Local Euler-Maclaurin formula for polytopes

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1 Introduction

By the name Euler-Maclaurin, one refers to formulas which relate discrete sums to integrals, in particular in the following framework. Let $\mathfrak{p}$ be a rational convex polytope in $\mathbb{R}^d$ and let $h(x)$ be a polynomial function on $\mathbb{R}^d$. The sum of the values $h(x)$ at integral points of $\mathfrak{p}$ is written as a sum of terms indexed by the set $\mathcal{F}(\mathfrak{p})$ of faces of $\mathfrak{p}$,

$$\sum_{x\in\mathfrak{p}\cap\mathbb{Z}^d} h(x) = \sum_{\mathfrak{f}\in\mathcal{F}(\mathfrak{p})} \int_{\mathfrak{f}} D(\mathfrak{p},\mathfrak{f})\cdot h \tag{1}$$

where, for each face $\mathfrak{f}$ of $\mathfrak{p}$, $D(\mathfrak{p},\mathfrak{f})$ is a differential operator (of infinite order) with constant coefficients on $\mathbb{R}^d$. The basic example is the historical Euler-Maclaurin summation formula in dimension 1: for $a_1 \leq a_2 \in \mathbb{Z}$,

$$\sum_{a_1}^{a_2} h(x) = \int_{a_1}^{a_2} h(t)dt - \sum_{n\geq 1}\frac{b(n)}{n!}h^{(n-1)}(a_1) + \sum_{n\geq 1}(-1)^n\frac{b(n)}{n!}h^{(n-1)}(a_2)$$

where $b(n)$ are the Bernoulli numbers.

When $\mathfrak{p}$ is an integral polytope, the existence of such operators is the combinatorial counterpart of a homological property of the associated toric variety: the invariant cycles corresponding to the faces of $\mathfrak{p}$ generate the equivariant homology module. An Euler-Maclaurin formula amounts to an explicit Riemann-Roch theorem, as obtained by Khovanskii and Pukhlikov [17] for integral polytopes corresponding to smooth toric varieties, and extended by Cappell and Shaneson [11] to any integral polytope. Furthermore, by transforming Cappell-Shaneson homological methods into purely combinatorial techniques, valid for any rational (not necessary integral) polytope, Brion and Vergne [9] obtained various expressions for the sum $\sum_{x\in\mathfrak{p}\cap\mathbb{Z}^d} h(x)$, either as an integral over a deformed polytope followed by differentiation with respect to the deformation parameter, or formulas of type (1).

In this article, we construct differential operators $D(\mathfrak{p},\mathfrak{f})$, with rational coefficients, which satisfy Formula (1) and which moreover enjoy two essential properties: they are *local* and they are *computable*. The existence of operators with these properties was conjectured in [3].

By local, one means that $D(\mathfrak{p},\mathfrak{f})$ depends only on the equivalence class - modulo integral translations - of the *transverse cone* $\mathfrak{t}(\mathfrak{p},\mathfrak{f})$ of $\mathfrak{p}$ along $\mathfrak{f}$ (see Definition 2 and Figure 4). In particular, if $\mathfrak{p}$ is an integral polytope, the operator $D(\mathfrak{p},\mathfrak{f})$ depends only on the cone of feasible directions of $\mathfrak{p}$ along $\mathfrak{f}$.

By computable, one means that there exists an algorithm which computes the m lowest order terms of $D(\mathfrak{p},\mathfrak{f})$, with running time polynomial with respect to the size of the data defining $\mathfrak{p}$, at least when the dimension d and the number m are fixed.

On the contrary, Cappell-Shaneson and Brion-Vergne operators are neither local nor computable.

When applied to the constant polynomial $h(x) = 1$, Formula (1) takes the form

$$\operatorname{Card}(\mathfrak{p}\cap\mathbb{Z}^d) = \sum_{\mathfrak{f}\in\mathcal{F}(\mathfrak{p})} \nu_0(\mathfrak{p},\mathfrak{f})\operatorname{vol}(\mathfrak{f}) \tag{2}$$

where the coefficients $\nu_0(\mathfrak{p},\mathfrak{f})$ are rational numbers. In the context of toric varieties, Danilov [14] asked the question of existence of coefficients $\nu_0(\mathfrak{p},\mathfrak{f})$ with the local property, in the case of an integral polytope. This result was proven by Morelli [18] and McMullen [19]. In [20], Pommersheim and Thomas gave a canonical construction of rational coefficients $\nu_0(\mathfrak{p},\mathfrak{f})$ which satisfy Formula (2), as a consequence of their expression for the Todd class of a toric variety. In a companion article, we will similarly obtain a local formula for the **equivariant** Todd class of any toric variety. The result is stated in Theorem 24.

The computability of our operators $D(\mathfrak{p},\mathfrak{f})$ extends the following remarkable result of Barvinok [2]: when the dimension d is fixed, the number of integral points $\operatorname{Card}(\mathfrak{p}\cap\mathbb{Z}^d)$ can be computed by a polynomial time algorithm. This result is a consequence of Brion's theorem [7], according to which the computation can be distributed over the tangent cones at the vertices, and of Barvinok's signed decomposition of a cone into unimodular cones by a polynomial time algorithm. Based on this method, efficient software packages for integer points counting problems and the effective computation of $\sum_{x\in\mathfrak{p}\cap\mathbb{Z}^d} h(x)$ have been developed [15], [16], [23], [24].

Let us now explain the construction of the operators $D(\mathfrak{p},\mathfrak{f})$. We define the differential operator $D(\mathfrak{p},\mathfrak{f})$ through its symbol, using a scalar product on $\mathbb{R}^d$. For this purpose, we first associate an analytic function $\mu(\mathfrak{a})$ (defined on a neighborhood of 0) on $\mathbb{R}^d$ to any rational affine cone $\mathfrak{a}\subset\mathbb{R}^d$.

For instance, for a half-line $s+\mathbb{R}_+$, we have

$$\mu(s+\mathbb{R}_+)(\xi) = \frac{e^{[[s]]\xi}}{1-e^{\xi}} + \frac{1}{\xi}.$$

where $[[s]] = n-s$, with $n\in\mathbb{Z}$, $n-1<s\leq n$. Remark that this function is analytic at 0, with value at 0 given by $\mu(s+\mathbb{R}_+)(0) = \frac{1}{2} - [[s]]$.

The assignment $\mathfrak{a}\mapsto\mu(\mathfrak{a})$ has beautiful geometric properties. The most important one is that it is a **valuation** when the vertex of $\mathfrak{a}$ is fixed (Theorem 20). For instance,

$$\mu(\mathfrak{a}_1\cup\mathfrak{a}_2) = \mu(\mathfrak{a}_1)+\mu(\mathfrak{a}_2)-\mu(\mathfrak{a}_1\cap\mathfrak{a}_2). \tag{3}$$

Moreover, $\mu(\mathfrak{a})$ is unchanged when $\mathfrak{a}$ is moved by a lattice translation, and the map $\mathfrak{a}\mapsto\mu(\mathfrak{a})$ is equivariant with respect to lattice-preserving isometries. In particular, with the standard scalar product, we thus get invariants of the group $\mathrm{O}(n,\mathbb{Z})$.

If $\mathfrak{a}$ contains a straight line, we set $\mu(\mathfrak{a})=0$. If $\mathfrak{a}$ is pointed with vertex s, we define $\mu(\mathfrak{a})$ recursively by the relation:

$$\mu(\mathfrak{a})(\xi) = e^{-\langle\xi,s\rangle}\left(\sum_{x\in\mathfrak{a}\cap\mathbb{Z}^d} e^{\langle\xi,x\rangle} + \sum_{\mathfrak{f},\dim(\mathfrak{f})>0}\mu(\mathfrak{t}(\mathfrak{a},\mathfrak{f}))(\xi)\int_{\mathfrak{f}} e^{\langle\xi,x\rangle}dm_{\mathfrak{f}}(x)\right) \tag{4}$$

where $\mathfrak{f}$ denotes a face of $\mathfrak{a}$ and $dm_{\mathfrak{f}}(x)$ denotes the canonical Lebesgue measure on $\mathfrak{f}$ defined by the lattice. In (4), the function $\mu(\mathfrak{t}(\mathfrak{a},\mathfrak{f}))$ is a priori defined only on a subspace of $\mathbb{R}^d$, namely the orthogonal to the face $\mathfrak{f}$. We extend it to $\mathbb{R}^d$ by orthogonal projection. Our main point is to show that Formula (4) actually defines an analytic function. Thanks to the valuation property, the proof is reduced to the case of a simplicial unimodular cone.

Then we define $D(\mathfrak{p},\mathfrak{f})$ as the differential operator with symbol $\mu(\mathfrak{t}(\mathfrak{p},\mathfrak{f}))(\xi)$. If the scalar product is rational, the Taylor series of $\mu(\mathfrak{a})$ has rational coefficients, in particular the numbers $\nu_0(\mathfrak{p},\mathfrak{f})$ in (2) are rational. Note that $D(\mathfrak{p},\mathfrak{f})$ involves only differentiation in directions perpendicular to $\mathfrak{f}$.

With this definition, Euler-Maclaurin formula (1) for any rational polytope $\mathfrak{p}$ follows easily from Brion's theorem. Indeed, the defining formula (4) is formally Formula (1) where the polytope $\mathfrak{p}$ is replaced by the cone $\mathfrak{a}$ and the polynomial $h(x)$ is replaced by the exponential $e^{\langle\xi,x\rangle}$.

The computability of the functions $\mu(\mathfrak{t}(\mathfrak{p},\mathfrak{f}))$ is also deduced from Barvinok's fast decomposition of cones, thanks to the valuation property.

Moreover, Barvinok proved recently [4] that, given an integer m, there exists a polynomial time algorithm which computes the m highest coefficients of the Ehrhart quasipolynomial of any rational simplex in $\mathbb{R}^d$, *when the dimension d is considered as an input.* We hope that our construction of the functions $\mu(\mathfrak{a})$ will lead to another polynomial time algorithm which would compute the m highest coefficients of the Ehrhart quasipolynomial for any simplex in $\mathbb{R}^d$ and any polynomial $h(x)$, when the dimension d and the degree of h are considered as input. We want to point out that our construction involves only cones of dimension less than m when computing the m highest order Ehrhart coefficients.

In the forthcoming article [1], we will compare our construction to the mixed valuation method of [4].

Let us illustrate the results in dimension 2. For an affine cone $\mathfrak{a}$ with integral vertex s and edges generated by two integral vectors v_1, v_2 with $\det(v_1, v_2) = 1$, (that is to say, $\mathfrak{a}$ is unimodular), we have:

$$\mu(\mathfrak{a})(0) = \frac{1}{4} + \frac{\langle v_1, v_2\rangle}{12}\left(\frac{1}{\langle v_1, v_1\rangle} + \frac{1}{\langle v_2, v_2\rangle}\right).$$

For a general cone, we compute $\mu(\mathfrak{a})(0)$ using the valuation property (3).

According to Pick's theorem, the number of integral points of an integral polygon $\mathfrak{p} \subset \mathbb{R}^2$ is given by

$$\mathrm{Card}(\mathfrak{p} \cap \mathbb{Z}^2) = \mathrm{area}(\mathfrak{p}) + \frac{1}{2}\mathrm{length}_{\mathbb{Z}^2}\,\partial(\mathfrak{p})\ +\ 1.$$

According to our local formula, as well as Pommersheim-Thomas's, we have:

$$\mathrm{Card}(\mathfrak{p} \cap \mathbb{Z}^2) = \mathrm{area}(\mathfrak{p}) + \frac{1}{2}\mathrm{length}_{\mathbb{Z}^2}\,\partial(\mathfrak{p})\ + \sum_s \nu_0(\mathfrak{p}, s)$$

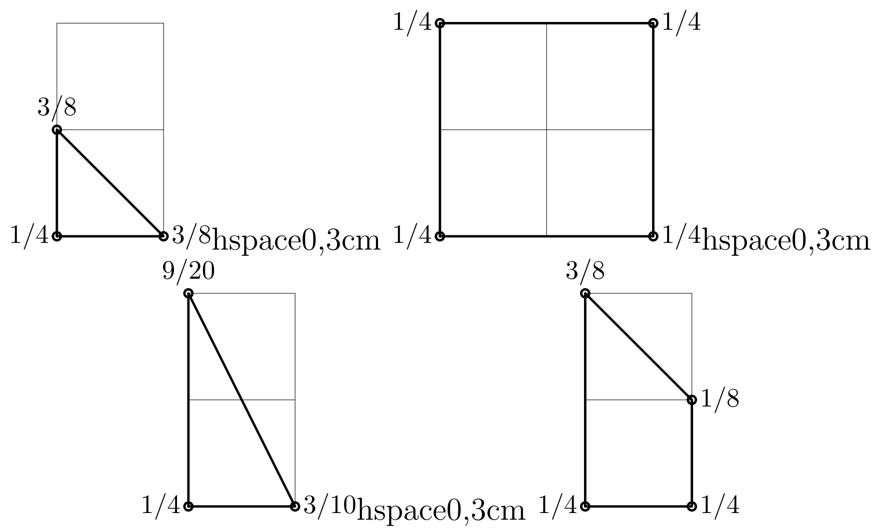

Figure 1: Coefficient $\nu_0(\mathfrak{p}, s) = \mu(\mathfrak{t}(\mathfrak{p}, s))(0)$ at the vertex s

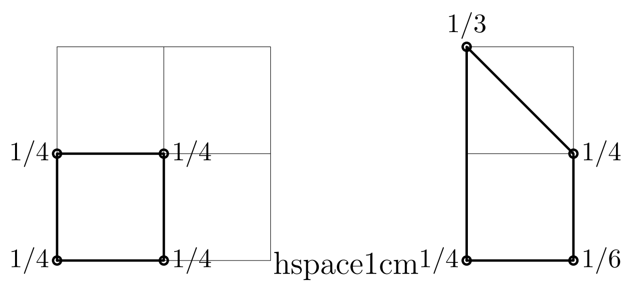

Figure 2: Cappell-Shaneson coefficients

with $\nu_0(\mathfrak{p}, s) = \mu(\mathfrak{t}(\mathfrak{p}, s))(0)$, where s runs over the vertices of $\mathfrak{p}$. Thus the constant 1 is canonically distributed over the vertices (Figure 1). Cappell-Shaneson coefficients [12] give a different, non local, distribution of the constant 1 over the vertices. For instance, the bottom right coefficients in the square and the trapezoid of Figure 2 are different, although the vertices have the same tangent cone.

