}{ll}
\text{(0)} & \text{hermiticity of the Hamiltonian:} \quad H^\dagger = H, \\
& [\text{or (0}'\text{) (pseudo-)unitarity of the total} \\
& S\text{-matrix:} \quad S^\dagger S = SS^\dagger = 1] \\
\text{(i)} & \text{invariance of } \mathcal{V}_{\rm phys} \text{ under the time development,} \\
& [\text{or (i}'\text{)} \quad S\mathcal{V}_{\rm phys} = S^{-1}\mathcal{V}_{\rm phys} = \mathcal{V}_{\rm phys}] \\
\text{(ii)} & \text{positive semi-definiteness of the inner product} \\
& \text{in } \mathcal{V}_{\rm phys},
\end{array}\right\} \tag{1·9}$$

then, the physical S-matrix $S_{\rm phys}$ can be defined consistently in the (completed) quotient space

$$H_{\rm phys} \equiv \overline{\mathcal{V}_{\rm phys}/\mathcal{V}_0} \quad (\mathcal{V}_0\text{: the zero-norm subspace of } \mathcal{V}_{\rm phys}) \tag{1·10}$$ *)

(which is a Hilbert space with positive definite metric), and the unitarity of $S_{\rm phys}$ holds:

$$S^\dagger_{\rm phys} S_{\rm phys} = S_{\rm phys} S^\dagger_{\rm phys} = 1. \tag{1·11}$$

Proof) First, by the Cauchy-Schwarz inequality due to (ii) of (1·9) [(1·7)], the zero-norm subspace $\mathcal{V}_0$ of $\mathcal{V}_{\rm phys}$ defined by

$$\mathcal{V}_0 \equiv \{|\chi\rangle \in \mathcal{V}_{\rm phys}; \langle\chi|\chi\rangle = 0\}, \tag{1·12a}$$

is orthogonal to every vector in $\mathcal{V}_{\rm phys}$ (see, (A·7) in Appendix A):

$$\mathcal{V}_0 \perp \mathcal{V}_{\rm phys}. \tag{1·12b}$$

Hence, two state vectors $|\Psi\rangle$ and $|\Psi\rangle + |\chi\rangle$ ($|\chi\rangle \in \mathcal{V}_0$) of $\mathcal{V}_{\rm phys}$ cannot be distinguished physically, because the difference $|\chi\rangle$ of them has no effect on any amplitude in $\mathcal{V}_{\rm phys}$. Then, by virtue of (ii), the (completed) quotient space (1·10), $H_{\rm phys} = \overline{\mathcal{V}_{\rm phys}/\mathcal{V}_0}$, of $\mathcal{V}_{\rm phys}$ with respect to $\mathcal{V}_0$ becomes a *Hilbert space*, equipped with *positive definite* metric defined by

$$\langle \hat{\Phi} | \hat{\Psi} \rangle = \langle \Phi | \Psi \rangle \tag{1·13}$$

*) The symbol $\overline{V}$ in (1·10) means the completed space of V including all the limiting states of Cauchy sequences in V.

for $|\hat{\Phi}\rangle = |\Phi\rangle + \mathcal{V}_0$, $|\hat{\Psi}\rangle = |\Psi\rangle + \mathcal{V}_0 \in \mathcal{V}_{\text{phys}}/\mathcal{V}_0$ ($|\Phi\rangle, |\Psi\rangle \in \mathcal{V}_{\text{phys}}$). Next, the condition (i′) [derivable from (i)] allows us to define the *physical S-matrix* S_{phys} in H_{phys} by the relation:

$$S_{\text{phys}}|\hat{\Phi}\rangle = \widehat{S|\Phi\rangle} \quad \text{for} \quad |\hat{\Phi}\rangle = |\Phi\rangle + \mathcal{V}_0 \in \mathcal{V}_{\text{phys}}/\mathcal{V}_0 . \tag{1·14}$$

One can easily check the *unitarity of* S_{phys} (1·11) by the condition (0′). In fact, e.g.,

$$\langle\hat{\Psi}|S^{\dagger}_{\text{phys}}S_{\text{phys}}|\hat{\Phi}\rangle = \langle\widehat{S\Psi}|\widehat{S\Phi}\rangle = \langle S\Psi|S\Phi\rangle = \langle\Psi|S^{\dagger}S|\Phi\rangle = \langle\Psi|\Phi\rangle = \langle\hat{\Psi}|\hat{\Phi}\rangle .$$

□

By Theorem 1.1, QED in the N.L. formalism, which satisfies the above physicality criteria, is assured to give a consistent theory succeeding in "confining" all the unphysical negative norm states. Here the crucial point in the N.L. (or G.B.) formalism resides in the *subsidiary condition* (1·5) (or (1·5′)): By the conditions (1·5) and (1·4b) together with the zero-norm property (1·4a) of the B-field, the observable photon modes are reduced to the transverse ones*) alone (with positive norms).

On the basis of the above observation, several attempts have been made at formulating the *YM theory* with such subsidiary conditions as $(\partial^{\mu}A_{\mu})^{(+)}|\text{phys}\rangle = 0$ or $B^{(+)}|\text{phys}\rangle = 0$ similarly to (1·5). In the YM theory, however, the corresponding B-fields no longer satisfy a free field equation because of non-linear self-coupling of the YM fields. Consequently, the requirement (1·5) in this case becomes inconsistent with the time evolution of the system and the condition (ii) is violated. The transversality condition like (1·5) can serve at best as the condition for the *one-particle* asymptotic physical states.[23] Once many-particle states are set up, transitions from the initial states consisting solely of physical particles into the final states containing such unphysical particles as FP ghost pairs**) can easily occur. Thus, in order to find out the correct subsidiary condition for non-Abelian gauge theories, we should reexamine the essence of the subsidiary condition (1·5) for QED, instead of imitating it in its outward appearance. The crux for this problem is provided by the remarkable symmetry found by Becchi, Rouet and Stora[24] in the quantum theoretical Lagrangian with the gauge fixing term as well as the FP ghost term—*BRS symmetry*. As will be shown in Chap. II, this symmetry is the

*) In the case of the Abelian Higgs model, the subsidiary condition (1·5) expels the Goldstone boson instead of longitudinal component of A_{μ} from the physical world as an unphysical particle.[9]

**) Someone claims that FP ghosts do not appear in the initial and final states because they go round only internal lines. But this is merely a tautology. In an *operator formalism*, every particle appearing in the intermediate states of the unitarity relations has its own *asymptotic field and state* appearing in both of initial and final states. The *reason why* the amplitudes with FP ghosts in the initial and final states make no contribution to the physical processes, should be *clarified*.

invariance under the *BRS transformation* obtained, essentially, from the local gauge transformation by replacing the infinitesimal transformation parameter $\theta^a(x)$ with $\lambda c^a(x)$, and can be viewed as the quantum theoretical version of the local gauge invariance, because it reproduces all the W.T. identities which compensates the local gauge invariance lost through the quantization procedure. Hinted by the fact that Eq. (1·3) crucial for the consistency of the subsidiary condition (1·5) also reproduces all the W.T. identities in the Abelian gauge theory, we can adopt the following subsidiary condition[3),25)] for the non-Abelian cases in terms of the generator Q_B of the BRS transformation:

$$Q_B|\text{phys}\rangle = 0 . \tag{1·15}$$

As will be shown in Chap. II, this subsidiary condition (1·15) reproduces the one, (1·5), in the case of the Abelian gauge theory; thus (1·15) is a natural extension of (1·5).

The next crux is the condition (0), which has been believed incorrectly to be violated in the YM theory.[23)~26)] Without this condition (0),[*)] however, it is almost impossible to prove the unitarity of the physical S-matrix S_{phys}. This mistake has arised from the following hermiticity assignment for the FP ghosts:[23)~26),**)]

$$C^\dagger = \overline{C} \quad \text{and} \quad \overline{C}^\dagger = C , \tag{1·16}$$

which is easily shown to lead to a non-hermitian Lagrangian and Hamiltonian:

$$\mathcal{L}^\dagger \neq \mathcal{L} \quad \text{and} \quad H^\dagger \neq H .$$

It is this very hermiticity assignment that has hindered us to construct a covariant canonical formalism of the YM theory in a consistent and transparent manner. What we have found is that, if we adopt the following assignment:[3),**)]

$$C^\dagger = C \quad \text{and} \quad \overline{C}^\dagger = -\overline{C} \tag{1·17}$$

or equivalently

$$c^\dagger = c \quad \text{and} \quad \bar{c}^\dagger = \bar{c} \tag{1·18}$$

with the redefinition of FP ghosts as

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*) Such a theory that (0) is not satisfied shows several pathological features, for instance, the time dependence of the hermiticity character of operators: $(e^{iHt}\varphi e^{-iHt})^\dagger = e^{iH^\dagger t}\varphi^\dagger e^{-iH^\dagger t} \neq e^{iHt}\varphi^\dagger e^{-iHt}$. Thus, without this condition (0), we cannot obtain a consistent Heisenberg operator formalism, from the beginning.

**) The authors of Ref. 26) insisted on the equivalence of the assignment (1·16) and another one, $C^\dagger = -C$ and $\overline{C}^\dagger = \overline{C}$, similar to (1·17) on the basis of their "R-transformation" (FP-ghost charge conjugation). This is not the case because the "R-transformation" is *not* a symmetry transformation of the non-Abelian gauge theory.

$$C=c \quad \text{and} \quad \overline{C}=i\bar{c}\,, \tag{1·19}$$

instead of the conventional (but wrong) one (1·16), the condition (0) is, in fact, verified. As will be shown in Chap. III, it is, more strongly, indispensable for the whole consistency of the theory. Then, on the basis of these important results (1·15) and (1·18), we can formulate and develop the local covariant quantum theory of non-Abelian gauge fields (the YM fields as well as the gravitational field) in quite a consistent and general manner, as will be done in this paper in its full detail.

Chapter II

Basic Ingredients of the Formalism

§ 2.1. Local Gauge Invariance

We present here our general formalism explicitly for the Yang-Mills type gauge theories based on a compact Lie group G. [As an example of other type of gauge theory, quantum gravity will be discussed in § 4.3.] Compact Lie groups are reductive, and hence G is given by a direct product

$$G=\prod_{\alpha} G_{\alpha}, \tag{2·1}$$

where the G_{α}'s are simple groups or, otherwise, one-dimensional Abelian groups. Corresponding to the decomposition (2·1), the generators X^a of the Lie algebra $\mathcal{G}$ of G are given by the totality of the generators of the Lie algebra $\mathcal{G}_{\alpha}$ of G_{α}. Therefore the structure constant f^{abc} of $\mathcal{G}$, defined by

$$[X^a, X^b]=if^{abc}X^c, \tag{2·2}$$

is given as

$$f^{abc}=\begin{cases} f_{\alpha}{}^{abc} & \text{if } X^a, X^b \text{ and } X^c \in \mathcal{G}_{\alpha} \\ 0 & \text{otherwise} \end{cases} \tag{2·3}$$

by the structure constant $f_{\alpha}{}^{abc}$ of the "component" algebra $\mathcal{G}_{\alpha}$.

Now we consider the system of gauge fields $A_{\mu}{}^{a}$ interacting with arbitrary matter fields denoted generically as φ_i which may consist of fermion and/or boson components. The Lagrangian density $\mathcal{L}_s(A, \varphi)$ [or, more weakly the action $A_s=\int d^4x\mathcal{L}_s$] of the system has an invariance under the local gauge transformation, the infinitesimal form of which is written as

$$\delta_{\Lambda}A_{\mu}{}^{a}=\partial_{\mu}\Lambda^a+g^{ab}f^{bcd}A_{\mu}{}^{c}\Lambda^d\equiv D_{\mu}{}^{ab}\Lambda^b, \tag{2·4a}$$

$$\delta_{\Lambda}\varphi_i=ig^{ab}\Lambda^a T^b_{ij}\varphi_j\,. \tag{2·4b}$$

Here $\Lambda^a=\Lambda^a(x)$ is a space-time dependent parameter of the Lie group G, and the T^a's stand for the (reducible, in general,) representation matrices on φ of the generators X^a. Since one coupling constant g_{α} can be associated with each "component" group G_{α} in (2·1), the coupling constant matrix g^{ab} in (2·4) has a diagonal form:

$$g^{ab}=\delta^{ab}g_{\alpha} \quad (\text{when } X^a\in\mathcal{G}_{\alpha})\,. \tag{2·5}$$

Hereafter we will often use the matrix- and vector-notation in order to avoid cumbersome indices a or i:

$$(A\times B)^a \equiv f^{abc}A^bB^c,\ A\cdot B\equiv A^aB^a\,,$$

$$(M\varphi)_i\equiv M_{ij}\varphi_j,\ (gA)^a\equiv g^{ab}A^b,\ \text{etc}. \tag{2·6}$$

So, for example, the gauge transformation (2·4) can be rewritten concisely as

$$\delta_\Lambda A_\mu=\partial_\mu\Lambda+gA_\mu\times\Lambda=D_\mu\Lambda\,,$$

$$\delta_\Lambda\varphi=i\Lambda\cdot gT\varphi\,.$$

Note also the relation $(gA)\cdot B=A\cdot(gB)$ due to (2·5).

§ 2.2. Lagrangian of YM Theory and Its Canonical Quantization

The Lagrangian density for quantum gauge theory should include a gauge fixing term accompanied by the well-known gauge-compensating FP ghosts. We, therefore, take it as

$$\mathcal{L}=\mathcal{L}_s(A,\varphi)+\mathcal{L}_{\rm GF}+\mathcal{L}_{\rm FP}\,, \tag{2·7}$$

$$\mathcal{L}_s(A,\varphi)=-\tfrac{1}{4}F_{\mu\nu}\cdot F^{\mu\nu}+\mathcal{L}_{\rm matter}(\varphi,\mathcal{D}_\mu\varphi)\,, \tag{2·7a}$$

$$\mathcal{L}_{\rm GF}=-\partial^\mu B\cdot A_\mu+(\alpha_0/2)B\cdot B\,, \tag{2·7b}$$

$$\mathcal{L}_{\rm FP}=-i\partial^\mu\bar{c}\cdot D_\mu c\,, \tag{2·7c}$$

where

$$F_{\mu\nu}\equiv\partial_\mu A_\nu-\partial_\nu A_\mu+gA_\mu\times A_\nu\,, \tag{2·8a}$$

$$\mathcal{D}_\mu\varphi\equiv(\partial_\mu-iA_\mu\cdot gT)\varphi\,. \tag{2·8b}$$

Since (2·7b) is rewritten as

$$\mathcal{L}_{\rm GF}=-\frac{1}{2\alpha_0}(\partial^\mu A_\mu)^2+\frac{\alpha_0}{2}\Big(B+\frac{1}{\alpha_0}\partial^\mu A_\mu\Big)^2-\partial^\mu(B\cdot A^\mu),$$

the present gauge fixing is equivalent to a more familiar covariant gauge fixing term $-(\partial^\mu A_\mu)^2/2\alpha_0$ at the level of equation of motion and of Feynman diagrams. The introduction of the multiplier fields B^a, however, will play an important role below in assuring the nilpotency of the BRS charge without use of equations of motion. We have introduced the factor i in front of the FP ghost term (2·7c) in order to treat both FP ghost c and $\bar{c}$ as hermitian fields under our new hermiticity assignment[1)] (1·18):

$$c^\dagger=c\,,\quad \bar{c}^\dagger=\bar{c}\,. \tag{1·18}$$

Hermiticity assignment for other fields are taken as usual; i.e., $A_\mu{}^\dagger=A_\mu$, $B^\dagger=B$

and so on. Note here that, only when this hermiticity assignment (1·18) for the FP ghosts is adopted, the Lagrangian $\mathcal{L}$ of (2·7) becomes hermitian and, hence, the total S-matrix is (pseudo-) unitary:

$$\mathcal{L}^\dagger = \mathcal{L}\,, \quad SS^\dagger = S^\dagger S = 1\,. \tag{2·9}$$

The Euler-Lagrange equations of motion for A_μ, B, c and $\bar{c}$ are

$$D^\mu F_{\mu\nu} = \partial_\nu B - gj_\nu - ig(\partial_\nu \bar{c} \times c)\,, \tag{2·10a}$$

$$\partial^\mu A_\mu + \alpha_0 B = 0\,, \tag{2·10b}$$

$$\partial^\mu D_\mu c = D^\mu \partial_\mu \bar{c} = 0\,, \tag{2·10c}$$

where the matter current j_μ is defined by

$$j_\mu{}^a \equiv -i(T^a\varphi)_i \frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi_i)}\,. \tag{2·10d}$$

In order to give canonical (anti-) commutation relations (CCR or CAR), we need the canonical conjugate momenta, which we define as

$$\pi_B{}^a \equiv \partial\mathcal{L}/\partial\dot{B}^a = -A_0{}^a, \tag{2·11a}$$

$$\pi_{A_k}^a \equiv \partial\mathcal{L}/\partial\dot{A}_k{}^a = F_{0k}^a\,, \quad (k=1,2,3) \tag{2·11b}$$

$$\pi_{\varphi_i} \equiv (\partial/\partial\dot{\varphi}_i)\,\mathcal{L}\,, \tag{2·11c}$$

$$\pi_c^a \equiv (\partial/\partial\dot{c}^a)\,\mathcal{L} = +i\dot{\bar{c}}^a, \tag{2·11d}$$

$$\pi_{\bar{c}}{}^a \equiv (\partial/\partial\dot{\bar{c}}^a)\,\mathcal{L} = -i(\dot{c} + gA_0 \times c)^a. \tag{2·11e}$$

For these momentum variables $\pi_\Phi{}^I$ conjugate to $\Phi_I (= B^a, A_k{}^a, \varphi_i, c^a, \bar{c}^a)$, we take the following CCR or CAR as usual:

$$[\pi_\Phi{}^I(\boldsymbol{x},t), \Phi_J(\boldsymbol{y},t)]_\mp = -i\delta^I{}_J \delta^3(\boldsymbol{x}-\boldsymbol{y})\,,$$
$$[\pi_\Phi{}^I(\boldsymbol{x},t), \pi_\Phi{}^J(\boldsymbol{y},t)]_\mp = [\Phi_I(\boldsymbol{x},t), \Phi_J(\boldsymbol{y},t)]_\mp = 0\,, \tag{2·12}$$

where the anti-commutators (+) are taken only between fermion fields. Especially we regard the *FP ghosts c and $\bar{c}$* as *fermion fields.* For simplicity, however, we adopt the convention of taking *commutators* (−) in (2·12) between any one of FP ghost fields and any one of fermion matter fields in φ_i.*)

Here we should note the following two points: First, concerning (2·11a), we have considered that the time-component gauge fields $A_0{}^a$ are *not* canonical coordinates but the momentum variables conjugate to the B^a's. If

*) Either commutator or anti-commutator can be adopted between two different species of fermion fields without altering physics by virtue of the well-known Klein transformation.

we treated the A_0^a's also as independent coordinate variables, Eqs. (2·11a), $\phi_1^a \equiv \pi_B^a + A_0^a = 0$, together with $\phi_2^a \equiv \pi_0^a \equiv \partial\mathcal{L}/\partial\dot{A}_0^a = 0$, would become the *second class constraints* of Dirac's classification. Then, CCR (or CAR) should be obtained by the help of the Dirac bracket, defined as

$$\{\alpha, \beta\}_D \equiv \{\alpha, \beta\}_P - \{\alpha, \phi_i^a\}_P (C^{-1})_{ij} \{\phi_j^a, \beta\}_P ,$$
$$\delta^{ab} C_{ij} \equiv \{\phi_i^a, \phi_j^b\}_P , \quad (i, j = 1, 2) \tag{2·13}$$

instead of the usual Poisson bracket $\{\alpha, \beta\}_P$. This Dirac's procedure, however, gives just the same CCR (or CAR)'s as those obtained by the above simplified treatment.*) Further, even if our starting gauge fixing Lagrangian $\mathcal{L}_{\mathrm{GF}}$ (2·7b) was changed by the following replacement:

$$-\partial^\mu B \cdot A_\mu \to \omega B \cdot \partial^\mu A_\mu + (\omega - 1) \partial^\mu B \cdot A_\mu \tag{2·14}$$

with an arbitrary real ω, this Dirac's method would produce the same results. Second, in (2·11), we have adopted the left-differentiation convention with respect to the anti-commuting number ξ such as the FP ghosts or fermion components in φ_i; that is, the differential operator $(\partial/\partial\xi)$ has a property

$$(\partial/\partial\xi) AB = [(\partial/\partial\xi) A] B + (-1)^{p_A} A [(\partial/\partial\xi) B], \tag{2·15}$$

where A and B are any monomials in the commuting and anti-commuting numbers and p_A is the number of factors anti-commuting with ξ contained in A. [Remember that we are taking the convention that the fermion matter fields in φ_i, if any, *commute* with the FP ghosts.] Correspondingly to this convention of left-differentiation, the Hamiltonian density $\mathcal{H}$ should be constructed as

$$\mathcal{H} = \dot{\Phi}_I \pi_\Phi^I - \mathcal{L}$$

but not as $\mathcal{H} = \pi_\Phi^I \dot{\Phi}_I - \mathcal{L}$. This is because the variation of the Hamiltonian density $\delta\mathcal{H} = \delta\dot{\Phi}_I \pi_\Phi^I + \dot{\Phi}_I \delta\pi_\Phi^I - \delta\mathcal{L}$ has to be independent of the velocity variation $\delta\dot{\Phi}_I$ while $\delta\mathcal{L} = \delta\Phi_I (\partial/\partial\Phi_I)\mathcal{L} + \delta\dot{\Phi}_I (\partial/\partial\dot{\Phi}_I)\mathcal{L}$ by the left-differentiation rule (2·15).

§ 2.3. BRS Symmetry as a "Local Gauge Invariance" in Quantum Theory

Due to the presence of $\mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{FP}}$, the Lagrangian density $\mathcal{L}$ (2·7) is no longer invariant under the local gauge transformation (2·4). The essence of the local gauge invariance is, however, inherited by the quantum theory in the form of the following global symmetry. Namely, Becchi, Rouet and Stora[2] have found a remarkable invariance of the quantum system $\mathcal{L}$ (2·7) under a *global* transformation, called the BRS transformation nowadays, from

*) An analogous situation indeed occurs in the CAR of Dirac field ψ: In the usual treatment, the variable $i\psi^*$ is not considered a coordinate but a momentum π_ψ conjugate to ψ.

which all the Ward-Takahashi (W.T.) identities can be derived very simply. The BRS transformation is given by replacing the parameter $\Lambda^a(x)$ in (2·4) by $\lambda c^a(x)$ for the ordinary fields $A_\mu{}^a$ and φ_i and by supplementing the transformation properties of c, $\bar{c}$ and B as follows:

$$\delta A_\mu(x)=\lambda D_\mu c(x), \tag{2·16a}$$

$$\delta\varphi(x)=i\lambda c(x)\cdot gT\varphi(x), \tag{2·16b}$$

$$\delta c(x)=-\lambda g(c(x)\times c(x))/2, \tag{2·16c}$$

$$\delta\bar{c}(x)=i\lambda B(x), \tag{2·16d}$$

$$\delta B(x)=0, \tag{2·16e}$$

where λ is an x-independent parameter anti-commuting with the FP ghosts c and $\bar{c}$. The BRS invariance of the Lagrangian density $\mathcal{L}$ (2·7) in fact follows directly from the local gauge invariance of $\mathcal{L}_s(A,\varphi)$ and the properties

$$\delta(D_\mu c)=0, \quad \delta(c\times c)=0. \tag{2·17}$$

Equations (2·17), which provide examples of the nilpotency of the BRS charge stated below, can be easily checked by noting that the structure-constant matrices $(if^a)_{bc}\equiv if^{bac}$ satisfy the commutation relation (2·2) with X^a substituted by if^a and that the FP ghosts c and $\bar{c}$ obey Fermi-statistics.

The Noether current of the BRS transformation given by

$$J_\mu{}^B=D^\nu c\cdot\frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\nu)}+i(c\cdot gT\varphi)_i\frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi_i)}-\frac{1}{2}g(c\times c)\cdot\frac{\partial\mathcal{L}}{\partial(\partial^\mu c)}+iB\cdot\frac{\partial\mathcal{L}}{\partial(\partial^\mu\bar{c})} \tag{2·18}$$

can be shown to be conserved ($\partial^\mu J_\mu{}^B=0$) and to be rewritten as[1)]

$$J_\mu{}^B=B\cdot D_\mu c-\partial_\mu B\cdot c+(i/2)g\partial_\mu\bar{c}\cdot(c\times c)-\partial^\nu(F_{\mu\nu}c), \tag{2·19}$$

by the help of the equations of motion (2·10). The corresponding *conserved charge* Q_B (BRS charge)

$$Q_B\equiv\int d^3x[B\cdot\overleftrightarrow{\partial}_0 c+gB\cdot(A_0\times c)+(i/2)g\partial_0\bar{c}\cdot(c\times c)], \tag{2·20}$$

generates the BRS transformation; that is,

$$[i\lambda Q_B,\Phi_I(x)]=\delta\Phi_I(x), \tag{2·21}$$

where Φ_I stands for A_μ, φ_i, B, c and $\bar{c}$, and $\delta\Phi_I$ is given by (2·16). It will be convenient for later use to introduce the *renormalized* BRS charge $Q_B{}^r$ defined as

$$Q_B{}^r\equiv(\tilde{Z}_3/Z_B)^{1/2}Q_B \tag{2·22}$$

which in fact generates a renormalized BRS transformation

$$[i\lambda Q_B{}^r, \bar{c}^{\text{ren}}] = i\lambda B^{\text{ren}}$$

for the renormalized fields $\bar{c}^{\text{ren}} = \tilde{Z}_3{}^{-1/2} c$ and $B^{\text{ren}} = Z_B{}^{-1/2} B$.

We have another important conserved charge in this system (2·7). One easily notices that the FP ghost number is conserved. Unlike the conservation of the usual fermion number, however, this FP ghost number conservation is not attributed to the invariance under the *phase* transformation, $c \to e^{i\theta} c$ and $\bar{c} \to e^{-i\theta} \bar{c}$. In fact, such a phase transformation is incompatible with our fundamental hermiticity assignment (1·18) to c and $\bar{c}$. Instead, an invariance exists under the *scale* transformation, $c \to e^{\theta} c$ and $\bar{c} \to e^{-\theta} \bar{c}$, consistent with the hermiticity of c and $\bar{c}$. The corresponding conserved current and charge are given by

$$J_\mu{}^c = i(\bar{c} \cdot D_\mu c - \partial_\mu \bar{c} \cdot c), \tag{2·23a}$$

$$Q_c = i \int d^3x [\bar{c} \cdot \overleftrightarrow{\partial}_0 c + g\bar{c} \cdot (A_0 \times c)]. \tag{2·23b}$$

This charge Q_c, called FP ghost charge, indeed generates the above scale transformation on the FP ghost fields:

$$[iQ_c, c(x)] = c(x),$$

$$[iQ_c, \bar{c}(x)] = -\bar{c}(x). \tag{2·24}$$

The FP ghost number is identified with the eigenvalue of the operator iQ_c.

The following simple algebra of Q_B and Q_c can be obtained from their definitions:

$$\tfrac{1}{2}\{Q_B, Q_B\} = Q_B{}^2 = 0, \tag{2·25a}$$

$$[iQ_c, Q_B] = Q_B, \tag{2·25b}$$

$$[Q_c, Q_c] = 0. \tag{2·25c}$$

Equation (2·25a) expresses the remarkable *nilpotency* property of the BRS transformation and is easily confirmed as follows:

$$2i\lambda(Q_B)^2 = \left[i\lambda Q_B, \int d^3x \left(B \cdot D_0 c - \partial_0 B \cdot c + \frac{i}{2} g\partial_0 \bar{c} \cdot (c \times c)\right)\right]$$

$$= \int d^3x \left(-\partial_0 B \cdot \delta c + \partial_0(\delta \bar{c}) \cdot \frac{ig}{2}(c \times c)\right) = 0,$$

where $\delta B = \delta(D_\mu c) = \delta(c \times c) = 0$ [(2·16e), (2·17)] and (2·16c, d) are used. Equations (2·25b) and (2·25c) say only that the charges Q_B and Q_c carry the FP ghost numbers $iQ_c = 1$ and $iQ_c = 0$, respectively.

Now some remarks are in order. The hermiticity assignment (1·18) to FP ghosts plays an important role also here in assuring the consistency of the formulation: First, the charges Q_B and Q_c are hermitian only when our assigment (1·18) is adopted:

$$Q_B{}^\dagger = Q_B,\ Q_c{}^\dagger = Q_c\,. \tag{2·26}$$

Under the conventional hermiticity assignment $c^\dagger = i\bar{c}$ [(1·15)], $Q_B{}^\dagger$ and $Q_c{}^\dagger$ become quite other quantities having no simple algebraic relations to Q_B and Q_c, respectively, and further are not assured by equations of motion to be conserved. This contradicts the conservation of $Q_B{}^\dagger$ following from $(d/dt)\langle\alpha|Q_B{}^\dagger|\beta\rangle = \langle\beta|(d/dt)Q_B|\alpha\rangle^* = 0$, which should hold *as far as* Q_B *is well-defined.* Further the conventional assignment (1·15) contradicts also the BRS transformation (2·16): In fact, since the transformed field $c' = c + \delta c$ should have the same hermiticity property as the original one c, the relation $(\delta c)^\dagger = i\delta\bar{c}$ is required to hold for the conventional case, while

$$(\delta c)^\dagger = -\frac{g}{2}(c\times c)^\dagger\lambda^\dagger \neq -\lambda B(x) = i\delta\bar{c}\,.$$

As for the assignment $\bar{c}^\dagger = \bar{c}$, $c^\dagger = c$, the BRS transformation is quite consistent with it, if the "anti-commuting number" λ obeys the rule

$$(\lambda\mathcal{O})^\dagger = \mathcal{O}^\dagger\lambda^\dagger \tag{2·27a}$$

for arbitrary operator $\mathcal{O}$ and is "pure-imaginary":

$$\lambda^\dagger = -\lambda\,. \tag{2·27b}$$

We assume these properties (2·27).*) Of course, if one wants, one can avoid the explicit use of such "anti-commuting numbers" by rewriting the BRS transformation (2·21) as

$$[iQ_B, \Phi_I(x)]_\mp = \delta'\Phi_I(x)\,, \tag{2·28a}$$

where $\delta'\Phi_I$ are the BRS transform $\delta\Phi_I(x)$ with λ factored out (i.e., $\delta\Phi_I = \lambda\delta'\Phi_I$) and the anti-commutator (+) is understood if Φ_I contains odd number of FP ghost fields. From such a standpoint, the use of the "anti-commuting number" λ may be understood as being purely for convenience' sake to write (2·28a) compactly as

$$[i\lambda Q_B, \Phi_I(x)] = \lambda\delta'\Phi_I(x) = \delta\Phi_I(x)\,. \tag{2·28b}$$

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*) One should also assume that there are an infinite number of such "anti-commuting numbers" anti-commuting with one another, in order to perform the BRS transformations successively.

§ 2.4. Subsidiary Condition, "Maxwell" Equation and Some Other Symmetries of the Theory

As explained in Chap. I, the total state vector space $\mathcal{V}$ in the covariant gauge theories necessarily contains negative norm states; i.e., $\mathcal{V}$ has an indefinite metric. In order to obtain a physically meaningful theory, we should specify the *physical subspace* $\mathcal{V}_{\text{phys}}=\{|\text{phys}\rangle\}$ so that it satisfies the physicality criteria (1·9), as explained in the Introduction. In Abelian gauge theories, $\mathcal{V}_{\text{phys}}$ was specified by a concise subsidiary condition (1·5): $B^{(+)}(x)|\text{phys}\rangle=0$ (or $(\partial A)^{(+)}(x)|\text{phys}\rangle=0$). Happily and surprisingly enough, we can really specify the physical subspace $\mathcal{V}_{\text{phys}}$ by a very simple *subsidiary condition* also in case of general (non-Ableian) gauge theories: $\mathcal{V}_{\text{phys}}=\{|\text{phys}\rangle\}$ is specified by[1),3)]

$$Q_B|\text{phys}\rangle=0\,. \tag{2·29}$$

This condition, intuitively speaking, expresses the gauge-invariance of the physical states in $\mathcal{V}_{\text{phys}}$. It is indeed analogous to the G.B.-N.L. condition $B^{(+)}(x)|\text{phys}\rangle=0$ where the $B(x)$ field [or more precisely, $\int d^3x\Lambda(x)\overleftrightarrow{\partial}_0 B(x)$ with $\Box\Lambda=0$] in Abelian case represents a generator of local gauge transformtion, while Q_B is a generator of the BRS's version of local (non-Abelian) gauge transformation. In fact, we can show here that the condition (2·29) really reproduces[1)] the subsidiary condition $B^{(+)}(x)|\text{phys}\rangle=0$ under the special circumstances of Abelian gauge theories: Peculiar points to the Abelian case are that the structure constant vanishes and no group indices appear, and hence the multiplier field B and the FP ghost fields c and $\bar{c}$ become completely *free* as is seen from Eqs. (2·10): $\Box B=\Box c=\Box\bar{c}=0$. Further the BRS charge (2·20) becomes quite simple as follows:

$$Q_B=\int d^3x\colon B\overleftrightarrow{\partial}_0 c\colon=i\sum_k(c_k{}^\dagger B_k-B_k{}^\dagger c_k)\,, \tag{2·30}$$

where $B_k{}^\dagger(B_k)$ and $c_k{}^\dagger(c_k)$ are creation (annihilation) operators of the B and c fields, respectively, referring to some wave packet system $\{g_k\}$. This is possible because B and c are free. The free property of FP ghosts c and $\bar{c}$ implies that the total state vector space $\mathcal{V}$ can be decomposed persistently into a direct product $\mathcal{V}=\mathcal{V}'\otimes\mathcal{V}_{\text{FP}}$ where $\mathcal{V}'$ is the usual state vector space consisting of particles other than FP ghosts and $\mathcal{V}_{\text{FP}}$ is the Fock space spanned by c and $\bar{c}$ alone. Further, since the FP ghosts are redundant from the beginning in the Abelian case, we can restrict ourselves to the sector containing neither c nor $\bar{c}$ ghosts: $\mathcal{V}'\otimes|0\rangle_{\text{FP}}$, which is isomorphic to the usual state vector space $\mathcal{V}'$ of Abelian gauge theory. Thus by using (2·30), the subsidiary condition (2·29) reduces to

$$Q_B(|\text{phys}\rangle\otimes|0\rangle_{\text{FP}})=i\sum_i B_k|\text{phys}\rangle\otimes|c_k\rangle_{\text{FP}}=0\,. \tag{2·31}$$

Here $|c_k\rangle = c_k^\dagger|0\rangle$, of course. By the linear independence of $|c_k\rangle$, we obtain

$$B_k|\text{phys}\rangle = 0 \quad \text{for all } k,$$

which is nothing but the Nakanishi-Lautrup condition $B^{(+)}(x)|\text{phys}\rangle = 0$ (for all x) for the Abelian case.

Interestingly, the present subsidiary condition (2·29) turns out to give, in a much simpler form, a natural extension of that of the Abelian cases, applicable to any gauge theories. The condition (2·29) provides an important basis in our formalism on which we develop all the discussions hereafter.

We should notice here that we have already implicitly assumed that *the BRS charge Q_B is well-defined.* As is evident from the general discussions in Appendix B, this is equivalent to any one of the following statements:

(i) $Q_B|0\rangle = 0$, (2·32a)

(ii) The vacuum is physical; $|0\rangle \in \mathcal{V}_{\text{phys}}$, (2·32b)

(iii) The BRS symmetry corresponding to Q_B suffers from no spontaneous symmetry breakdown. (2·32c)

The first equation (2·32a) will be often utilized henceforth.

[Digression: Historically, such a type of subsidiary condition as (2·29) was first discussed by Curci and Ferrari.[3] Unfortunately, however, they adopted the conventional (and hence *incorrect*) hermiticity assignment (1·16) for the FP ghosts and did not introduce the gauge-fixing multiplier field $B(x)$. These defects have caused difficulties in giving an explicit expression for the BRS generator Q_B satisfying hermiticity and nilpotency. Hence they gave up to construct the generator Q_B explicitly and simply *assumed* the very existence of Q_B as well as many BRS transformation properties of the asymptotic fields without any justifications. However these assumptions contradict one another as was seen before. Although they observed in Ref. 4) that the Lagrangian becomes hermitian under a similar hermiticity assignment to (1·17), they did not adopt it in Ref. 3). This fact shows that they did not recognize the fundamental importance of the assignment (1·17).]

We shall see in Chap. III how the physical S-matrix unitarity is assured generally by the subsidiary condition (2·29). Explicit examples will be discussed in Chap. IV. The condition (2·29) is really sufficient to prove the physical S-matrix unitarity. However, if one prefers specifying the physical subspace as small as possible, then one can add one more subsidiary condition:[1),3)]

$$Q_c|\text{phys}\rangle = 0, \tag{2·33}$$

where Q_c is, of course, the conserved FP ghost charge (2·23). This condition works only in reducing the physical subspace to the vanishing-FP-ghost-number sector and hence will not be imposed in this paper unless it is explicitly mentioned.

By virtue of the symmetric gauge-fixing choice (2·7b), the invariance of $\mathcal{L}_s(A, \varphi)$ under the *global* gauge transformation is preserved by $\mathcal{L}$ (2·7): that is, the Noether currents of the global gauge transformation,

$$J_\mu{}^a = (A^\nu \times F_{\nu\mu})^a + j_\mu{}^a + (A_\mu \times B)^a - i(\bar{c} \times D_\mu c)^a + i(\partial_\mu \bar{c} \times c)^a, \qquad (2\cdot34)$$

are conserved ($\partial^\mu J_\mu{}^a = 0$) because of the invariance of $\mathcal{L}$ under the following global gauge transformation:

$$[i\delta\varepsilon\cdot Q, \Phi(x)] = \delta\varepsilon \times \Phi(x) \qquad \text{for } \Phi = A_\mu, B, c \text{ and } \bar{c},$$
$$[i\delta\varepsilon\cdot Q, \varphi(x)] = -i\delta\varepsilon\cdot T\varphi(x), \qquad (2\cdot35)$$

where the global ('color' or 'flavor') charges Q^a are of course given by $Q^a = \int d^3x J_0{}^a$ and the matter currents $j_\mu{}^a$ are defined by (2·10d). Note that this global invariance would not be manifest (if any in physical sector) under asymmetric gauge-fixing choices such as R_ξ-gauge. As was first noted in Ref. 5), the equation of motion (2·10a) is rewritten into the following remarkable form by the use of the BRS charge Q_B and Eqs. (2·16), (2·21) and (2·34):

$$\partial^\nu F_{\nu\mu} + gJ_\mu = \{Q_B, D_\mu \bar{c}\}. \qquad (2\cdot36)$$

This equation clearly shows that the "classical" Maxwell-type equation

$$\langle f_1 | (\partial^\nu F_{\nu\mu} + gJ_\mu) | f_2 \rangle = 0 \qquad (2\cdot37)$$

holds for any physical states $|f_1\rangle, |f_2\rangle \in \mathcal{V}_{\text{phys}}$ specified by (2·29). Equation (2·36) will play an important role in the discussions of observables (Chap. V) and of the spontaneous symmetry breakdown and color confinement (Chap. VI), and will be referred to as "Maxwell" equation there. It should be kept in mind that the $J_\mu{}^a$'s are the currents of global color transformation in QCD usually supposed unbroken.

Aside from other possible continuous symmetries such as chiral symmetry, flavor symmetry, etc., the present system $\mathcal{L}$ (2·7) has also the basic discrete symmetries P, C and T if the original Lagrangian $\mathcal{L}_s(A, \varphi)$ has. We only note here the PCT symmetry: If the Lagrangian $\mathcal{L}_{\text{matter}}(\varphi, \partial_\mu\varphi)$ defines a PCT invariant theory with a suitable PCT transformation law for the matter fields φ_i, then $\mathcal{L}$ (2·7) is invariant under the following *anti-linear* PCT transformation:[6)]

$$\text{PCT:} \quad \begin{aligned} A_\mu{}^{a,\text{PCT}}(x) &= -A_\mu{}^a(-x), \\ B^{a,\text{PCT}}(x) &= B^a(-x), \\ c^{a,\text{PCT}}(x) &= -c^a(-x), \\ \bar{c}^{a,\text{PCT}}(x) &= \bar{c}^a(-x). \end{aligned} \qquad (2\cdot38)$$

[An extra minus sign in front of FP ghost field c^a should be noticed.] Then, we can safely suppose the existence of the anti-unitary PCT operator Θ satisfying

$$\Theta|0\rangle = |0\rangle\,, \tag{2·39a}$$

$$\Theta^2 = 1\,, \tag{2·39b}$$

$$\Theta\Phi_I(x)\,\Theta = \Phi_I{}^{\mathrm{PCT}}(x)\,, \tag{2·39c}$$

where $\Phi_I{}^{\mathrm{PCT}}(x)$ stands generically for the fields (2·38) supplemented by the PCT-transformed matter fields $\varphi_i{}^{\mathrm{PCT}}(x)$. Under the PCT transformation, the BRS charge Q_B, the FP ghost charge Q_c and the generators Q^a of the global gauge transformation behave as follows:[6]

$$\Theta Q_B\Theta = Q_B\,, \tag{2·40a}$$

$$\Theta Q_c\Theta = -Q_c\,, \tag{2·40b}$$

$$\Theta Q^a\Theta = -Q^a. \tag{2·40c}$$

Note that the BRS transformation (2·16) [or (2·21)] is consistent with the PCT transformation (2·38) and (2·40a).

§ 2.5. BRS Invariance and Ward-Takahashi Identities

The Ward-Takahashi (W.T.) identities for Green's functions can be derived quite simply by use of the BRS charge Q_B: Since $Q_B|0\rangle = 0$ (2·32a), we obtain

$$\begin{aligned} 0 &= \langle 0|\,[i\lambda Q_B, T(\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n))]\,|0\rangle \\ &= \sum_{k=1}^{n}\langle 0|T(\mathcal{O}_1(x_1)\cdots\mathcal{O}_{k-1}(x_{k-1})\,\delta\mathcal{O}_k(x_k)\,\mathcal{O}_{k+1}(x_{k+1})\cdots\mathcal{O}_n(x_n))\,|0\rangle\,, \end{aligned} \tag{2·41}$$

where the $\mathcal{O}_k(x)$'s are arbitrary field operators or their local products. All the W.T. identities for Green's functions are exhausted by Eq. (2·41).

In order to obtain the W.T. identities for the generating functional of one-particle-irreducible (1PI) vertices, consider a source functional $\mathcal{S}$:

$$\begin{aligned} \mathcal{S}[J,K] \equiv \int d^4x\,(&J_\mu\cdot A^\mu + J_i\varphi_i + \bar{J}_c\cdot c + J_{\bar{c}}\cdot\bar{c} + J_B\cdot B \\ &+ K_\mu\cdot D^\mu c + iK_i(c\cdot gT)_{ij}\varphi_j - \tfrac{1}{2}K_c\cdot g\,(c\times c))\,, \end{aligned} \tag{2·42}$$

where $\bar{J}_c$, $J_{\bar{c}}$, K_μ and K_i are "anti-commuting c-number sources" and J_μ, J_i, J_B and K_c are c-number sources. [For fermion matter components, K_i are commuting and J_i are anti-commuting. The use of "anti-commuting c-number sources" is purely for convenience' sake also here.] By the nilpotency of Q_B, (2·25a), we have

$$[\lambda Q_B, D^\mu c] = [\lambda Q_B, c\cdot gT\varphi] = [\lambda Q_B, g(c\times c)] = 0\,, \tag{2·43}$$

and hence obtain

$$\begin{aligned} 0 &= \langle 0|[i\lambda Q_B, T\exp i\mathscr{S}[J,K]]|0\rangle \\ &= i\lambda\int d^4x\langle 0|T(J_\mu\cdot D^\mu c + iJ_i(c\cdot gT)_{ij}\varphi_j \\ &\qquad + \tfrac{1}{2}\bar{J}_c\cdot g(c\times c) - iJ_{\bar{c}}\cdot B)\exp i\mathscr{S}[J,K]|0\rangle\,. \end{aligned} \tag{2·44}$$

Differentiations with respect to sources J reproduce the W.T. identities (2·41) for Green's functions of field operators. [The differentiations with respect to "anti-commuting c-number sources" obey the left-differentiation rule (2·15).] The generating functionals W and Γ of connected Green's functions and of 1PI vertices are defined, respectively, as

$$\exp iW[J,K] \equiv \langle 0|T\exp i\mathscr{S}[J,K]|0\rangle\,, \tag{2·45a}$$

$$\Gamma[\Phi,K] \equiv W[J,K] - J_I\Phi_I\,, \tag{2·45b}$$

$$\Phi_I \equiv (\delta/\delta J_I)\,W[J,K]\,, \tag{2·45c}$$

where J_I stands generically for J_μ, J_i, $\bar{J}_c$, $J_{\bar{c}}$ and J_B, and the c-number arguments Φ_I of Γ should not be confused with the corresponding Heisenberg operator. By using the dual relations in the Legendre transformation

$$(\delta/\delta\Phi_I)\Gamma = \begin{cases} -J_I & \text{for the commuting sources}\,, \\ +J_I & \text{for the anti-commuting sources}\,, \end{cases} \tag{2·46}$$

we can derive an identity for Γ from (2·44):[7]

$$\frac{\delta\Gamma}{\delta A_\mu}\cdot\frac{\delta\Gamma}{\delta K^\mu} + \frac{\delta\Gamma}{\delta\varphi_i}\frac{\delta\Gamma}{\delta K_i} + \frac{\delta\Gamma}{\delta c}\cdot\frac{\delta\Gamma}{\delta K_c} + i\frac{\delta\Gamma}{\delta\bar{c}}\cdot B = 0\,. \tag{2·47}$$

Here the cumbersome integration symbol $\int d^4x$ is omitted.