More examples are given at the end of the paper, for polygons $\mathfrak{p}$ with rational (non integral) non unimodular vertices. Based on our Euler-Maclaurin operators, we wrote a Maple program which computes the value of the sum $\sum_{x\in\mathfrak{p}\cap\mathbb{Z}^2} x_1^{m_1}x_2^{m_2}$ and also the (periodic) coefficients of the corresponding

Ehrhart quasipolynomial.

2 Definitions and notations

We consider a rational vector space V, that is to say a finite dimensional real vector space with a lattice denoted by Λ_V or simply Λ. By lattice, we mean a discrete additive subgroup of V which generates V as a vector space. Hence, a lattice is generated by a basis of the vector space V. A basis of V which is a $\mathbb{Z}$-basis of Λ_V is called an integral basis.

We will need to consider subspaces and quotient spaces of V, this is why we cannot just let $V = \mathbb{R}^d$ and $\Lambda = \mathbb{Z}^d$. The points of Λ are called integral. A point $x \in V$ is called rational if $qx \in \Lambda$ for some integer $q \neq 0$. The space of rational points in V is denoted by $V_{\mathbb{Q}}$. A subspace W of V is called rational if $W \cap \Lambda$ is a lattice in W. If W is a rational subspace, the image of Λ in V/W is a lattice in V/W, so that V/W is a rational vector space.

A rational space V, with lattice Λ, has a canonical Lebesgue measure, for which V/Λ has measure 1. An affine subspace W of V is called rational if it is a translate of a rational subspace by a rational element. It is similarly provided with a canonical Lebesgue measure. We will sometimes denote this measure by dm_W. For example, let W be a rational line of the form $W = s + \mathbb{R}v$. Assume that v is a generator of the group $\mathbb{R}v \cap \Lambda$ (we say that v is a *primitive* vector). Then $dm_W(s + tv) = dt$.

If $v_i \in V_{\mathbb{Q}}$ are linearly independent vectors, we denote by $\square(v_1, \ldots, v_k)$ the semi-open parallelepiped generated by the v_i's:

$$\square(v_1, \ldots, v_k) = \sum_{i=1}^{k} [0, 1[v_i.$$

We denote by $\mathrm{vol}(\square(v_1, \ldots, v_k))$ its relative volume, that is to say its volume with respect to the canonical measure on the subspace generated by $v_1, \ldots, v_k$.

We denote by V^* the dual space of V. We will denote elements of V by latin letters $x, y, v, \ldots$ and elements of V^* by greek letters $\xi, \alpha, \ldots$. We denote the duality bracket by $\langle \xi, x \rangle$.

V^* is equipped with the dual lattice Λ^* of Λ :

$$\Lambda^* = \{\xi \in V^* \,;\, \langle \xi, x \rangle \in \mathbb{Z} \text{ for all } x \in \Lambda\}.$$

If S is a subset of V, we denote by $S^\perp$ the subspace of V^* orthogonal to S:

$$S^\perp = \{\xi \in V^* \ ; \ \langle \xi, x\rangle = 0 \text{ for all } x \in S\}.$$

If W is a subspace of V, the dual space $(V/W)^*$ is canonically identified with the subspace $W^\perp \subset V^*$.

If S is a subset of V, we denote by $< S >$ the affine subspace generated by S. If S consists of rational points, then $< S >$ is rational. Remark that $< S >$ may contain no integral point. We denote by $\operatorname{lin}(S)$ the vector subspace of V parallel to $< S >$.

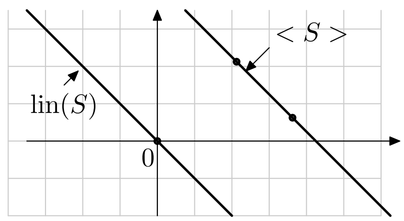

Figure 3: The affine space $< S >$ and the linear space $\operatorname{lin}(S)$

The set of non negative real numbers is denoted by $\mathbb{R}_+$. A convex rational cone $\mathfrak{c}$ in V is a closed convex cone $\sum_{i=1}^k \mathbb{R}_+ v_i$ which is generated by a finite number of elements v_i of $V_\mathbb{Q}$. In this article, we simply say cone instead of convex rational cone.

An affine (rational) cone $\mathfrak{a}$ is, by definition, the translate of a cone in V by an element $s \in V_\mathbb{Q}$. This cone is uniquely defined by $\mathfrak{a}$; it is called the cone of directions of $\mathfrak{a}$ and denoted by $\operatorname{dir}(\mathfrak{a})$. Thus $\mathfrak{a} = s + \operatorname{dir}(\mathfrak{a})$.

A cone $\mathfrak{c}$ is called simplicial if it is generated by independent elements of $V_\mathbb{Q}$. A simplicial cone $\mathfrak{c}$ is called unimodular if it is generated by independent integral vectors $v_1, \ldots, v_k$ such that $\operatorname{vol}(\Box(v_1, \ldots, v_k)) = 1$. An affine cone $\mathfrak{a}$ is called simplicial (resp. simplicial unimodular) if $\operatorname{dir}(\mathfrak{a})$ is simplicial (resp. simplicial unimodular).

An affine cone $\mathfrak{a}$ is called pointed if it does not contain any straight line.

The set of faces of an affine cone $\mathfrak{a}$ is denoted by $\mathcal{F}(\mathfrak{a})$. If $\mathfrak{a}$ is pointed, then the vertex of $\mathfrak{a}$ is the unique face of dimension 0, while $\mathfrak{a}$ is the unique face of maximal dimension $\dim \mathfrak{a}$.

The dual cone $\mathfrak{c}^*$ of a cone $\mathfrak{c}$ is the set of $\xi \in V^*$ such that $\langle \xi, x\rangle \geq 0$ for any $x \in \mathfrak{c}$.

A convex rational polyhedron $\mathfrak{p}$ in V (we will simply say polyhedron) is, by definition, the intersection of a finite number of half spaces with boundary a rational affine hyperplane.

Definition 1 *We say that* $\mathfrak{p}$ *is solid (in* V*) if* $<\mathfrak{p}>= V$.

The set of faces of $\mathfrak{p}$ is denoted by $\mathcal{F}(\mathfrak{p})$ and the set of vertices of $\mathfrak{p}$ is denoted by $\mathcal{V}(\mathfrak{p})$.

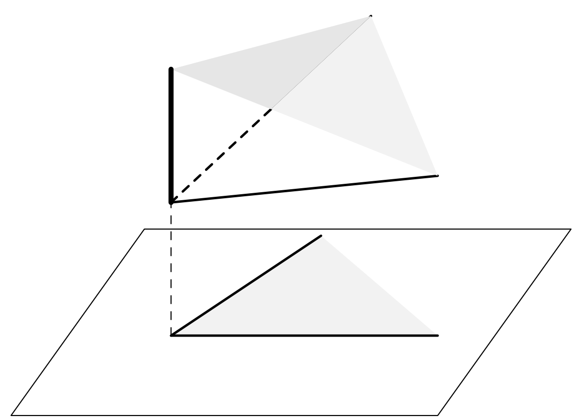

Figure 4: The transverse cone along an edge in dimension 3

We now introduce the main geometrical object in our study, the *transverse cone of a polyhedron* $\mathfrak{p}$ along one of its faces $\mathfrak{f}$ (see Figure 4). Let x be a point in the relative interior of $\mathfrak{f}$. Recall that the cone of feasible directions of $\mathfrak{p}$ at x is the set $\mathfrak{c}(\mathfrak{p},\mathfrak{f}) := \{v \in V \,; x+\epsilon v \in \mathfrak{p} \text{ for } \epsilon > 0 \text{ small enough } \}$. It does not depend on the choice of x and contains the linear space $\operatorname{lin}(\mathfrak{f})$. The supporting cone of $\mathfrak{p}$ along $\mathfrak{f}$ is the affine cone $<\mathfrak{f}> + \mathfrak{c}(\mathfrak{p},\mathfrak{f})$. We denote the projection $V \to V/\operatorname{lin}(\mathfrak{f})$ by $\pi_{\mathfrak{f}}$.

Definition 2 *Let* $\mathfrak{p}$ *be a polyhedron and* $\mathfrak{f}$ *a face of* $\mathfrak{p}$*. The transverse cone* $\mathfrak{t}(\mathfrak{p},\mathfrak{f})$ *of* $\mathfrak{p}$ *along* $\mathfrak{f}$ *is the image* $\pi_{\mathfrak{f}}(<\mathfrak{f}> + \mathfrak{c}(\mathfrak{p},\mathfrak{f}))$ *of the supporting cone in* $V/\operatorname{lin}(\mathfrak{f})$.

We will often write simply *transverse cone along* $\mathfrak{f}$, when $\mathfrak{p}$ is understood.

If v is a vertex of $\mathfrak{p}$, the transverse cone $\mathfrak{t}(\mathfrak{p}, v)$ coincides with the supporting cone $v + \mathfrak{c}(\mathfrak{p}, v) \subset V$.

The transverse cone $\mathfrak{t}(\mathfrak{p},\mathfrak{f})$ is a pointed **affine** cone in the quotient space $V/\operatorname{lin}(\mathfrak{f})$. Its dimension is equal to the codimension of $\mathfrak{f}$ in $<\mathfrak{p}>$. Its vertex is the projection $\pi_{\mathfrak{f}}(x)$ of any point x of $\mathfrak{f}$ on $V/\operatorname{lin}(\mathfrak{f})$.

If $\mathfrak{a}$ is an affine cone and $\mathfrak{f}$ is a face of $\mathfrak{a}$, then the supporting cone is $\mathfrak{a}+\operatorname{lin}(\mathfrak{f})$ and the transverse cone $\mathfrak{t}(\mathfrak{a},\mathfrak{f})$ along $\mathfrak{f}$ is just the projection $\pi_{\mathfrak{f}}(\mathfrak{a})$ of $\mathfrak{a}$ on $V/\operatorname{lin}(\mathfrak{f})$.

We shall make use of subdivisions of cones.

Definition 3 *A subdivision of a cone $\mathfrak{c}$ is a finite collection $\mathcal{C}$ of cones in V such that:*

(a) The faces of any cone in $\mathcal{C}$ are in $\mathcal{C}$.

(b) If $\mathfrak{d}_1$ and $\mathfrak{d}_2$ are two elements of $\mathcal{C}$, then the intersection $\mathfrak{d}_1\cap\mathfrak{d}_2$ is a face of both $\mathfrak{d}_1$ and $\mathfrak{d}_2$.

(c) We have $\mathfrak{c}=\cup_{\mathfrak{d}\in\mathcal{C}}\mathfrak{d}$.

If furthermore the elements of $\mathcal{C}$ are simplicial cones, the subdivision will be called simplicial.

Example 4 *The basic example is the subdivision $\{\mathbb{R}_+,(-\mathbb{R}_+),\{0\}\}$ of the one-dimensional cone $\mathbb{R}$.*

It is easy to see that any pointed cone admits a subdivision into simplicial unimodular cones.

If S is a subset of V, we denote by $\chi(S)$ the characteristic function of S (also called the indicator function of S).

Explicit expression of operators $D(\mathfrak{p},\mathfrak{f})$ involves the Bernoulli polynomials $b(n,t)$, defined by the generating series

$$\sum_{n=0}^{\infty}\frac{b(n,t)}{n!}X^n=\frac{e^{tX}X}{e^X-1}. \tag{5}$$

The Bernoulli number is $b(n,0)$ and is denoted by $b(n)$.

3 Meromorphic functions associated to polyhedra

By a meromorphic function on V^* with rational coefficients, we mean a meromorphic function on the complexification of V^* which can be written as the quotient of two holomorphic functions with rational Taylor coefficients with respect to an integral basis of V^*.

We recall the construction of two meromorphic functions with rational coefficients on V^*, associated to any polyhedron $\mathfrak{p}$ in V (see the survey [3]). The first function $I(\mathfrak{p})$ is defined via integration over $\mathfrak{p}$, the second function $S(\mathfrak{p})$ via summation over the set of integral points of $\mathfrak{p}$.

We denote by $dm_{<\mathfrak{p}>}$ the relative Lebesgue measure on the affine space spanned by $\mathfrak{p}$.

Proposition 5 *There exists a map I which to every polyhedron $\mathfrak{p} \subset V$ associates a meromorphic function with rational coefficients $I(\mathfrak{p})$ on V^*, so that the following properties hold:*

(a) If $\mathfrak{p}$ contains a straight line, then $I(\mathfrak{p})$=0.

(b) If $\xi \in V^*$ is such that $|e^{\langle \xi, x\rangle}|$ is integrable over $\mathfrak{p}$, then

$$I(\mathfrak{p})(\xi) = \int_{\mathfrak{p}} e^{\langle \xi, x\rangle} dm_{<\mathfrak{p}>}(x).$$

(c) For every point $s \in V_{\mathbb{Q}}$, we have

$$I(s+\mathfrak{p})(\xi) = e^{\langle \xi, s\rangle} I(\mathfrak{p})(\xi).$$

(d) The map I is a solid valuation*: if the characteristic functions $\chi(\mathfrak{p}_i)$ of a family of polyhedra $\mathfrak{p}_i$ satisfy a linear relation $\sum_i r_i \chi(\mathfrak{p}_i) = 0$, then the functions $I(\mathfrak{p}_i)$ satisfy the relation*

$$\sum_{\{i, <\mathfrak{p}_i> = V\}} r_i I(\mathfrak{p}_i) = 0.$$

Example 6 *.*

- *If $\mathfrak{p} = \{s\}$ is a point, then $I(\mathfrak{p})(\xi) = e^{\langle \xi, s\rangle}$.*
- *In dimension 1, if $\mathfrak{p} = s + \mathbb{R}^+$, where $s \in \mathbb{Q}$, then $I(\mathfrak{p})(\xi) = -\frac{e^{\xi s}}{\xi}$.*
- *If $\mathfrak{c}$ is a simplicial cone generated by $k \leq d$ independent vectors $v_1, \ldots, v_k$, we have*

$$I(\mathfrak{c})(\xi) = (-1)^k \frac{\mathrm{vol}(\square(v_1, \ldots, v_k))}{\prod_{i=1}^k \langle \xi, v_i \rangle}. \tag{6}$$

These formulas follow immediately from the computation in dimension 1, $\int_0^\infty e^{\xi x}dx = \frac{-1}{\xi}$ for $\xi < 0$. Thus, for a simplicial cone, the function $I(\mathfrak{c})$ can indeed be extended as a rational function on the whole space V^*.