On the other hand, the equations of motion (2·10b) and (2·10c) and CCR (2·12) lead to the equations for Γ:

$$\frac{\delta\Gamma}{\delta B} = \partial^\mu A_\mu + \alpha_0 B\,, \tag{2·48a}$$

$$\partial^\mu\frac{\delta\Gamma}{\delta K^\mu} + i\frac{\delta\Gamma}{\delta\bar{c}} = 0\,. \tag{2·48b}$$

These are the well-known results. These equations (2·47) and (2·48) will be much used in the analysis of the asymptotic fields in Chap. IV, and will be referred to as the Γ-W.T. identities.

Chapter III

Proof of Unitarity of the Physical S-Matrix

——"Confinement" of Unphysical Particles by Quartet Mechanism——

§ 3.1. Representations of the Algebra of Q_B and Q_c

In order to establish the physical S-matrix unitarity, it is sufficient to prove that the three *physicality criteria* (1·9) are all satisfied in our present formulation, as has been stated in Theorem 1.1 in § 1.2: First, (0′) of the physicality criteria (1·9), i.e., the (pseudo-) unitarity of the total S-matrix, $SS^\dagger = S^\dagger S = 1$, holds as noted in (2·9) by virtue of our correct hermiticity assignment (1·18) to the FP ghosts. Second, (i′) of the physicality criteria (1·9), i.e., the invariance of physical subspace under the time evolution, $\mathcal{V}^{\text{out}}_{\text{phys}} = \mathcal{V}^{\text{in}}_{\text{phys}}$, also holds. This is because the present physical subspace $\mathcal{V}_{\text{phys}}$ is specified by the subsidiary condition (2·29),

$$Q_B|\text{phys}\rangle = 0 \tag{3·1}$$

in terms of the *conserved* (and *scalar*) charge Q_B, and hence, is manifestly invariant under the time evolution as well as under the Lorentz transformation. Thus we have only to prove the third criterion (ii) of (1·9), i.e., the positive semi-definiteness of metric in $\mathcal{V}_{\text{phys}}$ which is not so trivial as the others. For this purpose, we should analyze the metric structure of the total state vector space $\mathcal{V}$ and the physical subspace $\mathcal{V}_{\text{phys}}$ explicitly for each of concrete models. We can, however, discuss the general feature of the metric structure to a considerable extent solely by analyzing the irreducible representations of algebra (2·25) of Q_B and Q_c:

$$\tfrac{1}{2}\{Q_B, Q_B\} = Q_B{}^2 = 0\,, \tag{3·2a}$$

$$[iQ_c, Q_B] = Q_B\,, \tag{3·2b}$$

$$[Q_c, Q_c] = 0\,. \tag{3·2c}$$

On the basis of such analysis, we will find quite a general mechanism, called "quartet mechanism", by which the unphysical particles having non-positive norms are made undetectable completely in the physical world ($H_{\text{phys}} = \overline{\mathcal{V}_{\text{phys}}/\mathcal{V}_0}$).

We assume that the BRS charge Q_B as well as the FP ghost charge Q_c does not suffer from spontaneous symmetry breaking, and hence $Q_B|0\rangle = Q_c|0\rangle = 0$, of course. So any one-particle states, physical or unphysical, are classified into the irreducible representations of the algebra (3·2). We assume also that these one-particle states are created (or annihilated) by the asymptotic fields which correspond to certain (interpolating) Heisenberg fields, composite or elementary. Because of the nilpotency (3·2a) of Q_B, the irreducible representation is at most two dimensional, and hence, singlet or doublet. As will be seen in Chap. V, the charge Q_B (as well as Q_c) is commutative with other conserved quantities such as the energy momentum P^μ, the angular momentum $M^{\mu\nu}$, the global ('color' or 'flavor') charges Q^a and other possible charges, if any. Therefore, the particle multiplet of an irreducible representation can be simultaneously assigned such quantum numbers. Taking account of these points, we find *only three* types for the structure of particle multiplet realizing the algebra (3·2):

$$\left.\begin{array}{ll}\text{(I)} & \text{physical particle} = \text{BRS-singlet,} \\ \text{(II)} & \text{singlet pair} = \text{“FP-conjugate” pair of two BRS-singlets,} \\ \text{(III)} & \text{quartet} = \text{“FP-conjugate” pair of two BRS-doublets.}\end{array}\right\} \qquad (3\cdot3)$$

Singlet representations. In order to show (3·3) explicitly, we begin with the analysis of singlet representations of (3·2). Let us denote one-particle states by $|k, N\rangle$ where N represents the eigenvalue of FP ghost charge iQ_c and k stands for all other quantum numbers, e.g., mass, momentum (or wave packet states), spin and internal quantum numbers, etc. If a state $|k, N\rangle$ satisfies

$$Q_B|k, N\rangle = 0 \qquad (3\cdot4)$$

and there exists no state $|*\rangle$ such that $Q_B|*\rangle = |k, N\rangle$, then, $|k, N\rangle$ trivially forms a basis of singlet representation of (3·2), and is called BRS-singlet. Now we discuss two cases $N=0$ and $N\neq0$, separately.

(I) *Genuine physical particle* ($N=0$). The creation operator $\phi_k{}^\dagger$, defined by $|k, N=0\rangle = \phi_k{}^\dagger|0\rangle$, commutes with Q_B by Eq. (3·4):*)

$$[Q_B, \phi_k{}^\dagger] = [Q_B, \phi_k] = 0\,, \qquad (3\cdot5)$$

where we have used the hermiticity of Q_B. Thus, since ϕ_k is gauge- (or BRS-) invariant and has vanishing FP ghost number, ϕ_k represents genuine physical particle which freely appears in the physical subspace $\mathcal{V}_{\text{phys}}$ specified by (3·1), $Q_B|\text{phys}\rangle = 0$. Therefore, for a consistency of theory, ϕ_k should have positive norm:

$$[\phi_k, \phi_l{}^\dagger]_\mp = +\delta_{kl}\,, \qquad (3\cdot6)$$

*) Since Eq. (3·4) leads only to $Q_B\phi_k{}^\dagger|0\rangle = [Q_B, \phi_k{}^\dagger]|0\rangle = 0$, one may suspect that $[Q_B, \phi_k{}^\dagger]$ might be a linear combination of some annihilation operators. This is, however, impossible because $\langle0|[Q_B, \phi_k{}^\dagger] = -\langle0|\phi_k{}^\dagger Q_B = 0$.

where l is another possible value for k, and the anti-commutator $(+)$ is understood for fermions of course. We assume here the positivity (3·6), which should be assured in each explicit model.

(II) *Singlet pair* ($N\neq 0$). If $N\neq 0$, then $\langle k, N|k, N\rangle=0$ due to the conservation of FP ghost number iQ_c. So there should exist some "FP-conjugate" state, say $|k, -N\rangle$, which has the FP ghost number $-N$ and $\langle k, -N|k, N\rangle\neq 0$, because, otherwise, the state $|k, N\rangle$ by itself cannot produce any poles in any Green's function as the intermediate state and hence cannot appear from the first as a one-particle asymptotic state, which should exhibit its existence in the poles of some Green's functions. Such "FP-conjugate" state $|k, -N\rangle$ is *unique* under the normalization

$$\langle k, -N|k, N\rangle=1\,, \tag{3·7}$$

as far as k contains a maximal set of quantum numbers by which particle states are discriminated. [Indeed, even if one finds many states $\{|l, -N\rangle\}$ satisfying $\langle l, -N|k, N\rangle\neq 0$, one can construct only one state satisfying (3·7) by Schmidt's orthogonalization method.] Further, this "FP-conjugate" state must also be a BRS-singlet:

$$Q_B|k, -N\rangle=0 \tag{3·8}$$

with no state $|*\rangle$ satisfying $Q_B|*\rangle=|k, -N\rangle$. Indeed, otherwise, $|k, -N\rangle$ belongs to a BRS-doublet and then the original $|k, N\rangle$ also turns out to fall into another BRS-doublet as will be seen in the next case (III) soon below. This contradicts the first assumption that $|k, N\rangle$ is a BRS-singlet. So, in this case, the representation becomes a singlet pair, "FP-conjugate" pair of two BRS-singlets. Introducing the creation (and annihilation) operators by

$$|k, N\rangle\equiv\sigma_k^\dagger|0\rangle, \quad |k, -N\rangle\equiv\bar{\sigma}_k^\dagger|0\rangle,$$

we obtain by (3·4) and (3·8)

$$[Q_B, \sigma_k]_\mp=[Q_B, \bar{\sigma}_k]_\mp=0 \tag{3·9a}$$

and their hermitian conjugates. Further, Eq. (3·7) and the equations $\langle k, N|k, N\rangle=\langle k, -N|k, -N\rangle=0$ due to FP ghost number conservation lead to the following (anti-)commutation relations:

$$[\bar{\sigma}_k,\sigma_l^\dagger]_\mp=\delta_{kl}\,,$$

$$[\sigma_k, \sigma_l^\dagger]_\mp=[\bar{\sigma}_k, \bar{\sigma}_l^\dagger]_\mp=0\,, \tag{3·9b}$$

and their hermitian conjugates. Here, in (3·9a) and (3·9b), the commutator $(-)$ [the anti-commutator $(+)$] should be taken for even [odd] N. Precisely speaking, in deriving the commutation relations (3·9b), we can obtain only the vacuum expectation values of them. However, by the present assumption that the operators σ_k, $\sigma_k^\dagger$, $\bar{\sigma}_k$ and $\bar{\sigma}_k^\dagger$ belong to the asymptotic fields corresponding to certain Heisenberg fields, we can use the Greenberg-Robinson theorem [Theorem C.1 in Appendix C] which assures that the (anti-) commutators of asymptotic fields are c-numbers, and hence conclude (3·9b).

If this type of multiplet would appear in the theory, a consistent formulation would not be possible. Despite that the particles σ_k and $\bar{\sigma}_k$ have *non-vanishing FP ghost numbers* $N(\neq 0)$, they appear freely in the physical subspace $\mathcal{V}_{\text{phys}}$ because of (3·9a), if they exist. They not only break spin-statistics connection (for odd N), but also violate the positive semi-definiteness of metric also in the physical subspace; for example, the states $(\sigma_k^\dagger - \bar{\sigma}_k^\dagger)|0\rangle \equiv |1\rangle$ satisfying $Q_B|1\rangle = 0$ have negative norm $\langle 1|1\rangle = -1$. Even if we further restrict the physical subspace by imposing one more subsidiary condition (2·33) of vanishing FP ghost number, $Q_c|\text{phys}\rangle = 0$, the examples of negative norm states are easily constructed for odd N; e.g., $\sigma_k^\dagger \bar{\sigma}_k^\dagger |0\rangle \equiv |2\rangle$ satisfies $Q_B|2\rangle = Q_c|2\rangle = 0$ and has negative norm, $\langle 2|2\rangle = -1$! Therefore, although this singlet-pair representation seems quite admissible from the algebraic consideration alone, we cannot construct a physically meaningful theory if such particles appear: The physical S-matrix unitarity breaks down and the probability-interpretation becomes quite impossible. Fortunately, in the explicit models (Yang-Mills, $SU(2)$ Higgs-Kibble and gravity) discussed in Chap. IV, we will find no evidence for such singlet pairs to exist. Especially, the "elementary" FP ghosts fall into the quartet considered in the next case (III) but not into this singlet pair. Further, if we do not stick to the covariance, such gauges as axial gauges *with no FP ghosts* are possible, so, from this fact, we may expect that a general proof of the absence of singlet pairs will be given in the near future. So we here simply assume that no such singlet pairs appear, and henceforth discard them.

Doublet representations. If a state $|k, N\rangle$ satisfies $Q_B|k, N\rangle \neq 0$, then, $|k, N\rangle$ and $|k, N+1\rangle \equiv Q_B|k, N\rangle$ form a basis of BRS-doublet representation of the algebra (3·2):

$$Q_B|k, N\rangle = |k, N+1\rangle\,. \tag{3·10}$$

The nilpotency (3·2a), $Q_B^2 = 0$, together with (3·10) leads to

$$Q_B|k, N+1\rangle = 0\,. \tag{3·11a}$$

Since this state $|k, N+1\rangle$ has vanishing norm,

$$\langle k, N+1|k, N+1\rangle = \langle k, N|Q_B|k, N+1\rangle = 0\,, \tag{3·11b}$$

by using the same reasoning as the above in the case (II), we can conclude that the state $|k, N+1\rangle$ must have a *unique* "FP-conjugate" state, say

$|k, -(N+1)\rangle$, which has the FP ghost number $-(N+1)$ and satisfies

$$\langle k, -(N+1)|k, N+1\rangle = 1\,.$$

And further, this "FP-conjugate" state $|k, -(N+1)\rangle$ also belongs to another BRS-doublet, because the state $|k, -N\rangle$ defined by

$$Q_B|k, -(N+1)\rangle = |k, -N\rangle\,, \tag{3·12}$$

does not vanish due to a "W.T. identity":

$$\begin{aligned}\langle k, N|k, -N\rangle &= \langle k, N|Q_B|k, -(N+1)\rangle \\ &= \langle k, N+1|k, -(N+1)\rangle = 1\,.\end{aligned} \tag{3·13}$$

Thus, the BRS-doublet representations are always realized *in pairs*: $(\{|k, N\rangle, |k, N+1\rangle\}, \{|k, -(N+1)\rangle, |k, -N\rangle\})$. This provides the third and final type of possibility for the representations of algebra (3·2). We call this "FP-conjugate" pair of two BRS-doublets simply a *quartet* and discuss its metric structure and BRS transformation property because it gives quite an interesting and general mechanism of "confinement" of unphysical particles.[1]

(III) *Quartet.* Let us introduce the creation operators defined by

$$\begin{aligned}&|k, N\rangle \equiv \chi_k^\dagger|0\rangle\,, \quad -|k, -N\rangle \equiv \beta_k^\dagger|0\rangle\,, \\ &i|k, N+1\rangle \equiv \gamma_k^\dagger|0\rangle\,, \quad -|k, -(N+1)\rangle \equiv \bar{\gamma}_k^\dagger|0\rangle\,,\end{aligned} \tag{3·14}$$

and the annihilation operators by their hermitian conjugates. We can regard the FP ghost number N as even without loss of generality, by exchanging one BRS-doublet of the pair for another if necessary. Then, from the definitions (3·10) and (3·12), we find the BRS transformation properties, in the same way as before,

$$[Q_B, \chi_k] = -i\gamma_k\,, \tag{3·15a}$$

$$\{Q_B, \bar{\gamma}_k\} = \beta_k\,, \tag{3·15b}$$

and hence, from the nilpotency $Q_B^2 = 0$,

$$[Q_B, \beta_k] = \{Q_B, \gamma_k\} = 0\,. \tag{3·15c}$$

The "W.T. identity" (3·13) and the FP ghost number conservation, together with Greenberg-Robinson theorem (Appendix C), are sufficient for us to conclude the following (anti-)commutation relations:

$$\begin{array}{c} \\ \chi_k \\ \beta_k \\ \\ \gamma_k \\ \bar{\gamma}_k \end{array}
\begin{array}{c} \begin{array}{cccc} \chi_l^\dagger & \beta_l^\dagger & \gamma_l^\dagger & \bar{\gamma}_l^\dagger \end{array} \\ \left(\begin{array}{cc|cc} \omega_{kl} & -\delta_{kl} & 0 & 0 \\ -\delta_{kl} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & i\delta_{kl} \\ 0 & 0 & -i\delta_{kl} & 0 \end{array}\right) \end{array} \equiv \eta_{kl}\,, \tag{3·16}$$

where the anti-commutators are understood only in the sector of γ and $\bar{\gamma}$ which have odd FP ghost number $\pm(N+1)$. All the vanishing matrix elements, except for $[\beta_k, \beta_l^\dagger]=0$ which is derived from the nilpotency $Q_B^2=0$, are the results of FP ghost number conservation. ω_{kl} in (3·16) represents the value of the commutator $[\chi_k, \chi_l^\dagger]$ which remains undetermined from the algebraic considerations alone,*) but it need not be specified for our purpose fortunately.

We have thus shown from the algebraic consideration that there *can exist* the quartet representations, i.e., the BRS-doublet pairs, satisfying the BRS transformation property (3·15) and having the metric structure (3·16). Here, we note an example of the quartets, present always in the theory, containing each "elementary" FP ghost pair as their members, for each group index 'a'. We call these quartets the "elementary quartets".[2] Now, note the following two W.T. identities:

$$\langle 0|TB^a(x)B^b(y)|0\rangle = \langle 0|\{Q_B, T(B^a(x)\bar{c}^b(y))\}|0\rangle = 0\,, \tag{3·17a}$$

$$\begin{aligned}\langle 0|TA_\mu^a(x)B^b(y)|0\rangle - i\langle 0|T(D_\mu c)^a(x)\bar{c}^b(y)|0\rangle \\ = \langle 0|\{Q_B, T(A_\mu^a(x)\bar{c}^b(y))\}|0\rangle = 0\,,\end{aligned} \tag{3·17b}$$

where use has been made of $Q_B|0\rangle=0$ and the BRS transformation (2·21) with (2·16). We recall the equations of motion (2·10c) and (2·10b),

$$\partial^\mu (D_\mu c)^a = 0\,, \tag{3·18a}$$

$$\partial^\mu A_\mu^a + \alpha_0 B^a = 0\,, \tag{3·18b}$$

and the equal-time commutation relations,

$$i\{(D_0 c)^a(x), \bar{c}^b(y)\}_{x_0=y_0} = i\delta^{ab}\delta^3(\boldsymbol{x}-\boldsymbol{y})\,, \tag{3·19a}$$

$$[A_0^a(x), B^b(y)]_{x_0=y_0} = i\delta^{ab}\delta^3(\boldsymbol{x}-\boldsymbol{y})\,, \tag{3·19b}$$

which are nothing but CAR and CCR (2·12) with (2·11). It follows from these (3·18a) and (3·19a) that

$$\begin{aligned}\partial_x^\mu \langle 0|T(D_\mu c)^a(x)\bar{c}^b(y)|0\rangle &= \delta(x_0-y_0)\langle 0|\{(D_0 c)^a(x), \bar{c}^b(y)\}|0\rangle \\ &= i\delta^{ab}\delta^4(x-y)\,.\end{aligned}$$

Hence we obtain, taking account of (3·17b) also,

$$\text{F.T.}\langle 0|T(D_\mu c)^a(x)\bar{c}^b(y)|0\rangle = i\delta^{ab}p_\mu/p^2, \tag{3·20a}$$

$$\text{F.T.}\langle 0|TA_\mu^a(x)B^b(y)|0\rangle = -\delta^{ab}p_\mu/p^2, \tag{3·20b}$$

where the Fourier transform F.T. is defined by the operation $\int d^4x e^{ip(x-y)}$.

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*) Of course, if $N\neq 0$, $\omega_{kl}=0$ by the FP ghost number conservation. In all important cases, however, $N=0$.

Equation (3·20b) can be directly obtained also by utilizing Eqs. (3·18b) and (3·19b) together with the help of (3·17a).

The pole structures of Eqs. (3·20) imply the existence of massless asymptotic fields (for each group index a) which are defined as follows:

$$A_\mu{}^a(x)^{\mathrm{as}} = \partial_\mu \chi^a(x) + \cdots, \tag{3·21a}$$

$$B^a(x)^{\mathrm{as}} = \beta^a(x) + \cdots, \tag{3·21b}$$

$$(D_\mu c)^a(x)^{\mathrm{as}} = \partial_\mu \gamma^a(x) + \cdots, \tag{3·21c}$$

$$\bar{c}^a(x)^{\mathrm{as}} = \bar{\gamma}^a(x) + \cdots, \tag{3·21d}$$

where use has been made of the notation

$$\Phi(x)^{\mathrm{as}} = \underset{x_0 \to \mp\infty}{w\text{-limit}}\, \Phi(x) \tag{3·22}$$

with the superscript 'as' representing 'in' or 'out', and the dots $(\cdots)$ stand for the other possible asymptotic fields irrelevant to the poles in (3·20). The BRS transformation (2·21) with (2·16) for the Heisenberg fields $A_\mu{}^a$, B^a, $(D_\mu c)^a$ and $\bar{c}^a$ determine the following BRS transformation properties for the asymptotic fields in (3·21) [see Theorem C.3 and (C·25) in Appendix C]:

$$[iQ_B, \chi^a(x)] = \gamma^a(x), \quad \{iQ_B, \bar{\gamma}^a(x)\} = i\beta^a(x), \tag{3·23a}$$

$$\{iQ_B, \gamma^a(x)\} = 0, \quad [iQ_B, \beta^a(x)] = 0. \tag{3·23b}$$

These equations (if rewritten in terms of the creation and annihilation operator) are nothing but the transformation properties (3·15) which we have found above for the quartet representations. Therefore, also their (anti-) commutation relations are proved to be identical to (3·16) by essentially the same reasonings as above: Equations (3·20) as a result of the present W.T. identity (3·17b), which corresponds to the "W.T. identity" (3·13) above, conclude

$$[\chi^a(x), \beta^b(y)] = i\delta^{ab}D(x-y), \quad \{\gamma^a(x), \bar{\gamma}^b(y)\} = -\delta^{ab}D(x-y), \tag{3·24a}$$

by virtue of the Greenberg-Robinson theorem (Appendix C). All the other (anti-) commutators among χ, β, γ and $\bar{\gamma}$ except for $[\chi^a(x), \chi^b(y)]$ are found to vanish from the FP ghost number conservation and (3·17a); e.g.,

$$[\beta^a(x), \beta^b(y)] = \{\gamma^a(x), \gamma^b(y)\} = \{\bar{\gamma}^a(x), \bar{\gamma}^b(y)\} = 0. \tag{3·24b}$$

These equations (3·24) just coincide with (3·16) when they are rewritten by introducing the creation and annihilation operators. Thus we have found generally the existence of "elementary quartet" for each group index a: β^a, γ^a and $\bar{\gamma}^a$ represent "elementary" asymptotic fields of Heisenberg operators of the multipliers $B^a(x)$, the FP ghosts $c^a(x)$ and anti-ghosts $\bar{c}^a(x)$. In the examples in Chap. IV, χ^a will be found to be identified with the longitudinal component of $A_\mu{}^a(x)$ for the symmetry unbroken Yang-Mills case, and with the Goldstone mode of Higgs scalar field for the case of $SU(2)$ Higgs-Kibble model. These "elementary quartets" will play an important role in the general discussion of spontaneous breakdown of global gauge symmetries in Chap. VI.

§ 3.2. Quartet Mechanism

Consider the case where exist a variety of quartets all of which satisfy the BRS transformation property (3·15) and have the metric structure (3·16), as well as the usual genuine physical particles with positive norm. Then, the total Fock space $\mathcal{V}$ spanned by those particles is full of negative metric. Even then, however, we can prove that, in the physical subspace $\mathcal{V}_{\text{phys}}$ specified by the condition $Q_B|\text{phys}\rangle=0$, *any members of any quartets always appear only in zero-norm combinations* and hence that *$\mathcal{V}_{\text{phys}}$ has positive semi-definite metric.* Thus, any quartet members can never be detected with finite probability in our world $\mathcal{V}_{\text{phys}}$: Quartets are always confined! To prove this is the subject of this section.

We call any quartet members unphysical particles (states). As for the BRS-singlet particle states, they are assumed to be made orthogonal to the unphysical particle states, which is always possible. The sector of states which contain n unphysical particles in sum aside from arbitrary number of genuine physical particles, is called the n-unphysical-particle sector. Since we are considering the case in which there exist arbitrary variety of quartets aside from many kinds of genuine physical particles, we should understand that the index k (or l) in (3·15) and (3·16) stands also for the kind of the quartets as well as other quantum numbers. By the help of the inverse of the metric matrix η of (3·16), the projection operator $P^{(n)}$ onto the n-unphysical-particle sector can be written inductively as[3)]

$$P^{(n)}=(1/n)\,(-\beta_k{}^\dagger P^{(n-1)}\chi_k-\chi_k{}^\dagger P^{(n-1)}\beta_k-\omega_{kl}\beta_k{}^\dagger P^{(n-1)}\beta_l$$
$$+i\gamma_k{}^\dagger P^{(n-1)}\bar{\gamma}_k-i\bar{\gamma}_k{}^\dagger P^{(n-1)}\gamma_k) \qquad (3\cdot25)$$

for $n=1, 2, \cdots$, where the summations over the repeated indices k and l are understood. $P^{(0)}$ is of course defined as the projection operator onto the zero-unphysical-particle sector [i.e., the subspace spanned solely by the genuine physical particles, say ϕ_α], which has positive metric by assumption and is denoted as $\mathcal{H}_{\text{phys}}$. $P^{(0)}$ is explicitly given by

$$P^{(0)}=\sum_m(1/m!)\,(\phi_{\alpha_1}^\dagger\phi_{\alpha_2}^\dagger\cdots\phi_{\alpha_m}^\dagger|0\rangle\langle0|\phi_{\alpha_m}\cdots\phi_{\alpha_2}\phi_{\alpha_1}), \qquad (3\cdot26)$$

because of their diagonal metric structure, $[\phi_\alpha, \phi_\beta{}^\dagger]_\mp=\delta_{\alpha\beta}$ [(3·6)]. Note that the $P^{(n)}$'s are orthogonal projection operators which are orthogonal to one another and complete (on the assumption of asymptotic completeness):

$$(P^{(n)})^2 = P^{(n)} = P^{(n)\dagger}, \tag{3·27a}$$

$$P^{(n)}P^{(m)} = P^{(m)}P^{(n)} = \delta_{mn}P^{(n)}, \tag{2·27b}$$

$$\sum_{n=0}^{\infty} P^{(n)} = 1. \tag{3·27c}$$

By using the BRS transformation properties (3·15) of quartets, we can prove the following important properties of $P^{(n)}$:[3]

$$[Q_B, P^{(n)}] = 0 \quad \text{for } n = 0, 1, 2, \cdots. \tag{3·28}$$

The proof goes by induction. First, $[Q_B, P^{(0)}] = 0$ is trivial because the commutativity of Q_B with the genuine physical particles ϕ_α, $[Q_B, \phi_\alpha] = 0$, and the explicit form (3·26) of $P^{(0)}$ lead to $Q_B P^{(0)} = P^{(0)} Q_B = 0$ by $Q_B|0\rangle = 0$. Next we calculate the commutator $[Q_B, P^{(n)}]$ by using (3·15), (3·25) and the induction assumption $[Q_B, P^{(n-1)}] = 0$,

$$\begin{aligned} -n[Q_B, P^{(n)}] &= \beta_k{}^\dagger P^{(n-1)}(-i\gamma_k) + (-i\gamma_k{}^\dagger)P^{(n-1)}\beta_k \\ &\quad + i\gamma_k{}^\dagger P^{(n-1)}\beta_k + i\beta_k{}^\dagger P^{(n-1)}\gamma_k = 0. \end{aligned}$$

This finishes the proof of (3·28). By virtue of this equation (3·28) together with the BRS transformation property (3·15), we can rewrite $P^{(n)}$ into the following remarkable form:

$$\begin{aligned} &P^{(n)} = \{Q_B, R^{(n)}\} \quad \text{for } n \geqq 1; \\ &R^{(n)} = -(1/n)(\bar{\gamma}_k{}^\dagger P^{(n-1)}\chi_k + \chi_k{}^\dagger P^{(n-1)}\bar{\gamma}_k + \omega_{kl}\beta_k{}^\dagger P^{(n-1)}\bar{\gamma}_l). \end{aligned} \tag{3·29 *)}$$

From this it follows directly that[3]

$$\langle f | P^{(n)} | g \rangle = 0 \quad \text{for } n \geqq 1 \tag{3·30}$$

for any physical states $\forall|f\rangle, \forall|g\rangle \in \mathcal{V}_{\text{phys}}$ satisfying the subsidiary condition (3·1), $Q_B|f\rangle = Q_B|g\rangle = 0$. From Eq. (3·30), we can see the following things:

(i) In the orthogonal decomposition of $\forall|f\rangle \in \mathcal{V}_{\text{phys}}$ according to the number n of unphysical particles,

$$|f\rangle = P^{(0)}|f\rangle + \sum_{n=1}^{\infty} P^{(n)}|f\rangle, \tag{3·31}$$

the norm of $|f\rangle$ is determined solely by the first component $P^{(0)}|f\rangle \in \mathcal{H}_{\text{phys}}$ and all the others have zero-norm by (3·30):

$$\langle f|f\rangle = \langle f|P^{(0)}|f\rangle = \langle P^{(0)}f, P^{(0)}f\rangle \geqq 0. \tag{3·32}$$

*) It was pointed out by Professor K. Fujikawa that $P^{(n)}$ can be rewritten into this concise form (3·29).

Thus, the physical subspace $\mathcal{V}_{\text{phys}}$ has positive semi-definite metric, as far as the positivity of physical-particle norm is assured. This finishes the proof of the third physicality criterion (ii) of (1·9), and hence the physical S-matrix is proved to be unitary. We note here also the following isomorphism:

$$\mathcal{V}_{\text{phys}}/\mathcal{V}_0 \cong \mathcal{H}_{\text{phys}} (\equiv P^{(0)}\mathcal{V}), \tag{3·33}$$

where $\mathcal{V}_0$ is the zero-norm subspace of $\mathcal{V}_{\text{phys}}$ defined by (1·12a). Further, since (3·27c), (3·29) and $Q_B{}^2=0$, together with the help of $P^{(0)}\mathcal{V}_0=0$ (by definition), lead to

$$\mathcal{V}_0 = \sum_{n=0}^{\infty} P^{(n)}\mathcal{V}_0 = \sum_{n=1}^{\infty} P^{(n)}\mathcal{V}_0$$

$$\subset \sum_{n=1}^{\infty} P^{(n)}\mathcal{V}_{\text{phys}} = \{Q_B, \sum_{n=1}^{\infty} R^{(n)}\}\,\mathcal{V}_{\text{phys}} \subset Q_B\mathcal{V} \subset \mathcal{V}_0\,,$$

we find the following equalities:

$$\mathcal{V}_0 = \sum_{n=1}^{\infty} P^{(n)}\mathcal{V}_{\text{phys}} = Q_B\mathcal{V}\,. \tag{3·34}$$

(ii) Since it is instructive to see explicitly what type of combinations of unphysical particles appear in the physical subspace $\mathcal{V}_{\text{phys}}$, we present a complete list of them up to 2-unphysical-particle sector in Table I. [$(\chi_k, \beta_k, \gamma_k, \bar{\gamma}_k)$ in Table I stands for the operators of quartet satisfying (3·15) and (3·16).] Notice that any states in $P^{(n)}\mathcal{V}_{\text{phys}}$ $(n\geqq 1)$ can easily be constructed as $Q_B|n\rangle$ from arbitrary n-unphysical-particle states $|n\rangle$; indeed, when $n\geqq 1$,

$$P^{(n)}\mathcal{V}_{\text{phys}} = P^{(n)}\,(\sum_{m=1}^{\infty} P^{(m)})\,\mathcal{V}_{\text{phys}} = P^{(n)}Q_B\mathcal{V} = Q_B\,(P^{(n)}\mathcal{V})$$

by the orthogonality (3·27b) of the $P^{(m)}$'s and (3·34). All the states in Table I are constructed by this method; e.g., $Q_B\bar{\gamma}_k{}^\dagger\bar{\gamma}_l{}^\dagger|\alpha\rangle = (\beta_k{}^\dagger\bar{\gamma}_l{}^\dagger - \bar{\gamma}_k{}^\dagger\beta_l{}^\dagger)\,|\alpha\rangle$. Also from this form $Q_B|n\rangle$, we see very clearly that the unphysical-particle states in $\mathcal{V}_{\text{phys}}$ have indeed zero-norm by the nilpotency $Q_B{}^2=0$:

$$(\langle m|Q_B{}^\dagger)\,(Q_B|n\rangle) = \langle m|Q_B{}^2|n\rangle = 0\,. \tag{3·35}$$

(iii) If we start from an initial state $|i\rangle \in \mathcal{V}_{\text{phys}} (= \mathcal{V}^{\text{in}}_{\text{phys}})$, the final state

Table I. List of unphysical-particle states contained in $\mathcal{V}_{\text{phys}}$. $|\alpha\rangle$ $(\in\mathcal{H}_{\text{phys}})$ stands for an arbitrary state consisting of physical particles alone.

	$iQ_c=-1$	$Q_c=0$	$iQ_c=1$	$iQ_c=2$
$P^{(1)}\mathcal{V}_{\text{phys}}$	—	$\beta_k{}^\dagger\|\alpha\rangle$	$\gamma_k{}^\dagger\|\alpha\rangle$	—
$P^{(2)}\mathcal{V}_{\text{phys}}$	$(\beta_k{}^\dagger\bar{\gamma}_l{}^\dagger-\bar{\gamma}_k{}^\dagger\beta_l)\|\alpha\rangle$	$\beta_k{}^\dagger\beta_l{}^\dagger\|\alpha\rangle$, $(\beta_k{}^\dagger\chi_l{}^\dagger+i\bar{\gamma}_k{}^\dagger\gamma_l{}^\dagger)\|\alpha\rangle$	$\beta_k{}^\dagger\gamma_l{}^\dagger\|\alpha\rangle$, $(\chi_k{}^\dagger\gamma_l{}^\dagger+\gamma_k{}^\dagger\chi_l{}^\dagger)\|\alpha\rangle$	$\gamma_k{}^\dagger\gamma_l{}^\dagger\|\alpha\rangle$

$|f\rangle$*) after scattering also remains in the physical subspace $\mathcal{V}_{\rm phys}(=\mathcal{V}_{\rm phys}^{\rm out})$ by the conservation of Q_B. Note, however, that the subspace $\mathcal{H}_{\rm phys}=P^{(0)}\mathcal{V}_{\rm phys}$ is *not* invariant under the time-evolution: $\mathcal{H}_{\rm phys}^{\rm in}=P_{\rm in}^{(0)}\mathcal{V}_{\rm phys}\neq P_{\rm out}^{(0)}\mathcal{V}_{\rm phys}=\mathcal{H}_{\rm phys}^{\rm out}$. Namely, even when we start from an initial state $|i\rangle\in\mathcal{H}_{\rm phys}^{\rm in}$ containing no unphysical particles, the final state $|f\rangle$ generally has non-vanishing components $P_{\rm out}^{(n)}|f\rangle$ of n-unphysical-particle states which are really produced by such processes as FP-ghost pair creations. The important point is that such unphysical particles are produced *only in zero-norm combinations* as is assured by (3·30) or (3·35). If the conventional wrong hermiticity assignment (1·16) was adopted, this was not the case. We here show a crucial example indicating the incorrectness of (1·16). Since the "elementary" quartets are shown to always exist, the state $(\beta_k{}^\dagger\chi_l{}^\dagger+i\bar{\gamma}_k{}^\dagger\gamma_l{}^\dagger)|\alpha\rangle\equiv|2\rangle$ in Table I with χ, β, γ and $\bar{\gamma}$ of the elementary quartet for instance, will be produced by interactions (and have zero-norm $\langle 2|2\rangle=0$ in our hermiticity assignment). Note the following correspondence between the FP ghosts with our hermiticity assignment (1·18) and the conventional one (1·16) (the conventional FP ghosts are denoted by capital letters for distinction),

$$c^{\rm as}(x)=\sum_k(\gamma_k g_k(x)+\gamma_k{}^\dagger g_k{}^*(x))\,;\quad i\bar{c}^{\rm as}(x)=i\sum_k(\bar{\gamma}_k g_k(x)+\bar{\gamma}_k{}^\dagger g_k{}^*(x)),$$
$$\updownarrow \qquad\qquad\qquad\qquad \updownarrow$$
$$C^{\rm as}(x)=\sum_k(\Gamma_k g_k(x)+\bar{\Gamma}_k{}^\dagger g_k{}^*(x))\,;\quad \bar{C}^{\rm as}(x)=\sum_k(\bar{\Gamma}_k g_k(x)+\Gamma_k{}^\dagger g_k{}^*(x)),\tag{3·36a}$$
**)

in conformity with the corresponding hermiticity assignments (1·16) and (1·18), respectively, and hence

$$\gamma_k\leftrightarrow\Gamma_k,\quad \gamma_k{}^\dagger\leftrightarrow\bar{\Gamma}_k{}^\dagger,\quad i\bar{\gamma}_k\leftrightarrow\bar{\Gamma}_k,\quad i\bar{\gamma}_k{}^\dagger\leftrightarrow\Gamma_k{}^\dagger,\tag{3·36b}$$

$$\{i\bar{\gamma}_k,\gamma_l{}^\dagger\}=-\{\gamma_k,i\bar{\gamma}_l{}^\dagger\}=\delta_{kl}\leftrightarrow\{\bar{\Gamma}_k,\bar{\Gamma}_l{}^\dagger\}=-\{\Gamma_k,\Gamma_l{}^\dagger\}=\delta_{kl}\,.\tag{3·36c}$$

Therefore, correspondingly to the above state $|2\rangle$, the state

$$|\tilde{2}\rangle=(\beta_k{}^\dagger\chi_l{}^\dagger+\Gamma_k{}^\dagger\bar{\Gamma}_l{}^\dagger)|\alpha\rangle\tag{3·37}$$

is produced in the conventional case, which, surprisingly, has a negative norm (for $k\neq l$)!:

$$\langle\tilde{2}|\tilde{2}\rangle=\langle\alpha|\bar{\Gamma}_l\Gamma_k\Gamma_k{}^\dagger\bar{\Gamma}_l{}^\dagger|\alpha\rangle=-1\langle\alpha|\alpha\rangle<0\,.\tag{3·38}$$

Although this could be seen more easily in (3·35) which manifestly shows that the zero-norm property breaks down if $Q_B{}^\dagger\not\propto Q_B$, we have preferred to

*) For definiteness, we should say that the initial and final states are understood to be written in the in-state and out-state bases, respectively: $|i\rangle=|i\ {\rm in}\rangle$ and $|f\rangle=|f\ {\rm out}\rangle$. Of course, in the Heisenberg picture, any state vector $|\alpha\rangle$ does not change in itself through time evolution: $|\alpha\rangle=|f\ {\rm out}\rangle=|i\ {\rm in}\rangle=\sum_{n=0}^{\infty}P_{\rm out}^{(n)}|i\ {\rm in}\rangle$.

**) Here $\{g_k(x)\}$ is a suitable wave packet system of massless particle. [See Chap. IV.]

give an explicit example state with negative norm which is really produced in the scattering process.

(iv) For any Green's functions of gauge-(BRS-) invariant operators, any intermediate states containing unphysical particles cannot contribute at all to their spectral functions. In fact, for example, if $\Phi_i(x)$ $(i=1,2)$ are gauge-invariant, i.e.,

$$[Q_B, \Phi_i(x)]=0\,, \tag{3·39}$$

then

$$Q_B\Phi_i(x)\,|\text{phys}\rangle=0 \;\Rightarrow\; \Phi_i(x)\,|\text{phys}\rangle\in\mathcal{V}_{\text{phys}}\,,$$

and therefore, the contribution of n-unphysical-particle intermediate states to the spectral function vanishes by (3·30):

$$\langle\text{phys}'|\Phi_1(x)\,P^{(n)}\Phi_2(y)\,|\text{phys}\rangle=0 \qquad \text{for } n\geqq 1\,. \tag{3·40}$$

Thus we have proved that the unphysical particles, i.e., any quartet members cannot be detected at all in the physical subspace due to their zero-norm combinations. This mechanism that a particle essentially decouples from the physical sector by forming a quartet is called *quartet mechanism*.[1] This mechanism is supposed to take place rather generally, not only in the usual unphysical particles of the longitudinal and scalar components of gauge fields: Indeed it should take place in the famous $U(1)$ problem (§ 7.1), and it can give a key mechanism even for the color confinement problem in QCD[1] (§ 6.2).

§ 3.3. Comments on Subsidiary Conditions for the Case of Non-Simple Gauge Groups

Now we discuss some arbitrariness in setting the subsidiary conditions for the cases of non-simple gauge group G. In such cases G can be decomposed into two factor groups:

$$G=G_1\times G_2\,, \tag{3·41}$$

where we need not assume that G_1 and G_2 are simple. [So, if one wants complete reduction of the form (2·1), one can iterate the procedure (3·41).] Consider the cases where the gauge fixing terms for the gauge groups G_1 and G_2 are chosen to *decouple* to each other, i.e., the gauge fixing terms for G_1 are invariant under the G_2 gauge-transformations and vice versa. [This is the case in our symmetric gauge fixing (2·7b).] Then, our BRS charge Q_B and FP ghost charge Q_c for the total group G are decomposed into those for each group G_1 and G_2 as

$$Q_B=Q_B{}^{(1)}+Q_B{}^{(2)},\quad Q_c=Q_c{}^{(1)}+Q_c{}^{(2)}, \tag{3·42}$$

and these $Q_{B,c}^{(1)}$ and $Q_{B,c}^{(2)}$ are separately conserved. As has been seen in the preceding section, the single subsidiary condition (3·1), $Q_B|\text{phys}\rangle=0$, is sufficient to specify such physical subspace $\mathcal{V}_{\text{phys}}$ that the isomorphism (3·33), $\mathcal{V}_{\text{phys}}/\mathcal{V}_0\cong\mathcal{H}_{\text{phys}}$, holds. However, here corresponding to the decomposition (3·42), there exist a variety of choices of subsidiary conditions if one prefers to make the physical subspace smaller; for examples, one can define

$$\mathcal{V}_{\text{phys}}^{B_1,B_2}\ni|\text{phys}\rangle \Leftrightarrow Q_B{}^{(1)}|\text{phys}\rangle=Q_B{}^{(2)}|\text{phys}\rangle=0\,,$$

$$\mathcal{V}_{\text{phys}}^{B_1,B_2,c_1,c_2}\ni|\text{phys}\rangle \Leftrightarrow Q_B{}^{(1)}|\text{phys}\rangle=Q_B{}^{(2)}|\text{phys}\rangle=Q_c{}^{(1)}|\text{phys}\rangle=Q_c{}^{(2)}|\text{phys}\rangle=0\,,$$

and so on, where the superscripts attached to $\mathcal{V}_{\text{phys}}$'s for distinction indicate the subsidiary conditions imposed. It is not difficult to see the following inclusion relations of them:

$$\mathcal{V}_{\text{phys}}\equiv\mathcal{V}_{\text{phys}}^{B}\supset\left\{\begin{array}{l}\mathcal{V}_{\text{phys}}^{B,c}\\ \mathcal{V}_{\text{phys}}^{B_1,B_2}\supset\left\{\begin{array}{l}\mathcal{V}_{\text{phys}}^{B_1,B_2,c}\ (\text{IV}+\text{VII})\\ \mathcal{V}_{\text{phys}}^{B_1,B_2,c_1}\ (\text{V}+\text{VII})\\ \mathcal{V}_{\text{phys}}^{B_1,B_2,c_2}\ (\text{VI}+\text{VII})\end{array}\right.\end{array}\right\}\supset\mathcal{V}_{\text{phys}}^{B_1,B_2,c_1,c_2}(\text{VII})\supset\mathcal{H}_{\text{phys}}\,. \tag{3·43}$$

More explicit relations are shown in Fig. 1. Note here that

$$\mathcal{V}_{\text{phys}}^{B,c_i}=\mathcal{V}_{\text{phys}}^{B_1,B_2,c_i}\quad\text{for } i=1,2\,, \tag{3·44}$$

because by virtue of the commutation relations

$$[iQ_c{}^{(1)},Q_B]=Q_B{}^{(1)},\quad [iQ_c{}^{(2)},Q_B]=Q_B{}^{(2)}, \tag{3·45}$$

the condition $Q_B|\text{phys}\rangle=Q_c{}^{(1)}|\text{phys}\rangle=0$, for instance, necessarily implies $Q_B{}^{(1)}|\text{phys}\rangle=[iQ_c{}^{(1)},Q_B]|\text{phys}\rangle=0$ and $Q_B{}^{(2)}|\text{phys}\rangle=(Q_B-Q_B{}^{(1)})|\text{phys}\rangle=0$, also. By way of illustration, we cite simple states contained in the regions I~VII of Fig. 1:

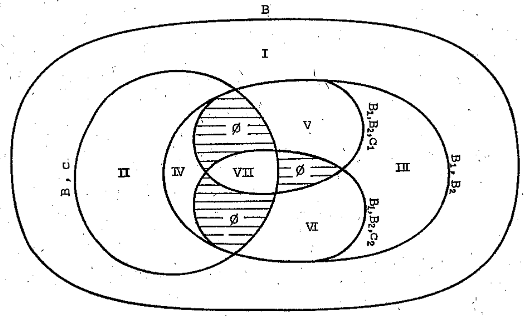

Fig. 1. Inclusion relations (3·43). Three empty regions denoted by Ø appear due to the equality $Q_c=Q_c{}^{(1)}+Q_c{}^{(2)}$.

$$
\begin{aligned}
&\mathrm{I} \ni (\chi_k^{(1)\dagger}\gamma_l^{(2)\dagger} + \gamma_k^{(1)\dagger}\chi_l^{(2)\dagger})\,|\alpha\rangle\,,\\
&\mathrm{II} \ni (\beta_k^{(1)\dagger}\chi_l^{(2)\dagger} + i\bar{\gamma}_k^{(1)\dagger}\gamma_l^{(2)\dagger})\,|\alpha\rangle\,,\\
&\mathrm{III} \ni \gamma_k^{(1)\dagger}\gamma_l^{(2)\dagger}|\alpha\rangle\,,\\
&\mathrm{IV} \ni \gamma_k^{(1)\dagger}(\beta_l^{(2)\dagger}\bar{\gamma}_m^{(2)\dagger} - \bar{\gamma}_l^{(2)\dagger}\beta_m^{(2)\dagger})\,|\alpha\rangle\,, \qquad (3\cdot 46)\\
&\mathrm{V} \ni \beta_k^{(1)\dagger}\gamma_l^{(2)\dagger}|\alpha\rangle\,,\\
&\mathrm{VI} \ni \gamma_k^{(1)\dagger}\beta_l^{(2)\dagger}|\alpha\rangle\,,\\
&\mathrm{VII} \ni \beta_k^{(1)\dagger}\beta_l^{(2)\dagger}|\alpha\rangle\,.
\end{aligned}
$$

We should note that these varieties of choices of subsidiary conditions are relevant only to the size of zero-norm subspace $\mathcal{V}_0$ in each $\mathcal{V}_{\text{phys}}$. Indeed, it is evident from the arguments in the preceding section that the isomorphism

$$\mathcal{V}^X_{\text{phys}}/\mathcal{V}_0{}^X \cong \mathcal{H}_{\text{phys}} \qquad (3\cdot 47)$$

holds for any physical subspaces in (3·43), denoted by $\mathcal{V}^X_{\text{phys}}$ generally. We prefer, however, our original physical subspace $\mathcal{V}_{\text{phys}}$ ($=\mathcal{V}^B_{\text{phys}}$) specified by a single subsidiary condition (3·1), $Q_B|\text{phys}\rangle = 0$. Aside from the fact that it is the simplest choice and makes the theoretical analysis easy, it allows us to take a wide variety of gauge fixings: The gauge fixing terms for G_1 gauge-group need not be invariant under the G_2 gauge-transformations and vice versa. In fact, in the cases when the groups G_1 and G_2 are mixed with each other by the gauge fixing terms, the charges $Q^{(1)}_{B,c}$ and $Q^{(2)}_{B,c}$ can no longer be defined separately. For instance, this is the case in the 't Hooft-Feynman gauge in Weinberg-Salam model.[4] Therefore, we always take only one subsidiary condition (3·1) hereafter.