In a similar way, one defines the second meromorphic function, which is the discrete analogue of $I(\mathfrak{p})$.

Proposition 7 *There exists a map S which to every polyhedron $\mathfrak{p} \subset V$ associates a meromorphic function with rational coefficients $S(\mathfrak{p})$ on V^* so that the following properties hold:*

(a) If $\mathfrak{p}$ contains a straight line, then $S(\mathfrak{p})$=0.

(b) If $\xi \in V^*$ is such that $\sum_{x\in\mathfrak{p}\cap\Lambda} |e^{\langle\xi,x\rangle}| < \infty$, then

$$S(\mathfrak{p})(\xi) = \sum_{x\in\mathfrak{p}\cap\Lambda} e^{\langle\xi,x\rangle}. \tag{7}$$

(c) For every integral point $s \in \Lambda$, we have

$$S(s + \mathfrak{p})(\xi) = e^{\langle\xi,s\rangle} S(\mathfrak{p})(\xi).$$

(d) The map S is a valuation: if the characteristic functions $\chi(\mathfrak{p}_i)$ of a family of polyhedra $\mathfrak{p}_i$ satisfy a linear relation $\sum_i r_i\chi(\mathfrak{p}_i) = 0$, then the functions $S(\mathfrak{p}_i)$ satisfy the same relation

$$\sum_i r_i S(\mathfrak{p}_i) = 0.$$

Example 8 .

- *If $\mathfrak{p} = \{s\}$ is a point, then we have two cases. If s is an integral point, then $S(\{s\})(\xi) = e^{\langle\xi,s\rangle}$, otherwise $S(\{s\}) = 0$.*
- *In dimension 1, if $\mathfrak{p} = s + \mathbb{R}^+$, where $s \in \mathbb{Q}$, then*

$$S(s + \mathbb{R}_+)(\xi) = \frac{e^{k\xi}}{1 - e^{\xi}}$$

where k is the smallest integer greater or equal than s.

- *Let $\mathfrak{a}$ be a solid simplicial affine cone with vertex $s \in V_{\mathbb{Q}}$. Let $v_1, \ldots, v_d$ be integral generators of the edges of $\mathfrak{a}$. Then*

$$S(\mathfrak{a})(\xi) = \left(\sum_{x\in(s+\Box(v_1,\ldots,v_d))\cap\Lambda} e^{\langle\xi,x\rangle} \right) \prod_{i=1}^{d} \frac{1}{1 - e^{\langle\xi,v_i\rangle}}. \tag{8}$$

We obtain this formula by observing that any element x of the affine cone $\mathfrak{a}$ can be written in a unique way as a sum $y + \sum_{i=1}^{d} n_i v_i$ where y lies in the semi-open parallelepiped $s + \square(v_1, \ldots, v_d)$ and the coefficients n_i are non negative integers, and that the point x is integral if and only if y is.

Thus indeed $S(s + \mathfrak{c})$ can be extended to a meromorphic function on the whole of V^*.

Let us check the valuation property in dimension one:

Example 9 *In dimension one, let $s \in \mathbb{Q}$. The relation $\chi(s + \mathbb{R}_+) + \chi(s - \mathbb{R}_+) - \chi(\{s\}) = \chi(\mathbb{R})$ must imply $S(s+\mathbb{R}_+) + S(s-\mathbb{R}_+) = S(\{s\})$. If $s = 0$, we have indeed*

$$\frac{1}{1-e^{\xi}} + \frac{1}{1-e^{-\xi}} = 1 = S(\{s\})(\xi),$$

while, if $0 < s < 1$, then

$$S(s + \mathbb{R}_+)(\xi) = \frac{e^{\xi}}{1-e^{\xi}},$$

$$S(s - \mathbb{R}_+)(\xi) = \frac{1}{1-e^{-\xi}},$$

thus $S(s + \mathbb{R}_+) + S(s - \mathbb{R}_+) = 0 = S(\{s\})$.

The valuation property of the maps $\mathfrak{p} \to S(\mathfrak{p})$ and $\mathfrak{p} \to I(\mathfrak{p})$ have the following important corollary. This was first obtained by Brion [7] using toric varieties:

Theorem 10 *(Brion) Let $\mathfrak{p}$ be a polyhedron in V. Then*

$$\begin{aligned} I(\mathfrak{p}) &= \sum_{v \in \mathcal{V}(\mathfrak{p})} I(\mathfrak{t}(\mathfrak{p}, v)), \\ S(\mathfrak{p}) &= \sum_{v \in \mathcal{V}(\mathfrak{p})} S(\mathfrak{t}(\mathfrak{p}, v)). \end{aligned}$$

The singularities of the functions $I(\mathfrak{a})$ and $S(\mathfrak{a})$ are easy to compute for a pointed affine cone:

Lemma 11 *Let $\mathfrak{a} = s + \mathfrak{c}$ be a pointed affine cone with vertex s and let $v_1, \ldots, v_k$ be rational generators of the edges of the cone $\mathfrak{c}$. The products*

$$\left(\prod_{i=1}^{k} \langle \xi, v_i \rangle\right) I(s + \mathfrak{c})(\xi) \text{ and } \left(\prod_{i=1}^{k} \langle \xi, v_i \rangle\right) S(s + \mathfrak{c})(\xi)$$

are analytic near 0.

Proof. It is easy to see that the cone $\mathfrak{c}$ admits a subdivision into simplicial cones whose edges are already edges of $\mathfrak{c}$. Thus, thanks to the valuation properties of Proposition 7, it is enough to prove the lemma when $\mathfrak{c}$ itself is a simplicial cone. In this case it follows immediately from Formulas (6) and (8). □

4 The main construction

In this section, we will perform the main construction of this article: to any affine cone $\mathfrak{a}$ in V or in a rational quotient V/L of V, we will associate an **analytic** function $\mu(\mathfrak{a})$ defined in a neighborhood of 0 in V^*. In the next section, if $\mathfrak{p}$ is a convex rational polytope in V, we will obtain a local Euler-Maclaurin formula for $\mathfrak{p}$ in terms of the functions $\mu(\mathfrak{t}(\mathfrak{p},\mathfrak{f}))$ associated to the transverse cones $\mathfrak{t}(\mathfrak{p},\mathfrak{f})$ of $\mathfrak{p}$ along its various faces $\mathfrak{f}$.

We will denote the ring of analytic functions with rational coefficients, defined in a neighborhood of 0 in V^*, by $\mathcal{H}(V^*)$ and the ring of meromorphic functions with rational coefficients, defined in a neighborhood of 0 in V^*, by $\mathcal{M}(V^*)$

We will need to extend to the space V^* some meromorphic functions which are a priori defined only on a subspace of the form $(V/L)^* = L^\perp$ of V^*. For that purpose, we fix a scalar product $Q(x, y)$ on V. We assume that Q is rational, meaning that $Q(x, y)$ is rational for $x, y \in V_{\mathbb{Q}}$. We denote also by $Q(\xi, \eta)$ the dual scalar product on V^* and we use the orthogonal projection $\mathrm{proj}_{L^\perp} : V^* \to L^\perp$. If ϕ is a meromorphic function (with rational coefficients) on $L^\perp$, we still write ϕ for the function on V^* defined by $\xi \mapsto \phi(\mathrm{proj}_{L^\perp}(\xi))$. It is meromorphic with rational coefficients.

Actually, we will do this not only for the space V itself, but also when V is replaced by a rational quotient space W. The dual W^* is a subspace of V^*, thus it inherits the scalar product of V^*.

A word of caution. *Let L be a rational subspace of V. By means of the scalar product Q, we can identify V/L with $L^{\perp_Q} \subset V$, the orthogonal of L with respect to Q. However these two spaces are not isomorphic as rational spaces. The lattice of V/L corresponds to the orthogonal projection of Λ on $L^{\perp_Q}$; it contains the lattice $L^{\perp_Q} \cap \Lambda$, and the inclusion is strict in general, see Figure 5.*

Let $\mathfrak{a}$ be an affine cone in V and let $\mathfrak{f}$ be a face of $\mathfrak{a}$. Recall that the transverse cone $\mathfrak{t}(\mathfrak{a},\mathfrak{f})$ is the projection $\pi_{\mathfrak{f}}(\mathfrak{a})$ of $\mathfrak{a}$ in $V/\operatorname{lin}(\mathfrak{f})$. When we

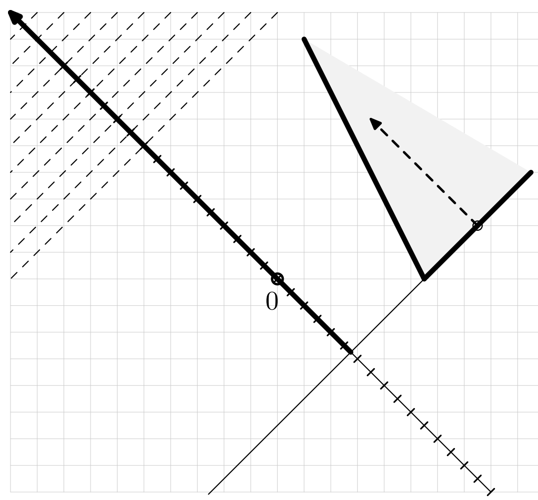

Figure 5: In dimension 2, the transverse cone along an edge with its lattice

identify $V/\operatorname{lin}(\mathfrak{f})$ with the orthogonal $\operatorname{lin}(\mathfrak{f})^{\perp_Q}$, the transverse cone $\mathfrak{t}(\mathfrak{a},\mathfrak{f})$ is a pointed affine cone in $\operatorname{lin}(\mathfrak{f})^{\perp_Q}$.

We denote by $\mathcal{C}_{pointed}(V)$ the set of pointed affine cones in V.

Proposition 12 *Let V be a rational space and let Q be a rational scalar product on V. There exists a unique family of maps μ_W, indexed by the rational quotient spaces W of V, such that the family enjoys the following properties.*

(a) μ_W maps $\mathcal{C}_{pointed}(W)$ to $\mathcal{M}(W^*)$.

(b) If $W = \{0\}$, then $\mu_W(\{0\}) = 1$.

(c) For any pointed affine cone $\mathfrak{a}$ in W, one has

$$S(\mathfrak{a}) = \sum_{\mathfrak{f}\in\mathcal{F}(\mathfrak{a})} \mu_{W/\operatorname{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{a},\mathfrak{f}))I(\mathfrak{f}) \tag{9}$$

where the sum is over the set of faces of $\mathfrak{a}$.

In Formula (9), the transverse cone $\mathfrak{t}(\mathfrak{a},\mathfrak{f})$ is a pointed affine cone in the quotient space $W/\operatorname{lin}(\mathfrak{f})$. The function $\mu_{W/\operatorname{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{a},\mathfrak{f}))$ is a meromorphic function on a neighborhood of 0 in the dual $(W/\operatorname{lin}(\mathfrak{f}))^*$. We give a meaning to the formula by extending this function to a neighborhood of 0 in the whole space W^* by means of orthogonal projection. The function $I(\mathfrak{f})$ is defined as a meromorphic function on W^* as in Section 3.

Proof. The result is easily obtained by induction on the dimension of W. If $W = \{0\}$, the only cone is $\mathfrak{a} = \{0\}$ and Formula (9) is true. Let $\mathfrak{a}$ be a pointed affine cone in W. Let s be the vertex of $\mathfrak{a}$. The transverse cone at the zero-dimensional face s is $\mathfrak{a}$ itself. Formula (9) gives

$$S(\mathfrak{a})(\xi) = e^{\langle \xi, s\rangle}\mu_W(\mathfrak{a})(\xi) + \sum_{\mathfrak{f}, \dim \mathfrak{f} > 0} \mu_{W/\operatorname{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{a}, \mathfrak{f}))(\xi) I(\mathfrak{f})(\xi). \tag{10}$$

For a face $\mathfrak{f}$ of positive dimension, the transverse cone $\mathfrak{t}(\mathfrak{a}, \mathfrak{f})$ is a pointed affine cone in the vector space $W/\operatorname{lin}(\mathfrak{f})$. Therefore, $\mu_{W/\operatorname{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{a}, \mathfrak{f}))$ being defined by the induction hypothesis, Formula (10) defines $\mu_W(\mathfrak{a})$ in a unique way, as a meromorphic function on W^*.□

The following property follows immediately from the definition:

Proposition 13 *If $V_1 \subset V_2$ and $\mathfrak{a}$ is an affine cone contained in V_1, then the function $\mu_{V_2}(\mathfrak{a})$ is the lift to V_2^* of $\mu_{V_1}(\mathfrak{a})$, by the natural restriction map $V_2^* \to V_1^*$.*

In the rest of this article, we will omit the subscript W in the notation $\mu_W(\mathfrak{a})$.

Proposition 14 *The functions defined in Proposition 12 have the following properties:*

(a) For any $s \in \Lambda$, one has $\mu(s + \mathfrak{a}) = \mu(\mathfrak{a})$.

(b) The map $\mathfrak{a} \mapsto \mu(\mathfrak{a})$ is equivariant with respect to lattice-preserving linear isometries. In other words, let g be a linear isometry of W which preserves the lattice Λ and denote its transpose by ${}^t g$, then $\mu(g(\mathfrak{a}))({}^t g^{-1}\xi) = \mu(\mathfrak{a})(\xi)$.

(c) The map $\mathfrak{a} \mapsto \mu(\mathfrak{a})$ is multiplicative with respect to orthogonal sums of cones. More precisely, if W is an orthogonal sum $W = W_1 \oplus W_2$ and $\mathfrak{a}_i$ is an affine cone in W_i for $i = 1, 2$, then

$$\mu(\mathfrak{a}_1 + \mathfrak{a}_2) = \mu(\mathfrak{a}_1)\mu(\mathfrak{a}_2).$$

(d) If $\mathfrak{a} \in \mathcal{C}_{pointed}(W)$ is such that $< \mathfrak{a} > \cap \Lambda_W = \emptyset$, then $\mu_W(\mathfrak{a}) = 0$.

Proof. The invariance in *(a)* and *(b)* and the multiplication rule in *(c)* follow immediately from the definition, by induction. To prove *(d)* , assume that $<\mathfrak{a}> \cap \Lambda_V = \emptyset$. Then $S(\mathfrak{a}) = 0$ and, for any face $\mathfrak{f}$ of $\mathfrak{a}$, the transverse cone $\mathfrak{t}(\mathfrak{a}, \mathfrak{f})$ does not contain any integral point of $V/\operatorname{lin}(\mathfrak{f})$, therefore, by induction, $\mu(\mathfrak{t}(\mathfrak{a}, \mathfrak{f})) = 0$ for $\dim \mathfrak{f} > 0$, hence $\mu(\mathfrak{a}) = 0$. $\square$

If $\mathfrak{a}$ is an affine cone in V which contains a straight line, we define $\mu(\mathfrak{a}) = 0$. Since all faces $\mathfrak{f}$ of $\mathfrak{a}$ contain a straight line, Formula (9) still holds in this case. Thus we have defined $\mu(\mathfrak{a})$ for any rational affine cone in any quotient space of V.

Our objective is to show that $\mu(\mathfrak{a})$ is indeed analytic near 0, but we will first observe some further properties of this family of functions.

It is easy to compute μ in dimension 1. Let $t \in \mathbb{Q}$ such that $0 \leq t < 1$. Then $\mu(-t + \mathbb{R}_+)(\xi)$ is defined by

$$S(-t + \mathbb{R}_+)(\xi) = \frac{1}{1 - e^{\xi}} = e^{-t\xi} \mu(-t + \mathbb{R}_+)(\xi) + \int_{-t}^{\infty} e^{x\xi} dx$$

hence

$$\mu(-t + \mathbb{R}_+)(\xi) = \frac{e^{t\xi}}{1 - e^{\xi}} + \frac{1}{\xi}. \tag{11}$$

We may write this in terms of the Bernoulli polynomials $b(n, t)$, defined by the generating series (5). We obtain

$$\mu(-t + \mathbb{R}_+)(\xi) = -\sum_{n=0}^{\infty} \frac{b((n+1), t)}{(n+1)!} \xi^n. \tag{12}$$

Let $\mathfrak{a}$ be a 1-dimensional pointed cone in V, that is to say a half-line. If $\mathfrak{a}$ does not contain any integral point, then $\mu(\mathfrak{a}) = 0$. If $\mathfrak{a}$ contains integral points, then there exists an integral point $a \in \mathfrak{a}$ such that the translated half-line $\mathfrak{a} - a$ is of the form

$$(-t + \mathbb{R}_+) v$$

where v is a primitive integral vector and $t \in [0, 1[$.

By a similar computation, we get, for $\xi \in V^*$,

$$\mu(\mathfrak{a})(\xi) = \frac{e^{t\langle \xi, v \rangle}}{1 - e^{\langle \xi, v \rangle}} + \frac{1}{\langle \xi, v \rangle}. \tag{13}$$

The next step is crucial to our construction; we will prove that the map $\mathfrak{c} \to \mu(s + \mathfrak{c})$ enjoys the valuation property:

Proposition 15 *Let $\mathfrak{c}_i$ be a finite family of cones in V. Assume that there exists a linear relation between their characteristic functions $\sum_i r_i\chi(\mathfrak{c}_i) = 0$. Then, for any $s \in V_{\mathbb{Q}}$, we have the corresponding relation $\sum_i r_i\mu(s+\mathfrak{c}_i) = 0$.*

Example 16 *Consider the subdivision of Example 4 in dimension one. Let $s \in \mathbb{Q}$. Then*

$$0 = \mu(\mathbb{R}) = \mu(s+\mathbb{R}_+) + \mu(s-\mathbb{R}_+) - \mu(\{s\}).$$

Indeed, if s is an integer, we have

$$\mu(s+\mathbb{R}_+)(\xi)+\mu(s-\mathbb{R}_+)(\xi) = (\frac{1}{1-e^{\xi}}+\frac{1}{\xi})\ +(\frac{1}{1-e^{-\xi}}-\frac{1}{\xi}) = 1 = \mu(\{s\})(\xi)$$

while, if $s \in]-1,0[$, we have

$$\mu(s+\mathbb{R}_+)(\xi)+\mu(s-\mathbb{R}_+)(\xi) = (\frac{e^{-s\xi}}{1-e^{\xi}}+\frac{1}{\xi})\ +(\frac{e^{-(s+1)\xi}}{1-e^{-\xi}}-\frac{1}{\xi}) = 0 = \mu(\{s\})(\xi).$$

Proof.

We will prove the proposition by induction on $\dim V$. By a standard argument ([21], see also [8]), it suffices to prove the result in the following particular case (see Figure 6). Let $\mathfrak{a}$ be a pointed solid affine cone in V with vertex s, let H be an affine hyperplane through s. Denote by $H^{\pm}$ the closed half-spaces separated by H. Then we have

$$\chi(\mathfrak{a}) = \chi(\mathfrak{a}\cap H^+) + \chi(\mathfrak{a}\cap H^-) - \chi(\mathfrak{a}\cap H)$$

and we must prove:

$$\mu(\mathfrak{a}) = \mu(\mathfrak{a}\cap H^+) + \mu(\mathfrak{a}\cap H^-) - \mu(\mathfrak{a}\cap H). \tag{14}$$

We proceed to prove (14).

The functions $S(\mathfrak{a})$ have the valuation property

$$S(\mathfrak{a}) - S(\mathfrak{a}\cap H^+) - S(\mathfrak{a}\cap H^-) + S(\mathfrak{a}\cap H) = 0.$$

By applying Formula (9), we obtain the following expansion of the left hand side

$$\begin{aligned}\sum_{\mathfrak{f}\in\mathcal{F}(\mathfrak{a})}\mu(\mathfrak{t}(\mathfrak{a},\mathfrak{f}))I(\mathfrak{f}) - \sum_{\mathfrak{f}\in\mathcal{F}(\mathfrak{a}\cap H^+)}\mu(\mathfrak{t}(\mathfrak{a}\cap H^+,\mathfrak{f}))I(\mathfrak{f})\\ - \sum_{\mathfrak{f}\in\mathcal{F}(\mathfrak{a}\cap H^-)}\mu(\mathfrak{t}(\mathfrak{a}\cap H^-,\mathfrak{f}))I(\mathfrak{f}) + \sum_{\mathfrak{f}\in\mathcal{F}(\mathfrak{a}\cap H)}\mu(\mathfrak{t}(\mathfrak{a}\cap H,\mathfrak{f}))I(\mathfrak{f}).\end{aligned} \tag{15}$$

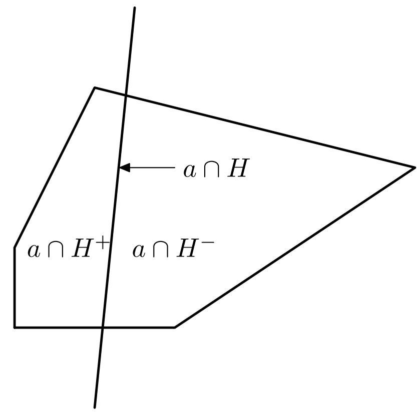

Figure 6: The cut of a 3-dimensional cone with 5 edges by a 2-dimensional plane H. The cone is represented by its slice in the figure plane. Thus H is represented by a line.

Let L be a affine subspace of V of dimension > 0. We will show

$$
\begin{aligned}
(16) \qquad & \sum_{\mathfrak{f}\in\mathcal{F}(\mathfrak{a}),<\mathfrak{f}>=L} \mu(\mathfrak{t}(\mathfrak{a},\mathfrak{f}))I(\mathfrak{f}) - \sum_{\mathfrak{f}\in\mathcal{F}(\mathfrak{a}\cap H^+),<\mathfrak{f}>=L} \mu(\mathfrak{t}(\mathfrak{a}\cap H^+,\mathfrak{f}))I(\mathfrak{f}) \\
& - \sum_{\mathfrak{f}\in\mathcal{F}(\mathfrak{a}\cap H^-),<\mathfrak{f}>=L} \mu(\mathfrak{t}(\mathfrak{a}\cap H^-,\mathfrak{f}))I(\mathfrak{f}) + \sum_{\mathfrak{f}\in\mathcal{F}(\mathfrak{a}\cap H),<\mathfrak{f}>=L} \mu(\mathfrak{t}(\mathfrak{a}\cap H,\mathfrak{f}))I(\mathfrak{f}) \\
= \; & 0.
\end{aligned}
$$

From the relation (16), it follows that the terms in (15) corresponding to the faces $\mathfrak{f}$ of positive dimension add up to 0. Therefore the contribution of the 0-dimensional vertex $\{s\}$ to (15) is also equal to 0, which proves the relation (14).

We fix L and we proceed to prove (16).

Remark that all the transverse cones which appear in (16) are affine cones in $V/\operatorname{lin}(L)$. We will apply the induction hypothesis to $V/\operatorname{lin}(L)$.

I) First we consider the case where there is a face $\mathfrak{f}$ of $\mathfrak{a}$ such that $<\mathfrak{f}>= L$. There are three cases, according to whether the relative interior of $\mathfrak{f}$ meets both the interiors of $H^\pm$, or only one, or none of them (in the third case, $\mathfrak{f}$ is contained in H).

• Case I.1: The relative interior of $\mathfrak{f}$ meets both the interiors of $H^\pm$. Then

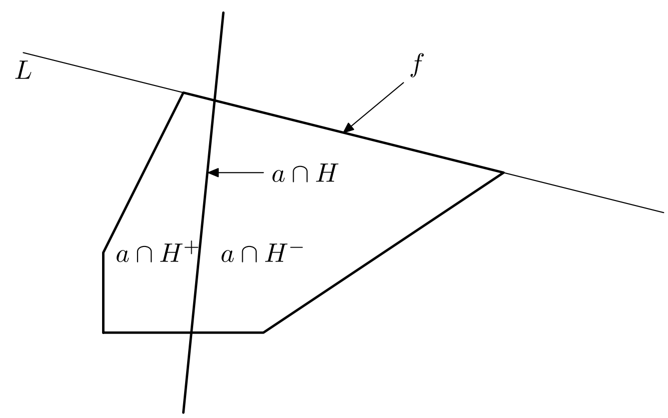

Figure 7: Case I.1

$\mathfrak{f} \cap H^{\pm}$ is a face of $\mathfrak{a} \cap H^{\pm}$ and $< \mathfrak{f} \cap H^{\pm} >= L$. Thus we have to prove

$$\mu(\mathfrak{t}(\mathfrak{a},\mathfrak{f}))I(\mathfrak{f}) - \mu(\mathfrak{t}(\mathfrak{a}\cap H^+, \mathfrak{f}\cap H^+))I(\mathfrak{f}\cap H^+) - \mu(\mathfrak{t}(\mathfrak{a}\cap H^-, \mathfrak{f}\cap H^-))I(\mathfrak{f}\cap H^-) = 0.$$

The three transverse cones $\mathfrak{t}(\mathfrak{a},\mathfrak{f})$, $\mathfrak{t}(\mathfrak{a} \cap H^+, \mathfrak{f} \cap H^+)$ and $\mathfrak{t}(\mathfrak{a} \cap H^-, \mathfrak{f} \cap H^-)$ coincide. The integrals add up:

$$I(\mathfrak{f}) = I(\mathfrak{f} \cap H^+) + I(\mathfrak{f} \cap H^-),$$

thus we get

$$\mu(\mathfrak{t}(\mathfrak{a},\mathfrak{f}))\left(I(\mathfrak{f}) - I(\mathfrak{f} \cap H^+) - I(\mathfrak{f} \cap H^-)\right)$$

which is equal to 0 as required.

- Case I.2: The relative interior of $\mathfrak{f}$ is contained in the interior of, say, H^+. Then $\mathfrak{f} = L \cap \mathfrak{a} \cap H^+$ is also a face of $\mathfrak{a} \cap H^+$, but $L \cap \mathfrak{a} \cap H^-$ and $L \cap \mathfrak{a} \cap H$ are smaller dimensional, or empty. This time we have to prove

$$\mu(\mathfrak{t}(\mathfrak{a},\mathfrak{f}))I(\mathfrak{f}) - \mu(\mathfrak{t}(\mathfrak{a} \cap H^+, \mathfrak{f} \cap H^+))I(\mathfrak{f} \cap H^+) = 0.$$

The transverse cones $\mathfrak{t}(\mathfrak{a},\mathfrak{f})$ and $\mathfrak{t}(\mathfrak{a} \cap H^+, \mathfrak{f})$ coincide, so we get

$$\mu(\mathfrak{t}(\mathfrak{a},\mathfrak{f}))\left(I(\mathfrak{f}) - I(\mathfrak{f} \cap H^+)\right).$$

As $\mathfrak{f} = \mathfrak{f} \cap H^+$, this is equal to 0 as required.