Chapter IV

Scattering Theoretical Analysis in Some Examples of Gauge Theories

In this chapter we discuss explicitly the following model theories: I $SU(2)$ Higgs-Kibble model with spontaneous symmetry breaking, II pure Yang-Mills theory of a simple group without spontaneous symmetry breaking and III gravity. Throughout this chapter we assume the *asymptotic completeness* in terms of the "*elementary*" fields. We analyze the properties of asymptotic fields of the "elementary" particles as follows: First, the general forms of the 2-point functions of Heisenberg fields are obtained by the requirements of Lorentz covariance and BRS invariance alone. The discrete spectrum parts of them uniquely determine the commutation relations of the asymptotic fields by virtue of the Greenberg-Robinson theorem (Appendix C). The commutation relations lead to the equations of motion of the asymptotic fields, in view of which we choose a complete set of the mutually independent modes, physical ones and unphysical ones. The properties required in the previous chapter are explicitly shown to be satisfied; that is, all the physical modes have positive norm and are orthogonal to all the unphysical modes which fall into quartets satisfying the BRS transformation property (3·15) and the metric structure (3·16). Further we will obtain the explicit asymptotic form of the BRS charge Q_B which will clarify once more the relationship of the present subsidiary condition $Q_B|\text{phys}\rangle=0$ and the Gupta-Bleuler condition $(\partial^\mu A_\mu)^{(+)}|\text{phys}\rangle=0$ or its generalization $B^{(+)}(x)|\text{phys}\rangle=0$ by Nakanishi and Lautrup[1] for the Abelian case.

§ 4.1. $SU(2)$ Higgs-Kibble Model

We discuss, following Ref. 2), the $SU(2)$ Higgs-Kibble model[3] as a typical and the simplest Yang-Mills theory with spontaneous breaking of the gauge symmetry. The Lagrangian density is given in (2·7), in which the group index a runs over $a=1, 2, 3$ and the matter Lagrangian density $\mathcal{L}_{\text{matter}}$ is explicitly given by

$$\mathcal{L}_{\text{matter}}=\left|\partial_\mu\Psi-\frac{i}{2}g\tau^a A_\mu{}^a\Psi\right|^2-V(\Psi^\dagger\Psi). \tag{4·1}$$

Here Ψ is a complex isospinor scalar field and the potential part $V(\Psi^\dagger\Psi)$ is adjusted so that the vacuum expectation value of Ψ becomes

$$\langle\Psi\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}v\\0\end{pmatrix}.$$

So it is convenient to parametrize the field Ψ as follows in terms of ψ, called (real) Higgs scalar, and $\chi^a (a=1, 2, 3)$, called Goldstone bosons:

$$\Psi(x) = \frac{1}{\sqrt{2}}[(v+\psi(x)) + i\chi^a(x)\tau^a]\begin{pmatrix}1\\0\end{pmatrix}. \tag{4·2}$$

The BRS transformations for these matter fields ψ and χ^a are given by

$$\delta\psi = [i\lambda Q_B, \psi] = -\lambda(g/2)\chi\cdot c\,, \tag{4·3a}$$

$$\delta\chi = [i\lambda Q_B, \chi] = \lambda(g/2)[(v+\psi)c + \chi\times c]. \tag{4·3b}$$

We should note here a peculiar property of this $SU(2)$ Higgs-Kibble model that it retains an unbroken global $SU(2)$ symmetry which is different from the spontaneously broken global one corresponding to the local gauge symmetry. With respect to this remaining $SU(2)$ symmetry, ψ and χ^a are iso-singlet and triplet, respectively. One of the Γ-W.T. identities, (2·47), is rewritten in this case as

$$\frac{\delta\Gamma}{\delta A_\mu}\cdot\frac{\delta\Gamma}{\delta K^\mu} + \frac{\delta\Gamma}{\delta\psi}\frac{\delta\Gamma}{\delta K_\psi} + \frac{\delta\Gamma}{\delta\chi}\cdot\frac{\delta\Gamma}{\delta K_\chi} + \frac{\delta\Gamma}{\delta c}\cdot\frac{\delta\Gamma}{\delta K_c} + i\frac{\delta\Gamma}{\delta\bar{c}}\cdot B = 0\,, \tag{4·4a}$$

and the others (2·48a) and (2·48b) remain unchanged:

$$\delta\Gamma/\delta B = \partial^\mu A_\mu + \alpha_0 B, \tag{4·4b}$$

$$\partial^\mu\frac{\delta\Gamma}{\delta K^\mu} + i\frac{\delta\Gamma}{\delta\bar{c}} = 0\,. \tag{4·4c}$$

The source functional (2·39) has the following form for the present case:

$$\mathscr{S}[J, K] = \int d^4x[J_\mu\cdot A^\mu + J_\psi\psi + J_\chi\cdot\chi + \bar{J}_c\cdot c + J_{\bar{c}}\cdot\bar{c} + J_B\cdot B + K_\mu\cdot D^\mu c$$
$$+ (g/2)\{-K_\psi\chi\cdot c + K_\chi\cdot[(v+\psi)c + \chi\times c] - K_c\cdot(c\times c)\}]. \tag{4·5}$$

Now we begin the analysis of 2-point functions. From (4·4b) we obtain

$$\Gamma^{(2)ab}_{A_\mu, B}(k) \equiv \int d^4x e^{ik(x-y)}\frac{\delta^2\Gamma}{\delta A_\mu{}^a(x)\delta B^b(y)}\Big|_0 = \delta^{ab}ik_\mu\,, \tag{4·6a}$$ *)

$$\Gamma^{(2)ab}_{B,B}(k) = \alpha_0\,, \qquad \Gamma^{(2)ab}_{B,\chi}(k) = 0\,. \tag{4·6b}$$

Taking account of Lorentz covariance, the remaining global $SU(2)$ symmetry, the FP ghost number conservation and (4·6), we define the one-particle-irreducible (1PI) 2-vertices (i.e., the inverse propagators) as follows:

---

*) $\cdots|_0$ represents to take the value setting all the arguments equal to zero.

$$\Gamma^{(2)ab}_{\Phi_i,\Phi_j}(k)=\delta^{ab}\times$$

$$\begin{array}{c} \\ A_\mu \\ B \\ \chi \\ c \\ \bar{c} \end{array}\left(\begin{array}{ccc|cc} A_\nu & B & \chi & c & \bar{c} \\ \left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)A(k^2)+\frac{k_\mu k_\nu}{k^2}B(k^2) & ik_\mu & ik_\mu C(k^2) & & \\ -ik_\nu & \alpha_0 & 0 & \multicolumn{2}{c}{0} \\ -ik_\nu C(k^2) & 0 & k^2F(k^2) & & \\ \hline & 0 & & 0 & ik^2\gamma(k^2) \\ & & & -ik^2\gamma(k^2) & 0 \end{array}\right). \tag{4·7}$$

We have omitted here the parts containing ψ which is decoupled from others. Further we define

$$\text{F.T.}\left.\frac{\delta^2\Gamma}{\delta K_\mu{}^a\delta c^b}\right|_0=-\delta^{ab}ik_\mu\omega(k^2), \tag{4·8a}$$

$$\text{F.T.}\left.\frac{\delta^2\Gamma}{\delta K_\chi{}^a\delta c^b}\right|_0=\delta^{ab}\zeta(k^2), \tag{4·8b}$$

where F.T. means Fourier transforms. The identity

$$\partial_\mu{}^x(\delta^2\Gamma/\delta K_\mu{}^a(x)\delta c^b(y)|_0)=i\delta^2\Gamma/\delta\bar{c}^a(x)\delta c^b(y)|_0, \tag{4·9}$$

which follows from (4·4c), indicates

$$\omega(k^2)=\gamma(k^2). \tag{4·10}$$

Equation (4·4a) with operation $\delta/\delta c|_{\substack{c=\bar{c}=0\\B=\psi=0}}$ is

$$\left.\left(\frac{\delta\Gamma}{\delta A_a{}^\mu}\frac{\delta^2\Gamma}{\delta K_\mu{}^a\delta c^b}+\frac{\delta\Gamma}{\delta\chi_a}\frac{\delta^2\Gamma}{\delta K_\chi{}^a\delta c^b}\right)\right|_{\substack{c=\bar{c}=0\\B=\psi=0}}=0. \tag{4·11}$$

We obtain the following two equations, differentiating (4·11) with respect to $A_\mu{}^a$ and χ^a:

$$B(k^2)\omega(k^2)=C(k^2)\zeta(k^2), \tag{4·12a}$$

$$C(k^2)\omega(k^2)=F(k^2)\zeta(k^2). \tag{4·12b}$$

And hence

$$B(k^2)F(k^2)=C^2(k^2). \tag{4·13}$$

Inverting the matrix $\Gamma^{(2)}$, (4·7), we obtain the propagators:

$$i^{-1}\text{F.T.}\langle 0|T(\Phi_i{}^a\Phi_j{}^b)|0\rangle=(\Gamma^{(2)})^{-1\,ab}_{ij}$$

$$
=\delta^{ab}\times
\begin{array}{c}
 \\ A_\mu \\ B \\ \chi \\ c \\ \bar{c}
\end{array}
\left(\begin{array}{ccc:cc}
A_\nu & B & \chi & c & \bar{c} \\
\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)A^{-1}-\alpha_0\frac{k_\mu k_\nu}{k^4} & i\frac{k_\mu}{k^2} & i\alpha_0 k_\mu\frac{C}{k^4F} & & \\
-ik_\nu/k^2 & 0 & -C/k^2F & & 0 \\
-i\alpha_0 k_\nu\frac{C}{k^4F} & -\frac{C}{k^2F} & \frac{1}{k^2F}-\frac{\alpha_0 B}{k^4F} & & \\
\hdashline
 & & & 0 & \frac{i}{k^2\gamma} \\
 & 0 & & & \\
 & & & \frac{-i}{k^2\gamma} & 0
\end{array}\right),
\tag{4·14}
$$

where use has been made of the W.T. relation (4·13), and $k^4\equiv(k^2)^2$.

From (4·14), one can deduce the vacuum expectation values of commutators:

$$
\begin{aligned}
\langle 0|[A_\mu{}^a(x), A_\nu{}^b(y)]|0\rangle &= \delta^{ab}\Big[-iZ_3(g_{\mu\nu}+m^{-2}\partial_\mu\partial_\nu)\Delta(x-y;m^2) \\
&\quad + iL\partial_\mu\partial_\nu D(x-y) - i\alpha_0\partial_\mu\partial_\nu E(x-y) \\
&\quad - i\int_{+0}^\infty ds\sigma(s)\,(g_{\mu\nu}+s^{-1}\partial_\mu\partial_\nu)\Delta(x-y;s)\Big], \\
\langle 0|[A_\mu{}^a(x), B^b(y)]|0\rangle &= -i\delta^{ab}\partial_\mu D(x-y), \\
\langle 0|[B^a(x), \chi^b(y)]|0\rangle &= \delta^{ab}\Big[-iM_1 D(x-y) - i\int_{+0}^\infty ds\sigma_{B\chi}(s)\Delta(x-y;s)\Big], \\
\langle 0|[A_\mu{}^a(x), \chi^b(y)]|0\rangle &= \delta^{ab}\alpha_0\partial_\mu\Big[-iM_2 D(x-y) + iM_1 E(x-y) \\
&\quad - i\int_{+0}^\infty ds\sigma_{A\chi}(s)\Delta(x-y;s)\Big], \\
\langle 0|[\chi^a(x), \chi^b(y)]|0\rangle &= \delta^{ab}\Big[(Z_\chi-\alpha_0 M_3)iD(x-y) + i\alpha_0 M_1{}^2 E(x-y) \\
&\quad + i\int_{+0}^\infty ds\sigma_{\chi\chi}(s)\Delta(x-y;s)\Big], \\
\langle 0|[B^a(x), B^b(y)]|0\rangle &= 0, \\
\langle 0|\{c^a(x), \bar{c}^b(y)\}|0\rangle &= \delta^{ab}\Big[-\tilde{Z}_3 D(x-y) - \int_{+0}^\infty ds\tilde{\sigma}(s)\Delta(x-y;s)\Big],
\end{aligned}
\tag{4·15a}
$$

where use has been made of the invariant dipole function $E(x)$, some properties of which are explained in Appendix D:

$$E(x) \equiv -(\partial/\partial m^2)\Delta(x;m^2)|_{m^2=0}, \quad \Box E(x) = D(x).$$

In addition, we can write

$$\langle 0|[\psi(x), \psi(y)]|0\rangle = iZ_\psi \Delta(x-y; m_\psi^2) + i\int ds\sigma_{\psi\psi}(s)\Delta(x-y;s), \tag{4·15b}$$

and all the vacuum expectation values of other commutators than (4·15) vanish owing to the global $SU(2)$ symmetry and/or the FP ghost number conservation. In (4·15), the vector mass m is defined by the pole position of $A^{-1}(s)$, i.e., $A(m^2)=0$, and the various quantities are defined as follows:*)

$$\begin{aligned}
&Z_3^{-1} \equiv -(d/ds)A(s)|_{s=m^2}, \quad \tilde{Z}_3^{-1} \equiv \gamma(0), \quad Z_\chi^{-1} \equiv F(0),\\
&L^{-1} \equiv A(0), \quad M_1 \equiv C(0)/F(0),\\
&M_2 \equiv (d/ds)\,\mathrm{Re}(C(s)/F(s))|_{s=0}, \quad M_3 \equiv (d/ds)\,\mathrm{Re}(B(s)/F(s))|_{s=0},\\
&\sigma(s) \equiv \pi^{-1}\,\mathrm{Im}(A^{-1}(s)) - Z_3\delta(s-m^2),\\
&\sigma_{B\chi}(s) \equiv -s\sigma_{A\chi}(s) \equiv -(\pi s)^{-1}\,\mathrm{Im}(C(s)/F(s)),\\
&\sigma_{\chi\chi}(s) \equiv -(\pi s)^{-1}\,\mathrm{Im}(F^{-1}(s)) + \alpha_0(\pi s^2)^{-1}\,\mathrm{Im}(B(s)/F(s)),\\
&\tilde{\sigma}(s) \equiv -(\pi s)^{-1}\,\mathrm{Im}(\gamma^{-1}(s)).
\end{aligned} \tag{4·16}$$

We assume that the LSZ asymptotic conditions hold, and hence the asymptotic fields are defined by the weak limit:**),***)

$$\begin{aligned}
&A_\mu(x) \xrightarrow[x_0\to\mp\infty]{} Z_3^{1/2}A_\mu^{\mathrm{as}}(x), \quad B(x) \to Z_B^{1/2}B^{\mathrm{as}}(x),\\
&\chi(x) \to Z_\chi^{1/2}\chi^{\mathrm{as}}(x), \quad \psi(x) \to Z_\psi^{1/2}\psi^{\mathrm{as}}(x),\\
&c(x) \to \tilde{Z}_3^{1/2}c^{\mathrm{as}}(x), \quad \bar{c}(x) \to \tilde{Z}_3^{1/2}\bar{c}^{\mathrm{as}}(x),
\end{aligned} \tag{4·17}$$

where 'as' stands for 'in' or 'out' and the renormalization constant Z_B is taken as $Z_B \equiv L^{-1} = A(0)$ for convenience. These asymptotic fields are, of course, supposed to have their supports in time-like and/or light-like regions in the momentum space, and hence their (anti-) commutation relations should be c-numbers according to the Greenberg-Robinson theorem [see Appendix C]. Thus the discrete spectrum parts of (4·15) determine their (anti-) commutation relations as follows:

$$\begin{aligned}
[A_\mu^{\mathrm{as}}(x), A_\nu^{\mathrm{as}}(y)] = &-i(g_{\mu\nu}+m^{-2}\partial_\mu\partial_\nu)\Delta(x-y;m^2)\\
&+iK\partial_\mu\partial_\nu D(x-y) - i\alpha\partial_\mu\partial_\nu E(x-y),
\end{aligned} \tag{4·18a}$$

---

*) We notice that $Z_3 = 1 - \int_{+0}^\infty ds\sigma(s)$ and $L = Z_3/m^2 + \int_{+0}^\infty ds\sigma(s)/s$.

**) Henceforth, we will omit the group index a of the fields A_μ^a, B^a, χ^a, c^a and $\bar{c}^a$.

***) We assume here that all Z-factors are positive. This is always true in the perturbation theory.

$$[A_\mu^{\rm as}(x), B^{\rm as}(y)] = -iK^{1/2}\partial_\mu D(x-y), \tag{4·18b}$$

$$[B^{\rm as}(x), B^{\rm as}(y)] = 0, \tag{4·18c}$$

$$[B^{\rm as}(x), \chi^{\rm as}(y)] = -iD(x-y), \tag{4·18d}$$

$$[A_\mu^{\rm as}(x), \chi^{\rm as}(y)] = -i\alpha N\partial_\mu D(x-y) + i(\alpha K^{-1/2})\partial_\mu E(x-y), \tag{4·18e}$$

$$[\chi^{\rm as}(x), \chi^{\rm as}(y)] = (1-2\alpha NK^{-1/2})iD(x-y) + i\alpha K^{-1}E(x-y), \tag{4·18f}$$

$$\{c^{\rm as}(x), \bar{c}^{\rm as}(y)\} = -D(x-y), \tag{4·18g}$$

$$[\phi^{\rm as}(x), \phi^{\rm as}(y)] = i\Delta(x-y; m_\phi^2) \tag{4·18h}$$

and all the commutators of other combinations vanish. Here we have defined

$$K \equiv L/Z_3 = (Z_3 Z_B)^{-1}, \quad \alpha \equiv \alpha_0/Z_3, \tag{4·19a}$$

$$N \equiv (Z_3/Z_\chi)^{1/2} M_2 = (K^{1/2} Z_3/Z_\chi) M_3/2, \tag{4·19b}$$

and have used the W.T. relation (4·13) for $k^2=0$, $B(0)F(0)=C^2(0)$, and the equalities

$$A(0) = B(0), \tag{4·20a}$$

$$\frac{d}{ds}\left(\mathrm{Re}\frac{B(s)}{F(s)}\right)\bigg|_{s=0} = 2\frac{C(0)}{F(0)}\frac{d}{ds}\left(\mathrm{Re}\ \frac{C(s)}{F(s)}\right)\bigg|_{s=0}. \tag{4·20b}$$

The equality (4·20a) is implied by the singularity-free assumption of $\Gamma^{(2)}_{A_\mu A_\nu}(k)$ at $k^2=0$. Equation (4·20b) can be derived, by the help of $B(k^2)/F(k^2) = (C(k^2)/F(k^2))^2$ from (4·13), on the reasonable assumption $\mathrm{Im}(C(s)/F(s))|_{s=0} = 0$. In fact these two assumptions are satisfied in any order of the perturbation theory. The last equality in (4·19b) is a consequence of (4·20b).

The present assumption of asymptotic completeness means that the asymptotic fields $A_\mu^{\rm as}$, $\phi^{\rm as}$, $B^{\rm as}$, $\chi^{\rm as}$, $c^{\rm as}$ and $\bar{c}^{\rm as}$ are complete without bound states. Therefore, we can deduce from the commutation relations (4·18) the following equations of motion for the asymptotic fields by the help of their irreducibility:

$$\Box B^{\rm as} = \Box c^{\rm as} = \Box \bar{c}^{\rm as} = (\Box + m_\phi^2)\phi = 0, \tag{4·21}$$

$$\Box \chi^{\rm as} = -\alpha K^{-1} B^{\rm as}. \tag{4·22}$$

This equation (4·22) indicates that $\chi^{\rm as}$ becomes a dipole ghost field except for the Landau gauge case ($\alpha=0$). The massive Proca field, say $U_\mu^{\rm as}$, contained in $A_\mu^{\rm as}$ can be separated from the unphysical modes as follows:[4)]

$$U_\mu^{\rm as} \equiv A_\mu^{\rm as} - (\sqrt{K} - \alpha N)\partial_\mu B^{\rm as} - \sqrt{K}\partial_\mu \chi^{\rm as}. \tag{4·23}$$

Then, in fact, we can easily convince ourselves that*)

*) In deriving (4·24b), we need the relation $L = Z_z/M_1^2$, which is guaranteed by the W.T. relation (4·13) and the equality (4·20a). Note also that Eq. (4·19b) is indispensable for the consistency of (4·24b) with (4·23), (4·18e) and (4·18f).

$$(\Box+m^2)\,U_\mu{}^{\rm as}(x)=0,\ \partial^\mu U_\mu{}^{\rm as}(x)=0\,, \tag{4·24a}$$

$$[U_\mu{}^{\rm as}(x),U_\nu{}^{\rm as}(y)]=-i(g_{\mu\nu}+m^{-2}\partial_\mu\partial_\nu)\Delta(x-y;m^2)\,, \tag{4·24b}$$

and the commutators of $U_\mu{}^{\rm as}$ with other fields $B^{\rm as}$, $\chi^{\rm as}$, $c^{\rm as}$, $\bar{c}^{\rm as}$ and $\psi^{\rm as}$ all vanish. Thus $A_\mu{}^{\rm as}$ satisfies

$$(\Box+m^2)\,A_\mu{}^{\rm as}=[(\sqrt{K}-\alpha N)\,m^2-\alpha\sqrt{K^{-1}}]\,\partial_\mu B^{\rm as}+\sqrt{K}m^2\partial_\mu\chi^{\rm as}\,,$$
$$\partial^\mu A_\mu{}^{\rm as}+\alpha\sqrt{K^{-1}}B^{\rm as}=0\,. \tag{4·25}$$

Now, all this information enables us to construct the Fock space of asymptotic fields, which is identified with the total state vector space $\mathcal{V}$ on the assumption of asymptotic completeness. The creation and annihilation operators for the Proca field $U_\mu{}^{\rm as}$ and the (real) Higgs scalar $\psi^{\rm as}$ are defined as usual by using suitable complete sets of wave packets, and are denoted as $U_\alpha{}^\dagger(U_\alpha)$ and $\psi_\rho{}^\dagger(\psi_\rho)$, respectively. Since the Goldstone boson $\chi^{\rm as}$ is a dipole ghost field now as is seen in (4·22), we need some manipulation to treat it. Let us define $\tilde{\chi}^{\rm as}$ field as follows:

$$\tilde{\chi}^{\rm as}=\chi^{\rm as}+\alpha K^{-1}\mathcal{D}^{(1/2)}B^{\rm as}\,, \tag{4·26}$$ *)

where the operator $\mathcal{D}^{(1/2)}$ is defined by (D·20) or (D·3) in Appendix D. By the equality (D·4), $\Box\mathcal{D}^{(1/2)}f(x)=f(x)$ if $\Box f(x)=0$, and (4·22), the $\tilde{\chi}^{\rm as}$ also becomes a simple pole field:

$$\Box\tilde{\chi}^{\rm as}=0\,. \tag{4·27}$$

Since all the four fields $B^{\rm as}$, $\tilde{\chi}^{\rm as}$, $c^{\rm as}$ and $\bar{c}^{\rm as}$ become massless simple pole fields now, we can define their creation and annihilation operators in the usual manner by using a common wave packet system $\{g_k(x)\}$ [see Appendix D for the property of $\{g_k(x)\}$]; especially, the annihilation operators for $\tilde{\chi}^{\rm as}$ field are defined by

$$\chi_k{}^{\rm as}\equiv(g_k,\tilde{\chi}^{\rm as})\equiv\int d^3x g_k{}^*(x)\overleftrightarrow{\partial_0}\tilde{\chi}^{\rm as}(x)\,. \tag{4·28}$$

Then, one can easily show by the help of (D·22) and (4·26) that

$$\chi_k{}^{\rm as}=(g_k,\chi^{\rm as})-\alpha K^{-1}(h_k,B^{\rm as})\,, \tag{4·29a}$$

$$\chi^{\rm as}(x)=\sum_k[(\chi_k{}^{\rm as}g_k(x)-\alpha K^{-1}B_k{}^{\rm as}h_k(x))+\text{h.c.}]\,, \tag{4·29b}$$

where the dipole wave packet system $\{h_k\}$ is defined in (D·20) as $h_k=\mathcal{D}^{(1/2)}g_k$.

The commutation relations (4·18f~h) and (4·24b) which we have found

*) Of course, this $\tilde{\chi}^{\rm as}$ is non-covariant and non-local (with respect to the asymptotic fields), which is, however, harmless.

above lead to the following commutation relations of the creation and annihilation operators:

$$\begin{array}{c} \\ U_\alpha \\ \psi_\rho \\ \chi_k \\ B_k \\ c_k \\ \bar{c}_k \end{array}
\begin{array}{c}
\begin{array}{cc|cc|cc} U_\beta^\dagger & \psi_\sigma^\dagger & \chi_l^\dagger & B_l^\dagger & c_l^\dagger & \bar{c}_l^\dagger \end{array} \\
\left(\begin{array}{cc|cc|cc}
\delta_{\alpha\beta} & 0 & & & & \\
0 & \delta_{\rho\sigma} & & 0 & & \\ \hline
 & & (1-2\alpha NK^{-1/2})\delta_{kl} & -\delta_{kl} & & 0 \\
 & 0 & -\delta_{kl} & 0 & & \\ \hline
 & & & & 0 & i\delta_{kl} \\
 & & & 0 & -i\delta_{kl} & 0
\end{array}\right)
\end{array} . \tag{4·30}$$

In deriving (4·30), we have utilized Eq. (D·22). These commutation relations completely determine the metric structure of the total state vector space $\mathcal{V}$. We notice here that the modes U_α and ψ_ρ have positive norm and the other four $(\chi_k^{\rm as}, B_k^{\rm as}, c_k^{\rm as}, \bar{c}_k^{\rm as})$ have indefinite metric which is exactly the same form as is found in (3·16) generally for the quartet $(\chi_k, \beta_k, \gamma_k, \bar{\gamma}_k)$. We will in fact see that U_α and ψ_ρ are BRS-singlets and $(\chi_k^{\rm as}, B_k^{\rm as}, c_k^{\rm as}, \bar{c}_k^{\rm as})$ really belong to quartet representation.

In order to show this, we determine the BRS transformation of those asymptotic fields. As is explained in detail in Appendix C, any well-defined symmetry transformation on Heisenberg fields Φ_i, denoted by

$$[i\delta\theta Q, \Phi_i(x)] = \delta\Phi_i(x), \tag{4·31}$$

induces a *linear transformation* on the asymptotic fields. This asymptotic transformation is determined by

$$[i\delta\theta Q, \Phi_i^{\rm as}(x)] = (\delta\Phi_i)^{\rm as}(x), \tag{4·32}$$

where the asymptotic form $(\delta\Phi_i)^{\rm as}$ can be read by inspecting the discrete spectrum parts of 2-point functions:

$$\begin{aligned} &\text{Discrete spectrum part } \langle 0|T\delta\Phi_i(x)\Phi_j(y)|0\rangle \\ &= \langle 0|T(\delta\Phi_i)^{\rm as}(x)\Phi_j^{\rm as}(y)|0\rangle . \end{aligned} \tag{4·33}$$

In our case of BRS transformation, the original Heisenberg field transformations are given by (2·16) and (4·3). We easily see that

$$\begin{aligned} \delta A_\mu(x) &= [i\lambda Q_B, A_\mu^a(x)] = \lambda D_\mu c(x) \\ &\xrightarrow[|x_0|\to+\infty]{} (\delta A_\mu)^{\rm as}(x) = \lambda\omega(0)\partial_\mu \tilde{Z}_3^{1/2} c^{\rm as}(x), \end{aligned} \tag{4·34a}$$

$$\delta\psi(x) = -\lambda(g/2)\chi\cdot c \longrightarrow (\delta\psi)^{\rm as}(x) = 0, \tag{4·34b}$$

$$\delta\chi(x) = \lambda(g/2)[(v+\psi)c + \chi\times c] \to (\delta\chi)^{\rm as}(x) = \lambda\zeta(0)\tilde{Z}_3^{1/2}c^{\rm as}(x), \tag{4·34c}$$

$$\delta B(x)=0 \longrightarrow (\delta B)^{\mathrm{as}}(x)=0\,, \tag{4·34d}$$

$$\delta c(x)=-\lambda(g/2)\,(c\times c)\rightarrow(\delta c)^{\mathrm{as}}(x)=0\,, \tag{4·34e}$$

$$\delta\bar{c}(x)=i\lambda B(x) \longrightarrow (\delta\bar{c})^{\mathrm{as}}(x)=i\lambda Z_B{}^{1/2}B^{\mathrm{as}}(x)\,. \tag{4·34f}$$

Equations (4·34a) and (4·34c) are due to the fact that there are no (bound-state) single-particle poles in the channels δA_μ and $\delta\chi$ other than those caused by "elementary" FP ghost c^{as} by the present assumption, and their coefficients $\omega(0)$ and $\zeta(0)$ come from

$$\langle 0|T\delta A_\mu\,\bar{c}|0\rangle=\frac{\delta^2\Gamma}{\delta K_\mu\delta c}\langle 0|Tc\bar{c}|0\rangle\,,$$

$$\langle 0|T\delta\chi\,\bar{c}|0\rangle=\frac{\delta^2\Gamma}{\delta K_\chi\delta c}\langle 0|Tc\bar{c}|0\rangle$$

with the definitions (4·8). Equations (4·34b) and (4·34e) follow from the fact that there exist no bound state poles in the composite channels $\chi\cdot c$ and $c\times c$. Thus from the general formula (4·32), Eqs. (4·34) and the definitions of asymptotic fields (4·17), we obtain

$$[Q_B{}^r, U_\mu{}^{\mathrm{as}}(x)]=[Q_B{}^r,\psi^{\mathrm{as}}(x)]=0\,, \tag{4·35}$$

$$[Q_B{}^r,\chi^{\mathrm{as}}(x)]=-ic^{\mathrm{as}}(x)\,, \tag{4·36a}$$

$$\{Q_B{}^r,\bar{c}^{\mathrm{as}}(x)\}=B^{\mathrm{as}}(x)\,, \tag{4·36b}$$

$$[Q_B{}^r, B^{\mathrm{as}}(x)]=\{Q_B{}^r, c^{\mathrm{as}}(x)\}=0\,, \tag{4·36c}$$

where use has been made of the 'renormalized' BRS charge $Q_B{}^r=(\tilde{Z}_3/Z_B)^{1/2}Q_B$ defined in (2·22). In deriving (4·35) and (4·36), we have used the definition of $U_\mu{}^{\mathrm{as}}$, (4·23), and the relations

$$\tilde{Z}_3\zeta(0)/(Z_BZ_\chi)^{1/2}=\tilde{Z}_3\omega(0)/(Z_BZ_3K)^{1/2}=1\,, \tag{4·37}$$

which are assured by the W.T. relations (4·10) and (4·12) together with the definitions $Z_B=A(0)$, $Z_\chi{}^{-1}=F(0)$, $\tilde{Z}_3{}^{-1}=\gamma(0)$ in (4·16) and $K=(Z_BZ_3)^{-1}$ in (4·19a) and the relation $A(0)=B(0)$ stated in (4·20a).

As is expected, (4·35) indicates that the Proca field $U_\mu{}^{\mathrm{as}}$ and the real Higgs scalar field ψ are physical particles of BRS-singlet representations having positive norm by (4·30). The BRS transformation property (4·36) is nothing but that of quartet, (3·15), and hence the Goldstone boson χ^{as}, the scalar B^{as}, and the FP ghosts c^{as} and $\bar{c}^{\mathrm{as}}$ (for each of omitted group indices $a=1, 2, 3$) are found to belong to quartet representation. Thus we have finished the proof of the unitarity of physical S-matrix defined on $H_{\mathrm{phys}}\equiv\mathcal{V}_{\mathrm{phys}}/\mathcal{V}_0$ which is isomorphic to the Hilbert space $\mathcal{H}_{\mathrm{phys}}$ spanned solely by the physical particles U_μ and ψ. Just similarly to Abelian Higgs model,[4] the Higgs phenomenon in the present non-Abelian case also is understood without any inconsistency with the Goldstone theorem: The Goldstone bosons surely exist but become undetectable unphysical quartet members, while the gauge bosons acquire non-vanishing mass and are physical.

We add two comments here: (i) By the help of the present assumption of asymptotic completeness, the BRS charge Q_B is expressed in terms of asymptotic fields as

$$Q_B{}^r = \int d^3x : B^{\text{as}}(x) \cdot \overleftrightarrow{\partial}_0 c^{\text{as}}(x) : = i \sum_k (c_k{}^{\text{as}\dagger} \cdot B_k{}^{\text{as}} - B_k{}^{\text{as}\dagger} \cdot c_k{}^{\text{as}}). \qquad (4\cdot 38)$$

This form in fact reproduces the asymptotic transformations (4·35) and (4·36) and even the original transformations on Heisenberg fields as is proved in Appendix C generally. It is interesting to note that this expression (4·38) for $Q_B{}^r$ has just the *same form* as that of Abelian case (2·30). This clearly indicates that our present formulation provides a very natural extension of the Gupta-Bleuler (or the Nakanishi-Lautrup) formalism for the Abelian case. (ii) By comparing (4·23) with (3·21a), the χ-field of the "elementary" quartet found by the general discussions in the preceding chapter is given in the present model explicitly as

$$\chi \text{ in } (3\cdot 21\text{a}) = Z_3{}^{1/2}[\sqrt{K}\chi^{\text{as}} + (\sqrt{K} - \alpha N) B^{\text{as}}], \qquad (4\cdot 39)$$

where the χ^{as} representing the present Goldstone mode should not be confused with χ of (3·21a).

§ 4.2. Pure Yang-Mills Theory without Spontaneous Symmetry Breaking

We analyze here the pure Yang-Mills (YM) theory based on a simple group G suffering no spontaneous symmetry breaking, following Ref. 5), but in a little simpler manner. Of course, the massless YM theory suffers from serious infrared divergences which may have deep relevance to the confinement mechanism of quarks as is currently expected. We, however, disregard the infrared problem here for simplicity and make a formal analysis of asymptotic fields.

The Lagrangian density is given by (2·7) with $\mathcal{L}_{\text{matter}}$ discarded. The propagators are given as the same form as (4·14) in the preceding model, where the group index a should be understood to run over $a = 1, 2, \cdots, n = \dim(G)$, and the Goldstone field χ is simply discarded. The function $A^{-1}(k^2)$ in (4·14) now has a massless pole and hence we rewrite it as

$$1/A(k^2) = 1/k^2 \tilde{A}(k^2). \qquad (4\cdot 40)$$

By performing the same procedures as in the preceding section, we find, for the asymptotic fields

$$A_\mu(x) \xrightarrow[|x_0|\to\infty]{} Z_3^{1/2} A_\mu^{\rm as}(x), \quad B(x) \to Z_3^{-1/2} B^{\rm as}(x),$$
$$c(x) \to \tilde{Z}_3^{1/2} c^{\rm as}(x), \quad \bar{c}(x) \to \tilde{Z}_3^{1/2} \bar{c}^{\rm as}(x), \tag{4·41}$$

the following commutation relations:

$$[A_\mu^{\rm as}(x), A_\nu^{\rm as}(y)] = -i(g_{\mu\nu} - K\partial_\mu\partial_\nu) D(x-y) + i(1-\alpha)\partial_\mu\partial_\nu E(x-y), \tag{4·42a}$$
$$[A_\mu^{\rm as}(x), B^{\rm as}(y)] = -i\partial_\mu D(x-y), \tag{4·42b}$$
$$[B^{\rm as}(x), B^{\rm as}(y)] = 0, \tag{4·42c}$$
$$\{c^{\rm as}(x), \bar{c}^{\rm as}(y)\} = -D(x-y). \tag{4·42d}$$

Here we have defined

$$Z_3^{-1} \equiv \tilde{A}(0), \quad K \equiv \tilde{A}(0)\,(d/ds)\,{\rm Re}\,(1/\tilde{A}(s))\,|_{s=0} \tag{4·43}$$

and other quantities in the same way as before: $\tilde{Z}_3^{-1} \equiv \gamma(0)$ and $\alpha \equiv \alpha_0/Z_3$. Note that Z_B is taken in the present case as $Z_B \equiv Z_3^{-1}$. Equations (4·42) together with the asymptotic completeness assumption lead to the equations of motion:

$$\Box B^{\rm as} = \Box c^{\rm as} = \Box \bar{c}^{\rm as} = 0, \tag{4·44a}$$
$$\Box A_\mu^{\rm as} = (1-\alpha)\partial_\mu B^{\rm as}, \tag{4·44b}$$
$$\partial^\mu A_\mu^{\rm as} + \alpha B^{\rm as} = 0. \tag{4·44c}$$

We can now construct the asymptotic Fock space. Noting that the vector field $A_\mu^{\rm as}$ is dipole field generally except for Feynman gauge ($\alpha=1$), we extract a (non-covariant and non-local) simple pole field $\tilde{A}_\mu^{\rm as}$ from $A_\mu^{\rm as}$ in a way similar to (4·26):

$$\tilde{A}_\mu^{\rm as}(x) \equiv A_\mu^{\rm as}(x) - (1-\alpha)\partial_\mu \mathcal{D}^{(1/2)} B^{\rm as}(x). \tag{4·45}$$

Then, (4·44b) and (4·44c) become, by help of $\Box \mathcal{D}^{(1/2)} B = B$ [(D·4)],

$$\Box \tilde{A}_\mu^{\rm as} = 0, \tag{4·46a}$$
$$\partial^\mu \tilde{A}_\mu^{\rm as} + B^{\rm as} = 0. \tag{4·46b}$$

Now we define the creation and annihilation operators of vector field $\tilde{A}_\mu^{\rm as}$ by the wave packet system $\{f_{k,\sigma}{}^\mu\}$ and of scalar fields $B^{\rm as}$, $c^{\rm as}$ and $\bar{c}^{\rm as}$ by a common system $\{g_k\}$: For the annihilation (creation) operators $A_{k,\sigma}^{\rm as}$ $(A_{k,\sigma}^{\rm as\dagger})$,

$$A_{k,\sigma}^{\rm as} \equiv (f_{k,\sigma}{}^\mu, \tilde{A}_\mu^{\rm as}) = i\int d^3x f_{k,\sigma}{}^{\mu *}(x)\,\overleftrightarrow{\partial}_0 \tilde{A}_\mu^{\rm as}(x), \tag{4·47a}$$
$$\tilde{A}_\mu^{\rm as}(x) = \sum_{k,\sigma} (A_{k,\sigma}^{\rm as} f_{k,\mu}^{\sigma}(x) + {\rm h.c.}), \tag{4·47b}$$

and for B_k, c_k and $\bar{c}_k$,

$$B_k^{\text{as}} \equiv (g_k, B^{\text{as}}),\quad B^{\text{as}}(x) = \sum_k (B_k^{\text{as}} g_k(x) + \text{h.c.}), \tag{4·48}$$

and so on. These two wave packet systems $\{f_{k,\sigma}{}^{\mu}\}$ and $\{g_k\}$ are constructed in Appendix D so that their mutual relations simplify the formulas below. These modes $A^{\text{as}}_{k,\sigma}(\sigma=1, 2, L, S)$ and B_k^{as} are not all mutually independent, as is evident from (4·46b) which in fact says that

$$B_k^{\text{as}} = A_k^{\text{as},S} \equiv \sum_\sigma \tilde{\eta}^{S\sigma} A^{\text{as}}_{k,\sigma} (= A^{\text{as}}_{k,L}), \tag{4·49}$$ *)

where use has been made of the definition of $\tilde{\eta}$ [(D·15)], (D·19b) and (D·19c). Thus the scalar polarization modes $A_k^{\text{as},S}$ are nothing but B_k modes. So A_μ^{as} and B^{as} are fully described in terms of the transverse modes $A^{\text{as}}_{k,\sigma=1,2}$ $(= -A_k^{\text{as},\sigma=1,2})$,*) the longitudinal modes $A_k^{\text{as},L} (= A^{\text{as}}_{k,S})$ and B_k alone. Indeed, we find from (4·45), (4·47)∼(4·49) that

$$A_\mu^{\text{as}}(x) = \{ \sum_{k;\sigma=1,2} A^{\text{as}}_{k,\sigma} f_k{}^{\sigma}{}_{,\mu}(x) + \sum_k A_k^{\text{as},L} \partial_\mu g_k(x)$$
$$+ \sum_k B_k [f_{k,S,\mu}(x) + (1-\alpha)\partial_\mu h_k(x)]\} + \text{h.c.}, \tag{4·50}$$

where the dipole wave packets $h_k(x)$ are defined in (D·20): $\mathscr{D}^{(1/2)} g_k(x) \equiv h_k(x)$. Noting that the commutator (4·42a) is rewritten by help of the identity (D·8) as

$$[\tilde{A}_\mu^{\text{as}}(x), \tilde{A}_\nu^{\text{as}}(y)] = -i(g_{\mu\nu} - K\partial_\mu\partial_\nu) D(x-y), \tag{4·51}$$

we find from (4·51), (4·42b∼d) the following commutation relations for the creation and annihilation operators:

$$\begin{array}{c|c} & A^\dagger_{l,\tau=1,2} \quad A_l{}'^{L\dagger} \quad B_l^\dagger \quad c_l^\dagger \quad \bar{c}_l^\dagger \\ \begin{array}{c} A_{k,\sigma=1,2} \\ A_k{}'^{L} \\ B_k \\ c_k \\ \bar{c}_k \end{array} & \left(\begin{array}{c:cc:cc} \delta_{kl}\delta_{\sigma\tau} & & 0 & & \\ \hdashline & -K\delta_{kl} & -\delta_{kl} & & \\ & -\delta_{kl} & 0 & \multicolumn{2}{c}{0} \\ 0 & & & & \\ \hdashline & \multicolumn{2}{c:}{0} & 0 & i\delta_{kl} \\ & & & -i\delta_{kl} & 0 \end{array} \right) \end{array}, \tag{4·52}$$

where (D·18) has been utilized.

Beside the LSZ reduction formula[6] for the transverse modes of the usual form,

$$\langle A_{k,\sigma=1,2}\alpha \text{ out}|T(\cdots)|\beta \text{ in}\rangle$$
$$= \langle \alpha \text{ out}|T(\cdots) A^{\text{in}}_{k,\sigma=1,2}|\beta \text{ in}\rangle +$$

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*) Note the positions of the suffices $\sigma=1, 2, L$ and S; e.g., $A^{\text{as}}_{k,L} = A_k^{\text{as},S} \neq A^{\text{as}}_{k,S} = A_k^{\text{as},L}$.

$$+i\int d^4x Z_3^{-1/2} f_{k,\sigma}^{\mu *}(x)\,\square^x \langle \alpha\ \mathrm{out}|T(A_\mu(x)\cdots)|\beta\ \mathrm{in}\rangle\,, \quad (4\cdot 53\mathrm{a})$$

we should note the following formula for the longitudinal modes:[7),8)]

$$\begin{aligned}&\langle A_k^{\prime L}\alpha\ \mathrm{out}|T(\cdots)|\beta\ \mathrm{in}\rangle\\ &\quad=\langle \alpha\ \mathrm{out}|T(\cdots)A_k^{\mathrm{in},L}|\beta\ \mathrm{in}\rangle\\ &\qquad+i\int d^4x Z_3^{-1/2} f_k^{L,\mu}(x)^*\square^x\langle \alpha\ \mathrm{out}|T(A_\mu(x)\cdots)|\beta\ \mathrm{in}\rangle\\ &\qquad-i(1-\alpha)\int d^4x Z_3^{+1/2}[f_k^{L,\mu}(x)^*\partial_\mu^x - h_k^*(x)\square^x]\\ &\qquad\quad\times\langle \alpha\ \mathrm{out}|T(B(x)\cdots)|\beta\ \mathrm{in}\rangle\,. \end{aligned} \quad (4\cdot 53\mathrm{b})$$

The BRS transformations of asymptotic fields are determined in exactly the same way as has been done in (4·34) in the preceding section. We find, by using the renormalized charge $Q_B{}^r=\tilde{Z}_3^{1/2}Z_B^{-1/2}Q_B=\tilde{Z}_3^{1/2}Z_3^{1/2}Q_B$,

$$\begin{aligned}&[Q_B{}^r, A_\mu^{\mathrm{as}}(x)]=-i\partial_\mu c^{\mathrm{as}}(x)\,,\\ &\{Q_B{}^r, \bar{c}^{\mathrm{as}}(x)\}=B^{\mathrm{as}}(x)\,,\\ &[Q_B{}^r, B^{\mathrm{as}}(x)]=\{Q_B{}^r, c^{\mathrm{as}}(x)\}=0\,. \end{aligned} \quad (4\cdot 54)$$

This leads to

$$[Q_B{}^r, A_{k,\sigma=1,2}^{\mathrm{as}}]=0\,, \quad (4\cdot 55\mathrm{a})$$

$$\begin{aligned}&[Q_B{}^r, A_k^{\mathrm{as},L}]=-ic_k^{\mathrm{as}}\,,\quad \{Q_B{}^r, \bar{c}_k^{\mathrm{as}}\}=B_k^{\mathrm{as}}\,,\\ &[Q_B{}^r, B_k^{\mathrm{as}}]=\{Q_B{}^r, c_k^{\mathrm{as}}\}=0\,. \end{aligned} \quad (4\cdot 55\mathrm{b})$$

Thus we see from the commutation relations (4·52) and the BRS transformation property that the transverse modes $A_{k,\sigma=1,2}^{\mathrm{as}}$ are really physical particles of BRS-singlets having positive norm and the longitudinal mode $A_k^{\mathrm{as},L}$ together with the scalar modes B_k and FP ghosts c_k and $\bar{c}_k$ belong to quartet representations. Hence also in this pure YM case, the physical S-matrix unitarity has been proved.