- Case I.3: $\mathfrak{f}$ is contained in H, thus it is a face of all four cones, in other words

$$\mathfrak{f} = \mathfrak{f} \cap H^+ = \mathfrak{f} \cap H^- = \mathfrak{f} \cap H.$$

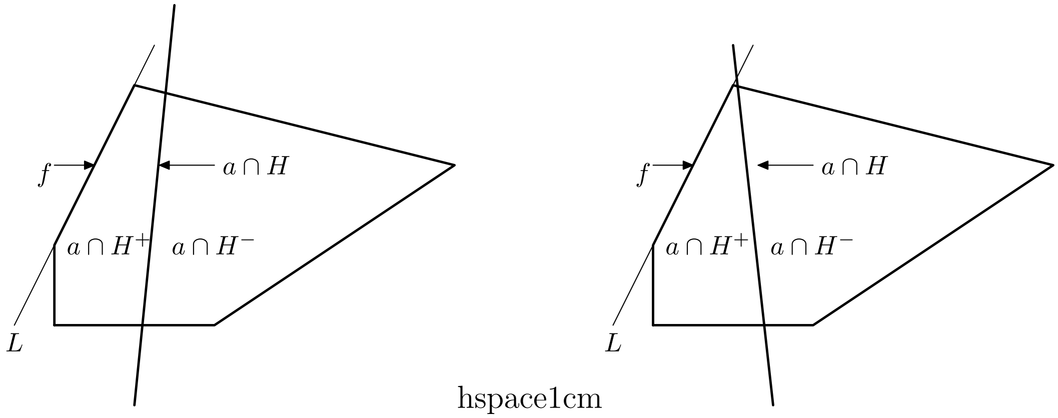

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Figure 8: Case I.2

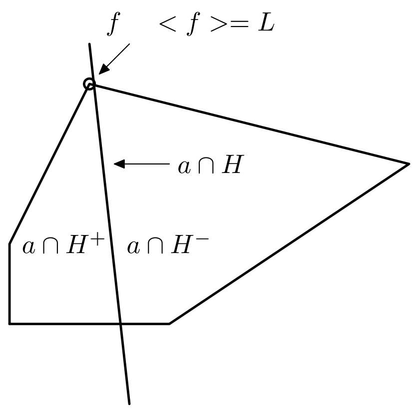

Figure 9: Case I.3

This time we have to prove

$$\begin{aligned} & \mu(\mathfrak{t}(\mathfrak{a}, \mathfrak{f}))I(\mathfrak{f}) - \mu(\mathfrak{t}(\mathfrak{a} \cap H^+, \mathfrak{f} \cap H^+))I(\mathfrak{f} \cap H^+) \\ - \quad & \mu(\mathfrak{t}(\mathfrak{a} \cap H^-, \mathfrak{f} \cap H^-))I(\mathfrak{f} \cap H^-) + \mu(\mathfrak{t}(\mathfrak{a} \cap H, \mathfrak{f}))I(\mathfrak{f} \cap H) = 0. \end{aligned}$$

In this case, the intersection of the transverse cones is

$$\mathfrak{t}(\mathfrak{a} \cap H^+, \mathfrak{f}) \cap \mathfrak{t}(\mathfrak{a} \cap H^-, \mathfrak{f}) = \mathfrak{t}(\mathfrak{a} \cap H, \mathfrak{f}),$$

the union of the transverse cones is

$$\mathfrak{t}(\mathfrak{a}, \mathfrak{f}) = \mathfrak{t}(\mathfrak{a} \cap H^+, \mathfrak{f}) \cup \mathfrak{t}(\mathfrak{a} \cap H^-, \mathfrak{f}).$$

Thus we get

$$\Big(\mu(\mathfrak{t}(\mathfrak{a},\mathfrak{f})) - \mu(\mathfrak{t}(\mathfrak{a}\cap H^+,\mathfrak{f})) - \mu(\mathfrak{t}(\mathfrak{a}\cap H^-,\mathfrak{f})) + \mu(\mathfrak{t}(\mathfrak{a}\cap H,\mathfrak{f}))\Big) I(\mathfrak{f}).$$

From the induction hypothesis applied to the space $V/\operatorname{lin}(L)$, we deduce that this is equal to 0.

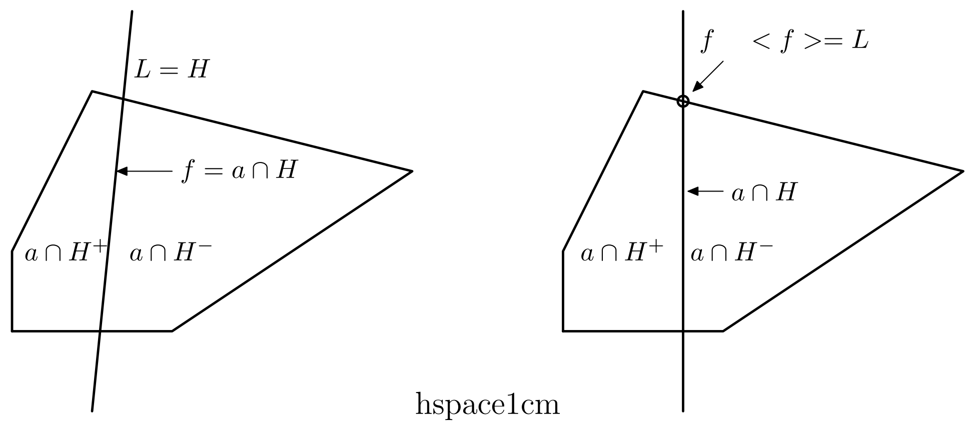

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Figure 10: Case II

II) Next, we consider an affine subspace L such that $L \cap \mathfrak{a}$ is not a face of $\mathfrak{a}$ but, say, $\mathfrak{f} = L \cap \mathfrak{a} \cap H^+$ is a face of $\mathfrak{a} \cap H^+$. Then we must have $L \subset H$ so that $\mathfrak{f}$ is a face of the three cones $\mathfrak{a} \cap H$, $\mathfrak{a} \cap H^+$ and $\mathfrak{a} \cap H^-$, but not a face of $\mathfrak{a}$. We have to show that

$$\begin{aligned}&\mu(\mathfrak{t}(\mathfrak{a}\cap H^+,\mathfrak{f}\cap H^+))I(\mathfrak{f}\cap H^+) + \mu(\mathfrak{t}(\mathfrak{a}\cap H^-,\mathfrak{f}\cap H^-))I(\mathfrak{f}\cap H^-)\\ &-\mu(\mathfrak{t}(\mathfrak{a}\cap H,\mathfrak{f}\cap H))I(\mathfrak{f}\cap H) = 0,\end{aligned}$$

with

$$\mathfrak{f}\cap H^+ = \mathfrak{f}\cap H^- = \mathfrak{f}\cap H = \mathfrak{f}.$$

In this case, the union $\mathfrak{t}(\mathfrak{a}\cap H^+,\mathfrak{f}) \cup \mathfrak{t}(\mathfrak{a}\cap H^-,\mathfrak{f})$ is the projection $\pi_{\operatorname{lin}(\mathfrak{f})}(\mathfrak{a})$ of $\mathfrak{a}$ on $V/\operatorname{lin}(\mathfrak{f})$; it is not pointed, therefore, applying again the induction hypothesis to the space $V/\operatorname{lin}(L)$, we have

$$\mu(\mathfrak{t}(\mathfrak{a}\cap H^+,\mathfrak{f})) + \mu(\mathfrak{t}(\mathfrak{a}\cap H^-,\mathfrak{f})) - \mu(\mathfrak{t}(\mathfrak{a}\cap H,\mathfrak{f})) = \mu(\pi_{\operatorname{lin}(\mathfrak{f})}(\mathfrak{a})) = 0$$

and the result follows. □

Corollary 17 *Let $\mathfrak{p}$ be a polytope in V and $s \in V_{\mathbb{Q}}$. Then*

$$\sum_{v \in \mathcal{V}(\mathfrak{p})} \mu(s + \mathrm{dir}(\mathfrak{t}(\mathfrak{p}, v)))$$

is equal to 1 *if the point s is integral and* 0 *otherwise.*

Proof. This follows immediately from the valuation property and the relation (see [3]) between the characteristic functions:

$$\sum_{v \in \mathcal{V}(\mathfrak{p})} \chi(\mathrm{dir}(\mathfrak{t}(\mathfrak{p}, v))) = \chi(\{0\}) \bmod \mathcal{L}$$

where $\mathcal{L}$ denotes the space of linear combinations of characteristic functions of cones with lines. □

Now we show that our functions are analytic near 0.

Proposition 18 *Let $\mathfrak{a}$ be an affine cone in V. The function $\mu(\mathfrak{a})$ is analytic near* 0*.*

Proof. The result is true when $V = 0$ (and the explicit computation shows that it is true also when $\dim V = 1$). We will prove it by induction on the dimension of V. Using the valuation property, it is enough to prove the analyticity when $\mathfrak{a}$ is a solid simplicial unimodular affine cone in V. Let $v_1, \ldots, v_k$ be primitive integral generators of the edges of $\mathrm{dir}\,\mathfrak{a}$. If Φ is a meromorphic function on V^* such that the product

$$\left(\prod_{i=1}^{k} \langle \xi, v_i \rangle\right) \Phi(\xi)$$

is analytic, we denote by $\mathrm{Res}_{v_1}(\Phi)$ the residue of Φ along $v_1 = 0$, that is to say the restriction to $v_1^{\perp} \subset V^*$ of

$$\langle \xi, v_1 \rangle \Phi(\xi).$$

From the properties of the functions S and I (Lemma 11) and the induction hypothesis, it follows that the product

$$\left(\prod_{i=1}^{k} \langle \xi, v_i \rangle\right) \mu(\mathfrak{a})(\xi)$$

is analytic. Thus we want to show that $\mathrm{Res}_{v_1}(\mu(\mathfrak{a})) = 0$. Starting from the defining formula (9), we want to prove that, for $\xi \in v_1^\perp$, we have

$$\mathrm{Res}_{v_1}(S(\mathfrak{a}))(\xi) = \sum_{\mathfrak{f};\dim\mathfrak{f}>0} \mu(\mathfrak{t}(\mathfrak{a},\mathfrak{f}))(\xi)\,\mathrm{Res}_{v_1}(I(\mathfrak{f}))(\xi). \tag{17}$$

Let us denote by π the projection $V \to V/<v_1>$. The cone $\pi(\mathfrak{a})$ is also a simplicial unimodular cone with primitive integral generators $\pi(v_2), \ldots, \pi(v_d)$. If $\mathfrak{a}$ is unimodular, then the parallelepiped $s + \Box(v_1, \ldots, v_d) \subset V$ contains exactly one integral point. Therefore, the explicit computations (6) and (8) of $I(\mathfrak{a})$ and $S(\mathfrak{a})$ imply immediately that the residues along $v_1 = 0$ of the functions $S(\mathfrak{a})$ and $I(\mathfrak{a})$ are given by:

$$\mathrm{Res}_{v_1}(S(\mathfrak{a})) = -S(\pi(\mathfrak{a})),$$

$$\mathrm{Res}_{v_1}(I(\mathfrak{a})) = -I(\pi(\mathfrak{a})).$$

In the sum (17), only the faces $\mathfrak{f}$ for which v_1 is an edge of $\operatorname{dir}\mathfrak{f}$ contribute, and these faces are in one to one correspondence with the faces of $\pi(\mathfrak{a})$. For such a face $\mathfrak{f}$, the transverse cone of $\pi(\mathfrak{a})$ along $\pi(\mathfrak{f})$ coincides with the transverse cone $\mathfrak{t}(\mathfrak{a},\mathfrak{f})$ and we have also

$$\mathrm{Res}_{v_1}(I(\mathfrak{f})) = -I(\pi(\mathfrak{f})),$$

whence (17), and the proposition. $\Box$

Next we will show that Formula (9) still holds when $\mathfrak{a}$ is replaced by any polyhedron $\mathfrak{p}$. This will be an easy consequence of Brion's theorem and the valuation property of I and S. In the following three theorems, we collect the results of this section.

Theorem 19 *Let V be a rational space and Q a rational scalar product on V^*. If $W = V/L$ is a rational quotient space of V, we denote by $\mathcal{C}(W)$ the set of affine cones in W. For $\mathfrak{a} \in \mathcal{C}(W)$, let $I(\mathfrak{a})$ and $S(\mathfrak{a})$ be the meromorphic functions with rational coefficients on W^* defined in Propositions 5 and 7.*

There exists a unique family of maps μ_W, indexed by the rational quotient spaces W of V, such that the family enjoys the following properties:

(a) μ_W maps $\mathcal{C}(W)$ to $\mathcal{H}(W^*)$, the space of analytic functions on W^*, with rational Taylor coefficients.

(b) If $W = \{0\}$, then $\mu_W(\{0\}) = 1$.

(c) If the affine cone $\mathfrak{a} \in \mathcal{C}(W)$ contains a straight line, then $\mu_W(\mathfrak{a}) = 0$.

(d) For any affine cone $\mathfrak{a}$ in W, one has

$$S(\mathfrak{a}) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{a})} \mu_{W/\operatorname{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{a}, \mathfrak{f})) I(\mathfrak{f})$$

where the sum is over all faces of the cone $\mathfrak{a}$.

As in all this section, in Formula *(d)*, the function $\mu_{W/\operatorname{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{a}, \mathfrak{f}))$ is considered as a function on W^* itself by means of the orthogonal projection $W^* \to (W/\operatorname{lin}(\mathfrak{f}))^* = (\operatorname{lin}(\mathfrak{f}))^\perp$ with respect to the scalar product on $W^* \subset V^*$.

Theorem 20 *The analytic functions defined in Theorem 19 have the following properties:*

(a) For any $x \in \Lambda$, one has $\mu(x + \mathfrak{a}) = \mu(\mathfrak{a})$.

(b) The map $\mathfrak{a} \mapsto \mu(\mathfrak{a})$ is equivariant with respect to lattice-preserving isometries. In other words, let g be an isometry of W which preserves the lattice Λ. Then $\mu(g(\mathfrak{a}))({}^t g^{-1}\xi) = \mu(\mathfrak{a})(\xi)$.

(c) If W is an orthogonal sum $W = W_1 \oplus W_2$ and $\mathfrak{a}_i$ is an affine cone in W_i for $i = 1, 2$, then

$$\mu(\mathfrak{a}_1 + \mathfrak{a}_2) = \mu(\mathfrak{a}_1)\mu(\mathfrak{a}_2).$$

(d) For a fixed $s \in W_{\mathbb{Q}}$, the map $\mathfrak{c} \to \mu(s + \mathfrak{c})$ is a valuation on the set of cones in W.

(e) Let $\mathfrak{p} \subset W$ be a polyhedron, then

$$S(\mathfrak{p})(\xi) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{p})} \mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi) I(\mathfrak{f})(\xi). \tag{18}$$

Proof. In Theorem 20, only point *(e)* has not yet been proven. If v is a vertex of $\mathfrak{p}$, let us denote by $\mathcal{F}(\mathfrak{p}, v)$ the set of faces of $\mathfrak{p}$ which contain v. For such a face $\mathfrak{f} \in \mathcal{F}(\mathfrak{p}, v)$, the intersection $\mathfrak{d} =< \mathfrak{f} > \cap \mathfrak{t}(\mathfrak{p}, v)$ is a face of the cone $\mathfrak{t}(\mathfrak{p}, v)$ and this correspondence is a bijection between $\mathcal{F}(\mathfrak{p}, v)$ and $\mathcal{F}(\mathfrak{t}(\mathfrak{p}, v))$ with inverse given by $\mathfrak{f} = \mathfrak{d} \cap \mathfrak{p}$. Moreover the transverse cone $\mathfrak{t}(\mathfrak{p}, v)$ of $\mathfrak{p}$ along its face $< \mathfrak{f} > \cap \mathfrak{t}(\mathfrak{p}, v)$ coincides with the transverse cone $\mathfrak{t}(\mathfrak{p}, \mathfrak{f})$ of $\mathfrak{p}$ along $\mathfrak{f}$. Therefore we have

$$S(\mathfrak{n}(\mathfrak{p}, v)) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{p}, v)} \mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))\; I(< \mathfrak{f} > \cap \mathfrak{t}(\mathfrak{p}, v)).$$

Replacing $S(\mathfrak{t}(\mathfrak{p}, v))$ with the right-hand side of this equality in Brion's formula, we obtain

$$S(\mathfrak{p}) = \sum_{v \in \mathcal{V}(\mathfrak{p})} \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{p}, v)} \mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))\, I(<\mathfrak{f}> \cap \mathfrak{t}(\mathfrak{p}, v)).$$

Then we reverse the order of summation and get

$$S(\mathfrak{p}) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{p})} \mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f})) \sum_{v \in \mathcal{V}(\mathfrak{f})} I(<\mathfrak{f}> \cap \mathfrak{t}(\mathfrak{p}, v)).$$

The last sum is equal to $I(\mathfrak{f})$. $\square$

Theorem 21 *Assume $V = \mathbb{R}^d$ with a fixed dimension d. Then, for m fixed, there exists a polynomial time algorithm which computes $\mu(\mathfrak{a})$ at order m for any affine cone $\mathfrak{a} \subset V$.*

Proof. By [2], there exist polynomial time algorithms which compute the functions $I(\mathfrak{c})$ and $S(\mathfrak{c})$ at order m for any affine cone $\mathfrak{c}$ in $\mathbb{R}^k$, if $k \leq d$. Therefore by induction we get an algorithm which computes $\mu(\mathfrak{a})$ for any $\mathfrak{a} \subset V$. $\square$

Let σ be a cone in the dual space V^*. The dual cone $\sigma^* \subset V$ contains the vector subspace $< \sigma >^{\perp}$. Let us denote by $\pi_{<\sigma>^{\perp}}$ the projection $V \to V/ < \sigma >^{\perp}$. For any $s \in V_{\mathbb{Q}}$, the projected cone $\pi_{<\sigma>^{\perp}}(s + \sigma^*)$ is a pointed cone in $V/ < \sigma >^{\perp}$. Thus $\mu(\pi_{<\sigma>^{\perp}}(s + \sigma^*))$ is an analytic function on $(V/ < \sigma >^{\perp})^* \cong < \sigma > \subset V^*$. We consider it as a function on V^* by means of orthogonal projection, as before. We obtain a map $\mu_s^* : \mathcal{C}(V^*) \to \mathcal{H}(V^*)$ defined by:

Definition 22

$$\mu_s^*(\sigma) = \mu(\pi_{<\sigma>^{\perp}}(s + \sigma^*)).$$

From the valuation behavior of μ, it follows that μ_s^* is a solid valuation. In other words, the following corollary holds.

Corollary 23 *Let σ be a cone in V^*, and let $\{\sigma'\}$ be a subdivision of σ. For any $s \in V_{\mathbb{Q}}$, we have*

$$\mu_s^*(\sigma) = \sum_{\dim \sigma' = \dim \sigma} \mu_s^*(\sigma').$$

Proof. As $\{\sigma'\}$ is a subdivision of σ, we have

$$\chi(\sigma) = \sum_{\dim\sigma'=\dim\sigma} \chi(\sigma') + \sum_{\dim\sigma'<\dim\sigma} \pm\chi(\sigma').$$

Let $L =< \sigma >^{\perp} \subset V$ and let π denote the projection $V \to V/L$. The map $\tau \mapsto \chi(\pi(s+\tau^*))$ is a valuation on the set of cones in V^* (see [3] for instance). Therefore

$$\chi(\pi(s+\sigma^*)) = \sum_{\dim\sigma'=\dim\sigma} \chi(\pi(s+\sigma'^*)) + \sum_{\dim\sigma'<\dim\sigma} \pm\chi(\pi(s+\sigma'^*)).$$

If $\dim\sigma' < \dim\sigma$, then the cone $\pi(s+\sigma'^*) \subset V/L$ contains a straight line , thus $\mu(\pi(s+\sigma'^*)) = 0$, and the corollary follows from the valuation property of μ. $\square$

In a companion paper [6], we will prove the following theorem, which extends to equivariant homology a result of Pommersheim-Thomas [20] by which they answered a question of Danilov [14].

Theorem 24 *Let $\mathcal{E}$ be a fan in V^* and let X be the corresponding toric variety. For $\sigma \in \mathcal{E}$, let $X(\sigma) \subseteq X$ be the corresponding orbit closure. Then the equivariant Todd class of X is equal to*

$$\sum_{\sigma\in\mathcal{E}} \mu_0^*(\sigma)[X(\sigma)]$$

in the equivariant homology ring of X.

5 Local Euler-Maclaurin formula

As in the previous section, V is a rational space and we fix a scalar product on V. Let $\mathfrak{p}$ be a (convex rational) polyhedron in V. To each face $\mathfrak{f}$ of $\mathfrak{p}$, we are going to associate a linear differential operator $D(\mathfrak{p},\mathfrak{f})$ on V.

To any analytic function $\Phi(\xi)$ on V^*, defined near 0, there corresponds a unique linear differential operator $D(\Phi)$ (of infinite degree) with constant coefficients on V such that $\Phi(\xi)$ is the symbol of $D(\Phi)$. More precisely, for $\xi \in V^*$, let us denote by e^{ξ} the function $x \mapsto e^{\langle\xi,x\rangle}$ on V, then $D(\Phi)$ is defined by the relation

$$D(\Phi)\cdot e^{\xi} = \Phi(\xi)e^{\xi} \text{ for } \xi \text{ small enough.}$$

Let $W = V/L$ be a quotient space of V and let $\mathfrak{a}$ be a pointed affine cone in W. In the previous section, we constructed an analytic function $\mu(\mathfrak{a})$ on $W^* = L^\perp \subset V^*$. By orthogonal projection, we consider $\mu(\mathfrak{a})$ as a function on V^* and we introduce the corresponding differential operator $D(\mu(\mathfrak{a}))$ on V:

$$D(\mu(\mathfrak{a})) \cdot e^\xi = \mu(\mathfrak{a})(\xi)e^\xi. \tag{19}$$

Let $\mathfrak{p}$ be a polyhedron in V.

Definition 25 *Let $\mathfrak{f}$ be a face of $\mathfrak{p}$. We denote by*

$$D(\mathfrak{p}, \mathfrak{f}) = D(\mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f})))$$

the differential operator on V associated to the transverse cone $\mathfrak{t}(\mathfrak{p}, \mathfrak{f})$ of $\mathfrak{p}$ along $\mathfrak{f}$. We denote its constant term by $\nu(\mathfrak{p}, \mathfrak{f})$. Thus

$$\nu(\mathfrak{p}, \mathfrak{f}) = D(\mathfrak{p}, \mathfrak{f}) \cdot 1 = \mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(0).$$

The operator $D(\mathfrak{p}, \mathfrak{f})$, as well as its constant term $\nu(\mathfrak{p}, \mathfrak{f})$, are **local** in the sense that they depend only on the class of $\mathfrak{t}(\mathfrak{p}, \mathfrak{f})$ modulo integral translations. In particular, if $\mathfrak{p}$ has integral vertices, then $D(\mathfrak{p}, \mathfrak{f})$ depends only on the cone of transverse feasible directions at a generic point of $\mathfrak{f}$. The operator $D(\mathfrak{p}, \mathfrak{f})$ involves only derivatives in directions orthogonal to the face $\mathfrak{f}$.

We are now ready to state the local Euler-Maclaurin formula for any polytope.

Theorem 26 *(Local Euler-Maclaurin formula)*

Let $\mathfrak{p}$ be a polytope in V. For any polynomial function $h(x)$ on V, we have

$$\sum_{x \in \mathfrak{p} \cap \Lambda} h(x) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{p})} \int_{\mathfrak{f}} D(\mathfrak{p}, \mathfrak{f}) \cdot h \tag{20}$$

where the integral on the face $\mathfrak{f}$ is taken with respect to the Lebesgue measure on $< \mathfrak{f} >$ defined by the lattice $\Lambda \cap \operatorname{lin}(\mathfrak{f})$.

Proof.

The method is to check equality (20) for a polynomial of the form $h(x) = \langle \xi, x \rangle^k$. Taking Taylor series, we may replace $h(x)$ by $e^{t\langle \xi, x \rangle}$ with t small. Then the equality (20) becomes the formula in Theorem 20, *(c)*

$$S(\mathfrak{p})(\xi) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{p})} \mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi) I_{\mathfrak{f}}(\xi).$$

$\square$

In dimension 1, when $\mathfrak{p}$ is an interval $[a_1, a_2]$, applying Formulas (13) and (12) for $\mu(a_1 + \mathbb{R}^+)$ and $\mu(a_2 - \mathbb{R}^+)$, we obtain

$$\sum_{a_1 \leq x \leq a_2, x \in \mathbb{Z}} h(x) = \int_{a_1}^{a_2} h(t)dt \quad - \quad \sum_{n \geq 0} \frac{b((n+1), t_1)}{(n+1)!}((\frac{d}{dt})^n h)(a_1)$$
$$- \quad \sum_{n \geq 0} (-1)^n \frac{b((n+1), t_2)}{(n+1)!}((\frac{d}{dt})^n h)(a_2),$$

where t_1 and t_2 in $[0, 1[$ are defined by $t_1 = k_1 - a_1$, $t_2 = a_2 - k_2$, with k_1 the smallest integer greater or equal than a_1 and k_2 the largest integer smaller or equal than a_2 (Figure 5). Of course, when a_1 and a_2 are integers, we recover the historical Euler-Maclaurin formula.

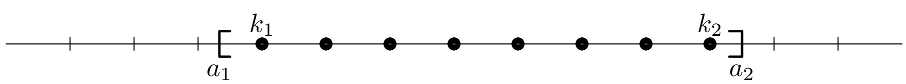

Figure 11: Euler-Maclaurin for the interval $[a_1, a_2]$

6 Ehrhart polynomial

Let $\mathfrak{p}$ be a rational polytope in a d-dimensional rational space V and let q be an integer such that $q\mathfrak{p}$ has integral vertices. Let $h(x)$ be a polynomial function of degree r on V. For any integer t, we consider the dilated polytope $t\mathfrak{p}$ and the corresponding sum

$$S(t\mathfrak{p}, h) = \sum_{x \in t\mathfrak{p} \cap \Lambda} h(x).$$

As a function of t, it is given by a quasipolynomial: there exist functions $t \mapsto E_i(\mathfrak{p}, h, t)$ on $\mathbb{Z}$ which are periodic with period q such that

$$S(t\mathfrak{p}, h) = \sum_{i=0}^{d+r} E_i(\mathfrak{p}, h, t) t^i \tag{21}$$

whenever t is a positive integer and even in a slightly larger range including negative values.

Definition 27 *The periodic functions* $E_i(\mathfrak{p}, h, t)$ *defined by Equation (21) are called the* Ehrhart coefficients for the polytope $\mathfrak{p}$ and the polynomial h.

When h *is the constant polynomial* $h(x) = 1$*, we denote the Ehrhart coefficients simply by* $E_i(\mathfrak{p}, t)$.

Our local Euler-Maclaurin formula gives an expression of the coefficients $E_i(\mathfrak{p}, h, t)$ in terms of the functions $\mu((\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))$, as we will now explain.

Let $\mathfrak{a}$ be a pointed affine cone with vertex s in V. If t is a nonzero integer, we define $\mu(\mathfrak{a}, t) = \mu(t\mathfrak{a})$. For $t = 0$, we define $\mu(\mathfrak{a}, 0) = \mu(\mathrm{dir}(\mathfrak{a}))$. Then $\mu(\mathfrak{a}, t + q) = \mu(\mathfrak{a}, t)$ for any integer $q \in \mathbb{N}$ such that the point qs is integral. Let $\mathfrak{p}$ be a polyhedron and $\mathfrak{f}$ a face of codimension m. We define $D(\mathfrak{p}, \mathfrak{f}, t) = D(\mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}), t))$ for any integer $t \in \mathbb{N}$.

Remark 28 *If the affine span* $< \mathfrak{f} >$ *of the face* $\mathfrak{f}$ *contains an integral point, then the vertex of the transverse cone* $\mathfrak{t}(\mathfrak{p}, \mathfrak{f})$ *is integral, therefore* $\mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}), t) = \mu(\mathrm{dir}(\mathfrak{t}(\mathfrak{p}, \mathfrak{f})))$ *and* $D(\mathfrak{p}, \mathfrak{f}, t)$ *do not depend on* t*. We have* $D(\mathfrak{p}, \mathfrak{f}, t) = D(\mathfrak{p}, \mathfrak{f})$.

Let $w_1, \ldots, w_m$ be an integral basis of the subspace $\mathrm{lin}(\mathfrak{f})^{\perp_Q} \subset V$. The operator $D(\mathfrak{p}, \mathfrak{f}, t)$ has the following expression:

$$D(\mathfrak{p}, \mathfrak{f}, t) = \nu_0(\mathfrak{p}, \mathfrak{f}, t) + \sum_{A, |A|=1}^{\infty} \nu_A(\mathfrak{p}, \mathfrak{f}, t) \partial^A$$

where $A = (a_1, \ldots, a_m)$, with $a_i \in \mathbb{N}$ and $\partial^A = D(w_1)^{a_1} \cdots D(w_m)^{a_m}$. The coefficients $\nu_A(\mathfrak{p}, \mathfrak{f}, t) \in \mathbb{Q}$ are periodic with respect to t, with period equal to the smallest integer $q_\mathfrak{f}$ such that $q_\mathfrak{f} < \mathfrak{f} >$ contains an integral point.

Proposition 29 *Let* $\mathfrak{p}$ *be a rational polytope and* h *a polynomial function of degree* r *on* V*. Then, for any integer* $t \geq 0$*, we have*

$$S(t\mathfrak{p}, h) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{p})} \int_{t\mathfrak{f}} D(\mathfrak{p}, \mathfrak{f}, t) \cdot h. \tag{22}$$

Furthermore we have

$$\int_{t\mathfrak{f}} D(\mathfrak{p}, \mathfrak{f}, t) \cdot h = \sum_{i=\dim \mathfrak{f}}^{\dim \mathfrak{f} + r} E_i(\mathfrak{p}, h, \mathfrak{f}, t)\, t^i$$

where the coefficients $E_i(\mathfrak{p}, h, \mathfrak{f}, t)$ *are periodic with period* $q_\mathfrak{f}$.