We should add a comment here. The above construction of physical transverse modes manifestly depends on the Lorentz frame to which we refer. So is the Hilbert space $\mathcal{H}_{\mathrm{phys}}$ spanned by the transverse modes alone. Our proof of unitarity of the physical S-matrix S_{phys} defined on $\overline{\mathcal{V}_{\mathrm{phys}}/\mathcal{V}_0}=H_{\mathrm{phys}}$, however, has a frame-independent meaning. Our physical subspace $\mathcal{V}_{\mathrm{phys}}$ is specified in a Lorentz invariant manner by scalar charge Q_B and hence its positive semi-definiteness of metric as well as the spaces $\mathcal{V}_{\mathrm{phys}}$, $\mathcal{V}_0$ and H_{phys} have Lorentz-invariant meanings, even when the proof is given by referring to a specific Lorentz frame.

§ 4.3. Quantum Theory of Gravity

The application of the present formalism to quantum gravity is essentially straightforward, although some complications occur in the kinematical calculations. Nakanishi initiated this task and has been researching the total structure of quantum gravity in detail in his succeeding papers.[9] Nishijima and Okawa[10] discussed the BRS transformation and charge in quantum gravity. Here we briefly summarize the results according to Ref. 11), only in which the properties of asymptotic fields are analysed in detail.

The Einstein Lagrangian density of the gravitational field is

$$\mathcal{L}_{\mathrm{E}}=\kappa^{-2}\sqrt{-g}\,R,\quad g=\det(g_{\mu\nu})\,. \tag{4·56}$$

Introducing the Goldberg variables

$$\tilde{g}^{\mu\nu}\equiv\sqrt{-g}\,g^{\mu\nu},\quad \tilde{g}_{\mu\nu}\equiv g_{\mu\nu}/\sqrt{-g};\quad \tilde{g}^{\mu\nu}\tilde{g}_{\nu\rho}=\delta_\rho{}^\mu, \tag{4·57}$$

we can rewrite (4·56) as follows as usual:

$$\mathcal{L}_{\mathrm{E}}=\mathcal{L}_{\mathrm{E}}'+\partial_\mu\mathcal{D}^\mu, \tag{4·58}$$

$$\begin{aligned}\mathcal{L}_{\mathrm{E}}'&\equiv\kappa^{-2}\sqrt{-g}\,g^{\mu\nu}(\Gamma^\rho_{\mu\nu}\Gamma^\lambda_{\rho\lambda}-\Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\nu\lambda})\\ &=(2\kappa)^{-2}(\tilde{g}^{\rho\sigma}\tilde{g}_{\lambda\mu}\tilde{g}_{\kappa\nu}-2\delta_\kappa{}^\sigma\delta_\lambda{}^\rho\tilde{g}_{\mu\nu}-\tfrac{1}{2}\,\tilde{g}^{\rho\sigma}\tilde{g}_{\mu\kappa}\tilde{g}_{\lambda\nu})\,\partial_\rho\tilde{g}^{\mu\kappa}\partial_\sigma\tilde{g}^{\nu\lambda},\end{aligned} \tag{4·59a}$$

$$\begin{aligned}\mathcal{D}^\mu&\equiv\kappa^{-2}\sqrt{-g}\,(g^{\mu\nu}\Gamma^\lambda_{\nu\lambda}-g^{\rho\sigma}\Gamma^\mu_{\rho\sigma})\\ &=\kappa^{-2}(\tfrac{1}{2}\,\tilde{g}^{\mu\nu}\tilde{g}_{\alpha\beta}\partial_\nu\tilde{g}^{\alpha\beta}+\partial_\nu\tilde{g}^{\mu\nu})\,.\end{aligned} \tag{4·59b}$$

In terms of $\tilde{g}^{\mu\nu}$, the (Lie derivative corresponding to) general coordinate transformation is written as

$$\delta\tilde{g}^{\mu\nu}(x)=\tilde{g}^{\mu\nu\prime}(x)-\tilde{g}^{\mu\nu}(x)=D^{\mu\nu}{}_\rho\delta x^\rho, \tag{4·60a}$$

$$D^{\mu\nu}{}_\rho\equiv\tilde{g}^{\mu\sigma}\delta_\rho{}^\nu\partial_\sigma+\tilde{g}^{\sigma\nu}\delta_\rho{}^\mu\partial_\sigma-\tilde{g}^{\mu\nu}\partial_\rho-(\partial_\rho\tilde{g}^{\mu\nu})\,. \tag{4·60b}$$

Then, as the Lagrangian density to be quantized, we adopt the following one:

$$\begin{aligned}&\mathcal{L}=\mathcal{L}_{\mathrm{E}}'+\mathcal{L}_{\mathrm{GF}}+\mathcal{L}_{\mathrm{FP}}\,,\\ &\mathcal{L}_{\mathrm{GF}}\equiv-(2\kappa)^{-1}\tilde{g}^{\mu\nu}(\partial_\mu B_\nu+\partial_\nu B_\mu)-(\alpha_0/2)\,\eta^{\mu\nu}B_\mu B_\nu\,,\\ &\mathcal{L}_{\mathrm{FP}}\equiv-(i/2)\,(\partial_\mu\bar{c}_\nu+\partial_\nu\bar{c}_\mu)\,D^{\mu\nu}{}_\rho c^\rho,\end{aligned} \tag{4·61}$$

where $\eta^{\mu\nu}$ is the Minkowski metric: $\eta^{\mu\nu}=\mathrm{diag.}\ (+1,-1,-1,-1)$. Due to the presence of $\mathcal{L}_{\mathrm{GF}}+\mathcal{L}_{\mathrm{FP}}$, the action $\int d^4x\mathcal{L}$ of the quantum system (4·61) is no longer invariant under the general coordinate transformation (4·60). Instead, this system (4·61) has the invariance under the (global) BRS transformation, which is defined by

$$\delta\tilde{g}^{\mu\nu}=\kappa\lambda D^{\mu\nu}{}_\rho c^\rho, \tag{4·62a}$$

$$\delta c^{\mu}=-\kappa\lambda c^{\lambda}\partial_{\lambda}c^{\mu}, \tag{4·62b}$$

$$\delta\bar{c}_{\mu}=i\lambda B_{\mu}, \tag{4·62c}$$

$$\delta B_{\mu}=0, \tag{4·62d}$$

and satisfies

$$\begin{aligned}&\delta\mathcal{L}_{\mathrm{E}}'=-\partial_{\mu}(\lambda\kappa c^{\mu}\mathcal{L}_{\mathrm{E}}+\delta\mathcal{D}^{\mu}),\\&\delta(\mathcal{L}_{\mathrm{GF}}+\mathcal{L}_{\mathrm{FP}})=0,\\&\delta\int d^4x\mathcal{L}=0.\end{aligned} \tag{4·63}$$

The Noether current $J_B{}^{\mu}$ corresponding to the BRS transformation (4·62) is[9)~11)]

$$\lambda J_B{}^{\mu}=\sum_{\Phi=\{\tilde{g},c,\bar{c}\}}\left(\delta\Phi\frac{\partial(\mathcal{L}-\partial_{\lambda}\mathcal{D}^{\lambda})}{\partial(\partial_{\mu}\Phi)}\right)+\kappa\lambda c^{\mu}\mathcal{L}_{\mathrm{E}}+\delta\mathcal{D}^{\mu}, \tag{4·64}$$

and the conserved charge $Q_B\equiv\int d^3xJ_B{}^0$ implements the transformation (4·62):[9)]

$$[i\lambda Q_B,\Phi]=\delta\Phi. \tag{4·65}$$

Now, in what follows, we consider the gravitational field $h^{\mu\nu}$ on the background Minkowski metric $\eta^{\mu\nu}$:

$$\tilde{g}^{\mu\nu}=\eta^{\mu\nu}+\kappa h^{\mu\nu}. \tag{4·66}$$

The analysis of commutators, equations of motion and BRS transformation properties of the asymptotic fields by using Lorentz covariance and W.T. identities, can be also performed for this case in exactly the same way as in the two preceding YM theories. Here, however, we only cite its brief outline. The interested readers should consult Ref. 11) for the detailed presentation of it.

We assume the following asymptotic condition:

$$\begin{aligned}&h^{\mu\nu}\xrightarrow[x_0\to\pm\infty]{}Z_3{}^{1/2}\varphi^{\mu\nu}_{\mathrm{in/out}}, \qquad B_{\mu}\longrightarrow Z_3{}^{-1/2}\beta_{\mu}{}^{\mathrm{in/out}},\\&c^{\mu}\longrightarrow\tilde{Z}_3{}^{1/2}\gamma^{\mu}_{\mathrm{in/out}}, \qquad \bar{c}_{\mu}\longrightarrow\tilde{Z}_3{}^{1/2}\bar{\gamma}_{\mu}{}^{\mathrm{in/out}}.\end{aligned} \tag{4·67}$$

The asymptotic fields $\varphi^{\mu\nu}$, and β_{μ} together with γ^{μ} and $\bar{\gamma}_{\mu}$ turn out to be a tripole field and dipole fields, respectively, in much contrast to the free theory described by the quadratic parts of the starting Lagrangian (4·61). In terms of the 4-dimensional momentum representation, such as

$$\varphi_{\mu\nu}(x)=(2\pi)^{-3/2}\int d^4p\theta(p^0)\,[\varphi_{\mu\nu}(p)\,e^{-ipx}+\varphi^{\dagger}_{\mu\nu}(p)\,e^{ipx}], \tag{4·68}$$

we decompose $\varphi_{\mu\nu}$ into the physical modes and the unphysical ones, in such a Lorentz frame that the 4-momentum p^μ takes the form $p^\mu = (p^0, 0, 0, p^3)$:

$$\varphi_1{}^T(p) = (\varphi_{11}(p) - \varphi_{22}(p))/2\,,$$

$$\varphi_2{}^T(p) = \varphi_{12}(p)\,, \tag{4·69}$$

$$\chi_0(p) = (i/2p_0)\left[\varphi_{00}(p) - \left(\frac{1-L_1+(L_2+2L_3)p_0{}^2}{2-3L_1}\right)\varphi^\lambda{}_\lambda(p)\right],$$

$$\chi_i(p) = i\varphi_{i0}(p)/p_0 \quad (i=1,2)\,,$$

$$\chi_3(p) = (i/2p_3)\left[\varphi_{33}(p) - \left(\frac{L_1-1+(L_2+2L_3)p_3{}^2}{2-3L_1}\right)\varphi^\lambda{}_\lambda(p)\right], \tag{4·70}$$

where L_1, L_2 and L_3 are some constants determined dynamically. The BRS transformation of the asymptotic fields is found by much use of W.T. identities to read

$$[Q_B{}^r, \varphi_i{}^T(p)] = 0\,, \quad (i=1,2) \tag{4·71}$$

$$[Q_B{}^r, \chi_\mu(p)] = -i\gamma_\mu(p)\,, \quad \{Q_B{}^r, \bar{\gamma}_\mu(p)\} = \beta_\mu(p)\,, \tag{4·72a}$$

$$[Q_B{}^r, \beta_\mu(p)] = \{Q_B{}^r, \gamma_\mu(p)\} = 0\,. \tag{4·72b}$$

The (anti-) commutation relations of these asymptotic fields are given by

$$\begin{array}{c} \\ \varphi_i{}^T(p) \\ \\ \chi_\mu(p) \\ \beta_\mu(p) \\ \\ \gamma_\mu(p) \\ \bar{\gamma}_\mu(p) \end{array}
\begin{array}{c}
\begin{array}{ccccc} \varphi_j{}^{T\dagger}(q) & \chi_\nu^\dagger(q) & \beta_\nu^\dagger(q) & \gamma_\nu^\dagger(q) & \bar{\gamma}_\nu^\dagger(q) \end{array} \\
\left(\begin{array}{c:cc:cc}
\delta_{ij}\delta(p^2) & & & 0 & \\ \hdashline
 & * & \mathcal{M}_{\mu\nu}(p) & & \\
0 & \mathcal{M}_{\mu\nu}(p) & 0 & 0 & \\ \hdashline
 & & & 0 & -i\mathcal{M}_{\mu\nu}(p) \\
 & 0 & & i\mathcal{M}_{\mu\nu}(p) & 0
\end{array}\right)
\end{array} \times \delta^4(p-q)\,, \tag{4·73}$$

where

$$\mathcal{M}_{\mu\nu}(p) = -(\eta_{\mu\nu} + L_3 p_\mu p_\nu)\delta(p^2) - (L_1/2)p_\mu p_\nu \delta'(p^2)\,. \tag{4·74}$$

This coincidence of the commutators, $[\chi_\mu(p), \beta_\nu^\dagger(q)] = i\{\gamma_\mu(p), \bar{\gamma}_\nu^\dagger(q)\} = \mathcal{M}_{\mu\nu}(p)\delta^4(p-q)$, is a direct consequence of BRS transformation law (4·72). We notice that the BRS transformation (4·72) and the commutators (4·73) for the fields $(\chi_\mu(p), \beta_\mu(p), \gamma_\mu(p), \bar{\gamma}_\mu(p))$ coincide in their *forms* with those of quartet presented in (3·15) and (3·16). Hence one can prove in quite the same way as in Chap. III that the present quartet $(\chi_\mu, \beta_\mu, \gamma_\mu, \bar{\gamma}_\mu)$ always conspires to form zero-norm combinations in the physical subspace $\mathcal{V}_{\text{phys}}$ specified by $Q_B|\text{phys}\rangle = 0$. Alternatively, if one wants, one can redefine the present dipole fields χ_μ, β_μ, γ_μ and $\bar{\gamma}_\mu$ such that they become simple pole fields, and then, can prove that the BRS transformation (4·72) and commutators (4·73) reduce to exactly the same ones as (3·15) and (3·16). Thus the physical S-matrix unitarity is established also in quantum gravity.

We should add a remark on the mode counting in the above. Among the ten components of $\varphi_{\mu\nu}(p)$, two are physical transverse modes $\varphi_i^T(i=1,2)$ of BRS-singlet and the other eight are unphysical ones falling into members of the quartet; that is, the four modes $\chi_\mu(p)$ represent essentially the "longitudinal" components of $\varphi_{\mu\nu}$ and the other four β_μ the redundant spin 1 components of $\varphi_{\mu\nu}$.

Finally we note that in Landau gauge the 10 components of the gravitational field are proved[12] to represent exactly *massless* particles identified as the 10 Goldstone bosons responsible for the spontaneous breakdown of $GL(4)$ invariance up to the Lorentz invariance, which is due to the background Minkowski metric $\eta_{\mu\nu}=\langle 0|g_{\mu\nu}(x)|0\rangle$.

Chapter V

Observables in the Yang-Mills Theory and Quark Confinement

——*Physical Contents Described in* H_{phys}——

§ 5.1. Concept of the Observable and Gauge Invariance as Its Criterion

So far, we have discussed the scattering theoretical aspects of the gauge theory, namely, the asymptotic states and the asymptotic fields. The physically meaningful quantity treated there is only the physical S-matrix, which has been proved in Chaps. III and IV to be a *unitary* operator in $H_{\text{phys}}=\overline{\mathcal{V}_{\text{phys}}/\mathcal{V}_0}$. Although such unphysical particles as FP ghosts may come out in the final states with non-vanishing S-matrix elements from the initial states containing no unphysical particles, they appear in $\mathcal{V}_{\text{phys}}$ *only in the zero-norm combination* $(\in\mathcal{V}_0)$, as has been shown explicitly in Chaps. III and IV. Since the zero-norm subspace $\mathcal{V}_0$ is orthogonal to $\mathcal{V}_{\text{phys}}$ ((A·7) in Appendix A),

$$\langle\chi|\Phi\rangle=0 \quad \text{for } |\Phi\rangle\in\mathcal{V}_{\text{phys}},\ |\chi\rangle\in\mathcal{V}_0, \tag{5·1}$$

those unphysical particles make no contribution to the scattering processes in $\mathcal{V}_{\text{phys}}$. Thus, *all physical scattering processes are completely described in* $H_{\text{phys}}=\overline{\mathcal{V}_{\text{phys}}/\mathcal{V}_0}$, where zero-norm physical states $|\chi\rangle\in\mathcal{V}_0$ containing unphysical particles are regarded as negligible objects:

$$|\hat{\chi}\rangle=|\chi\rangle+\mathcal{V}_0=\hat{0} \quad \text{in } H_{\text{phys}} \quad (|\chi\rangle\in\mathcal{V}_0). \tag{5·2}$$

These situations can be paraphrased in a rather general fashion as follows. Defining the *transition probability* $T(\Psi_1,\Psi_2)$ between two physical states $|\Psi_1\rangle, |\Psi_2\rangle\in\mathcal{V}_{\text{phys}}$ by

Definition 5.1. $$T(\Psi_1,\Psi_2)\equiv|\langle\Psi_1|\Psi_2\rangle|^2, \tag{5·3}$$

we obtain, from (5·1), the following relation:

$$T(\Psi_1+\chi_1,\Psi_2+\chi_2)=T(\Psi_1,\Psi_2) \quad \text{for } |\Psi_i\rangle\in\mathcal{V}_{\text{phys}},\ |\chi_i\rangle\in\mathcal{V}_0. \tag{5·4}$$

Namely, the transition probability $T(\Psi_1,\Psi_2)$ in $\mathcal{V}_{\text{phys}}$ is independent of the choice of the representative vectors $|\Psi_i\rangle\in\mathcal{V}_{\text{phys}}$ in the equivalence classes $|\hat{\Psi}_i\rangle$ with respect to $\mathcal{V}_0$ and it is really a function $\hat{T}$ depending on pairs of equivalence classes $|\hat{\Psi}_i\rangle\in H_{\text{phys}}=\overline{\mathcal{V}_{\text{phys}}/\mathcal{V}_0}$:

$$\hat{T}: H_{\text{phys}}\times H_{\text{phys}}\to\boldsymbol{R}_+$$
$$\hat{T}(\hat{\Psi}_1,\hat{\Psi}_2)=T(\Psi_1,\Psi_2)=|\langle\Psi_1|\Psi_2\rangle|^2=|\langle\hat{\Psi}_1|\hat{\Psi}_2\rangle|^2. \tag{5·5}$$

Then, as for the problems concerning the transition probabilities like the scattering theoretical problems, we can safely make all discussions in the Hilbert space $H_{\rm phys}=\overline{\mathcal{V}_{\rm phys}/\mathcal{V}_0}$, neglecting zero-norm states $|\chi\rangle\in\mathcal{V}_0$ as in (5·2).

Besides the transition probability, however, there are many physical quantities to be measured, for instance, the energy-momentum vector P_μ, and so on. If we want to describe, in the Hilbert space $H_{\rm phys}$, every physical process consistently according to the ordinary principles of quantum theory, any state $|\chi\rangle\in\mathcal{V}_0$ which corresponds to the null vector in $H_{\rm phys}$ should make no physical effects in the measurement of physical quantities. Now, since our *starting point* is not the quantum theory in $H_{\rm phys}$ but the field theory formulated in the state vector space $\mathcal{V}$ with an indefinite metric, we should write down the condition required for the consistent measurement in $H_{\rm phys}$ in terms of the operators A in $\mathcal{V}$. *If* a zero-norm physical state $|\chi\rangle\in\mathcal{V}_0$ were transformed by a physical quantity A into such a state $|\chi'\rangle=A|\chi\rangle$ that

$$\langle\Phi|\chi'\rangle=\langle\Phi|A|\chi\rangle\neq 0,\quad \text{for } some\ |\Phi\rangle\in\mathcal{V}_{\rm phys}\,, \tag{5·6}$$

then the measurement of A could *not* be described consistently in $H_{\rm phys}$, because the state $|\chi\rangle\in\mathcal{V}_0$ which is regarded as the *null* vector in $H_{\rm phys}$ makes a *non-vanishing* contribution in (5·6). So, we require a physical quantity A to satisfy the following equality:

$$\langle\Phi|A|\chi\rangle=\langle\chi|A|\Phi\rangle=0\quad \text{for } {}^\forall|\Phi\rangle\in\mathcal{V}_{\rm phys},\ {}^\forall|\chi\rangle\in\mathcal{V}_0\,. \tag{5·7}$$

As was shown in Ref. 1), the condition (5·7) agrees with the one which guarantees the usual connection between the transition probability and the expectation values of observables in the quantum theory. In the usual quantum theory formulated in the Hilbert space with a positive definite metric, we know that the relation

$$T(\Phi,\Psi)=|\langle\Phi|\Psi\rangle|^2=\langle\Psi|\Phi\rangle\langle\Phi|\Psi\rangle=E(P_\Phi;\Psi) \tag{5·8}$$

holds, where the expectation value of the observable A in the state $|\Psi\rangle$ is denoted as

$$E(A;\Psi)=\langle\Psi|A|\Psi\rangle \tag{5·9}$$

and P_Φ denotes the projection on the state $|\Phi\rangle$

$$P_\Phi=|\Phi\rangle\langle\Phi|\,. \tag{5·10}$$

Namely, the transition probability $T(\Phi,\Psi)$ is nothing but the expectation value $E(P_\Phi;\Psi)$ of a special type of observable P_Φ corresponding to the yes-no question about the state $|\Phi\rangle$. Conversely, since every observable A which is a self-adjoint operator admits the spectral resolution

$$A=\sum_n a_n P_{\Phi_n}=\sum_n a_n|\Phi_n\rangle\langle\Phi_n|^{*)} \tag{5·11}$$

with $A|\Phi_n\rangle=a_n|\Phi_n\rangle$, the expectation value $E(A,\Psi)$ can be reconstructed from the knowledge of the transition probabilities:

$$E(A;\Psi)=\sum_n a_n\langle\Psi|\Phi_n\rangle\langle\Phi_n|\Psi\rangle=\sum_n a_n T(\Phi_n,\Psi). \tag{5·12}$$

Thus, in order to maintain the relations (5·8) and (5·12) with (5·9) in our H_{phys} also, Eq. (5·4) should imply the equality

$$E(A;\Psi+\chi)=E(A;\Psi) \quad \text{for } |\Psi\rangle\in\mathcal{V}_{\text{phys}},\ |\chi\rangle\in\mathcal{V}_0, \tag{5·13}$$

which is really equivalent to (5·7). Thus, the conditions (5·7) and (5·13) are the equivalent expressions of a necessary condition for the consistent measurement of a physical quantity A. It can easily be checked that the energy-momentum operator P_μ satisfies the condition (5·7) as follows. First, since the BRS charge Q_B (2·20) is a translationally invariant Lorentz scalar, we obtain

$$[Q_B, P_\mu]=0, \tag{5·14}$$

as a consequence of which the state $P_\mu|\Phi\rangle$ with $|\Phi\rangle\in\mathcal{V}_{\text{phys}}$ belongs to $\mathcal{V}_{\text{phys}}$:

$$Q_B P_\mu|\Phi\rangle=[Q_B, P_\mu]|\Phi\rangle+P_\mu Q_B|\Phi\rangle=0 \Longrightarrow P_\mu|\Phi\rangle\in\mathcal{V}_{\text{phys}}. \tag{5·15}$$

Then, we obtain, from (5·1) and (5·15),

$$\langle\chi|P_\mu|\Phi\rangle=\langle\Phi|P_\mu|\chi\rangle=0 \quad \text{for } |\Phi\rangle\in\mathcal{V}_{\text{phys}},\ |\chi\rangle\in\mathcal{V}_0. \tag{5·16}$$

In what follows, we call an *observable* any operator A (hermitian or not) satisfying (5·7).

Definition 5.2. An operator A is called an *observable* if it satisfies

$$\langle\chi|A|\Phi\rangle=\langle\Phi|A|\chi\rangle=0 \tag{5·7}$$

or equivalently

$$\langle\Phi+\chi|A|\Phi+\chi\rangle=\langle\Phi|A|\Phi\rangle\equiv E(A;\Phi) \tag{5·13}$$

for any $|\Phi\rangle\in\mathcal{V}_{\text{phys}}$ and $|\chi\rangle\in\mathcal{V}_0$.

Now, as easily understood from the above argument of the "observability" of P_μ, the concept of the observable is closely related to the notion of *gauge invariance*, since the BRS charge Q_B in (5·14) is essentially a generator of

*) Precisely speaking, (5·11) should be written in general as

$$A=\int\lambda\, dP(\lambda), \qquad dP(\lambda): \text{spectral measure}$$

in order to treat A with continuous spectrum as well as discrete one.

"local" gauge transformation in quantum theory. In Ref. 1), the following four notions of the gauge invariance are introduced for the operators in QED:

(i) *gauge independence*: $\langle \Phi_1+\chi_1|A|\Phi_2+\chi_2\rangle=\langle \Phi_1|A|\Phi_2\rangle$

$$\text{for } |\Phi_i\rangle\in\mathcal{V}_{\text{phys}},\ |\chi_i\rangle\in\mathcal{V}_0 \tag{5·17}$$

(ii) *weak gauge invariance*: $A\mathcal{V}_0\subset\mathcal{V}_0,\ A^\dagger\mathcal{V}_0\subset\mathcal{V}_0$, (5·18)

(iii) *gauge invariance*: $A\mathcal{V}_{\text{phys}}\subset\mathcal{V}_{\text{phys}},\ A^\dagger\mathcal{V}_{\text{phys}}\subset\mathcal{V}_{\text{phys}}$, (5·19)

(iv) *strict gauge invariance*: gauge invariance (iii) augmented by the condition $[A,\partial^\nu F_{\nu\mu}-j_\mu]=0$. (5·20)

One can easily see, of these conditions, that the statement becomes stronger the latter it is in the list, and that the weakest one (i) agrees with the condition (5·7) for the observable. Note that the condition (iii) allows us to define an operator $\hat{A}$ in $H_{\text{phys}}=\overline{\mathcal{V}_{\text{phys}}/\mathcal{V}_0}$ by the equation

$$\hat{A}|\hat{\Phi}\rangle=\widehat{A|\Phi\rangle},\quad |\Phi\rangle\in\mathcal{V}_{\text{phys}},\quad |\hat{\Phi}\rangle=|\Phi\rangle+\mathcal{V}_0\in\overline{\mathcal{V}_{\text{phys}}/\mathcal{V}_0}\,. \tag{5·21}$$

In the N.L. formalism of QED where a physical state $|\text{phys}\rangle$ is specified by the condition

$$B^{(+)}(x)|\text{phys}\rangle=0\,, \tag{1·5}$$

the condition (iv) is equivalent to the following one:

(iv′) $$[A,B(x)]=0\,. \tag{5·20′}$$

This is due to the Maxwell equation

$$\partial^\nu F_{\nu\mu}=j_\mu+\partial_\mu B\,,$$

where $B(x)$ is the Lagrange multiplier field satisfying

$$\partial^\mu A_\mu+\alpha B=0\,. \tag{1·2}$$

Furthermore, the above four conditions in the Abelian cases are distinct from one another, namely, each latter one is truly stronger than the former one. For example, the energy-momentum P_μ satisfies (iii) but not (iv):

$$B^{(+)}(x)P_\mu\mathcal{V}_{\text{phys}}=[B^{(+)}(x),P_\mu]\mathcal{V}_{\text{phys}}=i\partial_\mu B^{(+)}(x)\mathcal{V}_{\text{phys}}=0\,, \tag{5·22a}$$

$$[B(x),P_\mu]=i\partial_\mu B(x)\neq0\,. \tag{5·22b}$$

Contrary to this case of QED, the weakest condition (i) in our formalism based upon the BRS symmetry is really equivalent to the stronger one (iii), which is rewritten equivalently as

$$[\bar{Q}_B,A]\mathcal{V}_{\text{phys}}=[\bar{Q}_B,A^\dagger]\mathcal{V}_{\text{phys}}=0\,. \tag{5·19′}$$

Here $\bar{Q}_B$ is the Klein transform of the BRS charge Q_B:

$$\bar{Q}_B = e^{\pm \pi Q_c} Q_B \,. \tag{5·23}$$

Proposition 5.3.[2] In the gauge theory with the BRS symmetry generated by the BRS charge Q_B, the condition (i) for an operator A implies the condition (iii).

Proof) If an operator A satisfies (i), or its equivalent (5·7), we obtain for any vector $|f\rangle \in \mathcal{V}$ and for any $|\Phi\rangle \in \mathcal{V}_{\text{phys}}$,

$$\langle f | (Q_B A |\Phi\rangle) = (\langle f | Q_B) A |\Phi\rangle = 0 \,, \tag{5·24}$$

because the state $|\chi\rangle \equiv Q_B |f\rangle$ belongs to $\mathcal{V}_0$:

$$Q_B |\chi\rangle = Q_B^2 |f\rangle = 0 \,, \tag{5·25a}$$

$$\langle \chi | \chi \rangle = \langle f | Q_B^2 | f \rangle = 0 \,. \tag{5·25b}$$

Since the inner product of $\mathcal{V}$ is assumed to be non-degenerate [(A·2) in Appendix A], (5·24) concludes

$$Q_B A \mathcal{V}_{\text{phys}} = 0 \,, \tag{5·26}$$

which is nothing but the condition $A\mathcal{V}_{\text{phys}} \subset \mathcal{V}_{\text{phys}}$. The condition $A^\dagger \mathcal{V}_{\text{phys}} \subset \mathcal{V}_{\text{phys}}$ follows in quite the same way, and hence, we arrive at the condition (iii). □

Thus, in our canonical formalism of the gauge theory, the three notions of gauge invariance (i) ~ (iii) are all equivalent. This criterion can be further sharpened for *local* observables in the following way.

Proposition 5.4.[2] If A is an *local observable*, namely, an operator $A \in \mathcal{F}(\mathcal{O})$*) satisfying one (and, consequently, all) of the conditions (i) ~ (iii) [i.e., (5·17), (5·18), (5·19), (5·19′)], then it satisfies the equality

$$[\bar{Q}_B, A] = 0 \,, \tag{5·27}$$

which implies, conversely, (i)~(iii). Namely, a *local* operator $A \in \mathcal{F}(\mathcal{O})$ is an *observable if and only if* it satisfies (5·27).

By comparing (5·27) with (5·20′) in view of the corresponding subsidiary conditions (2·29) and (1·5), the condition (5·27) should be interpreted as the one for strict gauge invariance. Namely, the condition for a *local* operator to be an *observable* agrees with the condition of *strict gauge invariance* in our formalism. The proof of the above proposition can be made easily by

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*) $\mathcal{F}(\mathcal{O})$ is the polynomial algebra generated by the field operators smeared with the testing functions with their compact supports in the finite space-time region $\mathcal{O}$. (See, (A·18) in Appendix A.)

using the following interesting lemma.

Lemma 5.5.[3] If A is a *local* operator $\in \mathcal{F}(\mathcal{O})$ in our theory satisfying the condition

$$Q_B A|0\rangle = 0\,, \tag{5·28}$$

then it satisfies (5·27), namely, A is a local *observable*. If we denote the set of local observables belonging to $\mathcal{F}(\mathcal{O})$ by $\mathcal{A}(\mathcal{O})$, the following equality holds:

$$\mathcal{F}(\mathcal{O})\,|0\rangle \cap \mathcal{V}_{\mathrm{phys}} = \mathcal{A}(\mathcal{O})\,|0\rangle\,. \tag{5·29}$$

Proof) Since we have assumed that the vacuum $|0\rangle$ is a physical state, namely,

$$Q_B|0\rangle = 0\,, \tag{2·29}$$

we obtain from (5·28)

$$e^{\mp\pi Q_c}[\bar{Q}_B, A]\,|0\rangle = Q_B A|0\rangle - e^{\mp\pi Q_c} A e^{\pm\pi Q_c} Q_B|0\rangle = 0\,. \tag{5·30}$$

By decomposing A, which is a *polynomial* of smeared field operators, into the part A_1 with even powers of FP ghosts and the one A_2 with odd powers, (5·30) can be written as

$$[Q_B, A_1]\,|0\rangle + \{Q_B, A_2\}\,|0\rangle = 0\,. \tag{5·30′}$$

Owing to the linear independence of the states with different eigenvalues of Q_c, (5·30′) is decomposed into the following two equations:

$$[Q_B, A_1]\,|0\rangle = 0\,, \tag{5·31a}$$

$$\{Q_B, A_2\}\,|0\rangle = 0\,. \tag{5·31b}$$

By (5·31), $[Q_B, A_1]$ and $\{Q_B, A_2\}$ are, respectively, local-anticommutative and local-commutative operators $\in \mathcal{F}(\mathcal{O})$ annihilating the vacuum, which vanish themselves by the well-known Reeh-Schlieder theorem (Corollary A·5 in Appendix A):

$$[Q_B, A_1] = \{Q_B, A_2\} = 0\,. \tag{5·32}$$

Thus, we obtain (5·27)

$$[\bar{Q}_B, A] = [e^{\pm\pi Q_c} Q_B, A_1 + A_2] = e^{\pm\pi Q_c}([Q_B, A_1] + \{Q_B, A_2\}) = 0\,. \qquad \square$$

Proof of Proposition 5.4) Let A be a local observable satisfying (5·19′), then it satisfies (5·28) because of (2·29). By Lemma 5.5, we obtain (5·27). □

In the above, it is worth while remarking that the very modest requirement (i) [(5·17) or (5·13)] for the natural relations (5·8) and (5·12) between the transition probabilities and the expectation values in the ordinary quantum theory to be preserved leads us to the condition (iii) of *gauge invariance* (5·19′) for a physical quantity A. Especially, this requirement for a *local* observable A is reduced to the salient *algebraic* condition (5·27) of gauge invariance:

$$[\bar{Q}_B, A]=0,$$

which can be examined directly by use of the canonical commutation relations without recourse to the dynamical information of Green's functions, etc. This shows the pertinence of our formalism of the gauge theory, especially, of the choice of the state vector space $H_{\text{phys}}=\overline{\mathcal{V}_{\text{phys}}/\mathcal{V}_0}$ in which every physical process should be described. In this context, it may be instructive to note another evidence for the consistency of the choice of the subsidiary condition

$$Q_B|\text{phys}\rangle=0. \tag{2·29}$$

Reversing the direction of the above arguments, let us select the observable A by the principle of gauge invariance:

$$[\bar{Q}_B, A]=0, \tag{5·27}$$

and require the observables to be represented in a Hilbert space H. Then, Q_B is an observable

$$[\bar{Q}_B, Q_B]=e^{\pm\pi Q_c}\{Q_B, Q_B\}=2e^{\pm\pi Q_c}Q_B{}^2=0, \tag{5·31}$$

whose representation $\hat{Q}_B$ in H with a *positive definite* metric is nothing but 0,

$$\hat{Q}_B=0 \tag{5·32}$$

because of the nilpotency of Q_B (2·25a):

$$\hat{Q}_B{}^\dagger\hat{Q}_B=\widehat{Q_B{}^\dagger Q_B}=\widehat{Q_B{}^2}=0. \tag{5·33}$$

Thus, the subsidiary condition (2·29) can almost be said to be demanded by the principle of gauge invariance (5·27) for the observable.

§ 5.2. "Maxwell" Equation and Structure of Local Observables
——*Local observables as group invariants*——

In the N.L. formalism of QED, the field strength $F_{\mu\nu}$ and the electromagnetic current j_μ satisfy the condition (iv) of strict gauge invariance which is equivalent to (5·20′):

$$[F_{\mu\nu}(x), B(y)]=-i(\partial_\mu\partial_\nu-\partial_\nu\partial_\mu)D(x-y)=0, \tag{5·34a}$$

$$[j_\mu(x), B(y)]=[\partial^\nu F_{\nu\mu}(x)-\partial_\mu B(x), B(y)]=0, \tag{5·34b}$$

and hence, they are (strictly gauge invariant) *observables.* For these observables, we obtain *the Maxwell equation* in H_{phys}:

$$\partial^\nu \hat{F}_{\nu\mu} = \hat{j}_\mu \,, \tag{5·35a}$$

from the equation of motion

$$\partial^\nu F_{\nu\mu} = j_\mu + \partial_\mu B \tag{5·35b}$$

or

$$\langle \Phi | (\partial^\nu F_{\nu\mu} - j_\mu) | \Psi \rangle = 0 \quad \text{for} \ |\Phi\rangle, |\Psi\rangle \in \mathcal{V}_{\text{phys}} \,. \tag{5·35c}$$

On the contrary, the equation of motion (2·10a) for the YM field

$$D^\nu F_{\nu\mu} = -g j_\mu + \partial_\mu B - ig \partial_\mu \bar{c} \times c \,, \tag{2·10a}$$

contains an unphysical term $-ig\partial_\mu \bar{c} \times c$ which cannot be neglected even in the matrix elements between physical states. The matter current $j_\mu{}^a$ defined in (2·10d), however, cannot be conserved by itself

$$\partial^\mu j_\mu{}^a = -g (A^\mu \times j_\mu)^a \neq 0 \tag{5·36}$$

and the conserved Noether current $J_\mu{}^a$ of the global gauge symmetry is given by

$$\begin{aligned} J_\mu{}^a &= j_\mu{}^a + (A^\nu \times F_{\nu\mu})^a + (A_\mu \times B)^a - i(\bar{c} \times D_\mu c)^a + i(\partial_\mu \bar{c} \times c)^a \\ &= [j_\mu{}^a + (A^\nu \times F_{\nu\mu})^a] - \{Q_B, (A_\mu \times \bar{c})^a\} + i(\partial_\mu \bar{c} \times c)^a, \end{aligned} \tag{5·37}$$

which also contains the same unphysical term $\partial_\mu \bar{c} \times c$ as the above. As noted in Chap. II, we obtain the "Maxwell" equation (2·36)

$$\partial^\nu F^a_{\nu\mu} + g J_\mu{}^a = \{Q_B, (D_\mu \bar{c})^a\}, \tag{5·38}$$

as a consequence of which the equation

$$\langle \Phi | (\partial^\nu F^a_{\nu\mu} + g J_\mu{}^a) | \Psi \rangle = 0 \tag{5·39}$$

holds for $|\Phi\rangle, |\Psi\rangle \in \mathcal{V}_{\text{phys}}$, similarly to (5·35c). It might seem, however, that the question how to interpret the unphysical term $i(\partial_\mu \bar{c} \times c)$ contained in (5·37) remains unsettled. The answer to this question is the following: In the non-Abelian case, the field strength $F^a_{\mu\nu}$ and the Noether current $J_\mu{}^a$ of the global gauge symmetry are *not* observables in contrast to the Abelian case,

$$[Q_B, F^a_{\mu\nu}] = ig(c \times F_{\mu\nu})^a \neq 0 \,, \tag{5·40}$$

$$[Q_B, J_\mu{}^a] = -i\partial^\nu (c \times F_{\nu\mu})^a \neq 0 \,, \tag{5·41}$$

and such a type of equation as (5·35a) does not hold in H_{phys} of this case. Consequently, the unphysical term $i(\partial_\mu \bar{c} \times c)^a$ makes no trouble, because the

J_μ^a's themselves are unphysical and cannot be observed in the physical world. This consequence is not an accidental situation, but can be viewed in a more general context. Namely, the following theorem asserting the *group invariance of local observables* holds in our formalism:

Theorem 5.6.[2),4)] Every local observable A commutes in H_{phys} with the global charge Q^a of the *unbroken**) global gauge symmetry:

$$[\hat{Q}^a, \hat{A}]=0 \quad \text{in } H_{\text{phys}}. \tag{5·42}$$

In order to prove this theorem, we should first examine the global charge operator Q^a of the global gauge symmetry defined (formally) by

$$Q^a \equiv \int d^3x J_0{}^a. \tag{5·43}$$

Lemma 5.7. If a conserved current J_μ of the form

$$J_\mu = \partial^\nu K_{\nu\mu} \tag{5·44}$$

with a local (or anti-local) operator $K_{\nu\mu}$ yields a *well-defined* charge Q

$$Q=\int d^3x J_0 = \int d^3x \partial^i K_{i0}, \tag{5·45}$$

then the charge Q is nothing but 0.

Proof) Let φ be a local operator belonging to $\mathfrak{F}(\mathcal{O}):\varphi\in\mathfrak{F}(\mathcal{O})$. Taking a sufficiently large $R>0$, we obtain

$$\begin{aligned} Q\varphi|0\rangle &= [Q, \varphi]_\mp|0\rangle \pm \varphi Q|0\rangle \\ &= [Q_R, \varphi]_\mp|0\rangle \end{aligned} \tag{5·46}$$

according to the general theory of the conserved charge in Appendix B, where a *well-defined* charge Q is shown to annihilate the vacuum

$$Q|0\rangle = 0. \tag{B·1}$$

Since, roughly speaking, Q_R is the volume integral of J_0 within the region $|\boldsymbol{x}|<R$,**) the commutator (or anti-commutator) $[Q_R, \varphi]_\mp$ vanishes for a sufficiently large $R>0$ owing to the local (anti-)commutativity of K_{i0} and φ:

$$[Q_R, \varphi]_\mp \sim \int d^3x \partial_x{}^i [K_{i0}(x), \varphi]_\mp = \int dS_x{}^i [K_{i0}(x), \varphi]_\mp = 0 \tag{5·47}$$

or precisely,

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*) The *whole* group symmetry may be broken spontaneously, in which case (5·42) holds for the charges of the *unbroken* subgroup of the *remaining symmetry*.

**) As for the precise definition of Q_R, see (B·9)～(B·11) in Appendix B.

$$[Q_B, \varphi] = \int d^4x \alpha_T(x^0) f_R(\boldsymbol{x}) [\partial^i K_{i0}(x), \varphi]_\mp$$
$$= -\int d^4x \alpha_T(x^0) \partial^i f_R(\boldsymbol{x}) [K_{i0}(x), \varphi]_\mp = 0, \quad (5\cdot 47')$$

because supp $\partial^i f_R \subset \{\boldsymbol{x} \in \boldsymbol{R}^3; R \leq |\boldsymbol{x}| \leq 2R\}$. Thus, we obtain for $\varphi|0\rangle \in \mathfrak{F}(\mathcal{O})|0\rangle$

$$Q(\varphi|0\rangle) = 0, \quad (5\cdot 48)$$

and hence,

$$\langle \Psi | Q(\varphi|0\rangle) = 0 \quad \text{for } \forall |\Psi\rangle \in \mathcal{V}. \quad (5\cdot 49)$$

By virtue of the Reeh-Schlieder theorem (Theorem A.4 in Appendix A)

$$\mathcal{V} = \overline{\mathfrak{F}(\mathcal{O})|0\rangle}^w, \quad (\text{A}\cdot 24')$$

we arrive at the conclusion

$$\langle \Psi | Q | \Phi \rangle = 0 \quad \text{for } \forall |\Phi\rangle, |\Psi\rangle \in \mathcal{V}, \quad (5\cdot 50)$$

which is nothing but the statement: $Q = 0$. □

Applying this Lemma 5.7 to the conserved current $[Q_B, J_\mu{}^a]$ in (5·41) $[K_{\nu\mu} \to -i(c \times F_{\nu\mu})^a]$, we conclude:

Corollary 5.8. The global charge $Q^a[(5\cdot 42)]$ is a (non-local) *observable*

$$[Q_B, Q^a] = 0, \quad (5\cdot 51)$$

as long as it is a *well-defined* charge, namely, as long as the global gauge symmetry corresponding to the charge Q^a is not broken spontaneously*).

Now we prove Theorem 5.6:

Proof of Theorem 5.6) Let A be a local observable $\in \mathfrak{F}(\mathcal{O})$:

$$[Q_B, A] = 0.$$

As is seen from the argument made in the proof of Lemma 5.5, A can be assumed without loss of generality to satisfy either $[Q_B, A] = 0$ or $\{Q_B, A\} = 0$. Then, the "Maxwell" equation (5·38) tells us the equality

$$[gJ_0{}^a, A] = [-\partial^i F_{i0}^a + \{Q_B, (D_0\bar{c})^a\}, A]$$
$$= -[\partial^i F_{i0}^a, A] + [Q_B, [(D_0\bar{c})^a, A]_\mp]_\pm + [(D_0\bar{c})^a, [Q_B, A]_\mp]_\pm$$

*) If the corresponding symmetry is broken spontaneously, the volume integrals

$$Q^a = \int d^3x J_0{}^a \quad \text{and} \quad [Q_B, Q^a] = -i \int d^3x \partial^i (c \times F_{i0})^a$$

become ill-defined owing to the massless contributions from the Goldstone particles and we cannot say anything definite about these charges.

$$= -[\partial^i F^a_{i0}, A] + [Q_B, [(D_0\bar{c})^a, A]_{\mp}]_{\pm} \tag{5·52}$$

corresponding to $[Q_B, A]_{\mp} = 0$. Thus, for any $R > 0$, we obtain

$$[gQ_R{}^a, A] = \int d^4x \alpha_T(x^0)\partial^i f_R(\boldsymbol{x})[F^a_{i0}(x), A]$$
$$+ \left[Q_B, \int d^4x \alpha_T(x^0) f_R(\boldsymbol{x})[(D_0\bar{c})^a(x), A]_{\mp}\right]_{\pm}. \tag{5·53}$$

By taking $R > 0$ sufficiently large, the first term vanishes by the local commutativity, and hence, we obtain

$$\langle\Phi|[gQ_R{}^a, A]|\Psi\rangle = \langle\Phi|\left[Q_B, \int d^4x \alpha_T(x^0) f_R(\boldsymbol{x})[(D_0\bar{c})^a(x), A]_{\mp}\right]_{\pm}|\Psi\rangle$$
$$= 0 \qquad \text{for} \quad |\Phi\rangle, |\Psi\rangle \in \mathcal{V}_{\text{phys}}. \tag{5·54}$$

Since we have assumed that the symmetry corresponding to Q^a is not broken spontaneously, we obtain from (5·54)

$$\langle\Phi|[gQ^a, A]|\Psi\rangle = 0 \quad \text{for} \ |\Phi\rangle, |\Psi\rangle \in \mathcal{V}_{\text{phys}}, \tag{5·55}$$

which concludes the following equality for the *observable* Q^a (see Corollary 5.8):

$$[g\hat{Q}^a, \hat{A}] = 0 \quad \text{in } H_{\text{phys}}. \tag{5·42} \ \square$$

Now, we investigate the structure of the observables in more detail. The *canonical* energy-momentum tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\Phi_j)}\partial_\nu\Phi_j - g_{\mu\nu}\mathcal{L}$$
$$= -F_{\mu\lambda}\cdot\partial_\nu A^\lambda - A_\mu\cdot\partial_\nu B - i\partial_\nu\bar{c}\cdot D_\mu c - i\partial_\mu\bar{c}\cdot\partial_\nu c$$
$$+ \frac{\partial\mathcal{L}_{\text{matter}}}{\partial(\partial^\mu\varphi_i)}\partial_\nu\varphi_i - g_{\mu\nu}\mathcal{L}, \tag{5·56}$$ *),**)

and satisfies the commutation relation

$$[iQ_B, T_{\mu\nu}] = \partial^\rho(\partial_\nu c\cdot F_{\rho\mu}) \neq 0. \tag{5·57}$$ **)

Hence, in view of the criterion (5·27) for the local observable $T_{\mu\nu}$ is *not* an

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*) In (5·56), the summation of Φ_j should include *all* the fields A_μ, B, c, $\bar{c}$ and φ_i. In particular, the contribution from the conjugate Dirac spinor $\bar{\psi}$ should be summed equally along with ψ, if they are contained in φ_i.

**) If we adopt the gauge fixing term $\mathcal{L}'_{\text{GF}} = B^a\partial^\mu A_\mu{}^a + (\alpha/2)B^aB^a$, Eqs. (5·43), (5·44) read as

$$T'_{\mu\nu} = (-F_{\mu\lambda} + g_{\mu\lambda})\cdot\partial_\nu A^\lambda - i\partial_\nu\bar{c}\cdot D_\mu c - i\partial_\mu\bar{c}\cdot\partial_\nu c + \frac{\partial\mathcal{L}_{\text{matter}}}{\partial(\partial^\mu\varphi_i)}\partial_\nu\varphi_i - g_{\mu\nu}\mathcal{L}' \tag{5·56'}$$

and

$$[iQ_B, T'_{\mu\nu}] = \partial^\rho(\partial_\nu c\cdot F_{\rho\mu}) + \partial_\nu(B\cdot D_\mu c) - g_{\mu\nu}\partial_\rho(B\cdot D^\rho c). \tag{5·57'}$$

observable. There exists, however, a physically more reasonable definition of the energy-momentum tensor, namely, the *symmetric* energy-momentum tensor $\Theta_{\mu\nu}$ defined[5] by

$$\Theta_{\mu\nu} \equiv -e_{a\mu}\frac{\delta \tilde{I}}{\delta e_a{}^\nu}\bigg|_{e_b{}^\lambda=\delta_b{}^\lambda} = -e_{a\mu}\left[\frac{\partial(\sqrt{-g}\tilde{\mathcal{L}})}{\partial e_a{}^\nu} - \partial_\lambda\left(\frac{\partial\sqrt{-g}\tilde{\mathcal{L}}}{\partial(\partial_\lambda e_a{}^\nu)}\right)\right]\bigg|_{e_b{}^\lambda=\delta_b{}^\lambda}$$

$$= \Theta_{\nu\mu}, \tag{5·58}$$

$$\partial^\mu \Theta_{\mu\nu} = 0, \tag{5·59}$$

where $\tilde{\mathcal{L}}$ is the local-Lorentz invariant Lagrangian density obtained from the original one $\mathcal{L}$ (2·7) by replacing the flat Minkowski metric $\eta_{\mu\nu}$ and the derivatives ∂_μ by

$$\eta_{\mu\nu} \to e_\mu{}^a \eta_{ab} e_\nu{}^b \tag{5·60a}$$

and

$$\partial_\mu \to \nabla_\mu \quad \text{(general-and local-Lorentz covariant-derivative)}, \tag{5·60b}$$

respectively. $e_\mu{}^a$ is the vierbein component and $\eta_{ab} = \mathrm{diag.}(+1, -1, -1, -1)$. Further, g in (5·58) is defined by

$$g = \det(e_\mu{}^a \eta_{ab} e_\nu{}^b). \tag{5·61}$$

In our case, $\Theta_{\mu\nu}$ defined from (2·7)*) according to (5·58) agrees with the one obtained by the Belinfante method adding to $T_{\mu\nu}$ the spin angular-momentum density term $S_{\mu\nu} \equiv (\partial\mathcal{L}/\partial(\partial^\mu\varphi_i))(\Sigma_{\mu\nu})_j{}^i\varphi^j$ ($\Sigma_{\mu\nu}$: spin matrix)

$$\Theta_{\mu\nu} = T_{\mu\nu} + \tfrac{1}{2}\partial^\rho(S_{\rho\mu\nu} + S_{\mu\nu\rho} + S_{\nu\mu\rho}), \tag{5·62}$$

and is found to be

$$\Theta_{\mu\nu} = \Theta_{\mu\nu}^{\text{phys}} - \{Q_B, \partial_\mu\bar{c}\cdot A_\nu + \partial_\nu\bar{c}\cdot A_\mu + g_{\mu\nu}(\tfrac{1}{2}\bar{c}\cdot B - \partial^\rho\bar{c}\cdot A_\rho)\}, \tag{5·63a}$$

$$\Theta_{\mu\nu}^{\text{phys}} = F_{\mu\rho}\cdot F^\rho{}_\nu + \tfrac{1}{4}g_{\mu\nu}F_{\rho\sigma}\cdot F^{\rho\sigma} + \Theta_{\mu\nu}^{\text{matter}}. \tag{5·63b}$$

$\Theta_{\mu\nu}^{\text{matter}}$ is the matter part obtained from $\mathcal{L}_{\text{matter}}$, for instance,

$$\Theta_{\mu\nu}^{\text{matter}} = \bar{q}\gamma_\mu(i\partial_\nu + g\tfrac{1}{2}\lambda^a A_\nu{}^a)q - g_{\mu\nu}\bar{q}[\gamma^\rho(i\partial_\rho + g\tfrac{1}{2}\lambda^a A_\rho{}^a) - m]q$$
$$+ \tfrac{1}{4}\partial^\rho(\bar{q}\gamma_\rho\sigma_{\mu\nu}q) + \tfrac{1}{4}i\partial^\rho[\bar{q}(2\gamma_\rho g_{\mu\nu} - \gamma_\nu g_{\mu\rho} - \gamma_\mu g_{\nu\rho})q], \tag{5·63c}$$

in the case of QCD:

$$\mathcal{L}_{\text{matter}} = \bar{q}\left(\frac{i}{2}\gamma^\mu\overleftrightarrow{\partial}_\mu + g\frac{\lambda^a}{2}A_\mu{}^a\gamma^\mu - m\right)q. \tag{5·64}$$

*) If one makes the replacement (2·14) in the Lagrangian density, the obtained result is not changed at all. $\Theta_{\mu\nu}$ for this case is quite the same as (5·62).

With (5·63a) and (5·63b), we can show that

$$[Q_B, \Theta_{\mu\nu}^{\text{phys}}] = [Q_B, \Theta_{\mu\nu}] = 0\,. \tag{5·65}$$

Thus, $\Theta_{\mu\nu}$ is an observable as it should be, as a trivial consequence of which, the Lorentz generators $M_{\mu\nu} \equiv \int d^3x(x_\mu\Theta_{0\nu} - x_\nu\Theta_{0\mu})$ as well as the energy-momentum vector $P_\mu = \int d^3xT_{0\mu} = \int d^3x\Theta_{0\mu}$ are observables:

$$[Q_B, P_\mu] = [Q_B, M_{\mu\nu}] = 0\,. \tag{5·66}$$

By (5·66), the Poincaré covariance of the theory in H_{phys} is ensured. The consequence (5·65) also guarantees the local measurement of the energy momentum:

$$(P_\mu)_R \equiv \int d^4x\alpha_T(x^0)f_R(\boldsymbol{x})\Theta_{0\mu}(x)\,,$$

$$[Q_B, (P_\mu)_R] = 0\,. \tag{5·67}$$

Here, we remark the structural feature of $\Theta_{\mu\nu}$ in (5·63) that $\Theta_{\mu\nu}$ consists of the following two parts. The first part $\Theta_{\mu\nu}^{\text{phys}}$ which is an observable in itself contains no such unphysical fields as c, $\bar{c}$ and B, and coincides with the energy-momentum tensor derived from the Lagrangian density $\mathcal{L}_s$ without the gauge fixing terms nor FP ghosts. The second one contains unphysical fields c, $\bar{c}$ and B in such a form that it vanishes in the physical subspace $\mathcal{V}_{\text{phys}}$. This is a physically reasonable result. It should be noted, however, that the unphysical second part $\Theta_{\mu\nu} - \Theta_{\mu\nu}^{\text{phys}}$, which makes no contribution in the physical world H_{phys}, plays an essential role in the conservation of $\Theta_{\mu\nu}$. Without this part, Θ^{phys} itself cannot be conserved as an operator in $\mathcal{V}$:

$$\partial^\mu\Theta_{\mu\nu}^{\text{phys}} = \{Q_B, \partial^\mu\bar{c}\cdot F_{\mu\nu}\} \neq 0\,. \tag{5·68}$$

As an operator in H_{phys}, $\hat{\Theta}_{\mu\nu}^{\text{phys}}$ coincides with $\hat{\Theta}_{\mu\nu}$:

$$\hat{\Theta}_{\mu\nu} = \hat{\Theta}_{\mu\nu}^{\text{phys}}\,, \tag{5·69}$$

and, of course, it is conserved: $\partial^\mu\hat{\Theta}_{\mu\nu}^{\text{phys}} = \widehat{\partial^\mu\Theta_{\mu\nu}} = 0$ *in the Hilbert space* H_{phys}. This situation is similar to the one encountered in the analysis of the S-matrix in Chap. III: Such unphysical particles as FP ghost pairs turn out to be produced easily even from initial states *without* unphysical particles, while they are contained in the final states *with zero norm*, and hence, they do not appear in the physical world H_{phys}. In short, if one simply neglects such unphysical fields as c, $\bar{c}$ and B in this formalism, the invariance under the time evolution or Lorentz transformations is violated *by zero norms*, which make no effects in H_{phys}. However, since one cannot attain H_{phys} directly without passing through the underlying $\mathcal{V}$ and $\mathcal{V}_{\text{phys}}$, such unphysical fields as c, $\bar{c}$ and B are indispensable for formulating the theory covariantly at every step.

In the case of QCD, some other examples of observables (hermitian or not) are given by

$$F^a_{\mu\nu}F^a_{\rho\sigma}\,,$$

$$\bar{q}\Gamma q\,,$$

$$\bar{q}\Gamma\left(i\partial_\mu+g\frac{\lambda^a}{2}A_\mu{}^a\right)q,\ \left(-i\partial_\mu\bar{q}+\bar{q}g\frac{\lambda^a}{2}A_\mu{}^a\right)\Gamma q; \tag{5·70a}$$

$$B^a=\{Q_B,\bar{c}^a\},\ (D_\mu c)^a=[iQ_B,A_\mu{}^a],\ -\frac{g}{2}(c\times c)^a=\{iQ_B,c^a\},$$

$$gc^a\frac{\lambda^a}{2}\Gamma q=[Q_B,\Gamma q],\ -g\bar{q}\Gamma\frac{\lambda^a}{2}c^a=[Q_B,\bar{q}\Gamma], \tag{5·70b}$$

where $[\Gamma,\lambda^a]=0$. The first group (5·70a) consists of local gauge invariant color singlets. Although the observables in the second group (5·70b) are color non-singlets, they are all *trivial* ones, that is, they reduce to 0 in $H_{\rm phys}$, in accordance with Theorem 5. 6.

From the examples (5·63), (5·68) and (5·70), we conjecture that every *trivial* local observable A has such a form as

$$A=[Q_B,M]_\pm \tag{5·71}$$

and that a *non-trivial* local observable A is written as the sum of some *trivial* observables and a local gauge invariant operator F composed of $A_\mu{}^a$ and φ_i without B^a, c^a, $\bar{c}^a$:

$$A=F(A_\mu,\varphi_i)+[Q_B,M(A_\mu,\varphi_i,B,c,\bar{c})]_\pm\,. \tag{5·72}$$

Although the general proof of (5·71) and (5·72) has not been given yet, we can prove the following proposition on the assumption of asymptotic completeness.

Proposition 5. 9.*) Let A be a local observable, then A can be written in the form

$$A=P^{(0)}AP^{(0)}+e^{\pm\pi Q_c}[\bar{Q}_B,R] \tag{5·73}$$

with some operator R. $P^{(0)}$ is the projection operator onto $\mathcal{H}_{\rm phys}$ defined in (3·26) of Chap. III.

Proof) As before, we can assume that either $[Q_B,A]=0$ or $\{Q_B,A\}=0$ holds: $[Q_B,A]_\mp=0$. Then, making use of the completeness relation (3·27c)

*) This proposition was found in discussions with Mr. H. Hata. The authors would like to thank him.

$$P^{(0)}+\sum_{n\geq 1}P^{(n)}=1 \tag{3·27c}$$

with $P^{(n)}=\{Q_B, R^{(n)}\}$ $(n\geq 1)$ (3·29), we obtain

$$\begin{aligned} A&=\sum_{m,n}P^{(m)}AP^{(n)} \\ &=P^{(0)}AP^{(0)}+\sum_{m\geq 1}\{Q_B, R^{(m)}\}AP^{(0)}+\sum_{\substack{m\geq 0\\ n\geq 1}}P^{(m)}A\{Q_B, R^{(n)}\} \\ &=P^{(0)}AP^{(0)}+[Q_B, \sum_{m\geq 1}R^{(m)}AP^{(0)}\pm\sum_{\substack{m\geq 0\\ n\geq 1}}P^{(m)}AR^{(n)}]_{\pm}\,. \end{aligned} \tag{5·74}$$

In (5·74), we have used the commutativity $[Q_B, P^{(n)}]=0$ $(n\geq 0)$ (3·28). □

Corollary 5.10. Let A be a *trivial* observable, then A has the following form:

$$A=e^{\pm\pi Q_c}[\bar{Q}_B, R] \tag{5·75}$$

with some operator R.

Proof) Since A satisfies the equation

$$\langle\Phi|A|\Psi\rangle=0 \quad \text{for } |\Phi\rangle, |\Psi\rangle\in\mathcal{V}_{\text{phys}}\,, \tag{5·76}$$

we obtain

$$\langle f|P^{(0)}AP^{(0)}|g\rangle=0 \quad \text{for } any\ |f\rangle, |g\rangle\in\mathcal{V}\,. \tag{5·77}$$

This is because $P^{(0)}\mathcal{V}=\mathcal{H}_{\text{phys}}\subset\mathcal{V}_{\text{phys}}$ holds. (5·77) is nothing but

$$P^{(0)}AP^{(0)}=0\,. \tag{5·78}$$

□

The above decomposition (5·73) tells us that such contributions from unphysical fields are cancelled in physically meaningful observable quantities and that our theory is really a *theory of the YM field* in spite of the presence of those auxiliary unphysical fields.

§ 5. 3. Characterization of localized physical states as group invariants ——*Absence of localized colored physical states*——

As a remarkable conclusion obtained from Theorem 5. 6, the following theorem stating the absence of *localized* colored (charged) physical states in QCD (QED) can be proved.

Theorem 5. 11.[3),6)] Let $|\Phi\rangle$ be a localized physical state, namely, a physical state $|\Phi\rangle\in\mathcal{V}_{\text{phys}}$ written by a suitable local operator $\varphi\in\mathfrak{F}(\mathcal{O})$ with finite space-time region $\mathcal{O}$ in such a form as

$$|\Phi\rangle=\varphi|0\rangle\,. \tag{5·79}$$

Then, $|\hat{\Phi}\rangle \in H_{\text{phys}}$ satisfies the condition

$$\hat{Q}^a|\hat{\Phi}\rangle = 0 \tag{5·80}$$

for the global charge Q^a of the *unbroken* global gauge symmetry.

Proof) Since $|\Phi\rangle = \varphi|0\rangle$ is a physical state: $Q_B|\Phi\rangle = Q_B\varphi|0\rangle = 0$, Lemma 5.5 tells us that φ is a *local observable*. Then, by virtue of Theorem 5.6, $\hat{\varphi}$ satisfies, for the unbroken charge Q^a,

$$[\hat{Q}^a, \hat{\varphi}] = 0\,,$$

and hence,

$$\hat{Q}^a|\hat{\Phi}\rangle = \hat{Q}^a\hat{\varphi}|\hat{0}\rangle = [\hat{Q}^a, \hat{\varphi}]\,|\hat{0}\rangle + \hat{\varphi}\hat{Q}^a|\hat{0}\rangle = 0\,. \qquad \square$$

Thus, every localized physical state in QCD (QED) is a color singlet (chargeless state) as long as the global color symmetry (global $U(1)$ symmetry) is not broken spontaneously. Needless to say, the above statement concerns only *localized* physical states ($\in \mathfrak{F}(\mathcal{O})\,|0\rangle \cap \mathcal{V}_{\text{phys}}$) and says nothing about *non-localized* physical states. In fact, if (5·80) held for *every* physical state without any restriction, then the electron in QED could not exist in this world. In the case of QED, what Theorem 5.11 tells us is that the charged physical state cannot be realized in a finite space-time region because of the long-range Coulomb tails.[3]

However, one should note that the gap between the localized state and the non-localized state is made subtle by the Reeh-Schlieder theorem [(A·24′) in Appendix A]:

$$\begin{aligned}\mathcal{V} &= \overline{\mathfrak{F}|0\rangle}^{\tau} = \mathfrak{F}|0\rangle^{\perp\perp}\\ &= \overline{\mathfrak{F}(\mathcal{O})\,|0\rangle}^{\tau} = \mathfrak{F}(\mathcal{O})\,|0\rangle^{\perp\perp}.\end{aligned} \tag{5·81}$$

Owing to this theorem, *every* state in $\mathcal{V}$ can be approximated by *localized* states as closely as one likes (in the sense of the arbitrary admissible topology τ, the weakest one of which is the weak topology (w)). So, one can surely approximate any non-localized physical state by *localized* states. If this approximation can be done for *every* physical state using *localized physical* states exclusively, then the color confinement is achieved. Namely:

Proposition 5.12.[3] If the equality

$$\overline{\mathfrak{F}(\mathcal{O})\,|0\rangle \cap \mathcal{V}_{\text{phys}}}^{\tau} = \overline{\mathfrak{F}(\mathcal{O})\,|0\rangle}^{\tau} \cap \overline{\mathcal{V}_{\text{phys}}}^{\tau}\;(=\mathcal{V}_{\text{phys}}) \tag{5·82}$$

holds, *every* physical state $|\Phi\rangle \in \mathcal{V}_{\text{phys}}$ satisfies the equality (5·80) for the global charge Q^a of the unbroken global gauge symmetry:

$$\hat{Q}^a|\hat{\Phi}\rangle = 0\,. \tag{5·80}$$

Remark: Note that, by (5·29) of Lemma 5.5, the equation (5·82) is equivalent to

$$\mathcal{V}_{\rm phys}=\overline{\mathcal{A}(\mathcal{O})|0\rangle}^{\tau}\,(=\mathcal{A}(\mathcal{O})|0\rangle^{\perp\perp}), \tag{5·82'}$$

from which the Reeh-Schlieder property in $H_{\rm phys}$ with respect to the local observable algebra $\hat{\mathcal{A}}(\mathcal{O})$ follows:

$$H_{\rm phys}=\overline{\hat{\mathcal{A}}(\mathcal{O})|\hat{0}\rangle}. \tag{5·83}$$

The conclusion (5·80) of Proposition 5.12 can really be obtained from this condition (5·83) *weaker* than the one (5·82):

Proof of the implication (5·83) ⇒ (5·80) By the assumption (5·83), for any given $|\Phi\rangle, |\Psi\rangle\in\mathcal{V}_{\rm phys}$*) and $\varepsilon>0$, there exists $\hat{\varphi}\in\hat{\mathcal{A}}(\mathcal{O})$ such that

$$|\langle\hat{\Psi}|\hat{Q}^a(|\hat{\Phi}\rangle-\hat{\varphi}|\hat{0}\rangle)|<\varepsilon. \tag{5·84}$$

Since $\hat{Q}^a\hat{\varphi}|\hat{0}\rangle=[\hat{Q}^a,\hat{\varphi}]|\hat{0}\rangle=0$ by Theorem 5.6, we obtain

$$|\langle\hat{\Psi}|\hat{Q}^a|\hat{\Phi}\rangle|<\varepsilon \quad \text{for } \forall\varepsilon>0, \tag{5·85}$$

which says

$$\langle\hat{\Psi}|\hat{Q}^a|\hat{\Phi}\rangle=0 \quad \text{for } |\hat{\Phi}\rangle, |\hat{\Psi}\rangle\in H_{\rm phys}, \tag{5·86}$$

or equivalently (5·80). □

From this result we know that the Reeh-Schlieder property (5·83) should *not* hold in the case of QED in order to secure the existence of the electron in this world. In this connection, it may be instructive to remark the role of the Reeh-Schlieder property played in the proof of Lemma 5.7. In fact, if this property in $H_{\rm phys}$ with respect to *any local* field algebra held for QED, then the *Maxwell equation* (5·35a) in $H_{\rm phys}$ would lead us again to the absurd conclusion:

$$\text{electric charge } \hat{Q}=0 \quad \text{in } H_{\rm phys}, \tag{5·80'}$$

according to Lemma 5.7, because the electromagnetic $U(1)$ symmetry should not suffer from spontaneous breakdown. Thus, in order to get rid of this pitfall, any type of the Reeh-Schlieder property should be invalidated in QED.

On the contrary, (5·83) is desirable for QCD. If it holds, the physical world $H_{\rm phys}$ of hadrons is described according to the principles of the ordinary local quantum field theory completely in terms of the color singlet local observable fields $(\hat{\mathcal{A}}(\mathcal{O}))$ identified with the *hadron* fields.

*) Precisely speaking, $|\Psi\rangle$ should belong to the domain, $\mathrm{Dom}(\hat{Q}^a)^\dagger$, of $(\hat{Q}^a)^\dagger$.

§ 5.4. Confining q-$\bar{q}$ Potential and Cluster Property

In the intuitive picture of the quark confinement, quark and antiquark q, $\bar{q}$ should be intermediated by a string-like object which produces a q-$\bar{q}$ potential not decreasing at infinity to confine quarks inside the hadron. This means the failure of the *cluster property*, while it was proved by Araki, Hepp and Ruelle[7] that the cluster property should hold in a Lorentz covariant *local* field theory with a *unique vacuum*. From these circumstances, one would like to conclude that quark confinement contradicts the usual framework of the local field theory. In such a quantum field theory with an indefinite metric as the present case, however, this is not the case as was pointed out by Strocchi.[4] It can be understood by the following generalization[4] of the cluster property theorem obtained by Araki, Hepp and Ruelle[7] to the indefinite metric case.

Theorem 5.13 [Strocchi]. On the assumption of

(i) covariance under translations,

(ii) local commutativity,

(iii) uniqueness of the vacuum

and (iv) the spectrum condition

$$\begin{cases} \text{a) with a mass gap } (0, M), \\ \text{or} \\ \text{b) without mass gap,} \end{cases}$$

one obtains an inequality:

$$|\langle 0|B_1(x_1)B_2(x_2)|0\rangle - \langle 0|B_1(x_1)|0\rangle\langle 0|B_2(x_2)|0\rangle|$$

$$\leq \begin{cases} C[\xi]^{-3/2}\exp(-M[\xi])[\xi]^{2N}(1+|\xi^0|/[\xi]), & \cdots \text{ a)} \\ \text{or} & \\ C'[\xi]^{-2}[\xi]^{2N}(1+|\xi^0|/[\xi]^2), & \cdots \text{ b)} \end{cases} \tag{5·87}$$

where

$$B_i(x_i) \equiv \int d^4x_1' \cdots d^4x'_{r(i)} f_i(x_1', \cdots, x'_{r(i)}) \Phi(x_1'+x_i) \cdots \Phi(x'_{r(i)}+x_i),$$

$f_i \in \mathscr{D}(\boldsymbol{R}^{4r(i)})$ [C^∞-functions with compact support],

$\xi \equiv x_1 - x_2$.

The above Φ is a generic notation for fields and N is a suitable non-negative integer dependent on the B_i's. $[\xi]$ is defined as the shortest space-like distance between ξ and a certain compact set which depends on the supports of the f_i's. If the Fourier transform $\tilde{h}_{12}(p)$ of

$$h_{12}(\xi) \equiv \langle 0|B_1(x_1)B_2(x_2)|0\rangle - \langle 0|B_1(x_1)|0\rangle\langle 0|B_2(x_2)|0\rangle \tag{5·88}$$

is a measure, as in the case with a positive metric,[7),8)] the integer N is 0 and thus the cluster property holds[7)] whether with or without mass gap.

Proof) [See, Ref. 4).] □

The above theorem tells us that the cluster property *may fail* without contradicting the usual axioms of quantum field theory only in the case *with an indefinite metric and without mass gap* ($N\neq0$ in the case b) of (5·87)). Here note that the mass spectrum in the assumption (iv) refers to the one in the *whole* state vector space $\mathcal{V}$ and that the *physical* spectrum in H_{phys} may have a mass gap also in the case b), as is expected for quark confinement in QCD as well as for cases with the Higgs phenomenon.[9)~11)]

As an example case of (5·87), we consider the case of the gauge field A_μ and its field strength $F_{\mu\nu}$ in QED. Setting

$$\begin{aligned}&\langle 0|A_\mu(x)\,A_\nu(y)\,|0\rangle\\&\quad=\langle 0|A_\mu(x)\,A_\nu(y)\,|0\rangle-\langle 0|A_\mu(x)\,|0\rangle\langle 0|A_\nu(y)\,|0\rangle\\&\quad\equiv g_{\mu\nu}F(x-y)+\partial_\mu\partial_\nu G(x-y)\,,\end{aligned}\qquad(5\cdot89)$$

we obtain

$$\begin{aligned}&\langle 0|F_{\mu\nu}(x)\,F_{\rho\sigma}(y)\,|0\rangle\\&\quad=-(g_{\mu\rho}\partial_\nu\partial_\sigma-g_{\nu\rho}\partial_\mu\partial_\sigma-g_{\mu\sigma}\partial_\nu\partial_\rho+g_{\nu\sigma}\partial_\mu\partial_\rho)\,F(x-y)\,.\end{aligned}\qquad(5\cdot90)$$

Since $F_{\mu\nu}$ in QED is an *observable*, we obtain

$$\langle 0|F(f)^*F(f)\,|0\rangle\geq 0\,,\qquad(5\cdot91)$$

where

$$F(f)\equiv\int d^4x F_{\mu\nu}(x)\,f^{\mu\nu}(x)\ \ \text{with}\ \ f^{\mu\nu}\in\mathcal{S}(\boldsymbol{R}^4).$$

Thus, the Fourier transform of $\langle 0|F_{\mu\nu}(x)\,F_{\rho\sigma}(y)\,|0\rangle$ is a measure and, by b) of (5·87) with $N=0$, $F(\xi)$ damps at infinity at least as $\sim[\xi]^{-2}$. (Note that $\partial_\mu\partial_\nu G(x-y)$ in (5·89) is an unphysical gauge part.)

On the contrary, since $F^a_{\mu\nu}$ in QCD is *not* an observable as has been shown in § 5. 2 (5·40), and since such a simple relation between (5·89) and (5·90) does not hold, no restrictions for $F^{ab}(x-y)$ in $\langle 0|A_\mu{}^a(x)\,A_\nu{}^b(y)\,|0\rangle$ corresponding to the $F(x-y)$ in (5·89) are obtained. In the case with $N\geq1$ for b) of (5·87), the q-$\bar{q}$ potential obtained from $\langle 0|A_\mu{}^a(x)\,A_\nu{}^b(y)\,|0\rangle$ may not decrease at infinity (confining q-$\bar{q}$ potential) and then, it could happen that the gluon fields $A_\mu{}^a$ have no asymptotic field (gluon confinement). The following theorem[12)] suggests more convincingly the failure of the cluster property for unobservable quantities.

Theorem 5. 14.[12)] Let A_1 and A_2 be local operators belonging, respectively, to $\mathfrak{F}(\mathcal{O}_1)$ and $\mathfrak{F}(\mathcal{O}_2)$ satisfying the following two conditions:

(i) $A_1(x)A_2(y)$ is an observable for any $x, y \in \boldsymbol{R}^4$, where $A_i(x) \equiv U(x)A_iU(x)^{-1}$ with the translation operator $U(x)$,

(ii) A_1 and A_2 satisfy the cluster property in the sense that

$$\lim_{-x^2\to+\infty} |\langle 0|\varphi_1 A_1 U(x) A_2\varphi_2|0\rangle - \langle 0|\varphi_1 A_1|0\rangle\langle 0|A_2\varphi_2|0\rangle| = 0 \qquad (5\cdot 92)$$

holds for any*) local operators φ_1 and φ_2,

then, both A_1 and A_2 are local *observables*.

Proof) Since $A_1(x)A_2(y) \in \mathfrak{F}((\mathcal{O}_1+x) \cup (\mathcal{O}_2+y))$ is a local observable according to (i), Proposition 5. 4 tells us the equality

$$[\bar{Q}_B, A_1(x)A_2(y)] = 0 \quad \text{for } \forall x, y \in \boldsymbol{R}^4, \qquad (5\cdot 93)$$

which is equivalent to

$$[\bar{Q}_B, A_1U(x)A_2] = 0 \quad \text{for } \forall x \in \boldsymbol{R}^4, \qquad (5\cdot 93')$$

because of $[\bar{Q}_B, U(x)] = 0$. Since $[\bar{Q}_B, \varphi_1]$ and $[\bar{Q}_B, \varphi_2]$ for *any* local operators φ_1, φ_2 are also local operators, we obtain, from (5·93′), (2·29) and (5·92),

$$\begin{aligned} 0 &= \lim_{-x^2\to+\infty} \langle 0|\varphi_1[\bar{Q}_B, A_1U(x)A_2]\bar{Q}_B\varphi_2|0\rangle \\ &= \lim_{-x^2\to+\infty} \langle 0|\varphi_1\bar{Q}_BA_1U(x)A_2\bar{Q}_B\varphi_2|0\rangle \\ &= \lim_{-x^2\to+\infty} \langle 0|[\varphi_1, \bar{Q}_B]A_1U(x)A_2[\bar{Q}_B, \varphi_2]|0\rangle \\ &= \langle 0|[\varphi_1, \bar{Q}_B]A_1|0\rangle\langle 0|A_2[\bar{Q}_B, \varphi_2]|0\rangle \\ &= \langle 0|\varphi_1\bar{Q}_BA_1|0\rangle\langle 0|A_2\bar{Q}_B\varphi_2|0\rangle, \end{aligned} \qquad (5\cdot 94)$$

from which at least one of the following equations holds:

$$\langle 0|\varphi_1\bar{Q}_BA_1|0\rangle = 0 \quad \text{for } \forall\varphi_1 \in \mathfrak{F}(\mathcal{O}_1), \qquad (5\cdot 95\text{a})$$

or

$$\langle 0|A_2\bar{Q}_B\varphi_2|0\rangle = 0 \quad \text{for } \forall\varphi_2 \in \mathfrak{F}(\mathcal{O}_2). \qquad (5\cdot 95\text{b})$$

Consider the case of (5·95a), then we obtain

$$\bar{Q}_BA_1|0\rangle = 0, \qquad (5\cdot 96)$$

*) It is, in fact, sufficient that this (5·92) holds *only* for local operators φ_1, φ_2 of the form $\varphi_i = [\bar{Q}_B, \varphi_i']$, as is seen from (5·94) in the following proof.

by virtue of the Reeh-Schlieder property (A·27):

$$(\mathcal{F}(\mathcal{O})\,|0\rangle)^{\perp} = (\mathcal{F}|0\rangle)^{\perp} = \mathcal{V}^{\perp} = 0\,. \tag{5·97}$$

By Lemma 5.5, Eq. (5·96) implies

$$[\bar{Q}_B, A_1] = 0\,, \tag{5·98}$$

which yields, by the help of (5·93),

$$0 = [\bar{Q}_B, A_1]\,A_2 + A_1[\bar{Q}_B, A_2] = A_1[\bar{Q}_B, A_2]\,. \tag{5·99}$$

Thus we have*)

$$[\bar{Q}_B, A_1] = [\bar{Q}_B, A_2] = 0\,.$$

The same arguments apply also to the case (5·95b). □

From this theorem, we know that, if local operators A_1 and A_2, at least one of which is *not* an observable, define an *observable* of the form $A_1(x)\,A_2(y)$, then the cluster property (5·92) for A_1 and A_2 is really broken down. The failure of the cluster property means that the correlation between $A_1(x)$ and $A_2(y)$ cannot be switched off, however far they are separated $(-(x-y)^2 \to +\infty)$. This implies that it is impossible for us to detect the quanta of A_1 and A_2 separately.

Thus, although the physical states are specified by a non-local condition (2·29): $Q_B|\mathrm{phys}\rangle = 0$, in terms of the volume-integrated charge Q_B, such a kind of difficulties as the "behind-the-moon" problem[13] does not arise in our formalism: Namely, we *need not* worry about the possibility of such a state that it is physical *as a whole* whereas it can be divided into *widely separated unphysical* two subsystems, like an FP ghost on the earth and an anti-FP ghost behind the moon. This is because, if such separation can be performed *satisfying the cluster property*, which would require these two subsystems to be *detected separately*, they should, by Theorem 5.14, be *physical in themselves* from the beginning. On the other hand, if those subsystems are *unphysical*, then the failure of the cluster property due to Theorem 5.14 prevents us from performing on the earth the detection of the FP ghost in a manner independent of the anti-FP ghost behind the moon, which physically means nothing but the impossibility of the detection of this FP ghost. In this way, the *failure of the cluster property in the unphysical world* operates to protect unphysical particles from being brought to light.

On the other hand, the cluster property for local observables is ensured in the *physical* space $H_{\mathrm{phys}} = \overline{\mathcal{V}_{\mathrm{phys}}/\mathcal{V}_0}$ with the *positive definite metric*, where the system of local observable algebras $\hat{\mathcal{A}}(\mathcal{O})$ will safely belong to the

*) Precisely speaking, we should assume that local field algebra $\mathcal{F}(\mathcal{O})$ does not contain a zero divisor, because, among the operators in $\mathcal{V}$, there exists such a nilpotent operator as Q_B: $Q_B{}^2 = 0$ but $Q_B \neq 0$.

category characterized by the usual Haag-Kastler-Araki axioms.[14),15),*)] In this situation, if (the representation of) the algebra $\hat{\mathcal{A}}$ of quasi-local observables*) is *reducible* in H_{phys}, the superselection rule holds[1),14),16)] and group non-invariant physical states may appear, as expected for the case of QED. On the contrary, the condition (5·83): $H_{\text{phys}}=\overline{\hat{\mathcal{A}}(\mathcal{O})|\hat{0}\rangle}$, from which the color confinement $\hat{Q}^a=0$ follows (Proposition 5.12), is equivalent to the condition of the *irreducibility* of (the representation of) $\hat{\mathcal{A}}$ in H_{phys}. In any case, if only we can show the color confinement: $\hat{Q}^a=0$ in H_{phys}, then the above Theorem 5.14 guarantees that every thing goes well about the quark confinement: The *failure of the cluster property in the unphysical colored sectors* prevents colored objects from coming out of colorless hadrons to be detected, permitting only the color singlets to appear in the physical world H_{phys}. In this colorless physical world H_{phys}, the *validity of the cluster property* enables us to perform physical measurements on the earth without worrying about the things behind the moon. In the next chapter, we will discuss the problem of color confinement: $\hat{Q}^a=0$ in H_{phys}, contrasting it with the Higgs phenomenon.

*) Precisely speaking, the Haag-Kastler-Araki axioms are formulated in terms of the *bounded* operators, which should be obtained, in a certain canonical manner, from the unbounded ones treated here. "Quasi-local observables" are defined, in the former version, as the operator-norm limits of local observables.

Chapter VI

Global Gauge Symmetry and Structure of the Associated Charges and Currents

§ 6. 1. Massive Gauge Bosons and Higgs Phenomenon

In this chapter, we discuss the consequences about the global gauge symmetry derivable from the "Maxwell" equation (2·36). This equation

$$gJ_\mu{}^a=\partial^\nu F^a_{\mu\nu}+\{Q_B,\ (D_\mu\bar{c})^a\}\tag{6·1}$$

tells us that the current $gJ_\mu{}^a$ consists of two parts,

$$\partial^\nu F^a_{\mu\nu}\equiv\mathcal{G}_\mu{}^a\tag{6·2}$$

and

$$\{Q_B,\ (D_\mu\bar{c})^a\}\equiv\mathcal{N}_\mu{}^a,\tag{6·3}$$

each of which is *conserved*:

$$\partial^\mu\mathcal{G}_\mu{}^a=\partial^\mu\mathcal{N}_\mu{}^a=0\,.\tag{6·4}$$

These currents, therefore, yield (formally) conserved charges G^a and N^a:

$$G^a\equiv\int d^3x\mathcal{G}_0{}^a=\int d^3x\partial^i F^a_{0i}\,,\tag{6·5}$$

$$N^a\equiv\int d^3x\mathcal{N}_0{}^a=\int d^3x\{Q_B,\ (D_0\bar{c})^a\},\tag{6·6}$$

and the global charge Q^a (5·43) of the global gauge symmetry is the sum of these two conserved charges:

$$gQ^a=\int d^3xgJ_0{}^a=G^a+N^a.\tag{6·7}$$

The characteristic forms (6·5) and (6·6) of these charges G^a and N^a reveal some interesting aspects of the global gauge symmetry with the charge Q^a, in the light of the *Goldstone theorem*[1]. The various versions of this theorem, Theorem B. 2, Corollaries B. 3 and B. 4 in Appendix B, state that the following three conditions concerning a conserved current J_μ and its global charge Q are all equivalent:

(a) $Q\equiv\int d^3xJ_0$ is a *well-defined* charge;

(b) Q does *not* suffer from *spontaneous symmetry breakdown*;

(c) J_μ contains *no discrete massless spectrum*: $\langle 0|J_\mu|\Psi(p^2=0)\rangle=0$.

Lemma 6.1.[3] If a linear combination, $\mathcal{G}_\mu{}^A \equiv \alpha_a{}^A \mathcal{G}_\mu{}^a = \alpha_a{}^A \partial^\nu F^a_{\mu\nu}$, of $\mathcal{G}_\mu{}^a$'s with some coefficients $\alpha_a{}^A$ contains *no discrete massless pole,*

$$\langle 0|\mathcal{G}_\mu{}^A|\Psi(p^2=0)\rangle = 0\,, \tag{6·8}$$

or equivalently, if the global charge G^A given by

$$G^A \equiv \int d^3x \mathcal{G}_0{}^A = \alpha_a{}^A G^a \tag{6·9}$$

is a *well-defined* charge, then G^A *is nothing but* 0:

$$G^A = 0\,. \tag{6·10}$$

Proof) The equivalence of the condition (6·8) to the well-definedness of G^A is stated above (Corollary B. 4 in Appendix B). Then, this proposition is nothing but the previous Proposition 5.7 applied to the case with $K_{\nu\mu} = \alpha_a{}^A F^a_{\mu\nu}$. □

Now, we utilize the information (3·21), (3·23) and (3·24) about the massless asymptotic fields constituting the "elementary" quartet. First, note that the $\chi^a(x)$ field in $A_\mu{}^a(x)^{\mathrm{as}}$ makes no contribution to $F^a_{\mu\nu}(x)^{\mathrm{as}}$ owing to the anti-symmetry of its suffices μ and ν, and that the contributions to $F^a_{\mu\nu}(x)^{\mathrm{as}}$ come from the (massless or massive) genuine *vector* fields $U_\mu{}^A$ which are assumed to be contained in $A_\mu{}^a$ with the weight $\tilde{\alpha}_A{}^a$

$$A_\mu{}^a(x)^{\mathrm{as}} = \partial_\mu \chi^a(x) + \tilde{\alpha}_A{}^a U_\mu{}^A(x) + \cdots\,. \tag{6·11}$$

The coefficients $\tilde{\alpha}_A{}^a$ should properly be taken into account, in the cases with particle mixing, so that the $U_\mu{}^A$'s represent particles with *definite masses.* We suppose that, for the *eigenchannel* of $U_\mu{}^A$, $F^A_{\mu\nu}(x)^{\mathrm{as}}$ is given, with some coefficients $\alpha_a{}^A$,*) by

$$F^A_{\mu\nu}(x)^{\mathrm{as}} = \alpha_a{}^A F^a_{\mu\nu}(x)^{\mathrm{as}} = \partial_\mu U_\nu{}^A(x) - \partial_\nu U_\mu{}^A(x) + \cdots\,. \tag{6·12}$$

Then, from this and Lemma 6.1, we obtain,

Corollary 6.2. If $U_\mu{}^A$ is *massive,* then $G^A = 0$.

Next, in order to examine the charges N^a, we have to introduce dynamical parameters $u_b{}^a$ defined as the pole residues of $gA_\mu \times \bar{c}$ at $p^2=0$:

$$g(A_\mu \times \bar{c})^a(x)^{\mathrm{as}} = u_b{}^a \partial_\mu \bar{\gamma}^b(x) + \cdots\,, \tag{6·13}$$

which can be estimated by the formula due to (3·21c) and (3·24):

*) Owing to the contributions from the term $gA_\mu \times A_\nu$, the matrix $\alpha_a{}^A$ is, in general, neither the inverse of $\tilde{\alpha}_A{}^a$ nor unitary.

$$\int d^4x e^{ip(x-y)} \langle 0|T[(D_\mu c)^b(x) g(A_\nu \times \bar{c})^a(y)]|0\rangle$$
$$= -u_b{}^a p_\mu p_\nu / p^2 + \cdots . \tag{6·14}$$

Then, we obtain

$$(D_\mu \bar{c})^a(x)^{\text{as}} = (\delta_b{}^a + u_b{}^a)\partial_\mu \bar{\gamma}^b(x) + \cdots .^{*)} \tag{6·15}$$

Lemma 6.3.[3] The conserved current $\xi^a \mathscr{N}_\mu^a$ yields a well-defined charge of the form $\xi^a N^a = \int d^3x \xi^a \mathscr{N}_0^a$, if and only if ξ^a satisfies

$$(\delta_b{}^a + u_b{}^a)\xi^a = 0 \quad \text{for all } b = 1, \cdots, n \tag{6·16}$$

or equivalently

$$(\mathbf{1} + u^T)\begin{pmatrix} \xi^1 \\ \vdots \\ \xi^n \end{pmatrix} = 0 . \tag{6·16'}$$

Proof) By Corollary B. 4 of the Goldstone theorem in Appendix B, a necessary and sufficient condition for $\xi^a N^a$ to be a well-defined charge is given as

$$\langle 0|\xi^a \mathscr{N}_\mu^a(x)|\Psi(p^2=0)\rangle = 0 \tag{6·17}$$

for any massless 1-particle state $|\Psi(p^2=0)\rangle$. Using the relation

$$\langle 0|\Phi(x)|\text{1-particle}\rangle = \langle 0|\Phi(x)^{\text{as}}|\text{1-particle}\rangle \tag{C·3}$$

which follows from the Yang-Feldman equation (C·2), we can obtain, from (6·3) and (6·15),

$$\langle 0|\xi^a \mathscr{N}_\mu^a(x)|\Psi(p^2=0)\rangle = \xi^a \langle 0|\{Q_B, (D_\mu \bar{c})^a(x)\}|\Psi(p^2=0)\rangle$$
$$= \xi^a \langle 0|(D_\mu \bar{c})^a(x) Q_B|\Psi(p^2=0)\rangle$$
$$= \xi^a \langle 0|(D_\mu \bar{c})^a(x)^{\text{as}} Q_B|\Psi(p^2=0)\rangle$$
$$= \xi^a(\delta_c{}^a + u_c{}^a)\partial_\mu^x \langle 0|\bar{\gamma}^c(x) Q_B|\Psi(p^2=0)\rangle . \tag{6·18}$$

Since the only non-vanishing matrix element $\langle 0|\bar{\gamma}^c(x) Q_B|\Psi(p^2=0)\rangle$ in (6·18) is given by taking $|\Psi(p^2=0)\rangle = \chi^b(y)|0\rangle$, with the help of (3·23a) and (3·24a):

$$\langle 0|\bar{\gamma}^c(x) Q_B(\chi^b(y)|0\rangle) = \langle 0|\bar{\gamma}^c(x)[Q_B, \chi^b(y)]|0\rangle$$
$$= \langle 0|\bar{\gamma}^c(x)(-i\gamma^b(y))|0\rangle$$
$$= -\delta^{cb} D_+(x-y) , \tag{6·19}$$

---

*) The dots (…) represent other possible *massive* components contained in $(D_\mu \bar{c})^a$.

the condition (6·17) in question turns out to be equivalent to the condition (6·16), as is easily seen from (6·18):

$$\langle 0|\xi^a \mathcal{N}_\mu^a(x)\chi^b(y)|0\rangle = -(\delta_b{}^a + u_b{}^a)\xi^a \partial_\mu{}^x D_+(x-y). \quad (6\cdot 20)$$

□

Corollary 6.4.[3] If $\det(\mathbf{1}+u)\neq 0$, every charge $\xi^a N^a$ is an ill-defined charge suffering from spontaneous breakdown, except for the trivial $\xi^a \equiv 0$. The Goldstone boson responsible for the spontaneous breakdown of N^a is given by χ^b corresponding to the suffix $\delta_b{}^a + u_b{}^a \neq 0$.