Hence the Ehrhart coefficients are given by

$$E_i(\mathfrak{p}, h, t) = \sum_{\mathfrak{f}, \dim \mathfrak{f} \leq i} E_i(\mathfrak{p}, h, \mathfrak{f}, t).$$

Proof. For $t > 0$, Formula (22) is just the Euler-Maclaurin formula of Theorem 26 and the second equation follows from obvious estimates on the polynomial behaviour of the integrals.

For $t = 0$, as both sides of (22) are quasipolynomials, they take the same value. We may also deduce the equality for $t = 0$ from Corollary 17. Indeed, for $t = 0$, the left hand side is $h(0)$ and the faces of dimension > 0 give a zero contribution to the right hand side. The equality becomes

$$h(0) = \sum_{v \in \mathcal{V}(\mathfrak{p})} (D(\mathrm{dir}(\mathfrak{t}(\mathfrak{p}, v))) \cdot h)(0),$$

and follows immediately from Corollary 17. □

For instance, let h be a monomial $h(x) = x^m$ with $m \in \mathbb{N}^d$. The coefficient of the highest degree term $t^{d+|m|}$ is the integral $\int_{\mathfrak{p}} x^m dx$. As the operator $D(\mathfrak{p}, \mathfrak{p}, t)$ is equal to 1, the face $\mathfrak{p}$ does not contribute to the coefficients of lower degree. The coefficient of $t^{d+|m|-1}$ involves only the faces of codimension 1, the coefficient of $t^{d+|m|-2}$ involves only the faces of codimension 1 and 2, etc.

When we apply the last proposition to the function $h(x) = 1$, we obtain a formula for the number of integral points in $t\mathfrak{p}$.

Corollary 30 *(a) The Ehrhart quasipolynomial of the polytope $\mathfrak{p}$ is given by*

$$\mathrm{Card}(t\mathfrak{p} \cap \Lambda) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{p})} \nu_0(\mathfrak{p}, \mathfrak{f}, t) \,\mathrm{vol}(\mathfrak{f}) t^{\dim \mathfrak{f}},$$

hence

$$E_k(\mathfrak{p}, t) = \sum_{\mathfrak{f}, \dim \mathfrak{f} = k} \nu_0(\mathfrak{p}, \mathfrak{f}, t) \,\mathrm{vol}(\mathfrak{f}).$$

(b) The rational number $\nu_0(\mathfrak{p}, \mathfrak{f}, t)$ depends only on the class modulo lattice translations of the transverse cone $t\mathfrak{t}(\mathfrak{p}, \mathfrak{f})$. Therefore, it is a periodic function of t with period at most equal to $q_{\mathfrak{f}}$, the smallest integer such that $q_{\mathfrak{f}} < \mathfrak{f} >$ contains integral points. In particular, if the affine span $< \mathfrak{f} >$ contains integral points for every k-dimensional face $\mathfrak{f}$ of $\mathfrak{p}$, then the Ehrhart coefficient $E_k(\mathfrak{p}, t)$ does not depend on t.

(c) When $\dim V$ *is fixed, there exists a polynomial time algorithm which computes* $\nu_0(\mathfrak{p}, \mathfrak{f}, t)$.

Proof. The last statement in *(b)* is due to Stanley [22]; it also follows immediately from Remark 28. The computability follows from the computability of the functions $\mu(\mathfrak{a})$. $\square$

Barvinok [4] proved recently that, given an integer m, there exists a polynomial time algorithm which computes the m highest coefficients of the Ehrhart quasipolynomial of any rational simplex in $\mathbb{R}^d$, *when the dimension* d *is considered as an input.* We hope that our construction of the functions μ leads to another polynomial time algorithm which would compute the m highest coefficients of the Ehrhart quasipolynomial for any simplex in $\mathbb{R}^d$ and any polynomial $h(x)$, when the dimension d and the degree of the polynomial are considered as input.

7 Computations in dimension 2

In this section, $V = \mathbb{R}^2$ with $\Lambda = \mathbb{Z}^2$. Let v_1 and v_2 be primitive integral vectors. Let $s \in V_{\mathbb{Q}}$. We are going to compute $\mu(\mathfrak{a})$ for the affine cone

$$\mathfrak{a} = s + \mathbb{R}_+ v_1 + \mathbb{R}_+ v_2.$$

We will use the following notations:
For $t \in \mathbb{R}$, we denote the smallest integer greater or equal than t by $\mathrm{ceil}(t)$ and we define $[[t]] \in [0, 1[$ by $[[t]] = \mathrm{ceil}(t) - t$.
We denote by B the function $B(y, t) = \frac{e^{[[t]]y}}{1-e^y} + \frac{1}{y}$. Recall that the function $\mu(\mathfrak{a})$ for a one-dimensional cone (half-line) $\mathfrak{a}$ is expressed in terms of B.
$s = s_1 v_1 + s_2 v_2$ with $s_i \in \mathbb{Q}$.
$C_i = \frac{Q(v_1,v_2)}{Q(v_i,v_i)}$, for $i = 1, 2$.
$q = \det(v_1, v_2)$. We assume $q > 0$.
$w \in \mathbb{Z}^2$ is a vector such that $\det(v_1, w) = 1$,
$p = \det(v_2, w)$. Thus p and q are coprime integers.
$r = (qs_1 + [[qs_1]]) + p(qs_2 + [[qs_2]])$.
ζ is a primitive q-th root of 1.
We observe that the lattice in $V/\mathbb{R}v_1$ is generated by $\bar{w} = \frac{1}{q}\bar{v}_2$ where $\bar{w}$, $\bar{v}_2$ denotes the image of w, v_2 in $V/\mathbb{R}v_1$. Thus the transverse cone $\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_1) \subset V/\mathbb{R}v_1$ is given by

$$\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_1) = (qs_2 + \mathbb{R}_+)\frac{1}{q}\bar{v}_2.$$

Proposition 31 *For $\xi \in V^*$, let $y_i = \langle \xi, v_i \rangle$, for $i = 1, 2$. Then the function $\mu(\mathfrak{a})(\xi)$ is given by*

(23) $\mu(\mathfrak{a})(\xi) =$

$$\frac{1}{q} e^{[[qs_1]]\frac{y_1}{q}} e^{[[qs_2]]\frac{y_2}{q}} \left(\frac{1}{(1 - e^{\frac{y_1}{q}})(1 - e^{\frac{y_2}{q}})} + \sum_{k=1}^{q-1} \frac{\zeta^{kr}}{(1 - \zeta^k e^{\frac{y_1}{q}})(1 - \zeta^{kp} e^{\frac{y_2}{q}})} \right)$$
$$+ \frac{1}{y_1} B(\frac{y_2 - C_1 y_1}{q}, [[qs_2]]) + \frac{1}{y_2} B(\frac{y_1 - C_2 y_2}{q}, [[qs_1]]) - \frac{q}{y_1 y_2}.$$

Its value at $\xi = 0$ is equal to

$$(24) \qquad \mu(\mathfrak{a})(0) = \frac{1}{q}((\frac{1}{2} - [[qs_1]])(\frac{1}{2} - [[qs_2]]))$$
$$+ \frac{Q(v_1, v_2)}{Q(v_1, v_1)}(\frac{1}{12} - \frac{1}{2}[[qs_2]] + \frac{1}{2}[[qs_2]]^2) + \frac{Q(v_1, v_2)}{Q(v_2, v_2)}(\frac{1}{12} - \frac{1}{2}[[qs_1]] + \frac{1}{2}[[qs_1]]^2)$$
$$+ \frac{1}{q} \sum_{k=1}^{q-1} \frac{\zeta^{kr}}{(1 - \zeta^k)(1 - \zeta^{kp})}.$$

When $\mathfrak{a}$ is unimodular, $\mu(\mathfrak{a})(\xi)$ is given by

$$(25) \quad \mu(a)(\xi) = \frac{\exp([[s_1]]y_1 + [[s_2]]y_2)}{(1 - e^{y_1})(1 - e^{y_2})}$$
$$+ \frac{1}{y_1} B(y_2 - C_1 y_1, [[s_2]]) + \frac{1}{y_2} B(y_1 - C_2 y_2, [[s_1]]) - \frac{1}{y_1 y_2}$$

and its constant term $\mu(\mathfrak{a})(0)$ is given by

$$(26) \quad \mu(\mathfrak{a})(0) = (\frac{1}{2} - [[s_1]])(\frac{1}{2} - [[s_2]])$$
$$+ \frac{Q(v_1, v_2)}{Q(v_1, v_1)}(\frac{1}{12} - \frac{1}{2}[[s_2]] + \frac{1}{2}[[s_2]]^2) + \frac{Q(v_1, v_2)}{Q(v_2, v_2)}(\frac{1}{12} - \frac{1}{2}[[s_1]] + \frac{1}{2}[[s_1]]^2).$$

Remark 32 *In actual Maple computations, we use only the unimodular case (25) which is computable in polynomial time at any given order. Thanks to the valuation property, for a non unimodular cone $\mathfrak{a}$, we compute $\mu(\mathfrak{a})$ by performing first a signed decomposition of $\mathfrak{a}$ into unimodular cones, similar to Barvinok's decomposition. As a result, by our local Euler-Maclaurin formula, we have fast algorithms which compute, for a polygon $\mathfrak{p} \subset \mathbb{R}^2$ and a monomial $h(x) = x_1^{m_1} x_2^{m_2}$, the sum of values at integral points $S(\mathfrak{p}, h) = \sum_{x \in \mathfrak{p} \cap \Lambda} h(x)$ and the coefficients of the corresponding Ehrhart quasipolynomial.*

Proof. We use the defining relation of Proposition 12. First, we obtain a summation formula for $S(\mathfrak{a})$ by using finite Fourier transform as in [9]. We observe that $\Lambda \subset M = \frac{1}{q}(\mathbb{Z}v_1 + \mathbb{Z}v_2)$. Let $\tilde{\mathfrak{a}} \subset \mathfrak{a}$ be the cone

$$\tilde{\mathfrak{a}} = \tilde{s} + \mathbb{R}_+ v_1 + \mathbb{R}_+ v_2$$

with vertex

$$\tilde{s} = \frac{1}{q}(\mathrm{ceil}(qs_1)v_1 + \mathrm{ceil}(qs_2)v_2).$$

Then $\tilde{\mathfrak{a}} \cap M = \mathfrak{a} \cap M$. As $\Lambda \subset M$, we have also $\tilde{\mathfrak{a}} \cap \Lambda = \mathfrak{a} \cap \Lambda$. Consider the dual lattice $M^* \subset \Lambda^*$. Let $x \in M$. We have

$$\sum_{\gamma \in \Lambda^*/M^*} e^{2i\pi\langle \gamma, x\rangle} = \begin{array}{l} 0 \text{ if } x \notin \mathbb{Z}^2 \\ q \text{ if } x \in \mathbb{Z}^2. \end{array}$$

Therefore we have

$$S(\tilde{\mathfrak{a}})(\xi) = \frac{1}{q} \sum_{\gamma \in \Lambda^*/M^*} \sum_{x \in \tilde{\mathfrak{a}} \cap M} e^{\langle 2i\pi\gamma + \xi, x\rangle}.$$

Since

$$\tilde{\mathfrak{a}} \cap M = \tilde{s} + \mathbb{Z}_+ \frac{v_1}{q} + \mathbb{Z}_+ \frac{v_2}{q},$$

we obtain

$$S(\mathfrak{a})(\xi) = S(\tilde{\mathfrak{a}})(\xi) = e^{\langle \xi, \tilde{s}\rangle} \frac{1}{q} \sum_{k=0}^{q-1} \frac{e^{\langle 2i\pi k\delta, \tilde{s}\rangle}}{(1 - e^{\langle 2i\pi k\delta + \xi, \frac{v_1}{q}\rangle})(1 - e^{\langle 2i\pi k\delta + \xi, \frac{v_2}{q}\rangle})}. \tag{27}$$

In this formula, δ is a generator of the group $\Lambda^*/M^* \cong \mathbb{Z}/q\mathbb{Z}$. By using the basis (v_1, w) of $\mathbb{Z}^2$, we obtain:

$$\langle \delta, v_1 \rangle = 1, \quad \langle \delta, v_2 \rangle = p.$$

Let (v_1^*, v_2^*) be the dual basis of (v_1, v_2). The orthogonal projection of ξ on $(V/\mathbb{R}v_1)^* = \mathbb{R}v_2^*$ is equal to $(-C_1 y_1 + y_2)v_2^*$, with

$$C_1 = -\frac{Q(v_1^*, v_2^*)}{Q(v_2^*, v_2^*)} = \frac{Q(v_1, v_2)}{Q(v_1, v_1)}.$$

Then the computation in dimension one (11) gives

$$\mu(\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_1))(\xi) = B(\frac{-C_1 y_1 + y_2}{q}, [[qs_2]])$$

and similarly

$$\mu(\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_2))(\xi) = B(\frac{y_1 - C_2 y_2}{q}, [[qs_1]])$$

with

$$C_2 = \frac{Q(v_1, v_2)}{Q(v_2, v_2)}.$$

We have

$$I(\mathfrak{f}_i)(\xi) = e^{y_1 s_1 + y_2 s_2}(\frac{-1}{y_i}),$$
$$I(\mathfrak{a})(\xi) = e^{y_1 s_1 + y_2 s_2}\frac{q}{y_1 y_2}.$$

Therefore, by (12), we have

$$\begin{aligned} \mu(\mathfrak{a})(\xi) &= e^{-(y_1 s_1 + y_2 s_2)} S(\mathfrak{a})(\xi) \\ &+ \frac{1}{y_1} B(\frac{-C_1 y_1 + y_2}{q}, [[qs_2]]) + \frac{1}{y_2} B(\frac{y_1 - C_2 y_2}{q}, [[qs_1]]) \\ &- \frac{q}{y_1 y_2}. \end{aligned} \tag{28}$$

In Formula (28), we replace $S(\mathfrak{a})$ with the right hand side of (27), taking in account the equality $\mathrm{ceil}(qs_i) - qs_i = [[qs_i]]$. This gives (23). □

If $\mathfrak{a}$ is not unimodular, then $\mu(\mathfrak{a})(\xi)$ involves the "extended" Fourier-Dedekind sum

$$\frac{1}{q}\sum_{k=1}^{q-1} \frac{\zeta^{kr}}{(1 - \zeta^k e^{\frac{y_1}{q}})(1 - \zeta^{kp} e^{\frac{y_2}{q}})},$$

and $\mu(\mathfrak{a})(0)$ involves the Fourier-Dedekind sum

$$D(q, 1, p, r) = \frac{1}{q}\sum_{k=1}^{q-1} \frac{\zeta^{kr}}{(1 - \zeta^k)(1 - \zeta^{kp})}.$$

One has (see for instance [5])

$$D(q, 1, p, r) = \sum_{k=0}^{q-1} ((-\frac{kp + r}{q}))((\frac{k}{q})) - \frac{1}{4q},$$

where the '"sawtooth" function $((a))$ is defined by

$$((a)) = a - [a] - \frac{1}{2}.$$

The Dedekind sum $D(q, 1, p, r)$ can also be computed in polynomial time by means of reciprocity relations (see for example [5]), but here we do not use this fact.