Corollary 6.5.[3] If $u=-\mathbf{1}$, $\xi^a N^a$ for any ξ^a is a well-defined charge.

Remark. In the Abelian case where the structure constant $f^{abc}\equiv 0$, the term $(gA_\mu \times \bar{c})^a = gf^{abc}A_\mu{}^b \bar{c}^c$ vanishes from the beginning, and hence, $u_b{}^a$ in (6·13) is identically zero:

$$u \equiv 0 .$$

Thus, the condition of Corollary 6.4 is always satisfied: $\det(\mathbf{1}+u) = \det \mathbf{1} = 1 \neq 0$, and the charge $N = \int d^3x\{Q_B, D_0\bar{c}\} = \int d^3x \partial_0 B$ is *not* well-defined.

In the case $\det(\mathbf{1}+u)\neq 0$, we obtain the following consequence:

Theorem 6.6[3] (Converse of the Higgs theorem). If $\det(\mathbf{1}+u)\neq 0$, for *each* eigenchannel of the *massive* gauge bosons $U_\mu{}^A$, the global gauge symmetries corresponding to the charges $Q^A = \alpha_a{}^A Q^a$ are *broken spontaneously*.

Proof) From Corollary 6.2, $G^A = 0$ follows for each massive $U_\mu{}^A$, and hence, we obtain

$$gQ^A = G^A + N^A = N^A = \alpha_a{}^A N^a. \quad (6\cdot 21)$$

Since $(\alpha_1{}^A, \cdots, \alpha_n{}^A) \not\equiv 0$ for the very existence of the massive gauge boson $U_\mu{}^A$, Corollary 6.4 asserts the spontaneous breakdown of the charge Q^A:

$$\begin{aligned}\langle 0|gJ_\mu{}^A(x)\chi^b(y)|0\rangle &= \langle 0|\mathcal{N}_\mu{}^A(x)\chi^b(y)|0\rangle \\ &= -\alpha_a{}^A(\delta_b{}^a + u_b{}^a)\partial_\mu{}^x D_+(x-y) \neq 0 . \quad (6\cdot 22)\ \square\end{aligned}$$

Here, a few remarks are in order:

Remark. i) If $\hat{\alpha} = (\hat{\alpha}_b{}^A) = (\alpha_a{}^A(\delta_b{}^a + u_b{}^a))$ $(A, b = 1, \cdots, n)$ is a non-singular (n, n)-matrix [where n is the dimension of the Lie algebra of the gauge group G], we can identify the Goldstone boson responsible for the spontaneous breakdown of Q^A with χ^A defined by

$$\chi^A \equiv (\hat{\alpha}^{-1})_A{}^a \chi^a . \quad (6\cdot 23)$$

In fact, it satisfies

$$\langle 0|gJ_\mu{}^A(x)\chi^B(y)|0\rangle = -\delta^{AB}\partial_\mu{}^x D_+(x-y) \tag{6·24}$$

for any channel B and the massive-gauge-boson channel A. In the example of the $SU(2)$ Higgs-Kibble model discussed in Chap. 4, we already know the explicit form of symmetry breaking:

$$A_\mu{}^{a,\mathrm{as}} = \sqrt{K}\partial_\mu\chi^{a,\mathrm{as}} + (\sqrt{K}-\alpha N)\partial_\mu B^{a,\mathrm{as}} + U_\mu{}^{a,\mathrm{as}}, \tag{6·25a}$$

$$\alpha_a{}^A \propto \tilde{\alpha}_a{}^A \propto \hat{\alpha}_a{}^A \propto \delta_a{}^A, \tag{6·25b}$$

where the $\chi^{a,\mathrm{as}}$'s are the asymptotic fields of the "elementary" unphysical Goldstone bosons. (6·25b) is a consequence of a certain global $SU(2)$ symmetry remaining even after the symmetry breaking. Equation (6·25a) agrees with the one (6·11) in the general context, by the following identification as is noted in (4·39):

$$\chi^a \to \sqrt{Z_3}[\sqrt{K}\chi^{a,\mathrm{as}} + (\sqrt{K}-\alpha N)B^{a,\mathrm{as}}], \tag{6·26}$$

since the mixture of $B^{a,\mathrm{as}}$ components in (6·26) causes no change in (3·23a) and (3·24a) by virtue of $[Q_B, B^{a,\mathrm{as}}] = [B^{a,\mathrm{as}}, B^{b,\mathrm{as}}] = 0$.

ii) Note that, in the above Theorem 6.6, the spontaneous breakdown of a charge Q^A is *in one-to-one correspondence* with the appearance of a *massive* gauge boson $U_\mu{}^A$. This should be compared with the result obtained in Ref. 4): There, occurrence of the spontaneous breakdown of *some* global charges has been proved *only* in the case where the gauge bosons acquire *different masses* within a group multiplet. By such a result, we can neither say anything about the case where the gauge bosons acquire a common mass within the group multiplet, nor even about Abelian case. In contrast to it, the above theorem asserts the spontaneous breakdown of a particular charge Q^A corresponding to each massive gauge boson $U_\mu{}^A$ irrespectively of its mass value. It is suitable to comment*) here on the usual misunderstanding about the "Schwinger mechanism" by which the gauge bosons are believed to become massive *without spontaneous symmetry breaking*[5]: In fact, the original Schwinger mechanism found in the Schwinger model[6] is nothing but a Higgs phenomenon as was explicitly shown by Ito[7] and Nakanishi.[7] Furthermore, since our Theorem 6.6 is always applicable irrespectively of the detailed mass generation mechanism, the massive gauge boson, which is caused by the "Schwinger mechanism" (if any) or by other ones, necessarily implies the spontaneous breakdown of global symmetries.

iii) One should *not* draw, from Corollary 6.2, such a conclusion that the spontaneously broken charge Q^A vanishes in H_{phys}:

$$g\hat{Q}^A = \widehat{G^A + N^A} = \hat{N}^A = \widehat{\left\{Q_B, \alpha_a{}^A \int d^3x(D_0\bar{c})^a\right\}} = 0,$$

*) This comment is due to Professor Nakanishi, to whom the authors are indebted.

because (unless $\det(\mathbf{1}+u)=0$) the limit $R\to\infty$ of the "local charges" $gQ_R{}^A$, $N_R{}^A$ does *not* yield a well-defined charge $gQ^A=N^A$, owing to the massless contributions χ^b to N^A (6·22). For a finite $R>0$, the local charge $N_R{}^A = \{Q_B, \int d^4x\alpha_T(x_0)f_R(\boldsymbol{x})\alpha_a{}^A(D_0\bar{c})^a(x)\}$ gives a trivial observable:

$$\hat{N}_R{}^A=0 \quad \text{in } H_{\text{phys}}, \tag{6·27}$$

whereas, in this case, the *local* charge $G_R{}^A$ cannot vanish contrary to the *global* one $G^A=0$, and hence, $gQ_R{}^A$ and $G_R{}^A$ are *not* observables separately.

Next, we consider the case where the global gauge symmetry of the charge Q^A remains unbroken. According to Corollary B. 4 in Appendix B, we should have, in this case,

$$0=\langle 0|gJ_\mu{}^A|\Psi(p^2=0)\rangle=\langle 0|\mathcal{G}_\mu{}^A|\Psi(p^2=0)\rangle+\langle 0|\mathcal{N}_\mu{}^A|\Psi(p^2=0)\rangle \tag{6·28a}$$

or equivalently,

$$\begin{aligned}\langle 0|\alpha_a{}^A\partial^\nu F^a_{\mu\nu}|\Psi(p^2=0)\rangle&=-\langle 0|\mathcal{N}_\mu{}^A|\Psi(p^2=0)\rangle\\&=-\alpha_a{}^A\langle 0|(D_\mu\bar{c})^aQ_B|\Psi(p^2=0)\rangle.\end{aligned} \tag{6·28b}$$

It may happen, however, that this equation (6·28) is not satisfied in spite of the absence of the spontaneous breakdown, as is seen explicitly in the case of QED.[3),8)] This is due to the massless contribution from $\alpha_a{}^A\partial_\mu\beta^a$ remaining in $gJ_\mu{}^A$ with some weight ζ. In the case $\det(\mathbf{1}+u)\neq 0$, the problem can be settled[3)] by the following modification of the definition of the current and the charge:

$$g\tilde{J}_0{}^A(x)\equiv gJ_0{}^A(x)-\zeta\mathcal{N}_0{}^A(x), \tag{6·29a}$$

$$g\tilde{Q}^A\equiv\int d^3x g\tilde{J}_0{}^A(x)=\int d^3x(gJ_0{}^A(x)-\zeta\mathcal{N}_0{}^A(x)), \tag{6·29b}$$

where the constant ζ is adjusted so that the massless contribution in $gJ_0{}^A$ is exactly cancelled out by the subtraction of $\zeta\mathcal{N}_0{}^A$. The constant ζ in QED is given by $\zeta=1-Z_3$. In this context, the case $\zeta=1$ corresponds to the spontaneous breakdown, which is equivalent to the appearance of the massive gauge boson $U_\mu{}^A$ as is shown by Theorem 6.6. Namely, if one wants to make the charge $g\tilde{Q}^A$ well-defined, then it is nothing but zero: $g\tilde{Q}^A=\int d^3x(gJ_0{}^A-\mathcal{N}_0{}^A)=G^A=0$, while non-vanishing $\tilde{Q}^A$ requires $\zeta\neq 1$, which makes $\tilde{Q}^A$ ill-defined owing to the remaining massless contribution.

Thus, if $\det(\mathbf{1}+u)\neq 0$, the unbroken symmetry should be realized with $\zeta\neq 1$. In this case, $\tilde{Q}^a/(1-\zeta)$ should be adopted as the generator of the global symmetry, in place of the original Q^a which is ill-defined by massless contributions. In fact, one can easily check the following commutation relations:[3)]

$$[g\widetilde{Q}^a, \varphi(x)] = -g(1-\zeta)T^a\varphi(x), \tag{6·30a}$$

$$[g\widetilde{Q}^a, g\widetilde{Q}^b] = ig(1-\zeta)f^{abc}g\widetilde{Q}^c. \tag{6·30b}$$

By this modification, the "Maxwell" equation (2·33) is merely changed as

$$g\tilde{J}_\mu{}^a/(1-\zeta) = \mathcal{G}_\mu{}^a/(1-\zeta) + \mathcal{N}_\mu{}^a, \tag{6·31}$$

which preserves the whole arguments made previously by the replacement of $J_\mu{}^a$ and $\mathcal{G}_\mu{}^a$ by $\tilde{J}_\mu{}^a/(1-\zeta)$ and $\mathcal{G}_\mu{}^a/(1-\zeta)$, respectively.

The condition for the charge $\widetilde{Q}^a$ to be well-defined is now given by

$$\langle 0|\partial^\nu F^a_{\nu\mu}|\Psi(p^2=0)\rangle/(1-\zeta) = \langle 0|(D_\mu\bar{c})^a Q_B|\Psi(p^2=0)\rangle, \tag{6·32}$$

which shows that $F^a_{\mu\nu}$ should contain massless component, in order to keep the global gauge symmetry unbroken, as long as $\det(\mathbf{1}+u)\neq 0$ [see, (6·17) and (6·19)]. This can be understood in the following way. According to Theorem 6.6, every gauge boson $U_\mu{}^A = U_\mu{}^a$ contained in $A_\mu{}^a$ should, in this unbroken case, remain massless and, in fact, we know already in (4·50) that the $U_\mu{}^a$ $(=A_\mu{}^{a,\text{as}})$ contains not only the transverse components but also the scalar components (certain combination of) β^a.*) Since $[\beta^a, \chi^b]\neq 0$ by (3·24a), we obtain

$$[U_\mu{}^a(x), \chi^b(y)]\neq 0, \tag{6·33}$$

which explains (6·32) for $|\Psi(p^2=0)\rangle = \chi^b(y)|0\rangle$,

$$\langle 0|\partial^\nu F^a_{\nu\mu}|\Psi(p^2=0)\rangle \propto \langle 0|\Box U_\mu{}^a - \partial_\mu\partial^\nu U_\nu{}^a|\Psi(p^2=0)\rangle \neq 0. \tag{6·34}$$

Thus, the two *ill-defined* charges $G^a/(1-\zeta)$ and N^a conspire to give a *well-defined* charge $g\widetilde{Q}^a$ in the combination $G^a/(1-\zeta)+N^a$ owing to the cancellation of the massless contributions of β^a. From this, we know that the term $\mathcal{G}_0{}^a = \partial^i F^a_{0i}$ in the integrand of $g\widetilde{Q}^a$ should not be discarded merely for the reason of its spatial divergence form. The massless particles can contribute on the surface at infinity, making the charge G^a broken spontaneously.

§ 6. 2. Color Confinement

Now, we discuss the case $\det(\mathbf{1}+u)=0$, which may occur only in the non-Abelian case. From Lemma 6.3, we obtain

Proposition 6. 7.[3] Let $U_\mu{}^A$ be a *massive* gauge boson. If the coefficients $\hat{\alpha}_a{}^A$ $(a=1,\cdots,n)$ vanishes in this channel A:

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*) For massless channel in (6·11), precisely speaking, neither $U_\mu{}^A$ is a genuine vector nor $\chi^A \equiv (\hat{\alpha}^{-1})_a{}^A\chi^a$ is a scalar, and only the sum $a_\mu{}^A \equiv U_\mu{}^A + \partial_\mu\chi^A$ transforms properly as a vector. In fact, Eq. (4·50) shows that $\chi^A(x) = \sum_k (a_k{}^L g_k(x) + \text{h.c.})$ is the longitudinal component of $a_\mu{}^A$.

$$\hat{\alpha}_a{}^A = \alpha_b{}^A (\delta_a{}^b + u_a{}^b) = 0 \quad \text{for } a = 1, \cdots, n, \tag{6·35}$$

the charge Q^A is well-defined in spite of the massiveness of $U_\mu{}^A$.

Proof) Since, by Corollary 6.2, $G^A = 0$ follows from the massiveness of $U_\mu{}^A$, Lemma 6.3 tells us that the charge

$$gQ^A = G^A + N^A = N^A = \alpha_a{}^A N^a \tag{6·36}$$

is well-defined, if (6·35) holds. □

In the case of QCD, we can obtain a concise criterion for the color confinement as follows:

Theorem 6.8.[3] If the following two conditions are satisfied in QCD,

(A) $u = -\mathbf{1}$, namely $u_b{}^a = -\delta_b{}^a$, (6·37)

and (B) the global color gauge symmetry with charges gQ^a is *unbroken*, or equivalently,

(A) together with (B′) $\partial^\nu F_{\mu\nu}^a$ contains *no massless discrete pole*,

then, the color confinement is realized:

$$\hat{Q}^a = 0 \quad \text{in } H_{\text{phys}}. \tag{6·38}$$

Proof) By Corollary 6.5, all the charges N^a are ensured to be well-defined by the condition (A). Then, from the condition (B) and the equation (6·7), every charge G^a becomes well defined, and hence, by Lemma 6.1, it is nothing but 0: $G^a = 0$. Thus, we obtain the equation for *the well-defined charges*

$$gQ^a = N^a = \left\{ Q_B, \int d^3x (D_0 \bar{c})^a (x) \right\}, \tag{6·39}$$

which asserts the consequence (6·38). The equivalence of the two conditions (B) and (B′) on the condition (A) is easily seen from the equation (6·7). □

Corollary 6.9.[3] If the condition (A) together with the following one holds,

(B″) all the gauge bosons $A_\mu{}^a$ become massive

then, the color confinement (6·38) holds.

Proof) (B″) implies (B′), and hence, the conclusion follows immediately from Theorem 6.8. □

In QCD, it is a very important problem whether the global color symmetry remains intact or is broken spontaneously. One usually supposes that this symmetry is not broken spontaneously, because, otherwise, one cannot imagine such simple quark-configurations as $q\bar{q}$ and qqq for hadrons, and further, the quark confinement, if any, cannot be assured by the color-singletness of physical states. So, it seems natural to *assume that the color symmetry is not broken spontaneously*.

If we adopt this assumption, we know from (6·32) that the *gluons* U_μ^a *should remain massless unless* $\det(\mathbf{1}+u)=0$. In this case, their transverse components may not be forbidden to appear as physical particles, *on the mass-shell* of which the infrared divergences arise inevitably in an formidable manner.[9),*)] Thus, as long as the color symmetry is assumed to be unbroken, the requirement of $\mathbf{1}+u=0$[**)] seems almost inevitable, and we have arrived at the two conditions (A) and (B) of Theorem 6.6, which leads us to the conclusion of the color confinement. Here, one may doubt the possibility of the condition $u=-\mathbf{1}$: By small changes of the number and/or interaction type of matter fields, the dynamically determined parameter u would be easily perturbed to shift from $-\mathbf{1}$, even if it was just set on the desired value $-\mathbf{1}$ at first. There are, however, some such possibilities as the following example[10)] of $u_b{}^a$ which may exhibit its stability against such perturbations as the above:

$$u_b{}^a(p^2)=\delta_b{}^a\bar{g}(p^2)/(r-\bar{g}^2(p^2)), \tag{6·40}$$

where $\bar{g}(p^2)$ is the effective coupling at p^2. The present parameter $u_b{}^a$ is given by $u_b{}^a=u_b{}^a(p^2=0)$. In the weak coupling limit of (6·40), it is reduced to the perturbatively reasonable form:

$$u_b{}^a\sim\delta_b{}^a\bar{g}^2/r. \tag{6·41}$$

If $\bar{g}(p^2)$ satisfies the condition expected in the renormalization group method:

$$\bar{g}^2(p^2)\to\infty \quad \text{as} \quad p^2\to 0, \tag{6·42}$$

then (6·40) will reproduce the desired form (6·37), irrespectively of the value of the constant r and of the rate of $\bar{g}^2$ approaching infinity. This arbitrariness in the constant r and in the behavior of $\bar{g}^2$ tending to infinity would endow the condition $\mathbf{1}+u=0$ with a considerable extent of stability. Of course, further detailed investigations are needed concerning the question whether $\mathbf{1}+u=0$ really satisfies the stability of this sort.

The assumption (6·42) crucial for the above example argument of (6·40) is the familiar anticipation known as the "*infrared slavery*".[11)] Since this means a *strong coupling at long distances*, it must have a tight connection with the *failure of the cluster property* discussed in § 5. 4 from the viewpoint of the confining q-$\bar{q}$ potential. For the sake of the failure of the cluster

*) In the perturbative approaches made in Ref. 9), the infrared divergences are shown to survive, in the *on-shell* renormalization scheme, after the cancellations of those appearing in the *off-shell* renormalization.

**) By the unbroken color symmetry, the condition $\det(\mathbf{1}+u)=0$ is strengthened to $\mathbf{1}+u=0$.

property in the channel $\langle 0|A_\mu^a(x)A_\nu^b(y)|0\rangle$, it is sufficient that F.T. $\langle 0|A_\mu^a(x)A_\nu^b(y)|0\rangle$ behaves as $\sim 1/(p^2)^\alpha$ with $\alpha \gtrsim 3/2$,

$$\text{F.T.}\ \langle 0|A_\mu^a(x)A_\nu^b(y)|0\rangle \sim 1/(p^2)^\alpha, \tag{6·43}$$

and the case $\alpha=2$ of a massless dipole $1/(p^2)^2$ corresponds to the "linear potential" case. In view of the condition (B′) of Theorem 6.8 necessary for the well-definedness of the charge $G^a=\int d^3x\partial^i F^a_{0i}$, we notice here the following contrast: The condition (B′) requires a *weaker* singularity at $p^2=0$ in (6·43) (with smaller α, $\alpha<1$), whereas the *stronger* one (with larger α, $\alpha \gtrsim 3/2$) is more favorable for the failure of the cluster property (and infrared slavery). These two requirements are really *compatible* to each other because the Green's function (6·43) in fact has two independent components, namely, the transverse and longitudinal parts: In this connection, one should first recall that the *isotropic* linear q-$\bar{q}$ potential is *not* so convenient for the *string*-like picture of the confinement which seems to have a directional dependence determined by the q-$\bar{q}$ configuration. Next it may be instructive to refer to the remark made by Frenkel and Taylor[12] that the components of YM field responsible for the peculiar property of asymptotic freedom is only the longitudinal one in Coulomb gauge whereas other transverse ones contribute destructively to the asymptotic freedom similarly to the usual matter fields. Thus, it seems likely that the components responsible for the failure of the cluster property differ from those which could contribute to $(\partial^\nu F^a_{\mu\nu})^{\text{as}}$ or to the charge $G^a=\int d^3x\partial^i F^a_{0i}$ which should vanish: The former components may be identified with the "longitudinal" one $\chi=A^L$ and the latter would be "transverse", if any. In other words, the failure of the cluster property in the *unphysical* world realizes the condition $u=-\mathbf{1}$ making the charge N^a well-defined, while the absence of the infrared tails in the *physical* world protects the charge G^a from being ill-defined. Both of these cooperate to achieve the color confinement. This contrast reminds us of the situation encountered in § 5. 4, where the former operates to confine unphysical particles and the latter guarantees the physical measurement independent of the "behind-the-moon". These observations may give us some clues to the understanding of the naive string picture and of the notion of complete anti-dielectricity of the vacuum.

As for the possibility of (A) and (B″) discussed in Corollary 6.9, some comments might be necessary. One might suspect that the very existence of massive gluons having colors is contradictory to the conclusion (6·38) of color confinement, which is, however, not the case. The colored particles can exist as asymptotic fields, but only in the quartet representations. It is the subject of the next section to discuss this point in some details.

§ 6. 3. Color Confinement from the Viewpoint of Quartet Mechanism

The quartet mechanism, by which particles essentially decouple from the physical sector as has been discussed in Chap. 3, can be found to take place by a simple criterion. Consider a BRS transformation

$$[iQ_B, X(x)] = \mathscr{C}(x), \tag{6·44}$$

where $X(x)$ is an arbitrary Heisenberg operator with vanishing FP ghost number $(iQ_c=0)$ and $\mathscr{C}(x)$ is its BRS transform having $iQ_c=+1$, then the following theorem holds.

Theorem 6.10.[13] If the operator $\mathscr{C}(x)$ in (6·44) has an asymptotic field $\gamma(x)$,

$$\mathscr{C}(x)^{\mathrm{as}} = Z^{1/2}\gamma(x) + \cdots, \tag{6·45}$$

then, the following holds:

(i) The operator $X(x)$ in (6·44) also has an asymptotic field, say $\chi(x)$, and the pair $(\chi(x), \gamma(x))$ forms a BRS-doublet:

$$[iQ_B, \chi(x)] = \gamma(x). \tag{6·46}$$

(ii) There exists another Heisenberg operator $\overline{\mathscr{C}}(x)$ with the FP ghost number $iQ_c=-1$, which has the asymptotic field $\bar{\gamma}(x)$ "FP-conjugate" to $\gamma(x)$. This $\bar{\gamma}(x)$ also forms another BRS-doublet:

$$\{Q_B, \bar{\gamma}(x)\} = \beta(x), \tag{6·47}$$

where $\beta(x)$ is supplied as the asymptotic field of the Heisenberg operator $\mathscr{B}(x)$ with $iQ_c=0$, defined by

$$\{Q_B, \overline{\mathscr{C}}(x)\} = \mathscr{B}(x). \tag{6·48}$$

(iii) These two BRS-doublets $(\{\chi(x), \gamma(x)\}, \{\bar{\gamma}(x), \beta(x)\})$ constitute a *quartet* having a common mass, spin and color indices, and hence these asymptotic fields appear in the physical subspace only in zero-norm combinations (confinement).

Proof) In view of the BRS transformation (6·44), the existence of asymptotic field γ(6·45) for $\mathscr{C}$ necessarily implies the existence of χ also for X which should satisfy (6·46), as is explained in (C·24) and (C·25). Next, since the assumed asymptotic field γ of $\mathscr{C}$ should appear as a pole in a certain two-point Green's function, there should exist (at least one) such Heisenberg operator $\overline{\mathscr{C}}$ 'conjugate' to $\mathscr{C}$ that the propagator $\langle 0|T\mathscr{C}(x)\overline{\mathscr{C}}(y)|0\rangle$ has the pole at the mass m of γ-field:

$$\text{F.T.}\ \langle 0|T\mathscr{C}(x)\overline{\mathscr{C}}(y)|0\rangle = -Z\frac{1}{p^2-m^2} + \cdots. \tag{6·49}$$ *)

*) Equations (6·49), (6·51) and (6·53) are explicitly written for the case of scalar fields, for simplicity.

This indicates the existence of asymptotic field $\bar{\gamma}$ for $\bar{\mathscr{C}}$. Note the following W.T. identity:

$$0=\langle 0|\{Q_B, T(X(x)\bar{\mathscr{C}}(y))\}|0\rangle$$
$$=\langle 0|TX(x)\mathscr{B}(y)|0\rangle-i\langle 0|T\mathscr{C}(x)\bar{\mathscr{C}}(y)|0\rangle, \qquad (6\cdot 50)$$

where use has been made of (6·44), $Q_B|0\rangle=0$ and the definition (6·48), $\mathscr{B}=\{Q_B, \bar{\mathscr{C}}\}$. Equations (6·50) and (6·49) lead to

$$\text{F.T.}\ \langle 0|TX(x)\mathscr{B}(y)|0\rangle=-iZ\frac{1}{p^2-m^2}+\cdots, \qquad (6\cdot 51)^{*)}$$

which says the existence of asymptotic field $\beta(x)$ for the operator $\mathscr{B}(x)$. Defining these asymptotic fields by

$$X(x)\to Z^{1/2}\chi(x)+\cdots,\quad \mathscr{B}(x)\to Z^{1/2}\beta(x)+\cdots,$$
$$\bar{\mathscr{C}}(x)\to Z^{1/2}\bar{\gamma}(x)+\cdots, \qquad (6\cdot 52)$$

we can easily see that (6·47) holds as in (C·15), and can derive

$$[\chi(x),\beta(y)]=iD(x-y)\quad\text{and}\quad \{\gamma(x),\bar{\gamma}(y)\}=-D(x-y) \qquad (6\cdot 53)^{*)}$$

from (6·49) and (6·51) by virtue of Greenberg-Robinson theorem (Appendix C). All other (anti-)commutators between χ, β, γ and $\bar{\gamma}$ except for $[\chi(x),\chi(y)]$ are found to vanish from the FP ghost number conservation and the nilpotency $Q_B^2=0$; e.g.,

$$[\beta(x),\beta(y)]=\{\gamma(x),\gamma(y)\}=\{\bar{\gamma}(x),\bar{\gamma}(y)\}=0. \qquad (6\cdot 54)$$

One may notice that these logics presented here are identical to those utilized in Chap. III in showing the presence of the "elementary" quartet. In fact, the present BRS transformation property (6·46) and (6·47), and the commutation relations (6·53) and (6·54) exactly coincide with those [(3·23) and (3·24)] for the "elementary" quartet and hence also with those [(3·15) and (3·16)] for the general quartet. □

Theorem 6.11. If the operator $\mathscr{C}(x)$ in (6·44) has no asymptotic field and $X(x)$ has its asymptotic field $X^{as}(x)$, then

$$[iQ_B, X^{as}(x)]=0, \qquad (6\cdot 55)$$

which implies that X^{as}-particle appears in the physical subspace $\mathcal{V}_{phys}$, and $X^{as}(x)$ becomes

(i) a BRS-singlet physical particle with positive norm,

or otherwise,

(ii) a zero-norm particle having such a BRS-doublet partner $\bar{\gamma}_X(x)$ with FP-ghost number $iQ_c=-1$ that

*) See the footnote on p. 87.

$$\{Q_B, \bar{\gamma}_X(x)\} = X^{as}(x). \tag{6·56}$$

Proof) Equation (6·55) directly follows from (6·44) as in (C·15). As we have seen in Chap. III, any asymptotic field becomes BRS-singlet or otherwise doublet. Hence $\chi(x)$ should represent either (i) or (ii) in the above. □

In the preceding section, we have shown in Theorem 6.8 that the color confinement really takes place in the case $u=-1$ and unbroken color symmetry: In fact, the color charges $\hat{Q}^a$ vanish in H_{phys}, or equivalently

$$\langle \Psi | Q^a | \Phi \rangle = 0 \quad \text{for } \forall |\Psi\rangle, |\Phi\rangle \in \mathcal{V}_{phys}, \tag{6·57}$$

by the remarkable equality (6·39),

$$Q^a = \{Q_B, M^a\} \text{ with } M^a \equiv g^{-1} \int d^3x (D_0 \bar{c})^a (x). \tag{6·58}$$

What does this color confinement imply on the character of asymptotic fields? We know already from the arguments in Chap. III that any asymptotic field is either a BRS-singlet or a quartet member. The color confinement (6·57) or (6·58) means that the *colored asymptotic fields*, if any, *should belong to quartet representations*: In fact, the BRS-singlet (and hence physical) particles, denoted by ϕ_i, are necessarily color singlets. This is because Eq. (6·57) with $|\Psi\rangle = \phi_i^\dagger|0\rangle$ and $|\Phi\rangle = \phi_j^\dagger|0\rangle$ leads to

$$\begin{aligned} 0 &= \langle 0|\phi_i Q^a \phi_j^\dagger|0\rangle = \langle 0|\phi_i [Q^a, \phi_j^\dagger]|0\rangle \\ &= \langle 0|\phi_i \phi_k^\dagger T^a_{kj}|0\rangle = T^a_{ij}, \end{aligned} \tag{6·59}$$

by using the normalization $[\phi_i, \phi_j^\dagger] = \delta_{ij}$, which says that all the representation matrices T^a_{ij} of color charges Q^a on the particles ϕ_i should vanish. On the other hand, the quartet particles denoted by $(\chi_i, \beta_i, \gamma_i$ and $\bar{\gamma}_i)$, can have color charges consistently to (6·58); e.g., $[Q^a, \chi_i] = -T^a_{ij}\chi_j$ with $T^a \neq 0$. Indeed, Eq. (6·58) only dictates the following forms for Q^a and M^a:

$$\begin{aligned} Q^a \sim \sum_{i,j} (\beta_i^\dagger T^a_{ij}\chi_j + \chi_i^\dagger T^a_{ij}\beta_j + \tfrac{1}{2}\beta_i^\dagger (\omega T^a + T^a\omega)_{ij}\beta_j \\ + i\bar{\gamma}_i^\dagger T^a_{ij}\gamma_j - i\gamma_i^\dagger T^a_{ij}\bar{\gamma}_j), \end{aligned} \tag{6·60a}$$

$$M^a \sim \sum_{i,j} (\bar{\gamma}_i^\dagger T^a_{ij}\chi_j + \chi_i^\dagger T^a_{ij}\bar{\gamma}_j + \tfrac{1}{2}[\beta_i^\dagger (\omega T^a)_{ij}\bar{\gamma}_j + \bar{\gamma}_i^\dagger (T^a\omega)_{ij}\beta_j]), \tag{6·60b}$$

where use has been made of the metric matrix (3·16) for the quartets, and the symbol $\sim$ indicates an equality restricted on the Fock space spanned by all the asymptotic fields. Hence both of (6·60) become exact equalities if the asymptotic completeness holds. Thus, these arguments explicitly show that the color confinement (6·38), $\hat{Q}^a = 0$ in H_{phys}, is really a confinement, the consistency of which is guaranteed by the quartet mechanism found in Chap. III.

We already know the existence of one color-octet of 'elementary' quartet. Further, there may exist asymptotic fields even for the quark fields $q(x)$ and the vector parts of gluon fields $A_\mu(x)$. In this case, we see from Theorem 6.11 and the BRS transformation

$$[iQ_B, q(x)] = igc^a(x)\frac{\lambda^a}{2}q(x),$$

$$[iQ_B, A_\mu{}^a(x)] = \partial_\mu c^a(x) + g(A_\mu(x)\times c(x))^a,$$

that the quark (vector gluon) asymptotic fields, if they exist, should necessarily be accompanied by bound-states of FP-ghosts and quarks (gluons) in the channels of $c^a(x)\lambda^a q(x)$ $(A_\mu(x)\times c(x))$. Note that the formations of such bound-states are possible only in non-Abelian cases, because the FP-ghosts are completely free in Abelian cases. This point agrees with the previous observation that the confinement condition (A) $u=-\mathbf{1}$ in Theorem 6.8 can be realized only in non-Abelian cases.

Some comments are in order: (i) Since the quartet mechanism takes place in this color confinement, the physical S-matrix unitarity is automatically assured to hold. (ii) The confinement by Theorem 6.8 holds irrespectively of the presence or the absence of quark asymptotic fields. One may, however, prefer the case where the quark fields have their own asymptotic fields, because the asymptotic completeness can hold only in such case. In fact, if the quark fields belonging to *fundamental representations* of $SU(3)$ color group do not have their asymptotic fields, then the quark Heisenberg fields themselves will not be expressed by other asymptotic fields of non-fundamental representations alone.

Chapter VII

Miscellany of Other Topics

§ 7.1. $U(1)$-Problem

Now many physicists believe that the strong interaction in the hadron world can be described by an $SU_c(3)$ color gauge theory, namely, QCD. The Lagrangian density of QCD is written as (2·7) with the matter part given by

$$\mathcal{L}^{\rm QCD}_{\rm matter}=\bar{q}\left[i\gamma^{\mu}\left(\frac{1}{2}\overleftrightarrow{\partial}_{\mu}-ig\frac{\lambda^a}{2}A_{\mu}{}^a\right)-m\right]q\,, \tag{7·1}$$

in terms of quark fields q. In the limit of vanishing mass matrix m, this system has a chiral $U(1)$ symmetry besides the desirable chiral $SU(N_f)$ flavor symmetry and the exact $SU_c(3)$ color symmetry. The Goldstone theorem[1] tells us that there appear N_f^2 Goldstone bosons after the spontaneous breaking of the $U(N_f)$ chiral symmetry. Further, Weinberg[2] has shown, by using the usual technique of current algebra in the case $m\neq 0$, that an isoscalar pseudoscalar Goldstone boson (say, $U(1)$ Goldstone boson) should exist with a mass comparable to the pion mass μ_π, if it is a physical particle. There is no such a particle in reality. So the $U(1)$ Goldstone boson must not appear as a physical particle in QCD. This is the $U(1)$ problem.[2),3)]

Hereafter, we restrict our considerations to the case of chiral symmetric limit $m=0$. As is well-known, the $U(1)$ axial-vector current suffers from an anomaly of the Adler-Bell-Jackiw type,[4] which modifies the conservation law of the gauge-invariant current $j_5{}^\mu$ to read

$$\partial_\mu j_5{}^\mu=(N_f g^2/32\pi^2)\,\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}\cdot F_{\rho\sigma}\,. \tag{7·2}$$

This, however, implies the existence of another conserved but *gauge-variant* current $J_5{}^\mu$, defined by

$$J_5{}^\mu=j_5{}^\mu+X^\mu, \tag{7·3a}$$

$$X^\mu\equiv-(N_f g^2/16\pi^2)\,\epsilon^{\mu\nu\rho\sigma}[A_\nu\cdot F_{\rho\sigma}-(g/3)\,A_\nu\cdot(A_\rho\times A_\sigma)]\,. \tag{7·3b}$$

Since the added anomalous term X^μ commutes with the quark field q at equal times in any covariant gauges, one can derive chiral $U(1)$ Ward identities of the usual form such as, e.g.,

$$\partial_\mu{}^x\langle 0|T(J_5{}^\mu(x)\,\bar{\psi}\gamma_5\psi(0))\,|0\rangle=2i\delta^4(x)\langle 0|\bar{\psi}\psi(0)\,|0\rangle\,. \tag{7·4}$$

Equation (7·4) clearly indicates that the $U(1)$ Goldstone boson must exist and produce a massless pole in this Green's function, because $\langle\bar{\psi}\psi\rangle \neq 0$ in our real world by which the chiral $SU(N_f)$ symmetry is spontaneously broken. Thus, the anomaly by itself does not necessarily give a way out of the $U(1)$ problem. The $U(1)$ Goldstone boson, say χ, the existence of which is enforced by the chiral $U(1)$ Ward identity (7·4) in any *covariant gauges,* is suitably defined by the following equation in the LSZ sense:

$$J_5^\mu(x) \underset{x_0\to\pm\infty}{\longrightarrow} Z^{1/2}\partial^\mu\chi^{\text{out/in}}(x) + \cdots . \tag{7·5}$$

Here the dots $(\cdots)$ indicates the other possible massive bound-states in this channel J_5^μ, which are commutative with massless pseudoscalar χ and hence need not be considered. The superscripts out and in are omitted for simplicity henceforth.

The above argument, essentially the same as the Goldstone theorem,[1] does not claim that the existing $U(1)$ Goldstone boson χ is a *physical* particle, and therefore, we have yet a good chance to solve the $U(1)$ problem in QCD: Especially, since the conserved current J_5^μ in (7·5) is *gauge-variant,* its asymptotic field χ can really be *unphysical.* Our present formalism developed so far provides us in fact a very suitable framework to discuss the fate of this $U(1)$ Goldstone boson χ. Especially the quartet mechanism explained in Chaps. III and VI is directly related to this problem. We will obtain below the following results[5] by analyzing the meaning of $SU_c(3)$-gauge-invariance in terms of the BRS charge Q_B:

(i) The $U(1)$ Goldstone boson χ does not appear as a physical particle at all *if* and *only if* the FP ghost forms a massless bound-state with the gauge-boson (gluon) in a pseudoscalar channel.

(ii) This decoupling of χ from the physical sector is caused by a mechanism of '*Goldstone quartet*' including the FP ghost-gluon bound state as a member of it.

(iii) If the strong interaction were described by *Abelian* gluon gauge theory, the chiral $U(1)$ Goldstone boson would necessarily appear as a physical particle.

(iv) The 'Goldstone quartet' mechanism become equivalent to the 'Goldstone dipole' one proposed by Kogut and Susskind[3] only in a special case, i.e., the *Abelian* gauge theory in *two dimension* (the Schwinger model[6]).

Now we begin the analysis. Define a new Heisenberg operator $\mathscr{C}^\mu(x)$ by the BRS transformation of the gauge-variant J_5^μ:

$$\mathscr{C}^\mu(x) \equiv [iQ_B, J_5^\mu(x)], \tag{7·6a}$$

$$= -(N_f g^2/8\pi^2)\,\epsilon^{\mu\nu\rho\sigma}\partial_\nu c\cdot\partial_\rho A_\sigma(x). \tag{7·6b}$$

Note that all the contributions to this commutator come from the anomalous term X^μ given by (7·3b) in J_5^μ: $[iQ_B, X^\mu] = \mathscr{C}^\mu$. In view of the commutator

(7·6a) and the definition (7·5) of $U(1)$ Goldstone boson χ, we see that the following two cases should be discussed separately: (A) The case when $\mathscr{C}^{\mu}(x)$ has a pseudo-scalar massless asymptotic field corresponding to a FP ghost-gluon bound-state, and (B) the case of no such massless asymptotic field in $\mathscr{C}^{\mu}$.

First consider the case (A). In this case, we have a pseudo-scalar massless asymptotic field $\gamma(x)$, defined by

$$\mathscr{C}^{\mu}(x) \xrightarrow[|x_0|\to\infty]{} Z^{1/2}\partial^{\mu}\gamma(x) + \cdots$$

with the same renormalization constant $Z^{1/2}$ as in (7·5), and obtain the following BRS transformation relation from (7·6a):

$$[Q_B, \chi(x)] = -i\gamma(x). \tag{7·7}$$

This relation is nothing but (6·46) which we have found in Theorem 6.10 in § 6.3 as a necessary and sufficient condition for the quartet mechanism to take place. Accordingly, this BRS-doublet (χ, γ) necessarily has a partner doublet, say $(\bar{\gamma}, \beta)$, and they all become unphysical particles by forming a quartet which cannot contribute to any physical quantities in the physical subspace $\mathcal{V}_{\text{phys}}$ specified by $Q_B|\text{phys}\rangle = 0$, as has been stated in detail in Chap. III below (3·30). The partner doublet $(\bar{\gamma}, \beta)$ can be found by following the reasoning in the proof of Theorem 6.10. Since $\mathscr{C}^{\mu}(x)$ has a massless pseudo-scalar asymptotic field, it must develop a massless pole in the 2-point Green's function as

$$\text{F.T.}\langle 0|T\mathscr{C}^{\mu}(x)\bar{\mathscr{C}}^{\nu}(y)|0\rangle = -Z\frac{p^{\mu}p^{\nu}}{p^2} + \cdots, \tag{7·8}$$

at least for some operators $\bar{\mathscr{C}}^{\nu}$ which create the same quantum numbers as $\mathscr{C}^{\nu}$ annihilates. It will be instructive to cite here some candidates for $\bar{\mathscr{C}}^{\nu}$; e.g.,

$$\bar{\mathscr{C}}^{\mu}(x) = -\frac{N_f g^2}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}\cdot\begin{cases}\partial_{\nu}\bar{c}\cdot\partial_{\rho}A_{\sigma}(x), \\ \partial_{\nu}\bar{c}\cdot F_{\rho\sigma}(x), \qquad \text{etc.}\end{cases} \tag{7·9}$$

The BRS transform of this $\bar{\mathscr{C}}^{\mu}$ defines another operator $\mathscr{B}^{\mu}$:

$$\mathscr{B}^{\mu}(x) = \{Q_B, \bar{\mathscr{C}}^{\mu}(x)\}, \tag{7·10}$$

which is explicitly written as

$$\mathscr{B}^{\mu}(x) = -\frac{N_f g^2}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}\cdot\begin{cases}(\partial_{\nu}B\cdot\partial_{\rho}A_{\sigma}(x) + i\partial_{\nu}\bar{c}\cdot\partial_{\rho}D_{\sigma}c(x)), \\ (\partial_{\nu}B\cdot F_{\rho\sigma}(x) + i\partial_{\nu}\bar{c}\cdot g(F_{\rho\sigma}\times c)),\end{cases} \tag{7·11}$$

correspondingly to the two examples of $\bar{\mathscr{C}}^{\mu}$ in (7·9). The BRS-doublet $(\bar{\gamma}, \beta)$ is the asymptotic fields of these operators $\bar{\mathscr{C}}^{\mu}$ and $\mathscr{B}^{\mu}$:

$$\mathscr{C}^{\mu}(x) \xrightarrow[|x_0|\to\infty]{} Z^{1/2}\partial^{\mu}\bar{\gamma}(x) + \cdots, \quad \mathscr{B}^{\mu}(x) \to Z^{1/2}\partial^{\mu}\beta(x) + \cdots . \tag{7·12}$$

This quartet $(\chi, \beta, \gamma, \bar{\gamma})$ is called 'Goldstone quartet', hereafter.

Next consider the case (B) where no massless pseudoscalar asymptotic fields is contained in the operator $\mathscr{C}^{\mu}(x)$ $(=[iQ_B, J_5^{\mu}(x)])$. Then, from the asymptotic form of J_5^{μ} (7·5), we conclude in this case that

$$[Q_B, \chi(x)] = 0 . \tag{7·13}$$

Note that this $U(1)$ Goldstone boson χ cannot be a member of BRS-doublet, namely, there is no field $\bar{\gamma}_\chi$ satisfying $\{iQ_B, \bar{\gamma}_\chi\} = \chi$. The reason is as follows: First, if some $\bar{\gamma}_\chi$ exists, then χ has vanishing norm, $[\chi(x), \chi(y)] = 0$. Second, since the operator $\bar{\psi}\gamma_5\psi$ clearly cannot contain the 1-particle state $\bar{\gamma}_\chi$ having FP ghost number $iQ_c = -1$, $\bar{\psi}\gamma_5\psi$ contains only (BRS-invariant) χ-field, if any. Since also J_5^{μ} contains only χ-field by the definition (7·5), the massless pole required in the chiral $U(1)$ Ward identity (7·4) cannot be produced by this zero-norm χ-field. This contradicts the first assumption that the χ-field is a $U(1)$ Goldstone boson responsible for the massless pole required in (7·4). Thus we see from (7·13) and from these arguments that the $U(1)$ Goldstone boson χ in this case become BRS-singlet (gauge-invariant)! As far as the $SU_c(3)$-gauge (BRS) invariance of the system is concerned, no other massless particle non-commutative with $\chi(x)$, is necessitated to exist. Thus we conclude that the $U(1)$ Goldstone boson $\chi(x)$ should have positive norm in order for the theory to be physically interpretable and has necessarily to appear as a *physical* particle in the world. This conclusion is inevitable as far as we take it for granted to assume that only the gauge invariance is relevant to the fate of the $U(1)$ Goldstone boson.

In Abelian gauge theories, the FP ghost and anti-ghost become *free* and have no interactions with other particles, as is apparent from the fact that they have the Lagrangian $-i\partial^{\mu}\bar{c}\cdot\partial_{\mu}c$ instead of $-i\partial^{\mu}\bar{c}^a\cdot D_{\mu}{}^{ab}c^b$. So they cannot form any bound-states at all, and hence the composite operator $\mathscr{C}^{\mu}(x)$ defined in (7·6b) has no asymptotic fields. This corresponds to the case (B) discussed above. Therefore, we obtain an important conclusion: *If the strong interaction is described by an Abelian gauge theory, the chiral $U(1)$ Goldstone boson has to appear as a physical particle in the world.* Since we have no chiral $U(1)$ Goldstone boson in this real world, the Abelian gauge theory of strong interaction should be rejected.

In two (space-time) dimensions, the situation is rather different. Since it is instructive to analyse the model in 2 dimensions by our machinery presented above, we here discuss the Schwinger model[6] briefly.

Although the Lagrangian has exactly the same form as in 4 dimensions, an essential difference appears in the form of the ABJ type anomaly:

$$\partial_{\mu}j_5^{\mu} = (N_f g/2\pi)\,\epsilon_{\mu\nu}F^{\mu\nu},$$

which is *linear* with respect to $F_{\mu\nu}$ in clear distinction from the quadratic one (7·2) in 4 dimensions. Hence the quartet of operators J_5^μ, $\mathscr{C}^\mu$, $\overline{\mathscr{C}}^\mu$ and $\mathscr{B}^\mu$ are given very simply in this case as

$$J_5^\mu = j_5^\mu - (N_f g/\pi)\,\epsilon^{\mu\nu} A_\nu\,, \tag{7·14a}$$

$$\mathscr{C}^\mu = [iQ_B, J_5^\mu] = -(N_f g/\pi)\,\epsilon^{\mu\nu}\partial_\nu c\,, \tag{7·14b}$$

$$\overline{\mathscr{C}}^\mu = -(N_f g/\pi)\,\epsilon^{\mu\nu}\partial_\nu \bar{c}\,, \tag{7·14c}$$

$$\mathscr{B}^\mu = \{Q_B, \overline{\mathscr{C}}^\mu\} = -(N_f g/\pi)\,\epsilon^{\mu\nu}\partial_\nu B\,. \tag{7·14d}$$

We notice that $\mathscr{C}^\mu$ really in this case has a massless asymptotic field supplied by the elementary (and free!) FP ghost c. Hence the Schwinger model realizes an example of the 'Goldstone quartet' corresponding to the case (A) above. The Goldstone quartet of asymptotic fields for the operators (7·14) can now be given explicitly:

$$\chi = \tilde{A}^L_{\text{in/out}}\,, \quad \beta = \tilde{B}\,,$$
$$\gamma = \tilde{c}\,, \quad \bar{\gamma} = \tilde{\bar{c}} \tag{7·15}$$

with the renormalization constant taken as $Z^{1/2} = N_f g/\pi$. Here $\tilde{\Phi}$ indicates the 'conjugate' field to Φ (peculiar to 2 dimensions) satisfying

$$\partial_\mu \tilde{\Phi} = -\epsilon_{\mu\nu}\partial^\nu \Phi\,, \quad \partial_\mu \Phi = -\epsilon_{\mu\nu}\partial^\nu \tilde{\Phi}\,, \tag{7·16}$$

and $A^L_{\text{in/out}}$ stands for the asymptotic fields of longitudinal component of A_μ. Thus we see from (7·15) that the present Goldstone quartet is nothing but the 'conjugate' of the elementary quartet $\{A^L, B, c, \bar{c}\}$. So here the decoupling of the latter quartet assures the physical S-matrix unitarity and simultaneously implies the decoupling of the Goldstone quartet at issue.

Here we should note: Due to the fact that the FP ghost is completely *free* in this *Abelian* case, the norm cancellations among the quartet $(A^L, B, c, \bar{c})$ in fact occurs in the subset (A^L, B), as is well-known since the Gupta-Bleuler formalism. Since $A^L(\boldsymbol{k}) \sim A_0(\boldsymbol{k}) - A_1(\boldsymbol{k})$ and $B(\boldsymbol{k}) \sim A_0(\boldsymbol{k}) + A_1(\boldsymbol{k})$ when $\boldsymbol{k} \| \boldsymbol{e}_1$, they form a pair of positive metric A_1 and negative metric A_0. Thus their 'conjugate' fields $\chi = \tilde{A}^L$ and $\beta = \tilde{B}$ also form a *Goldstone pair* between which norm cancellations occur. This is nothing but 'Goldstone dipole' called by Kogut-Susskind.[8] They further proposed that this 'Goldstone dipole' mechanism may take place even in QCD. We, however, know now that the 'Goldstone dipole' mechanism is just a special form realized only in the Schwinger model (*2-dimensional Abelian* model) of our general quartet mechanism.

Since we believe that the $U(1)$ problem can be understood in QCD, we strongly expect the existence of massless pseudoscalar bound-state of FP-ghost and gluon in the channel $\mathscr{C}^\mu$ (7·6b). If this is proved, then, this 'Goldstone quartet' provides us the first non-trivial (namely, dynamical) example of quartet, which may support the previous idea of quark and gluon confinement due to the formation of the FP ghost-quark and FP ghost-gluon bound states.[7]

Finally we should comment on the current belief that the $U(1)$ problem *was* resolved by instanton.[8] It is not correct; namely, the instanton by itself cannot assure the *unphysicalness* of the $U(1)$ Goldstone boson χ which *is* contained in J_5^μ because of the chiral $U(1)$ Ward identity (7·4). No one has ever proved in the framework of "instanton physics" in a satisfactory manner that the χ really contained in the gauge-variant current J_5^μ *is not* contained in the gauge-invariant one j_5^μ, although this problem has been discussed by many authors.[9]

§ 7. 2. Universality of Electric Charge in Weinberg-Salam Model

The Weinberg-Salam (W.S.) model based on $SU(2)\times U(1)$ shows remarkable agreements with (almost) all the experiments up to now.[10] Especially the recent neutral-current experimental data have excluded various variations of weak-interaction models[11] other than the original W.S. model with the GIM mechanism supplemented.

A little lengthy Lagrangian of W.S. model is now well known, and is omitted here. We remark that such *asymptotic-field analysis* as was done in Chap. IV explicitly for YM theories with symmetry broken and unbroken can be performed also for this W.S. model. The result is, however, trivial, although some complications occur owing to the mixing of two gauge bosons in the neutral channel: For the symmetry-broken parts and unbroken '$U(1)$' part of the $SU(2)\times U(1)$ group, the situations are quite similar to the broken $SU(2)$ Higgs-Kibble model (§ 4. 1) and the unbroken YM model (§ 4. 2), respectively.

Here we want to discuss only the problem of (*electric-*) *charge universality* in W.S. model. We mean by *charge* universality that the *on-shell coupling constants of photon* to the charged fields are universal. This should be proved also in the W.S. model because the absolute values of the charges of electron (or muon) and proton are known to coincide with quite a good accuracy. This problem was trivial in the case of QED. The Ward identity, which is very simple in the Abelian case, assures the proportionality of the bare coupling constants e_i^0 for the charged fields ϕ_i to the on-shell coupling constants e_i:

$$e_i = Z_3^{1/2} e_i^0 = Z_3^{1/2} e^0 q_i\,, \tag{7·17}$$

where we should recall that *quantum numbers* q_i of the charge operator Q, of course, determine the bare coupling constants e_i^0 as $e_i^0 = e^0 q_i$ and that

$$[Q, \phi_i^\dagger] = q_i \phi_i^\dagger\,. \tag{7·18}$$

Since, the proportionality constant $Z_3^{1/2}$ or $Z_3^{1/2}e^0$ is independent of the matters ϕ_i, Eq. (7·17) proves the charge universality.

In the W.S. model, the W.T. identities become quite complicated and the proof of such proportionality relation as (7·17) is made difficult by the non-Abelian character of this model. In fact, the proof of such proportionality by use of W.T. identities was accomplished only in the Landau gauge[12),13)] and the unitary gauge.*) In the other covariant gauges, only the charge conservation law in arbitrary scattering processes,

$$\sum_{i\in\{\text{out-going}\}} e_i = \sum_{j\in\{\text{in-coming}\}} e_j\,, \tag{7·19}$$

was derived from the W.T. identity as yet.[13)] This conservation law (7·19), which can be proved also from more general S-matrix theoretical arguments alone as was done by Weinberg,[14)] is not sufficient unfortunately for the charge universality proof; for instance, in the case where the muon (μ) and electron (e) numbers are separately conserved, one cannot derive the equality $e_\mu = e_e$ from (7·19) alone even when $e_\mu^0 = e_e^0$.

Now we give the proof of the proportionality

$$e_i = \text{constant}\cdot e_i^0 = \text{constant}\cdot e^0 q_i\,, \tag{7·20}$$

by the machinery of the present formalism, in the W.S. model in arbitrary covariant gauges respecting the original global symmetry $SU(2)\times U(1)$, namely in such type of gauges as (2·7b):

$$\mathcal{L}_{\text{GF}} = -(\partial^\mu B^a) A_\mu{}^a + \frac{\alpha_0}{2} B^a B^a\,, \tag{7·21}$$

where the index a runs over 0 (corresponding to $U(1)$) and 1, 2 and 3 (of $SU(2)$). This proof was first given by Aoki.[13),15)] We present it in a more complete form. Let us recall the "Maxwell" equations (2·36):

$$\partial^\nu F^a_{\mu\nu} = g J_\mu{}^a - \{Q_B, (D_\mu \bar{c})^a\} \quad \text{for } a = 1, 2, 3\,, \tag{7·22a}$$

$$\partial^\nu F^0_{\mu\nu} = \tfrac{1}{2} g' J_\mu{}^0 - \{Q_B, \partial_\mu \bar{c}^0\}\,. \tag{7·22b}$$

Owing to the spontaneous breakdown of $SU(2)\times U(1)$ to '$U(1)$', only one charge operator, say electric charge operator Q, can be well-defined and is usually (and formally) written as

$$Q = Q^3 + Q^0/2\,. \tag{7·23}$$

In view of this combination, the 0- and 3-components of "Maxwell" equations (7·22) are combined to produce

*) There exist, in the unitary gauge, some doubts in the renormalizability, and hence, in the well-definedness of it.

$$\partial^\nu F_{\nu\mu}=e^0(J_\mu{}^3+\tfrac{1}{2}J_\mu{}^0)+\mathcal{N}_\mu\,, \tag{7·24}$$

where the following definitions are used:

$$F_{\mu\nu}\equiv(g'F^3_{\mu\nu}+gF^0_{\mu\nu})/\sqrt{g^2+g'^2}\,, \tag{7·25}$$

$$\mathcal{N}_\mu\equiv\{Q_B,\,[g'(D_\mu\bar{c})^3+g\partial_\mu\bar{c}^0]/\sqrt{g^2+g'^2}\}, \tag{7·26}$$

$$e^0\equiv-gg'/\sqrt{g^2+g'^2}\,. \tag{7·27}$$

Both of the formal charges $\int d^3x(J_0{}^3+\frac{1}{2}J_0{}^0)$ and $\int d^3x\mathcal{N}_0/e^0$ may be identified with Q of (7·23), since both of them *formally* reproduce the commutation relations (7·18) as is easily assured by using the canonical commutation relations. As has been explained in § 6. 1, however, neither of them provides a well-defined charge because of the massless one-particle contribution from the 'elementary' quartet members β^a ($a=3$ and 0). By the assumption that one symmetry corresponding to the combination (7·23) remains unbroken, a certain linear combination of the above formal two charges

$$Q=\int d^3x\left[\left(J_0{}^3+\frac{1}{2}J_0{}^0\right)+\zeta\mathcal{N}_0/e_0\right]/(1-\zeta)$$

$$\equiv\int d^3xJ_0{}^{\text{e.m.}} \tag{7·28}$$

similarly to (6·29), must provide a well-defined charge, when the constant $\zeta(\neq1)$ is suitably adjusted so that the massless contribution in $(J_0^3+J_0^0/2)$ is cancelled by that in $\zeta\mathcal{N}_0/e^0$. Since (7·28) clearly reproduces the commutation relations (7·18), we see that the well-defined charge (7·28) gives a desired correct expression for the electric charge operator Q formally given by (7·23). According to (7·28), the "Maxwell" equation (7·24) is now rewritten as

$$\partial^\nu F_{\nu\mu}/(1-\zeta)=e^0J_\mu{}^{\text{e.m.}}+\mathcal{N}_\mu\,. \tag{7·29}$$

This equation sandwiched between two *physical* 1-particle states $|i\rangle$ and $|f\rangle$ ($\in\mathcal{V}_{\text{phys}}$) leads to

$$\langle f|\partial^\nu F_{\nu\mu}|i\rangle/(1-\zeta)=e^0\langle f|J_\mu{}^{\text{e.m.}}|i\rangle\,, \tag{7·30}$$

where use has been made of the following equation:

$$\langle f|\mathcal{N}_\mu|i\rangle=\langle f|\{Q_B,*\}|i\rangle=0 \tag{7·31}$$

due to (7·26) and $Q_B|f\rangle=Q_B|i\rangle=0$.

Applying $\int d^4xe^{ikx}$ to both sides of (7·30), we consider the limit $k_\mu\to0$. First, similarly to the soft-pion technique, all the contributions to the l.h.s. of (7·30) remaining in this $k_\mu\to0$ limit come from the massless one-particle in the channel $F_{\mu\nu}$, namely, the photon. The one-photon contribution can be estimated by the asymptotic form of $F_{\mu\nu}$:

$$(F_{\mu\nu})^{\mathrm{as}} = Y(\partial_\mu A_\nu^{\mathrm{ph.}} - \partial_\nu A_\mu^{\mathrm{ph.}}) + \cdots , \tag{7·32}$$

where $A_\mu^{\mathrm{ph.}}$ represents the renormalized photon asymptotic field and the dots $(\cdots)$ stand for the other massive particles irrelevant here. The constant Y can be evaluated by the pole residue, for example, of

$$\begin{aligned}&\mathrm{F.T.}\ \langle 0|TF_{\mu\nu}(x)\, F_{\rho\sigma}(y)\,|0\rangle \\ &\quad = -Y^2(p_\mu p_\rho g_{\nu\sigma} + p_\nu p_\sigma g_{\mu\rho} - p_\nu p_\rho g_{\mu\sigma} - p_\mu p_\sigma g_{\nu\rho})/p^2 + \cdots .\end{aligned} \tag{7·33}$$

Noting that (7·32) says

$$\mathrm{F.T.}\ \langle 0|TF_{\mu\nu}(x)\, A_\rho^{\mathrm{ph.}}(y)\,|0\rangle = iY(p_\mu g_{\nu\rho} - p_\nu g_{\mu\rho})/p^2,$$

we easily see that the l.h.s. of (7·30) becomes

$$\begin{aligned}&\lim_{k\to 0}\int d^4x e^{ikx}\langle f|\partial^\nu F_{\nu\mu}(x)\,|i\rangle/(1-\zeta) \\ &\quad = [Y/(1-\zeta)]\lim_{p^f\to p^i}(2\pi)^4\delta^4(p^f-p^i)\cdot e_{fi}(p^i+p^f)_\mu ,\end{aligned} \tag{7·34}$$

where e_{fi} and $(p^i+p^f)_\mu$ are the renormalized on-shell photon coupling constant and the kinematical factor of the proper vertex $\langle f|A_\mu^{\mathrm{ph.}}|i\rangle_{\mathrm{amp.}}$, respectively. Next, since the well-defined charge operator Q is given by (7·28), the time-component ($\mu=0$) of the r.h.s. of (7·30) produces

$$\int d^4x\langle f|e^0 J_0^{\mathrm{e.m.}}(x)\,|i\rangle = e^0 q_i \delta_{fi}\lim_{p^f\to p^i}(2\pi)^4\delta^4(p^f-p^i)\cdot 2p_0^i , \tag{7·35}$$

where use has been made of (7·18). The normalization convention adopted in (7·34) and (7·35) is $\langle f|i\rangle = (2\pi)^3 2p_0^i\delta_{fi}\delta^3(p^f-p^i)$. By comparing (7·34) and (7·35), we obtain

$$e_{fi} = [(1-\zeta)/Y]\delta_{fi}e^0 q_i . \tag{7·36}$$

This result indicates not only that the on-shell photon coupling constant is *diagonal* with respect to the types of matter, but also the desired proportionality (7·20) since the constants ζ and Y are manifestly independent of i and f. This finishes the proof of the charge universality.

The above result (7·36) is obtained for arbitrary covariant gauges of the type (7·21). In Landau gauge, only in which such proportionality as (7·36) is proved also by the W.T. identity method, our result (7·36) can easily be assured to coincide with that due to the W.T. identity method. We should finally note the crucial step (7·31) in the above proof. Equation (7·31) here has represented in a very concise form all the necessary information which is buried in many complicated W.T. identities in the case of usual proof by the W.T. identity method. From this example also, we see that the "Maxwell" equation (2·36) and the subsidiary condition $Q_B|\text{phys}\rangle=0$, in our canonical operator formalism, are really powerful and useful.

§ 7.3. Some Other Problems

There are some other problems (or topics) which have not been touched upon up to now in this paper. We pick up and discuss some of them briefly here.

Superfield treatment of BRS symmetry. This type of approach was initiated by the authors of Ref. 16), and is made more complete by Fujikawa in Ref. 17).

The basic idea of the superfield approach is to realize our fundamental algebra (2·25) of Q_B and Q_c as a kind of 'conformal' one on 'superfields', say $\Phi(x,\theta)$, defined in a fictitious five-dimensional (super-) space (x_μ,θ), where the coordinate θ as well as the transformation parameters $i\lambda$ of the BRS transformation should be taken as ('real') elements of the Grassmann algebra. Consider the following transformations similar to the usual conformal ones:

$$U_B(\Lambda)\,\Phi(x,\theta)\,U_B^\dagger(\Lambda)=\Phi(x,\theta+\Lambda), \tag{7·37a}$$

$$U_c(\rho)\,\Phi(x,\theta)\,U_c^\dagger(\rho)=e^{d\rho}\Phi(x,e^\rho\theta), \tag{7·37b}$$

where ρ is a usual real number, $\Lambda\equiv i\lambda$ is a 'real' Grassmann number, and

$$U_B(\Lambda)\equiv e^{\Lambda Q_B},\quad U_c(\rho)\equiv e^{i\rho Q_c}. \tag{7·38}$$

The parameter d in (7·37b) is called the BRS-dimension of $\Phi(x,\theta)$, which will be related to the FP ghost number soon below. Directly from the conformal-type definitions (7·37) of operations of the unitary operators (7·38) on the arbitrary superfield Φ, we can easily conclude the commutation relations,

$$[\Lambda_1 Q_B,\Lambda_2 Q_B]=0\,, \tag{7·39a}$$

$$[iQ_c,\Lambda Q_B]=\Lambda Q_B\,, \tag{7·39b}$$

$$[Q_c,Q_c]=0\,, \tag{7·39c}$$

which are quite equivalent to our fundamental algebra (2·25). Let us see this situation more explicitly. First, since $\theta^2=0$, the Taylor expansion of $\Phi(x,\theta)$ with respect to θ produces, generally,

$$\Phi(x,\theta)=X(x)+\theta\mathscr{C}(x)\,. \tag{7·40}$$

Then, the 'super-translation' (7·37a) induces the following transformation on these component fields X and $\mathscr{C}$:

$$U_B(\Lambda)X(x)U_B^\dagger(\Lambda)=X(x)+\Lambda\mathscr{C}(x),$$
$$U_B(\Lambda)\mathscr{C}(x)U_B^\dagger(\Lambda)=\mathscr{C}(x), \qquad (7\cdot41)$$

the 'infinitesimal' form of which is written by using (7·38) as

$$[\Lambda Q_B, X(x)]=\Lambda\mathscr{C}(x),$$
$$[\Lambda Q_B, \mathscr{C}(x)]=0. \qquad (7\cdot42)$$

The 'dilatation' transformation (7·37b) gives

$$U_c(\rho)X(x)U_c^\dagger(\rho)=e^{d\rho}X(x),$$
$$U_c(\rho)\mathscr{C}(x)U_c^\dagger(\rho)=e^{(d+1)\rho}\mathscr{C}(x), \qquad (7\cdot43)$$

or, infinitesimally,

$$[iQ_c, X(x)]=dX(x),$$
$$[iQ_c, \mathscr{C}(x)]=(d+1)\mathscr{C}(x). \qquad (7\cdot44)$$

From these, we explicitly see that the two Heisenberg fields X and $\mathscr{C}$ form a 'BRS-doublet' [by (7·42)] and have FP ghost numbers d and $d+1$ [by (7·44)], respectively. Noting this, we can now cast our ordinary fields into superfields; e.g.,

$$\boldsymbol{\alpha}_\mu{}^a(x,\theta)=A_\mu{}^a(x)+\theta(-iD_\mu c)^a(x), \quad (d=0)$$
$$\Phi_i(x,\theta)=\varphi_i(x)+\theta gc\cdot T_{ij}\varphi_j(x), \quad (d=0)$$
$$\bar{\boldsymbol{\gamma}}^a(x,\theta)=\bar{c}^a(x)+\theta B^a(x), \quad (d=-1)$$
$$\boldsymbol{\gamma}^a(x,\theta)=c^a(x)+\theta(g/2)(c\times c)^a(x). \quad (d=+1) \qquad (7\cdot45)$$

As is evident from these examples of superfields, the present super-conformal-symmetry (7·37) is realized as a *non-linear representation*. By this reason, superfield theoretical treatment is not quite useful in practical calculations. A formal simplicity attained by the introduction of BRS-superfields, however, is often proved useful[17] and brings us such convenience that we can retain *manifest BRS-covariance* in all the stages of calculations. So this technique may find its important applications in the future. As an example of such simplicity, we only note here that the gauge-fixing term $\mathscr{L}_{\mathrm{GF}}$, (2·7b) and the corresponding FP ghost term $\mathscr{L}_{\mathrm{FP}}$, (2·7c), can be combined and rewritten by the use of superfields (7·45) compactly as follows:

$$\int d^4x(\mathscr{L}_{\mathrm{GF}}+\mathscr{L}_{\mathrm{FP}})=\int d^4xd\theta\left[-\partial^\mu\bar{\boldsymbol{\gamma}}(x,\theta)\cdot\boldsymbol{\alpha}_\mu(x,\theta)+\frac{\alpha_0}{2}\bar{\boldsymbol{\gamma}}(x,\theta)\cdot\partial_\theta\bar{\boldsymbol{\gamma}}(x,\theta)\right]. \qquad (7\cdot46)$$

The gauge-fixing invariance of physical contents. It is very important and necessary to prove that the physical contents of theory are not altered by the choices of gauge-fixing conditions at all. In the scattering theoretical aspects, it should be proved that the physical S-matrix is independent of gauge-fixing conditions. We show in the following that the usual proof*) given by 't Hooft and Veltman[18] and Lee[19] can be easily transcribed into the present formalism.

Let us consider the response of arbitrary Green's functions under an infinitesimal change of gauge-fixing condition. Take a gauge-fixing with arbitrary gauge-fixing function F ($F=\partial_\mu A^\mu$ for our previous gauge-fixing (2·7b)],

$$\mathcal{L}_{\rm GF}+\mathcal{L}_{\rm FP}=B\cdot F+\frac{\alpha_0}{2}B\cdot B+i\bar{c}\cdot\delta' F\,, \tag{7·47}$$

where δ' means the BRS transformation with λ factored out: $\delta' F=[iQ_B, F]$, and consider its arbitrary infinitesimal change:

$$\Delta(\mathcal{L}_{\rm GF}+\mathcal{L}_{\rm FP})=B\cdot\Delta F+i\bar{c}\cdot\delta'(\Delta F)\,. \tag{7·48}$$

[Note that the change of gauge parameter α_0 to $\alpha_0+\Delta\alpha_0$ can be considered by simply taking $\Delta F=\Delta\alpha_0\cdot B/2$.] The (arbitrary) Green's function $\langle 0|T\Phi_1\Phi_2\cdots\Phi_n|0\rangle\equiv G$ containing no FP ghosts receives, under the change (7·48), the following infinitesimal change:

$$\begin{aligned}\Delta G&=\langle 0|T[B\cdot\Delta F+i\bar{c}\cdot\delta'(\Delta F)]\Phi_1\Phi_2\cdots\Phi_n|0\rangle\\&=\langle 0|T\{Q_B, \bar{c}\cdot\Delta F\}\Phi_1\Phi_2\cdots\Phi_n|0\rangle\\&=\sum_{i=1}^{n}\langle 0|T(\bar{c}\cdot\Delta F)\Phi_1\cdots\Phi_{i-1}(\delta'\Phi_i)\Phi_{i+1}\cdots\Phi_n|0\rangle\,,\end{aligned} \tag{7·49}$$

where the integrations over the argument x of $B\cdot\Delta F$, etc. are understood, and use has been made of $Q_B|0\rangle=0$, $\delta'\Phi_i=[Q_B, \Phi_i]$ and an important equality

$$B\cdot\Delta F+i\bar{c}\cdot\delta'(\Delta F)=\{Q_B, \bar{c}\cdot\Delta F\}\,. \tag{7·50}$$

In obtaining the on-shell S-matrix, the i-th leg of Green's function G is multiplied by the Klein-Gordon operator $\Box+m_i^2$ and by some polarization vector, and its momentum p_i is set on the mass-shell $p_i^2=m_i^2$. Note that $\delta'\Phi_i$ is a composite operator accompanied by the FP ghost, which does not have 1-particle pole exactly at $p_i^2=m_i^2$, in general. Only in the case $\Phi_i=A_\mu$, the BRS transform $\delta' A_\mu=D_\mu c$ contains the 1-particle pole term $\partial_\mu c^{\rm as}$, which, however, does not contribute to the *physical* S-matrix owing to transversality of physical polarization vectors. Accordingly, in order that ΔG can contribute to the on-shell S-matrix, $\delta'\Phi_i$ in (7·49) must be combined with the term $\bar{c}\cdot\Delta F$

*) As will be discussed later, such a type of proof may be criticised from the standpoint of the operator formalism.

of $iQ_c=-1$ and produce the original i-th particle pole; that is, the $(\bar{c}\cdot\varDelta F)\cdot(\delta'\Phi_i)$ operator in (7·49) is effectively replaced by the original one-particle field $\delta Z_i^{1/2}\Phi_i$ where

$$\delta Z_i^{1/2}=\langle 0|T(\bar{c}\cdot\varDelta F)(\delta'\Phi_i)|\Phi_i^{\mathrm{as}}\text{-1-particle}\rangle\,, \tag{7·51}$$

which is nothing but a change of wave-function renormalization constant $Z_i^{1/2}$ under (7·48) as is easily seen. Thus, we find that all the changes of on-shell amplitudes are absorbed into those of wave-function renormalization constants, and hence that the physical S-matrix remains unchanged. Here we cite another proof which is much simpler than the above one and interesting [although the present authors cannot be convinced of its correctness completely]: The S-matrix of physical particles is given by the matrix elements

$$S_{\alpha\beta}=\langle\alpha\ \mathrm{out}|\beta\ \mathrm{in}\rangle\,, \tag{7·52}$$

where $|\alpha\ \mathrm{out}\rangle$ and $|\beta\ \mathrm{in}\rangle$ are composed of physical particles alone, and hence,

$$Q_B|\alpha\ \mathrm{out}\rangle=Q_B|\beta\ \mathrm{in}\rangle=0\,.$$

Similarly to (7·49), the infinitesimal change of the matrix elements (7·52) under the gauge-fixing change (7·48) is evaluated as

$$\langle\alpha\ \mathrm{out}|B\cdot\varDelta F+i\bar{c}\cdot\delta'(\varDelta F)|\beta\ \mathrm{in}\rangle=\langle\alpha\ \mathrm{out}|\{Q_B,\bar{c}\cdot\varDelta F\}|\beta\ \mathrm{in}\rangle=0\,. \tag{7·53}$$

This finishes the proof.

Also for physical quantities other than those in scattering theory, like the expectation values of *observables* between two physical states, we can prove the gauge-fixing independence just similarly to the above. We should note, however, that such usual proofs as shown here *may* be incomplete. In order to explain this, let us reconsider the first one of the above proofs more carefully. We have implicitly understood there the following equality:

$${}_{F+\varDelta F}\langle 0|T\Phi_1^{F+\varDelta F}\cdots\Phi_n^{F+\varDelta F}|0\rangle_{F+\varDelta F}={}_F\langle 0|T\varDelta(\mathcal{L}_{\mathrm{GF}}+\mathcal{L}_{\mathrm{FP}})\Phi_1^F\cdots\Phi_n^F|0\rangle_F\,. \tag{7·54}$$

We are not quite sure of its validity. The reason why we have written the indices F and $F+\varDelta F$ carefully is that we should distinguish the fields and the vacuum in one gauge-fixing from those in another. Namely, to each gauge-fixing, there corresponds a set of field operators and state vector space quite different from one another. Thus the field operators in one gauge-fixing cannot be expressed by those in another gauge-fixing. Hence, in order to prove the gauge-fixing invariance in more satisfactory manner, it is necessary to enlarge the state vector space $\mathcal{V}$ so that we can consider the transformation of the gauge-fixing *within a given* $\mathcal{V}$. Such an enlargement is supposed to be accomplished by the introduction of many auxiliary fields. In fact, Yokoyama[20] has succeeded in doing this in the case of Abelian gauge theory. By introducing two auxiliary fields (other than the multiplier B) called *gaugeons,* he constructed a state vector space in which the change of the parameter α_0 of covariant gauge-fixings in (2·7b) can be realized as a transformation of field operators. He extended his gaugeon formalism to the non-Abelian case also.[21] In this case, however, it is successful at present only in a special family of gauge-fixings with a gauge parameter α^a as a group *vector* [in contrast to the scalar parameter α_0 in our gauge-fixing (2·7b)]. The introduction of such a group vector parameter has a serious disadvantage in violating the manifest group symmetry.

Applications of the present formalism to other gauge theories. The present formalism has been applied successfully not only to YM theories based on internal symmetries (hence, of *compact groups*) but also to gravity based on a non-compact group, as has been seen in Chap. VI. We expect that it is *always* applicable to any meaningful gauge-theory. Recently, it was applied in an elegant form by Nakanishi[22] to gravity based on vierbein formalism which, therefore, includes spinor matter fields. Later, the supergravity[23] was also formulated within the framework of our formalism in Ref. 24) and the physical S-matrix unitarity was established in any covariant gauges.

Chapter VIII

Discussion

We have presented the manifestly covariant and local canonical operator formalism in full detail. In many examples we have seen that the present formalism really provides us powerful tools with which we can reveal the possible structures of dynamics in arbitrary gauge theories. Among others, we should note the following two basic ingredients or 'tools':

(i) the subsidiary condition $Q_B|\text{phys}\rangle = 0$, (8·1a)

(ii) the "Maxwell" equation: $\partial^\nu F_{\mu\nu} = gJ_\mu - \{Q_B, D_\mu\bar{c}\}$, (8·1b)

both of which appear quite elementary. Nevertheless, the former (i) together with the nilpotency $Q_B^2 = 0$ led to finding of

(iii) quartet mechanism, (8·2)

and the latter (ii) combined with the former (i), with the help of consequences in general theory of local covariant quantum fields, made it possible to derive both of an interesting result stating

(iv) any local observables are color-singlets (group invariants),

and the following remarkable criterion of color (i.e., quark and gluon) confinement:

(v) $u = -\mathbf{1}$ with unbroken global color symmetry. (8·3)

These (iii), (iv) and (v) are quite non-trivial and important results which could not be obtained so easily in such other formalism as the path-integral formulation. We emphasize again here the generality of the quartet mechanism to confine any type of 'unphysical' particles; for example, apart from such trivial ones as the longitudinal and scalar components of gauge bosons, the Goldstone bosons in the presence of gauge bosons (Higgs phenomena), the $U(1)$ Goldstone boson in the $U(1)$ problem (irrelevant to Higgs phenomenon), and even the very quarks and gluons (if their asymptotic fields exist and the confinement is realized by (v)). Recall also that the peculiar form of our "Maxwell" equation was useful in the proof of electric-charge universality in the W.S. model.

Since the logical structure of non-Abelian gauge theories has been clarified in the present formalism to a large extent, we should examine whether or not the 'possibilities' proposed in this paper are realized in QCD. For example, the presence of massless pseudoscalar bound-state in the channel $\epsilon^{\mu\nu\rho\sigma}\partial_\nu c \cdot \partial_\rho A_\sigma$, which is a necessary and sufficient condition for the $U(1)$ problem to be solved, must be assured directly. As for the long outstanding problem of quark confinement, we want to prove that the criterion (8·3) is really satisfied. Of course, it will require a more detailed information of dynamics. As was explained in Chap. VI, the criterion $u=-\mathbf{1}$ is closely related to the infinite effective coupling constant at the infrared limit. In this connection, it will be interesting to clarify the logical relationship between our criterion (8·3) and the others based on more intuitive pictures of confinement (e.g., the Wilson criterion[1] based on the string picture[2]). To find out such relationship might help us also to prove our criterion directly. Further, if confinement is 'proved', for instance, by the Wilson criterion in the future, then, such relationship will be very helpful for us to convince ourselves of the logical consistency of the confinement theory, because such consistency, especially the physical S-matrix unitarity, is already assured in our present formalism.

The instanton physics,[3] which has been much developed recently, is not touched upon in this report. Although we are not sure that the quark confinement problem can be solved by the instanton technique alone, one should notice that the semi-classical approximation using instanton solutions is useful as a new computational method.[4] In order to solve various dynamical problems, such non-perturbative methods are absolutely necessary to be developed.

Another important problem which has not been discussed at all is the "flavor dynamics"; namely, how many quarks (and leptons) there are, how the structure of their interactions is and why they exist as they are in the nature.[5] We have no clear ideas at present. We, however, expect that all the interactions can be described by simple gauge theories, and our present formalism is applicable to any type of gauge theories. So, as is seen in the CVC (conserved vector current) hypothesis which has been successfully proposed on the basis of the knowledge of renormalization theory, the insight into the gauge theories attained by the present formalism may some day lead to a brilliant idea to determine the structure of flavor dynamics.

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Appendix

General Aspects of Indefinite-Metric Quantum Field Theory

In this appendix, we collect some useful consequences of the general theory of relativistic quantum fields[1),1a)] extended to the cases with an indefinite metric.

A. General Postulates of the Relativistic Quantum Field Theory with an Indefinite Metric and Their Consequences

The state vector space $\mathcal{V}$ of our indefinite-metric quantum field theory is required to satisfy the usual postulates[1)] of quantum field theory apart from the positivity assumption of the metric, namely,

(0) Principles of the quantum theory (apart from the positivity),
(i) Poincaré covariance,
(ii) Spectrum condition,
(iii) Local (anti-) commutativity.

In connection with the postulate (0), the space $\mathcal{V}$ is required to be a *topological vector space* with an (indefinite) inner product $\langle | \rangle$ *separately continuous* with respect to its topology τ. If this inner product $\langle | \rangle$ is degenerate, namely, if there exists some non-zero vector $|\omega\rangle \in \mathcal{V}$ orthogonal to $\mathcal{V}$,

$$\langle \omega | \Psi \rangle = 0 \quad \text{for } \forall |\Psi\rangle \in \mathcal{V}\,, \tag{A·1}$$

such a vector $|\omega\rangle$ as the above has no physical effect and is an irrelevant object according to the principles of the quantum theory. So, we can assume, without loss of generality, the inner product $\langle | \rangle$ is *non-degenerate*, namely, a vector $|\chi\rangle$ satisfying the condition (A·1) is nothing but the null vector:

$$\langle \omega | \Psi \rangle = 0 \quad \text{for } \forall |\Psi\rangle \in \mathcal{V} \quad \Rightarrow \quad |\omega\rangle = 0\,. \tag{A·2}$$

By the assumption (A·2), for any zero-norm vector ('neutral vector' in the mathematical terminology[2)]) $|\chi\rangle$ orthogonal to itself

$$\langle \chi | \chi \rangle = 0\,, \tag{A·3}$$

there should exist some vector $|\Psi\rangle \in \mathcal{V}$ not orthogonal to it:

$$\langle \chi | \Psi \rangle \neq 0\,. \tag{A·4}$$

It may be instructive to note that the coexistence of the above two conditions (A·3) and (A·4) implies[2)] the indefiniteness of the inner product $\langle | \rangle$:

$$\langle\Phi|\Phi\rangle \gtreqqless 0 \quad \text{in } \mathcal{V}.$$

While the inner product $\langle|\rangle$ is *non-degenerate in the whole* $\mathcal{V}$, it *may be degenerate in the subspace* of $\mathcal{V}$, for example,

*Lemma A.*1 Let $\mathcal{W}$ be a *positive* (semi-definite) subspace of $\mathcal{V}$. Then, every neutral vector in $\mathcal{W}$ is orthogonal to $\mathcal{W}$:

$$\mathcal{W}_0 \equiv \{|\chi\rangle \in \mathcal{W};\ \langle\chi|\chi\rangle = 0\} \quad \perp \mathcal{W}. \tag{A·5}$$

The same conclusion as (A·5) holds for the case with a negative (semi-definite) subspace $\mathcal{W}$.

Proof) By the semi-definiteness of the inner product in $\mathcal{W}$, the Cauchy-Schwarz inequality holds in $\mathcal{W}$:

$$|\langle\Phi|\Psi\rangle| \leq |\langle\Phi|\Phi\rangle|^{1/2}|\langle\Psi|\Psi\rangle|^{1/2}, \tag{A·6}$$

from which (A·5) follows. □

Thus, the zero-norm subspace $\mathcal{V}_0$ of the physical subspace $\mathcal{V}_{\text{phys}}$ (2·29) is orthogonal to $\mathcal{V}_{\text{phys}}$:

$$\mathcal{V}_0 \perp \mathcal{V}_{\text{phys}}, \tag{A·7}$$

because $\mathcal{V}_{\text{phys}}$ is positive semi-definite as is proved in Chap. III. The above (A·3) and (A·4) tell us that for any zero-norm physical state $|\chi\rangle \in \mathcal{V}_0$ there is some unphysical state $|\Psi\rangle \notin \mathcal{V}_{\text{phys}}$ not orthogonal to it:

$$\langle\chi|\Psi\rangle \neq 0 \quad \text{for } |\chi\rangle \in \mathcal{V}_0 \text{ and } \exists|\Psi\rangle \notin \mathcal{V}_{\text{phys}}. \tag{A·8}$$

By (A·8) we know that the subspace $\mathcal{V}_0$, and hence $\mathcal{V}_{\text{phys}}$, have *no orthogonal projection* to themselves.

The next problem is the topology of $\mathcal{V}$, which has been assumed to make the inner product $\langle|\rangle$ separately continuous. Such a topology is called a *partial majorant*.[2] From various points of view, it seems natural and convenient to impose the additional requirement on the topology τ that it should be an *admissible topology*,[2] the weakest one of which is the weak topology (w).[2] Namely, the linear functional φ on $\mathcal{V}$ is continuous with respect to τ if and only if φ is written in the form:

$$\varphi(|\Psi\rangle) = \langle\Phi|\Psi\rangle$$

by some vector $|\Phi\rangle \in \mathcal{V}$. The following lemma, which is familiar in the cases of the Hilbert space, holds:

Lemma A. 2 Let $\mathcal{W}$ be an arbitrary subspace of $\mathcal{V}$. Then, the equality

$$\mathcal{W}^{\perp\perp}=\overline{\mathcal{W}}^{\tau} \qquad \text{(A·9)}$$

holds for any admissible topology τ of $\mathcal{V}$, where $\mathcal{W}^{\perp}$ is defined by

$$\mathcal{W}^{\perp}\equiv\{|\Phi\rangle\in\mathcal{V};\langle\Phi|\Psi\rangle=0 \quad \text{for } {}^{\forall}|\Psi\rangle\in\mathcal{W}\} \qquad \text{(A·10)}$$

and $\overline{\mathcal{W}}^{\tau}$ is the closure of $\mathcal{W}$ with respect to the topology τ.

Proof) See, Ref. 2). □

Corollary A. 3 Let $\mathcal{W}$ be any dense subspace of $\mathcal{V}$ with respect to an admissible topology τ:

$$\overline{\mathcal{W}}^{\tau}=\mathcal{V}\,. \qquad \text{(A·11)}$$

Then, any vector orthogonal to $\mathcal{W}$ is 0:

$$\mathcal{W}^{\perp}=(\overline{\mathcal{W}}^{\tau})^{\perp}=\mathcal{V}^{\perp}=0\,. \qquad \text{(A·12)}$$

Proof) The first equality is due to the equality

$$\mathcal{W}^{\perp}=\mathcal{W}^{\perp\perp\perp} \qquad \text{(A·13)}$$

and (A·9). The third one follows from the assumption (A·2). □

As for the postulate (i) of the Poincaré covariance, we assume such 'ordinary ingredients[1] as the unitary representation $U(a,\Lambda)$ of the Poincaré group and as the fields $\varphi_i(x)$ covariant under the Poincaré transformations, and so on. In some respects, however, we can do so, for the time being, only *in a formal sense*, owing to lack of the positivity. For example, the unitarity of $U(a,\Lambda)$ means the *unitarity with respect to the indefinite inner product* $\langle|\rangle$, which, contrary to the unitarity with respect to the positive definite inner product, neither necessarily implies that the operator $U(a,\Lambda)$ is a bounded one nor that $U(a,\boldsymbol{1})$ can be written in the form

$$U(a,\boldsymbol{1})\equiv U(a)=e^{iP_\mu a^\mu}=\int e^{ip_\mu a^\mu}dE(p)\,. \qquad \text{(A·14)}$$

Since no correspondent of the spectral resolution theorem,*) valid in the Hilbert space, has been proved yet in the indefinite metric cases, the precise meaning of such an expression as $e^{iP_\mu a^\mu}$ is not so clear. Thus, the relation between the energy-momentum operator P_μ and the translation operator $U(a)$ is a symbolic one. These situations may seem to endanger the postulate (ii) of the spectrum condition, which can, however, be formulated in the following form without any difficulties.

First, let us recall that the fields $\varphi_i(x)$ are not operators by themselves but operator-valued (tempered) distributions which become operators by smear-

*) See, SNAG theorem in Ref. 1).

ing with test functions:

$$\varphi_i(f)=\int d^4x\varphi_i(x)f(x). \tag{A·15}$$

Precisely speaking, we should have a *common dense domain* Ω in $\mathcal{V}$ of any operator of the form (A·15) with $f\in\mathscr{S}(\boldsymbol{R}^4)$ *) and of any $U(a,\Lambda)$, stable under these operators $\varphi_i(f)$ and $U(a,\Lambda)$, where the term "dense" means "dense in any *admissible topology* τ of $\mathcal{V}$", namely,

$$\bar{\Omega}^\tau=\Omega^{\perp\perp}=\mathcal{V}\,. \tag{A·16}$$

Further, the linear functional

$$f\mapsto\langle\Phi|\varphi_i(f)|\Psi\rangle \tag{A·17}$$

should be continuous for any $|\Phi\rangle, |\Psi\rangle\in\Omega$ with respect to the topology of $\mathscr{S}(\boldsymbol{R}^4)$. We denote, as $\mathfrak{F}$ and $\mathfrak{F}(\mathcal{O})$, the polynomial algebras generated by the operators of the form

$$\int d^4x_1\cdots d^4x_r\varphi_{i_1}(x_1)\cdots\varphi_{i_r}(x_r)f(x_1,\cdots,x_r) \tag{A·18}$$

with $f\in\mathscr{S}(\boldsymbol{R}^{4r})$ and with $f\in\mathscr{D}(\overbrace{\mathcal{O}\times\cdots\times\mathcal{O}}^{r})$,*) respectively. In the case that $\mathcal{O}\subset\boldsymbol{R}^4$ is a finite space-time region, we call an element of $\mathfrak{F}(\mathcal{O})$ an *local* operator taking account of the local (anti-) commutativity postulate (iii). Here we add a further postulate (iv):

(iv) Existence of the cyclic vacuum: There exists a vector $|0\rangle\in\Omega$ (vacuum) invariant under any translations,

$$U(a)|0\rangle=|0\rangle \tag{A·19a}$$

or

$$P_\mu|0\rangle=0\,, \tag{A·19b}$$

which is *cyclic* with respect to $\mathfrak{F}$:

$$\mathcal{V}=\overline{\mathfrak{F}|0\rangle}^\tau. \tag{A·20}$$

Now, the spectrum condition (ii) is postulated in the form[3] as

$$\int d^4ae^{-ip_\mu a^\mu}\langle\Phi|U(a)|\Psi\rangle=0,\quad \text{if } p\notin\bar{V}_+\equiv\{q\in\boldsymbol{R}^4;\, q_0\geq0,\, q^2\geq0\} \tag{A·21}$$

for any $|\Phi\rangle, |\Psi\rangle\in\mathfrak{F}|0\rangle$, or equivalently as

*) $\mathscr{S}$ and $\mathscr{D}$ here represent the spaces of test functions decreasing rapidly and having compact supports, respectively. See the textbooks of the theory of distributions or Ref. 1).

$$\int d^4\xi_1\cdots d^4\xi_{r-1}e^{i(q_1\xi_1+\cdots+q_{r-1}\xi_{r-1})}W_{i_1\cdots i_{r-1}}(\xi_1,\cdots,\xi_{r-1})$$
$$\equiv\widetilde{W}_{i_1\cdots i_{r-1}}(q_1,\cdots,q_{r-1})=0\qquad \text{if } \exists q_j\notin\overline{V}_+\,, \tag{A·22}$$

where $W_{i_1\cdots i_r}(\xi_1,\cdots\xi_{r-1})$ is defined by

$$\langle 0|\varphi_{i_1}(x_1)\cdots\varphi_{i_r}(x_r)|0\rangle\equiv W_{i_1\cdots i_r}(x_1-x_2,\cdots,x_{r-1}-x_r) \tag{A·23}$$

upon the basis of the translational invariance of the vacuum (A·19a). According to the well-known techniques,[1] (A·22) combined with the postulates (i) and (iii) allows us to continue $W_{i_1\cdots i_r}(\xi_1,\cdots,\xi_{r-1})$ analytically to the complex analytic function $W_{i_1\cdots i_r}(\zeta_1,\cdots,\zeta_{r-1})$ in the permuted extended tube, in much the same way as the positive metric cases. This analyticity property furnishes us with powerful techniques, the well-known one of which is the Reeh-Schlieder theorem:[4),1)]

Theorem A. 4 (Reeh-Schlieder theorem) For any open set $\mathcal{O}$ of space-time, the equality

$$\overline{\mathfrak{F}(\mathcal{O})|0\rangle}^{\tau}=\overline{\mathfrak{F}|0\rangle}^{\tau} \tag{A·24}$$

holds for any *admissible topology* τ. On the assumption (iv) of the cyclicity of the vacuum (A·20), we obtain

$$\overline{\mathfrak{F}(\mathcal{O})|0\rangle}^{\tau}=\mathcal{V} \tag{A·24′}$$

or

$$(\mathfrak{F}(\mathcal{O})|0\rangle)^{\perp}=(\mathfrak{F}|0\rangle)^{\perp}=0\,. \tag{A·24″}$$

Proof) By the "Edge-of-the-Wedge" theorem, we obtain

$$\langle\Psi|\varphi_{i_1}(x_1)\cdots\varphi_{i_r}(x_r)|0\rangle=0\,, \tag{A·25}$$

from the equality

$$\langle\Psi|\int d^4x_1\cdots d^4x_r f(x_1,\cdots,x_r)\varphi_{i_1}(x_1)\cdots\varphi_{i_r}(x_r)|0\rangle=0 \tag{A·26}$$

for $f\in\mathcal{D}(\overbrace{\mathcal{O}\times\cdots\times\mathcal{O}}^{r})$, namely,

$$(\mathfrak{F}(\mathcal{O})|0\rangle)^{\perp}=(\mathfrak{F}|0\rangle)^{\perp}. \tag{A·27}$$

By virtue of Lemma A. 2, Eq. (A·27) tells us

$$\overline{\mathfrak{F}(\mathcal{O})|0\rangle}^{\tau}=(\mathfrak{F}(\mathcal{O})|0\rangle)^{\perp\perp}=(\mathfrak{F}|0\rangle)^{\perp\perp}=\overline{\mathfrak{F}|0\rangle}^{\tau}. \qquad \square$$

Combining the above theorem with the postulate (iii) of *local (anti-) commutativity,* we obtain the following corollary.