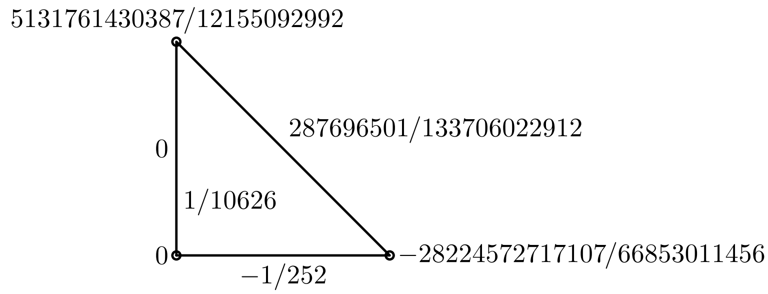

Figure 12: Le savant Cosinus

Example 33 *(Figure12) We compute the right hand side of Euler-Maclaurin formula in the case of the "dull triangle" with vertices* $(0, 0)$,$(1, 0)$,$(0, 1)$*, and the polynomial* $h(x) = x_1^{20}x_2$*. As expected, the contributions of the various faces of* $\mathfrak{p}$ *add up to* 0[1].
Contribution of vertices: $0, -\frac{28224572717107}{66853011456}, \frac{5131761430387}{12155092992}$.
Contribution of edges: $-\frac{1}{252}, \frac{287696501}{133706022912}, 0$.
Integral over triangle: $\frac{1}{10626}$.

Example 34 *Triangle with vertices* $s_1 = (\frac{1}{3}, \frac{1}{5})$*,* $s_2 = (\frac{16}{3}, \frac{1}{7})$*,* $s_3 = (\frac{37}{5}, \frac{92}{7})$*.*
Number of integral points: 31
Contribution of vertices: $\frac{89133678169939}{66088208614500}, -\frac{4281800310619}{2106396270216}, -\frac{401172431621091}{457987274773000}$
Contribution of edges : $\frac{1}{210}, -\frac{1}{210}, \frac{1}{1050}$
Area of triangle: $\frac{34187}{1050}$

Example 35 *Quadrangle with vertices* $(\frac{1}{3}, \frac{1}{5})$*,* $(\frac{16}{3}, \frac{1}{7})$*,* $(\frac{37}{5}, \frac{92}{7})$*,* $(3, 10)$*.*
Number of integral points: 49
Contribution of vertices: $\frac{210849514883}{127956322980}, -\frac{4281800310619}{2106396270216}, -\frac{179008247}{706816180}, -\frac{4382929}{6869864}$
Contribution of edges : $\frac{1}{210}, -\frac{1}{210}, \frac{11}{35}, \frac{1}{30}$
Area: $\frac{699}{14}$

[1]This computation delighted us, and it would have delighted Dr. Pancrace Eusèbe Zéphyrin Brioché alias "Dr. Cosinus" [13]

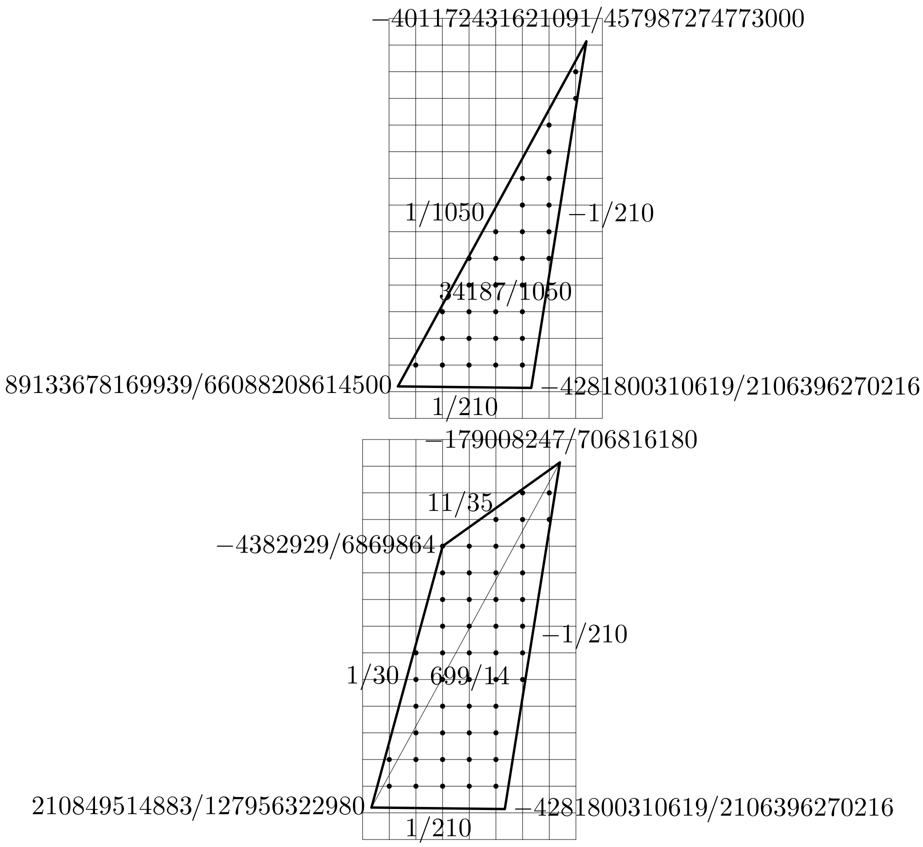

Figure 13: Triangle with vertices $(\frac{1}{3}, \frac{1}{5})$, $(\frac{16}{3}, \frac{1}{7})$, $(\frac{37}{5}, \frac{92}{7})$ and quadrangle with extra vertex (3,10)

Remark that, as expected, the contributions of the bottom right vertex $(\frac{16}{3}, \frac{1}{7})$ in the triangle or the trapezoid of Figure 13 are the same, as this vertex have the same tangent cone in both polygons.

Example 36 *We compute the Ehrhart quasipolynomial $E_2 t^2 + E_1(t)t + E_0(t)$ for the number of integral points of the triangle of Example 34. The highest coefficient is the area of the triangle, $E_2 = \frac{34187}{1050}$. The coefficient $E_1(t)$ is the sum of the contributions of the edges. The coefficient $E_0(t)$ is the sum of the contributions of the vertices.*

On this example, we can observe the periods of the contributions of the edges and vertices to the Ehrhart coefficients (Corollary 30). The period of a *vertex contribution is equal to the* lcm *of the denominators of its coordinates. For an edge starting from a vertex* (a_1, a_2) *and parallel to the primitive vector* (v_1, v_2)*, the period is the least integer* q *such that* $q(a_1v_2 - a_2v_1)$ *is an integer.*

Contribution of edges (periods 3, 5, 7 *respectively):*
$-\frac{1}{105}\, mod\,(t,3) + \frac{1}{70}, \quad -\frac{1}{105}\, mod\,(4\,t,7) + 1/30, \quad -\frac{1}{525}\, mod\,(2\,t,5) + \frac{1}{210}.$

Contribution of vertex $s_1 = (\frac{1}{3}, \frac{1}{5})$ *(period* 15*):*

$$\begin{aligned}
&\tfrac{1}{75}\, mod\,(t,5)\, mod\,(t,15) - \tfrac{1}{45}\, mod\,(t,3)\, mod\,(t,15) - \tfrac{1}{15}\, mod\,(t,5)\, mod\,(2\,t,3)\\
&+\tfrac{1765457034769}{293725371620} - \tfrac{44}{5}\, mod\,(t,5) - \tfrac{747989}{11987225}\, mod\,(2\,t,5) + \tfrac{1}{30}\, mod\,(7\,t,15)\\
&+\tfrac{3}{5}\, mod\,(8\,t,15) - \tfrac{1}{15}\, mod\,(t,15) + \tfrac{44}{25}\, (mod\,(t,5))^2\\
&-\tfrac{2}{45}\, (mod\,(8\,t,15))^2 + \tfrac{1}{225}\, (mod\,(t,15))^2 + \tfrac{1}{3}\, mod\,(2\,t,3)\\
&+\tfrac{15227}{183774}\, mod\,(t,3) - \tfrac{1}{9}\, (fmod\,(2\,t,3))^2 + \tfrac{2567}{91887}\, (mod\,(t,3))^2\\
&-\tfrac{901467}{119872250}\, (mod\,(2\,t,5))^2 - \tfrac{1}{150}\, (mod\,(7\,t,15))^2 + \tfrac{1}{75}\, mod\,(2\,t,5)\, mod\,(7\,t,15)\\
&+\tfrac{1}{45}\, mod\,(8\,t,15)\, mod\,(2\,t,3)
\end{aligned}$$

Contribution of vertex $s_2 = (\frac{16}{3}, \frac{1}{7})$ *(period* 21*):*

$$\begin{aligned}
&-\tfrac{32132693735}{4776408776} - \tfrac{13}{14}\, (mod\,(t,7))^2 + \tfrac{13}{2}\, mod\,(t,7) + \tfrac{1}{63}\, mod\,(t,3)\, mod\,(2\,t,21)\\
&-\tfrac{15227}{183774}\, mod\,(t,3) - \tfrac{1}{21}\, mod\,(2\,t,21) - \tfrac{2567}{91887}\, (mod\,(t,3))^2 + \tfrac{1}{9}\, (mod\,(2\,t,3))^2\\
&-\tfrac{2}{3}\, mod\,(2\,t,3) + \tfrac{1}{63}\, mod\,(2\,t,21)\, mod\,(2\,t,3)\\
&+\tfrac{1}{21}\, mod\,(2\,t,3)\, mod\,(4\,t,7) - \tfrac{30189}{545804}\, mod\,(4\,t,7)\\
&-\tfrac{8797}{3820628}\, (mod\,(4\,t,7))^2
\end{aligned}$$

Contribution of vertex $s_3 = (\frac{37}{5}, \frac{92}{7})$ *(period* 35*):*

$$
\begin{array}{l}
\frac{9}{1225}\,(mod(16\,t,35))^2 + \frac{3}{1225}\,(mod(23\,t,35))^2 + \frac{641856910509}{373867163080} \\
+\frac{30189}{545804}\,mod(4\,t,7) + \frac{1}{1225}\,(mod(34\,t,35))^2 \\
+1/10\,mod(3\,t,5) - 1/35\,mod(34\,t,35) - \frac{1}{2450}\,(mod(18\,t,35))^2 \\
+\frac{2}{1225}\,(mod(9\,t,35))^2 + \frac{901467}{119872250}\,(mod(2\,t,5))^2 \\
+\frac{1}{70}\,mod(18\,t,35) + \frac{1}{50}\,(mod(3\,t,5))^2 + \frac{1}{1225}\,mod(34\,t,35)mod(18\,t,35) \\
+\frac{1}{1225}\,fmod(23\,t,35)mod(16\,t,35) \\
+\frac{1}{175}\,mod(3\,t,5)mod(23\,t,35) - \frac{1}{1225}\,mod(34\,t,35)mod(16\,t,35) \\
-\frac{1}{175}\,mod(2\,t,5)mod(16\,t,35) \\
-1/35\,mod(3\,t,7)mod(3\,t,5) \\
-1/35\,mod(3\,t,5)mod(4\,t,7) - \frac{1}{245}\,mod(9\,t,35)mod(3\,t,7) \\
-\frac{1}{175}\,mod(3\,t,5)mod(9\,t,35) \\
-\frac{1}{1225}\,mod(16\,t,35)mod(18\,t,35) + \frac{747989}{11987225}\,mod(2\,t,5) \\
+\frac{8797}{3820628}\,(mod(4\,t,7))^2 - 1/35\,mod(9\,t,35) - \frac{4}{35}\,mod(23\,t,35) \\
-\frac{8}{35}\,mod(16\,t,35) + 1/49\,(mod(3\,t,7))^2
\end{array}
$$

Example 37 *(***Computation time***).*

We computed the full Ehrhart quasipolynomial corresponding to the triangle of Example 34 and the polynomial $h(x_1, x_2) = x_1^{k_1} x_2^{k_2}$*, with increasing exponents* k_1 *and* k_2*. Allowing a computation time of about one hour, we reached* $k_1 = k_2 = 24$*. The result is of course too big to write here.*

The sum of values $x_1^{48} x_2^{48}$ *at the integral points of the triangle of Example 34 dilated by the factor* $N = 11^5$ *took about the same time. The result is the following number*

55969247458735493271268368615238071121335974262337882261418363621
89704055956429496253759473056373507451253522021344188115187647607
84555431172202923756940824265247663088847763429436570335188702325
06644969965841257822711805056447218921550669146263582661876630783
21357671611262065293901983868557252464459832189159990869820527095
53646871654914800005753059422066576204781923454823934475242960034
42199041253798398004263030681714027295470241663946228744550160085
43856624239377702107746492579014275563017167813144052693763385569
75239252588060279466314599314734680953729093269435217987689840619
0740089242444014302.

As experiments showed, our method for the computation of $\sum_{x \in \mathfrak{p} \cap \mathbb{Z}^d} h(x)$ is very efficient for this small dimension, compared to other available softwares. Furthermore, as Example 36 shows, the Ehrhart polynomial is written as a sum of canonical contributions of all faces, once the scalar product is fixed. We will come back soon to the computational and complexity aspects of this problem for higher dimensions.

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