Corollary A. 5 If $\mathcal{O}$ is an open set of space-time whose causal complement $\mathcal{O}'$ defined by

$$\mathcal{O}' \equiv \text{the interior of the set } \{x \in \boldsymbol{R}^4;\ (x-y)^2 < 0 \ \text{ for } \forall y \in \mathcal{O}\} \qquad (\text{A}\cdot 28)$$

is not empty, and $\varphi \in \mathfrak{F}(\mathcal{O})$, then

$$\varphi|0\rangle = 0 \qquad (\text{A}\cdot 29)$$

implies $\varphi = 0$ on the assumption of (iv). In particular, since $\mathcal{O}'$ for a *bounded* open set $\mathcal{O}$ is not empty, any *local* operator $\varphi \in \mathfrak{F}(\mathcal{O})$ annihilating the vacuum is vanishing in itself.

Proof) It is sufficient to consider the case with fields satisfying the local commutativity, since the cases containing both local commutative and anti-commutative fields can be treated in a similar manner with slight modifications. From (A·29) and the local commutativity, we obtain

$$0 = \langle \Psi | \psi\varphi | 0\rangle = \langle \Psi | \varphi\psi | 0\rangle \qquad (\text{A}\cdot 30)$$

for any $|\Psi\rangle \in \Omega$ and any $\psi \in \mathfrak{F}(\mathcal{O}')$. Thus $\varphi^\dagger |\Psi\rangle$ belongs to $\mathfrak{F}(\mathcal{O}')|0\rangle^\perp$, which is nothing but 0:

$$\varphi^\dagger |\Psi\rangle \in \mathfrak{F}(\mathcal{O}')\,|0\rangle^\perp = 0\,, \qquad (\text{A}\cdot 31)$$

by virtue of Theorem A. 4:

$$\overline{\mathfrak{F}(\mathcal{O}')\,|0\rangle}^{\tau} = \mathcal{V}\,, \qquad (\text{A}\cdot 32)$$

and of Corollary A. 3. Then, we obtain, from (A·31),

$$\langle \Psi | \varphi | \Phi\rangle = \langle \varphi^\dagger \Psi | \Phi\rangle = 0 \qquad \text{for } \forall |\Phi\rangle \in \Omega\,, \qquad (\text{A}\cdot 33)$$

which says

$$\varphi|\Phi\rangle = 0 \quad \text{for } \forall |\Phi\rangle \in \Omega \qquad (\text{A}\cdot 34)$$

or

$$\varphi = 0\,, \qquad (\text{A}\cdot 35)$$

by virtue of the denseness of Ω (A·16) and of Corollary A. 3. □

Next, we comment on the postulate (iv) of the cyclicity of the vacuum, which is nothing but a natural requirement that every state in a field theory should be described in terms of fields. In the positive metric cases, it is well known[1), 5)~7)] that, on the assumptions (i) ~ (iii), this condition (iv)

(iv) cyclicity of the vacuum

is equivalent to the following three conditions equivalent to one another:

(iva) irreducibility*) of the field algebra $\mathfrak{F}$,
(ivb) uniqueness of the vacuum,
(ivc) cluster property.

The implication (iv) ⇒ (iva),[5),**)] which is a consequence of the spectrum condition, has not proved yet in the general cases with indefinite metrics. But this holds also in these cases on the assumption of asymptotic completeness, because one can prove (iv) ⇒ (iva) in a Fock space, as will be seen in Appendix C. The cluster property in the indefinite metric theory has already been discussed in § 5. 4 from the viewpoint of the quark confinement.

The problem that we want to discuss is the implication (iva) ⇒ (ivb) in the indefinite metric cases. The proof given by Borchers[6)] in the positive metric cases is based upon the consequence of a profound theorem—the *PCT theorem*.[1),8)] This theorem, obtained from the analyticity combined with the Lorentz invariance (ii), clarifies the relation between the locality (iii) and the PCT symmetry, the important discrete symmetry of the theory: The PCT invariance of the theory is equivalent to the weak local commutativity which is a weaker condition than the local commutativity. Since this theorem presupposes the validity of the spin-statistics theorem,[1)] however, it does not hold generally in the theory with an indefinite metric which invalidates the spin-statistics theorem allowing the existence of such *scalar fermions* as the Faddeev-Popov ghosts, for example. In the case in question of our Yang-Mills theory, the PCT symmetry does hold with a slight modification as has been shown in § 2. 4. Namely, the invalidity of the spin-statistics theorem due to the Faddeev-Popov ghosts is harmless except the minor change of their PCT transformation law (2·38). From this fact and the reconstruction theorem[9),10)] valid also in cases with indefinite metric, we can safely assert the existence of the antiunitary PCT operator Θ defined (at least in Ω) by

$$\Theta|0\rangle=0\,, \tag{A·36a}$$

$$\Theta\Phi_{i_1}(x_1)\cdots\Phi_{i_r}(x_r)\,|0\rangle=\Phi_{i_1}^{\mathrm{PCT}}(x_1)\cdots\Phi_{i_r}^{\mathrm{PCT}}(x_r)\,|0\rangle\,, \tag{A·36b}$$

and satisfying

$$\Theta^2=1\ \ (\text{in }\Omega)\,, \tag{A·36c}$$

$$\Theta\Phi_i(x)\,\Theta=\Phi_i^{\mathrm{PCT}}(x)\,. \tag{A·36d}$$

Using this fact, we can now conclude, from the irreducibility (iva), the *unique-*

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*) In the case with indefinite metric, the concept of irreducibility splits into the two concepts of "subspace irreducibility" and "operator irreducibility" [see I. M. Gelfand, et al., *Generalized Functions* (Academic Press, New York-London, 1966) Vol. 5, pp. 148~150], which are equivalent to each other in the Hilbert space. Here we understand the term "irreducibility" to mean that both of the above two irreducibilities hold.

**) The implication (iva) ⇒ (iv) is trivial.

ness of the vacuum (ivb) (in a rather restricted sense), which plays an important role in the discussions made in Appendix B about the well-definedness condition for charge operators.

Proposition A. 6[11] If the field algebra $\mathcal{F}$ is irreducible, there exists no such other vacuum*) $|0'\rangle$ (in Ω) linearly independent of $|0\rangle$ that

$$U(x)|0'\rangle=|0'\rangle \quad \text{for} \quad \forall x\in \boldsymbol{R}^4 \tag{A·37}$$

$$\Theta|0'\rangle=e^{i\omega}|0'\rangle. \quad (\omega\in \boldsymbol{R}) \tag{A·38}$$ *)

Proof) Let there exist such a vacuum as $|0'\rangle$, then we can obtain one more vacuum $|0\rangle_{\alpha\beta}$

$$|0\rangle_{\alpha\beta}\equiv\alpha|0\rangle+\beta|0'\rangle, \tag{A·39}$$

which is normalized by a suitable choice of complex numbers α, β and is cyclic because of the assumption of the irreducibility. Then, according to the (modified) PCT theorem in the Yang-Mills theory, the locality and the spectrum condition of theory with the vacuum $|0\rangle_{\alpha\beta}$ allow us to construct the PCT operator $\Theta_{\alpha\beta}$ referring to this vacuum $|0\rangle_{\alpha\beta}$:

$$\Theta_{\alpha\beta}|0\rangle_{\alpha\beta}=|0\rangle_{\alpha\beta}, \tag{A·40a}$$

$$\Theta_{\alpha\beta}\Phi_{i_1}(x_1)\cdots\Phi_{i_r}(x_r)|0\rangle_{\alpha\beta}=\Phi_{i_1}^{\text{PCT}}(x_1)\cdots\Phi_{i_r}^{\text{PCT}}(x_r)|0\rangle_{\alpha\beta}, \tag{A·40b}$$

$$\Theta_{\alpha\beta}(\lambda|\Psi\rangle+\mu|\Phi\rangle)=\lambda^*\Theta_{\alpha\beta}|\Psi\rangle+\mu^*\Theta_{\alpha\beta}|\Phi\rangle, \tag{A·40c}$$

$$\Theta_{\alpha\beta}^2=1 \quad (\text{in } \Omega) \tag{A·40d}$$

$$\Theta_{\alpha\beta}\Phi_i(x)\Theta_{\alpha\beta}=\Phi_i^{\text{PCT}}(x). \tag{A·40e}$$

By (A·36d) and (A·40e), one can easily check the commutativity of $\Theta\Theta_{\alpha\beta}$ with every $\Phi_i(x)$. On the other hand, $\Theta\Theta_{\alpha\beta}|0\rangle_{\alpha\beta}$ can be made not proportional to $|0\rangle_{\alpha\beta}$,

$$\begin{aligned}\Theta\Theta_{\alpha\beta}|0\rangle_{\alpha\beta}&=\Theta|0\rangle_{\alpha\beta}=\alpha^*|0\rangle+\beta^*e^{i\omega}|0'\rangle\\ &\not\propto\alpha|0\rangle+\beta|0'\rangle=|0\rangle_{\alpha\beta},\end{aligned} \tag{A·41}$$

by choosing α and β such that

$$\alpha^*/\alpha\neq\beta^*e^{i\omega}/\beta. \tag{A·42}$$

Then, $\Theta\Theta_{\alpha\beta}$ is not a c-number operator, whereas it commutes with every $\Phi_i(x)$. This contradicts the assumption of the irreducibility, so the vacuum $|0'\rangle$ satis-

*) In this context, "vacuum" means merely a translationally invariant (normalizable) state. In the cases with positive metric, we need not require (A·38), while, in our case, it is satisfied by the states $|0'\rangle\equiv Q|0\rangle$ with $Q=gQ^a$, G^a, N^a, etc., discussed in Chap. VI, because of (2·40).

fying (A·37) and (A·38) should be proportional to the original vacuum $|0\rangle$.

□

B. Symmetries, Currents and Charges

——*Well-definedness condition for charge operators and Goldstone theorem*——

According to Proposition A. 6, the vacuum $|0\rangle$ is assumed, in this section B, to be unique as the translationally invariant state [satisfying the condition of the form (A·38)]. As an immediate consequence of this assumption, we obtain:

Proposition B. 1 Let Q be a well-defined*) conserved charge associated with an internal symmetry. Then, Q annihilates the vacuum:

$$Q|0\rangle=0\,. \tag{B·1}$$

Proof) First, note that Q is invariant under the translation, and hence $Q|0\rangle$ is a translationally invariant state:

$$U(x)\,Q|0\rangle=QU(x)\,|0\rangle=Q|0\rangle \quad \text{for } \forall x\in \boldsymbol{R}^4. \tag{B·2}$$

Then the uniqueness**) of the vacuum implies that $Q|0\rangle$ is proportional to $|0\rangle$:

$$Q|0\rangle=q|0\rangle, \quad q=\langle 0|Q|0\rangle\,. \tag{B·3}$$

Since Q is obtained (formally***)) as the volume integral of the current j_μ

$$Q=\int d^3x\, j_0\,, \tag{B·6}$$ ***)

and j_μ satisfies

$$\langle 0|j_\mu|0\rangle=0 \tag{B·7}$$

owing to the Lorentz covariance, the coefficient $q=\langle 0|Q|0\rangle$ in (B·3) should vanish

$$\langle 0|Q|0\rangle=0\,,$$

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*) 'Well-defined' means being defined in a *dense* subspace of $\mathcal{V}$ containing the vacuum.

**) Owing to the PCT invariance of the YM theory, the Noether current j_μ of the symmetry generated by Q satisfies

$$\Theta j_\mu(x)\Theta=\mp j_\mu(-x), \tag{B·4}$$

and hence, $|0'\rangle\equiv Q|0\rangle$ satisfies (A·38) with $e^{i\omega}=\mp 1$:

$$\Theta Q|0\rangle=\mp Q|0\rangle. \tag{B·5}$$

Thus, (B·3) follows from (B·2) and (B·5), according to Proposition A. 6.

***) The precise meaning of the formal expression (B·6) will become clear in the following.

and we obtain

$$Q|0\rangle=0\,. \qquad \square$$

On the contrary, a spontaneously broken symmetry signalled by the formal expression

$$\langle 0|[Q,\chi(x)]|0\rangle\neq 0 \tag{B·8}$$

brings us a charge Q *not* satisfying the condition (B·1):

$$Q|0\rangle\neq 0\,.$$

By the above Proposition B. 1, therefore, the *global* charge Q as the total volume integral of the current (B·6) *cannot be well-defined* in this case, whereas a 'local charge' Q_R exists and generates the transformation 'locally':[12)]

$$Q_R\equiv\int d^4x\, j_0(x_0,\boldsymbol{x})\alpha_T(x_0)f_R(\boldsymbol{x}), \tag{B·9}$$

$$\frac{d}{ds}\langle\Phi|\tau_s(\varphi)|\Psi\rangle|_{s=0}=\langle\Phi|[iQ_R,\varphi]|\Psi\rangle \quad \text{for } \varphi\in\mathfrak{F}(\mathcal{O}). \tag{B·10}$$

In the above, τ_s is a (1-parameter subgroup of the) symmetry transformation of field operators and $\alpha_T\in\mathcal{D}(\boldsymbol{R})$, $f_R\in\mathcal{D}(\boldsymbol{R}^3)$ are such test functions that

$$\int dx_0\alpha_T(x_0)=1\,, \tag{B·11a}$$

$$f_R(\boldsymbol{x})=\begin{cases}1 & (|\boldsymbol{x}|\leq R),\\ 0 & (|\boldsymbol{x}|>2R).\end{cases} \tag{B·11b}$$

Equation (B·10) holds for any sufficiently large $R>0$ and independently of the choice of α_T,[12),13)] as a consequence of the locality and of the conservation law: $\partial^\mu j_\mu=0$. Now, the intuitive expression (B·8) of the spontaneous breakdown of the symmetry τ_s should properly replaced by the condition:[12)~14)]

$$\lim_{R\to\infty}\langle 0|[iQ_R,\varphi]|0\rangle\neq 0 \quad \exists\varphi\in\mathfrak{F}(\mathcal{O}). \tag{B·12}$$

The implication of (B·12) is well known as the Goldstone theorem[12),14)~16)] which asserts the existence of a massless Goldstone boson. In the neatest form, this theorem can be stated as a corollary of the following equation:[17),1a)]

$$\begin{aligned}\lim_{R\to\infty}\langle 0|[Q_R,\varphi]|0\rangle&=\lim_{R\to\infty}(\langle 0|Q_RE_1\varphi|0\rangle-\langle 0|\varphi E_1Q_R|0\rangle)\\&=2\lim_{R\to\infty}\langle 0|Q_RE_1\varphi|0\rangle=2\lim_{R\to\infty}\langle 0|Q_R\varphi|0\rangle\,,\end{aligned} \tag{B·13}$$

where E_1 is the projection of the states of mass zero. Equation (B·13) is proved on the assumption of the positive metric and the authors do not know whether it can be extended to the cases with indefinite metrics. To authors' knowledge, there exists the following statement for these cases made by Strocchi:[8]

Theorem B. 2 (Goldstone theorem) Spontaneous breakdown of the symmetry, (B·12), occurs, if and only if the Fourier transform of $\langle 0|[j_\mu(x),\varphi]|0\rangle$ contains a $\delta(p^2)$ singularity.

Proof) [See, Ref. 3).] □

We believe, from this theorem, that if the projection E_1 of the states of mass zero can be defined also in these cases in such a form as

$$E_1=\sum_{i,j} a_i^\dagger|0\rangle\eta_{ij}^{-1}\langle 0|a_j\,, \tag{B·14}$$

Eq. (B·13) might be verified in the indefinite metric cases. Here, $a_i^\dagger$ and a_j are the creation and annihilation operators of massless asymptotic fields and η_{ij}^{-1} is the inverse of the metric matrix $\eta_{ij}=[a_i, a_j^\dagger]_\mp$. In any case,

Corollary B. 3 *The criterion for the spontaneous breakdown of the symmetry* is given by

$$\langle 0|j_\mu|\Psi\rangle\neq 0 \quad \text{for } \exists|\Psi\rangle\text{: massless 1-particle state.} \tag{B·15}$$

Taking account of the fact that the weak topology is the (weakest) *admissible* topology,[2] we can verify the following:

Corollary B. 4 The necessary and sufficient condition for the global charge Q given by

$$\langle\Phi|Q|\Psi\rangle=\lim_{R\to\infty}\langle\Phi|Q_R|\Psi\rangle \tag{B·16}$$

to be a well-defined charge with the dense domain $\mathfrak{F}(\mathcal{O})|0\rangle$ is that one of the following conditions is valid:

(i) $\lim_{R\to\infty}\langle 0|[Q_R,\varphi]|0\rangle=0 \quad \text{for } \forall\varphi\in\mathfrak{F}(\mathcal{O})$, (B·17)

(ii) $\lim_{R\to\infty}\langle 0|\varphi Q_R|0\rangle=0 \quad \text{for } \forall\varphi\in\mathfrak{F}(\mathcal{O})$, (B·18)

(iii) $\langle 0|j_\mu|\Psi(p^2=0)\rangle=\langle 0|j_\mu^{\mathrm{as}}|\Psi(p^2=0)\rangle=0$, (B·19)

(iv) $Q|0\rangle=0$, (B·20)

(v) $Q\varphi|0\rangle=[Q_R,\varphi]|0\rangle \quad \text{for } \forall\varphi\in\mathfrak{F}(\mathcal{O}),\ \exists R_0>0,\ \forall R>R_0$. (B·21)

In (B·19) where the Yang-Feldman equation [(C·2) below] is used, j_μ^{as} is the asymptotic form of j_μ.

Proof) (Omitted.) □

These results show that every *well-defined* charge does *not* suffer from spontaneous symmetry breaking and contains no *discrete massless* spectrum and, conversely, that every charge suffering from *spontaneous breakdown* cannot be well defined owing to the discrete massless spectrum—Goldstone boson.

C. Asymptotic Fields, Asymptotic States and Their Behavior under the Symmetry Transformation

——*Greenberg-Robinson theorem and GLZ formula*——

In the theory with a *positive definite metric* and with *mass gap*, the notions of asymptotic states and asymptotic fields have their sound basis in the Haag-Ruelle scattering theory.[5),18)] Since the existence of a massless field*) and of an indefinite metric obstructs the extension of this theory to our present case, we cannot but take a naive attitude to interpret the asymptotic fields and states as the representatives of the *discrete poles* in Green's functions. Namely,

(ii′) *Characterization of asymptotic fields and spectrum condition for them*:
Corresponding to each discrete spectrum of $P_\mu P^\mu$ appearing as a discrete pole at $p^2=m_i^2$ (≥ 0) of time-ordered Green's functions in momentum space, an asymptotic field ϕ_i^{as} ('as'=in or out) satisfying

$$(\square+m_i^2)^{r_i}\phi_i^{\mathrm{as}}(x)=0 \tag{C·1}$$

with a positive integer r_i is assumed to exist.

As for the relation between the asymptotic fields and the original Heisenberg fields, we assume the validity of the *Yang-Feldman equation*, which gives an expression for the asymptotic fields in terms of Heisenberg fields: For example, in the case with $r_i=1$, it is written as

$$\phi_i^{\substack{\mathrm{in}\\ \mathrm{out}}}(x)=\Phi_i(x)-\int d^4y\,\Delta_{\substack{\mathrm{ret}\\ \mathrm{adv}}}(x-y\,;\,m_i^2)\,j_i(y) \tag{C·2}$$

with a Heisenberg field $\Phi_i(x)$ and with its source $j_i(x)=(\square+m_i^2)\Phi_i(x)$ containing no discrete spectrum at $p^2=m_i^2$. From this, we obtain a convenient equation

$$\langle 0|\Phi_i(x)|\Psi(p^2=m_i^2)\rangle=\langle 0|\phi_i^{\mathrm{as}}(x)|\Psi(p^2=m_i^2)\rangle. \tag{C·3}$$

In the cases with bound states and with multipole-ghosts ($r_i\geq 2$), due modifications to (C·2) are necessary.[20),21)]

The Haag-Ruelle scattering theory[22)] valid in the cases with positive definite metric and with mass gap tells us that all asymptotic fields are mutually (anti-) local. In the present case with indefinite metric and without mass gap,

*) As for an extension of the Haag-Ruelle theory to a certain type of massless theory with positive metric, see Ref. 19).

there has been no such proof.*) So, we simply assume

(iii′) *Locality of asymptotic fields:*
All asymptotic fields are mutually (anti-) local.

The correspondent of the postulate (iv) of the cyclicity of the vacuum in this case means nothing but the assumption of *asymptotic completeness.*

(iv′) *Asymptotic completeness*: The vacuum $|0\rangle$ is cyclic with respect to the totality of asymptotic fields: $\mathcal{V}=\mathcal{V}^{\rm in}=\mathcal{V}^{\rm out}$.

On the assumption of (ii′), (iii′) together with the Lorentz covariance of the asymptotic fields which follows from (C·2), we obtain the following theorem, from the Greenberg-Robinson theorem,[23] which can be extended[21] to the cases with indefinite metric.

Theorem C. 1 The (anti-) commutator $[\phi_i^{\rm as}(x), \phi_j^{\rm as}(y)]_\mp$ is a c-number.

Proof) [See, Ref. 21).] □

Thus, the space of asymptotic states is a *Fock space* of asymptotic fields, and hence, for the asymptotic fields, the cyclicity (iv′) of the vacuum implies the *irreducibility* of the asymptotic fields. This is an immediate consequence of the following (generalized) *Haag-GLZ expansion formula*[24),25)] which gives, on the assumption of (iv′), any linear operator L an expression in terms of the asymptotic fields:

$$L=\sum_{n=0}^{\infty}\frac{i^n}{n!}\sum_{\substack{i_1\cdots i_n\\ j_1\cdots j_n}}\int\Big(\prod_{a=1}^{n}d^3x_a\Big)\langle 0|[\cdots[L,\phi_{i_1}^{\rm as}(x_1)],\cdots,\phi_{i_n}^{\rm as}(x_n)]|0\rangle$$
$$\times\eta_{i_1j_1}^{-1}\cdots\eta_{i_nj_n}^{-1}\overleftrightarrow{\partial}_0^{x_1}\cdots\overleftrightarrow{\partial}_0^{x_n}:\phi_{j_n}^{\rm as}(x_n)\cdots\phi_{j_1}^{\rm as}(x_1): \qquad \text{(C·4)}$$

In (C·4), :⋯: means the Wick normal product. For simplicity, we have written the formula for the case of scalar fields satisfying the commutation relation

$$[\phi_i^{\rm as}(x), \phi_j^{\rm as}(y)]=i\eta_{ij}\delta_{m_im_j}\Delta(x-y;m_i^2). \qquad \text{(C·5)}$$

From (C·4), it follows trivially that any operator L commuting with all the asymptotic fields $\phi_i^{\rm as}$ is nothing but a c-number: $L=\langle 0|L|0\rangle\mathbf{1}$, namely, the totality of asymptotic fields is irreducible. This fact implies further the following consequence:

Proposition C. 2 The assumption (iv′) of asymptotic completeness implies the irreducibility of the (Heisenberg) field algebra $\mathcal{F}$.

*) In the proof of the locality of asymptotic fields made in Ref. 21), there is a mistake in the use of the Jost-Lehmann-Dyson representation, as a consequence of which the proof is invalidated.

Proof) Let L be any operator commuting with all the Heisenberg fields, then the Yang-Feldman equation (C·2) tells us that L commutes with all the asymptotic fields. Since the totality of asymptotic fields is irreducible on the assumption (iv′) of asymptotic completeness, L is nothing but a c-number. □

If we take L as a Heisenberg field $\Phi_i(x)$, the Haag-GLZ formula (C·4) gives us an expression for the Heisenberg fields in terms of the asymptotic fields. Taking account of the LSZ reduction formulae[26] which can be derived from the LSZ weak asymptotic conditions as a consequence of the Yang-Feldman equation (C·2), we can rewrite the above Haag-GLZ formula in the following way:

$$\begin{aligned} S\varphi = & \sum_{n=0}^{\infty} \frac{i^n}{n!} \sum_{i_1 \cdots i_n} \sum_{j_1 \cdots j_n} \int \prod_{a=1}^{n} d^4 x_a : \phi_{i_1}^{\mathrm{as}}(x_1) \cdots \phi_{i_n}^{\mathrm{as}}(x_n) : \\ & \times \prod_{b=1}^{n} [\eta_{i_b j_b}^{-1} (\Box^{x_b} + m_{j_b}^2)] \langle 0 | T[\varphi \Phi_{j_n}(x_n) \cdots \Phi_{j_1}(x_1)] | 0 \rangle \\ \equiv & : \exp(\phi^T \eta^{-1} K \delta/\delta J) : \langle 0 | T(\varphi \exp iJ^T \Phi) | 0 \rangle |_{J=0} \\ \equiv & : \mathcal{K} : \langle 0 | T\varphi \exp iJ^T \Phi | 0 \rangle , \end{aligned} \tag{C·6}$$

where K is the matrix of Klein-Gordon operators: $K \equiv (K_{ij}) = (\delta_{ij}(\Box + m_i^2))$. We call this formula (C·6) the (generalized) *GLZ formula*, which holds for any polynomial $\varphi(\in \mathfrak{F})$ of local (Heisenberg) operators. The operator S in (C·6) is the S-matrix operator, which is written as

$$\begin{aligned} S = & : \exp(\phi^T \eta^{-1} K \delta/\delta J) : \langle 0 | T(\exp iJ^T \Phi) | 0 \rangle |_{J=0} \\ = & : \mathcal{K} : \langle 0 | T \exp iJ^T \Phi | 0 \rangle \end{aligned} \tag{C·7}$$

by setting $\varphi = \mathbf{1}$ in (C·6). These GLZ formulae are useful in the discussion about the behavior of the asymptotic fields under the symmetry transformation.

First, we note that every asymptotic field is transformed *linearly* under *any* unbroken (nonlinear) transformation of an internal symmetry.

Theorem C. 3[27] Let $\delta\phi_i^{\mathrm{as}}$ be the infinitesimal transform of the asymptotic field ϕ_i^{as} by the symmetry transformation generated by a well-defined charge Q:

$$[iQ, \phi_i^{\mathrm{as}}(x)]_{\mp} = \delta\phi_i^{\mathrm{as}}(x). \tag{C·8}$$

Then, $\delta\phi_i^{\mathrm{as}}(x)$ depends *linearly* upon the asymptotic fields $\phi_j^{\mathrm{as}}(x)$:

$$\delta\phi_i^{\mathrm{as}}(x) = a_{ij} \phi_j^{\mathrm{as}}(x). \tag{C·9}$$

The coefficients a_{ij} may contain finite-order differential operators in ∂_μ.

Proof) We first note that the W.T. identity

$$\langle 0|TJ^T\delta\theta\delta\Phi \exp iJ^T\Phi|0\rangle=\langle 0|[i\delta\theta Q, T\exp iJ^T\Phi]|0\rangle=0\,, \qquad (\text{C}\cdot 10)$$

follows from the unbroken symmetry generated by Q:

$$[i\delta\theta Q, \Phi_i]=\delta\theta[iQ,\Phi_i]_{\mp}=\delta\theta\delta\Phi_i\,. \qquad (\text{C}\cdot 11)$$

Here we have inserted an "infinitesimal-transformation parameter" $\delta\theta$ in order to treat the ordinary and "super-type" charges on the same footing. Differentiating (C·10) with respect to $J_k(x)$, we obtain

$$\langle 0|T(\delta\Phi_k(x)+i\Phi_k(x)J^T\delta\theta\delta\Phi)\exp iJ^T\Phi|0\rangle=0\,. \qquad (\text{C}\cdot 12)$$

Next, these W.T. identities (C·10) and (C·12) with the operation $:\mathcal{K}:$ defined in (C·6) can be simplified as follows. When the operator $:\phi^T\eta^{-1}K\delta/\delta J:$ is applied to (C·10) and (C·12), the external sources J_i are replaced by the Klein-Gordon operator K with coefficient of on-shell quantity ϕ: $J_i\rightarrow(\phi^T\eta^{-1}K)_i$. Because of the presence of the operator $(\phi^T\eta^{-1}K)_i$ instead of $J_i(x)$, the fields $\delta\Phi_i(x)$, which generally contain non-linear terms of fields also, can be replaced by the *linear* combinations of fields with the *same mass* m_i:

$$\delta\Phi_i(x)\underset{\text{on-shell}}{\Longrightarrow}\delta\theta\, a_{ij}\Phi_j(x)\,, \qquad (\text{C}\cdot 13\text{a})$$

$$\begin{aligned} a_{ij}&=\sum_k\int d^4z\langle 0|T[\delta\Phi_i(x)\Phi_k(z)]|0\rangle\langle 0|T(\Phi_k(z)\Phi_j(y))|0\rangle^{-1}|_{\text{on-shell}}\\ &=i\langle 0|T[\delta\Phi_i(x)\Phi_k(y)]|0\rangle\overleftarrow{K}_{m_i}(y)\eta^{-1}_{kj}|_{\text{on-shell}}\,. \end{aligned} \qquad (\text{C}\cdot 13\text{b})$$

By this replacement, (C·10) and (C·12) lead to the "on-shell W.T. identities":

$$:\mathcal{K}:\langle 0|TJ^T\delta\theta a\Phi\exp iJ^T\Phi|0\rangle=0\,, \qquad (\text{C}\cdot 14)$$

$$:\mathcal{K}:\langle 0|T(\delta\Phi_k(x)+iJ^T\delta\theta a\Phi\Phi_k(x))\exp iJ^T\Phi|0\rangle=0\,, \qquad (\text{C}\cdot 15)$$

where a denotes the matrix (a_{ij}) commuting with $K=(K_{ij})$:

$$Ka=aK\,. \qquad (\text{C}\cdot 16)$$

(C·14) and (C·15) are further rewritten as

$$:\mathcal{K}\cdot\phi^T:\eta^{-1}K\delta\theta a\langle 0|T\Phi\exp iJ^T\Phi|0\rangle=0\,, \qquad (\text{C}\cdot 17)$$

$$\begin{aligned} &:\mathcal{K}:\langle 0|T\delta\theta\delta\Phi_k(x)\exp iJ^T\Phi|0\rangle\\ &\quad=-i:\mathcal{K}\cdot\phi^T:\eta^{-1}K\delta\theta a\langle 0|T\Phi\Phi_k(x)\exp iJ^T\Phi|0\rangle\,. \end{aligned} \qquad (\text{C}\cdot 18)$$

Now, we can determine the form of $\delta\phi_i^{\text{in}}(x)$. Since Q is the unbroken conserved charge, we should have

$$\begin{aligned} 0=[i\delta\theta Q, S]&=:[i\delta\theta Q,\phi^T]\eta^{-1}K\delta/\delta J\mathcal{K}:\langle 0|T\exp iJ^T\Phi|0\rangle\\ &=:\mathcal{K}\cdot\delta\theta\delta\phi^T:\eta^{-1}K\langle 0|Ti\Phi\exp iJ^T\Phi|0\rangle\,, \end{aligned} \qquad (\text{C}\cdot 19)$$

using (C·7). In view of (C·16) and of the on-shell W.T. identity (C·17), we find it sufficient for the validity of (C·19) to take $\delta\theta(\delta\phi)^T=-\phi^T\eta^{-1}a\eta\delta\theta$, i.e.,

$$\delta\theta\delta\phi_i^{\text{as}}(x)=-\eta_{ji}a_{kj}\eta_{lk}^{-1}\phi_l^{\text{as}}(x)\delta\theta\,. \tag{C·20}$$

Using (C·16) and the second on-shell W.T. identity (C·18), we can further verify that (C·20) really reproduces the original transformation law (C·11) of the Heisenberg fields:

$$\begin{aligned}[i\delta\theta Q, S\Phi_k(x)] &= :\mathcal{K}\cdot\delta\theta\delta\phi^T:\eta^{-1}K\langle 0|Ti\Phi\Phi_k(x)\exp iJ^T\Phi|0\rangle\\ &= :\mathcal{K}:\langle 0|T\delta\theta\delta\Phi_k(x)\exp iJ^T\Phi|0\rangle\\ &= S\delta\theta\delta\Phi_k(x)\,.\end{aligned} \tag{C·21}$$

In the above, we have used the GLZ formula (C·6) for $\varphi=\Phi_k(x)$ and $\varphi=\delta\theta\delta\Phi_k(x)$. The commutativity (C·19) of Q and S ensures the equivalence of (C·21) to (C·11). Finally, by virtue of the Jacobi identity

$$\begin{aligned}&-[[\phi_i(x),\phi_j(y)]_\mp, i\delta\theta Q]\\ &=[\phi_i(x),[i\delta\theta Q,\phi_j(y)]]_\mp+[[i\delta\theta Q,\phi_i(x)],\phi_j(y)]_\mp\,,\end{aligned} \tag{C·22}$$

we can check the equality

$$\delta\theta\delta\phi_i^{\text{as}}=-\eta_{ji}a_{kj}\eta_{lk}^{-1}\phi_l^{\text{as}}\delta\theta=\delta\theta a_{ij}\phi_j^{\text{as}}\,, \tag{C·23}$$

which proves (C·9) with the coefficients a_{ij} explicitly given by (C·13b).

At the end of this section, we note that, if the 'infinitesimal' transform $\delta\Phi_i(x)=[iQ,\Phi_i(x)]_\mp$ with a hermitian charge Q is shown to have a *discrete pole* at $p^2=m^2$ represented by an asymptotic field $(\delta\Phi_i)^{\text{as}}$, then the original Heisenberg field $\Phi_i(x)$ should necessarily have a discrete pole of the *same mass and spin* as $\delta\Phi_i(x)$. Namely, the existence of the discrete spectrum of $\delta\Phi_i(x)$ means the following:

$$\begin{aligned}0&=\langle 0|\delta\Phi_i(x)|\Psi(p^2=m^2)\rangle=\langle 0|[iQ,\Phi_i(x)]_\mp|\Psi(p^2=m^2)\rangle\\ &=\mp i\langle 0|\Phi_i(x)(Q|\Psi(p^2=m^2)\rangle)\,,\end{aligned} \tag{C·24}$$

which asserts the existence of the asymptotic field $\phi_i^{\text{as}}(x)$ of $\Phi_i(x)$ satisfying

$$[iQ,\phi_i^{\text{as}}(x)]_\mp=(\delta\Phi_i)^{\text{as}}(x)\,. \tag{C·25}$$

Since Q is a scalar quantity, we know from (C·15) that the mass and spin of ϕ_i^{as} coincide with those of $(\delta\Phi_i)^{\text{as}}$.

D. Properties of Dipole Functions and Wave Packet Systems

Here some invariant delta-functions and wave packets related with the massless dipole ghost are collected.

We begin with the definitions of E-functions:

$$E_{(\cdot)}(x)\equiv-\frac{\partial}{\partial m^2}\Delta_{(\cdot)}(x;m^2)|_{m^2=0} \tag{D·1}$$

$$=\frac{1}{2}(\nabla^2)^{-1}(x_0\partial_0-1)D_{(\cdot)}(x), \tag{D·2}$$

where $E_{(\cdot)}$ denotes E, E_1, $E_\pm$ and E_F corresponding to $D_{(\cdot)}=D$, D_1, $D_\pm$ and D_F, respectively. These E-functions, in fact, suffer from infrared divergences except for $E(x)$, and hence one should adopt a suitable infrared cutoff procedure to define them properly. Since such a procedure is fully described by Nakanishi in Ref. 28), we neglect this point here for simplicity.

First note that the integro-differential operator $\mathcal{D}^{(\omega)}$ defined by

$$\mathcal{D}_x^{(\omega)}=(1/2)\,(\nabla^2)^{-1}(x_0\partial_0-\omega) \tag{D·3}$$

for arbitrary constant ω works as an "inverse" of d'Alembertian $\Box$ in front of any *simple pole functions* $f(x)$:

$$\Box\mathcal{D}^{(\omega)}f(x)=f(x)\quad\text{if }\Box f(x)=0\,. \tag{D·4}$$

Hence, from (D·2), the equations

$$\Box E_{(\cdot)}(x)=\Box\mathcal{D}^{(1)}D_{(\cdot)}(x)=D_{(\cdot)}(x) \tag{D·5}$$

hold (except for Feynman's propagation functions $E_{(\cdot)}=E_F$ and $D_{(\cdot)}=D_F$ not satisfying $\Box D_F=0$). It is an easy task to prove the following useful formulae also from (D·2):

$$E_{(\cdot)}(x-y)=\mathcal{D}^{(1)}_{x-y}D_{(\cdot)}(x-y)=(\mathcal{D}_x^{(1/2)}+\mathcal{D}_y^{(1/2)})\,D_{(\cdot)}(x-y), \tag{D·6}$$

$$\partial_\mu^x E_{(\cdot)}(x-y)=(\partial_\mu^x\mathcal{D}_x^{(1/2)}-\mathcal{D}_y^{(1/2)}\partial_\mu^y)\,D_{(\cdot)}(x-y), \tag{D·7}$$

$$\partial_\mu^x\partial_\nu^x E_{(\cdot)}(x-y)=(\partial_\mu^x\mathcal{D}_x^{(1/2)}\partial_\nu^x+\partial_\nu^y\mathcal{D}_y^{(1/2)}\partial_\mu^y)\,D_{(\cdot)}(x-y)\,. \tag{D·8}$$

Next, we introduce wave packet systems for massless scalar and vector fields. Let $\{g_k\}$ be a complete set of positive frequency solutions of the d'Alembert equation:

$$g_k(x)=\int\frac{d^3p}{\sqrt{(2\pi)^3 2p_0}}\varphi_k(\boldsymbol{p})\,e^{-ipx},\quad p_0=|\boldsymbol{p}|\,, \tag{D·9}$$

where the following conditions should be satisfied,

$$\sum_k\varphi_k(\boldsymbol{p})\varphi_k{}^*(\boldsymbol{q})=\delta^3(\boldsymbol{p}-\boldsymbol{q}), \tag{D·10a}$$

$$\int d^3p\varphi_k{}^*(\boldsymbol{p})\varphi_l(\boldsymbol{p})=\delta_{kl}\,. \tag{D·10b}$$

Then the g_k's satisfy

$$\sum_k g_k(x) g_k^*(y) = D_+(x-y), \tag{D·11a}$$

$$i\int d^3x g_k^*(x) \overleftrightarrow{\partial}_0 g_l(x) \equiv (g_k, g_l) = \delta_{kl}, \tag{D·11b}$$

where

$$f\overleftrightarrow{\partial}_0 g \equiv f(\partial_0 g) - (\partial_0 f) g. \tag{D·12}$$

By the use of the *same* $\{\varphi_k(\boldsymbol{p})\}$, we define wave packet system $\{f_{k,\sigma}{}^\mu\}$ for the massless vector fields:

$$f_{k,\sigma}{}^\mu(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p_0}} \varphi_k(\boldsymbol{p}) \varepsilon_\sigma{}^\mu(\boldsymbol{p}) e^{-ipx}, \tag{D·13}$$

where the polarization vectors $\varepsilon_\sigma{}^\mu(\boldsymbol{p})$ $(\sigma=1,2,L,S)$ are defined as

$$\left.\begin{array}{l}\boldsymbol{p}\cdot\boldsymbol{\varepsilon}_\sigma(\boldsymbol{p})=0,\ \ \varepsilon_\sigma{}^0(\boldsymbol{p})=0\\ \boldsymbol{\varepsilon}_\sigma(\boldsymbol{p})\cdot\boldsymbol{\varepsilon}_\tau(\boldsymbol{p})=\delta_{\sigma\tau}\end{array}\right\}\ \text{for } \sigma,\tau=1,2, \tag{D·14a}$$

$$\varepsilon_L{}^\mu(\boldsymbol{p}) = -ip^\mu = -i(|\boldsymbol{p}|, \boldsymbol{p}), \tag{D·14b}$$

$$\varepsilon_S{}^\mu(\boldsymbol{p}) = -i\bar{p}^\mu/2|\boldsymbol{p}|^2 \equiv -i(|\boldsymbol{p}|, -\boldsymbol{p})/2|\boldsymbol{p}|^2. \tag{D·14c}$$

Defining a 'metric' $\tilde{\eta}^{\sigma\tau}$ by

$$\tilde{\eta}^{\sigma\tau} = \begin{array}{c|cccc} & 1 & 2 & L & S \\ \hline 1 & -1 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 & 0 \\ L & 0 & 0 & 0 & 1 \\ S & 0 & 0 & 1 & 0 \end{array}, \tag{D·15}$$

we introduce 'contravariant' polarization vectors $\varepsilon^{\sigma,\mu}(\boldsymbol{p})$:

$$\varepsilon^{\sigma,\mu}(\boldsymbol{p}) = \sum_{\tau=1,2,L,S} \tilde{\eta}^{\sigma\tau} \varepsilon_\tau{}^\mu(\boldsymbol{p}). \tag{D·16}$$

Then one can easily check the relations as

$$\sum_\sigma \varepsilon_\sigma{}^\mu(\boldsymbol{p}) \varepsilon^{\sigma,\nu}(\boldsymbol{p})^* = g^{\mu\nu}, \tag{D·17a}$$

$$\varepsilon_\sigma{}^\mu(\boldsymbol{p}) \varepsilon_{\tau,\mu}(\boldsymbol{p})^* = \tilde{\eta}_{\sigma\tau}. \tag{D·17b}$$

Then $\{f_{k,\sigma}{}^\mu\}$ satisfy

$$\sum_{k,\sigma} f_{k,\sigma}{}^{\mu}(x) f_k{}^{\sigma,\nu}(y)^* = g^{\mu\nu} D_+(x-y), \tag{D·18a}$$

$$(f_{k,\sigma}{}^{\mu}, f_l{}^{\tau}{}_{,\mu}) = \delta_\sigma{}^\tau \delta_{kl}. \tag{D·18b}$$

By virtue of the common use of $\{\varphi_k(\boldsymbol{p})\}$ both in $\{g_k(x)\}$ and $\{f_{k,\sigma}{}^{\mu}(x)\}$, we have the following useful relations:

$$f_{k,L}{}^{\mu}(x) = \partial^\mu g_k(x), \tag{D·19a}$$

$$\partial_\mu f_{k,S}{}^{\mu}(x) = -g_k(x), \tag{D·19b}$$

$$\partial_\mu f_{k,\sigma}{}^{\mu}(x) = 0 \quad \text{for } \sigma = 1, 2, \text{ and } L. \tag{D·19c}$$

Now the dipole wave packet system $\{h_k(x)\}$ is introduced by the definition as

$$h_k(x) = \mathscr{D}_x{}^{(1/2)} g_k(x) = (1/2)(\boldsymbol{\nabla}^2)^{-1}(x_0\partial_0 - 1/2) g_k(x). \tag{D·20}$$

This $\{h_k(x)\}$ satisfies

$$\sum_k (h_k(x) g_k{}^*(y) + g_k(x) h_k{}^*(y)) = E_+(x-y), \tag{D·21}$$

$$(g_k, h_l) + (h_k, g_l) = 0. \tag{D·22}$$

Here (D·21) follows at once from (D·6) with (D·11a). Equation (D·22) can be proved directly by using the definition (D·20), but it would be easier to utilize the identity

$$E_+(x-y) = i\int d^3z [D_+(x-z) \overleftrightarrow{\partial_0^z} E_+(z-y) + E_+(x-y) \overleftrightarrow{\partial_0^z} D_+(z-y)]$$

and the completeness relations (D·11a) and (D·21).

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