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Local Exact Controllability for the One-Dimensional Compressible Navier–Stokes Equation — Source link

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Local exact controllability for the 1-D compressible Navier-Stokes equation

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1 Introduction

In this article, we consider the compressible Navier-Stokes equation in one space dimension in a bounded domain $(0,L)$:

$$\begin{cases} \partial_t \rho_{\mathcal{S}} + \partial_x(\rho_{\mathcal{S}} u_{\mathcal{S}}) = 0 \text{ in } (0,T)\times(0,L), \\ \rho_{\mathcal{S}}(\partial_t u_{\mathcal{S}} + u_{\mathcal{S}}\partial_x u_{\mathcal{S}}) - \nu \partial_{xx} u_{\mathcal{S}} + \partial_x p(\rho_{\mathcal{S}}) = 0 \text{ in } (0,T)\times(0,L). \end{cases} \tag{1.1}$$

Here $\rho_{\mathcal{S}}$ is the density, $u_{\mathcal{S}}$ the velocity and p denotes the pressure, which follows the standard law:

$$p(\rho_{\mathcal{S}}) = c_p \rho_{\mathcal{S}}^{\gamma}, \tag{1.2}$$

for some constants $c_p > 0$ and $\gamma \geq 1$. This law is the classical one when considering isentropic flows ($\gamma = 1.4$ for perfect gases) or isothermal flows ($\gamma = 1$). We also impose the initial data:

$$(\rho_{\mathcal{S}}, u_{\mathcal{S}})_{|t=0} = (\rho_0, u_0) \text{ in } (0,L), \tag{1.3}$$

Let us emphasize that the boundary conditions do not appear in the equation (1.1), as frequently happens when controlling hyperbolic equations like the equation of the density. They will be used as the controls on the system. Our goal is to prove the local exact controllability to constant states $(\overline{\rho}, \overline{u})$, which of course satisfy (1.1), when the velocity part of the target does not vanish. To be more precise, given $(\overline{\rho}, \overline{u}) \in \mathbb{R}_+^* \times \mathbb{R}^*$, we want to prove that, for (ρ_0, u_0) close enough to $(\overline{\rho}, \overline{u})$, one can find a solution of (1.1) with initial data (1.3) connecting the initial state to the target $(\overline{\rho}, \overline{u})$ in some time T.

The goal of this article is to prove the following result.

Theorem 1.1. *Let $\overline{u} \in \mathbb{R}^*$ and $\overline{\rho} \in \mathbb{R}_+^*$. Let $T > 0$ satisfy*

$$T > \frac{L}{|\overline{u}|}. \tag{1.4}$$

Then there exists $\kappa > 0$ such that, for any $u_0 \in H^3(0,L)$ and $\rho_0 \in H^3(0,L)$ such that

$$\|u_0 - \overline{u}\|_{H^3(0,L)} + \|\rho_0 - \overline{\rho}\|_{H^3(0,L)} < \kappa, \tag{1.5}$$

there exists a solution $(\rho_{\mathcal{S}}, u_{\mathcal{S}})$ of (1.1)–(1.3) satisfying

$$(\rho_{\mathcal{S}}, u_{\mathcal{S}})(T) = (\overline{\rho}, \overline{u}) \text{ in } (0,L). \tag{1.6}$$

Besides, the controlled trajectory satisfies $\rho_{\mathcal{S}} \in H^1((0,T)\times(0,L))$ and $u_{\mathcal{S}} \in H^1((0,T); L^2(0,L)) \cap L^2((0,T); H^2(0,L))$.

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Remark 1.2. *It is likely that we can reduce the regularity asked on the initial data. However, as can be seen in the proof, our method requires information on the second derivative of ρ_S, which can be obtained using third derivatives of u_S.*

Remark 1.3. *Conditions $\overline{u} \neq 0$ and $T > L/|\overline{u}|$ appear natural if we want the velocity u to stay close to $\overline{u}$ as, for example, the waves of density which travel at velocity $\overline{u}$ have to reach the boundary (where the control acts) before time T. Actually, if we consider the linearized system around $(\overline{\rho}, 0)$, it appears that the density is not controllable as can be seen in [20]. But this does not necessarily imply that the nonlinear system is not controllable as the numerous examples of use of the so-called return method [3, 9, 5] show.*

Theorem 1.1 appears to be the first controllability result concerning a compressible and viscous fluid except for the recent result by Amosova [2], which deals with a controllability problem concerning compressible viscous fluids in dimension 1. In this paper, the author considers the equation in Lagrangian coordinates, with zero boundary condition for the velocity on the boundaries of the interval and an interior control on the velocity equation. She proves a result of local exact controllability to trajectories for the velocity, provided that the initial density is already on the "targeted trajectory". Our result differs because:

- We consider boundary controls for both equations, but have no assumption on the initial density except the smallness of $\overline{\rho} - \rho_0$,
- We suppose $\overline{u} \neq 0$ and obtain a local exact controllability result for both the density and the velocity to $(\overline{\rho}, \overline{u})$,
- The change of variable between Lagrangian and Eulerian coordinates (which consists in taking a primitive of the density as a new space variable) does not leave the domain (or the control zone) invariant.

Let us now give more references on control results for fluids.

Controllability problems for incompressible fluids have been extensively studied in the recent years. In [3], Coron obtained a global exact controllability result for Euler equations in the 2 dimensional case and Glass extended in [9] this result to the 3 dimensional case. Concerning incompressible Navier-Stokes equations and related systems, Fursikov and Imanuvilov gave in [8] the first local exact controllability result for boundary conditions on the normal velocity and on the curl. Then Imanuvilov in [14] gave a local result for the no-slip Dirichlet boundary conditions and this result was extended by Fernandez-Cara, Guerrero, Imanuvilov and Puel in [7]. Let us also mention the results and method of [11] where a fictitious control is introduced and which can be applied to coupled systems like Boussinesq system. Global controllability is here an open question and it is also the case for incompressible Navier-Stokes equations, except for controls acting on the whole boundary (see Coron and Fursikov [4]). For Burgers equation in 1-D Guerrero and Imanuvilov gave in [12] a counterexample for global controllability whereas for 2-D Burgers equations, the situation is more complex and Imanuvilov and Puel in [15] proved global controllability for a special geometry and gave a counterexample for another one.

Controllability problems have also been considered in the context inviscid compressible fluids. In dimension one, since the compressible Euler equation is a hyperbolic system of conservation laws, the general result of Li and Rao [16] applies to it and proves a local controllability result of classical solutions (of class C^1). For further results in this context, see the book [17] and the references therein. A local exact controllability result for the one dimensional isentropic Euler equation in the context of weak entropy solutions was established by Glass in [10]. Let us also mention a result of approximate controllability in the 3-dimensional case by means of a finite number of modes, see Nersisyan [19].

The rest of the paper is devoted to the proof of Theorem 1.1. In Section 2, we describe the structure of an operator, connected to a controllability problem which is still nonlinear but not as severely as the original one and whose fixed point will give a solution to the controllability problem. In fact we introduce a decoupling which gives a linear controllability problem for the velocity u and, once u is given, a linear controllability problem for the density ρ. In Section 3, we describe how we solve the part of the linear controllability problem concerning the velocity. In Section 4, we describe how we solve the part concerning the density. In Section 5, we prove that the operator that we constructed admits a fixed point, proving Theorem 1.1. Finally, the appendix gives the details of some tedious computations and some comments on the Cauchy problem for (1.1)–(1.3).

For the rest of the paper, we will assume, without loss of generality, that

$$\overline{u} > 0.$$

It is just a matter of using the change of coordinates $x \to L - x$.

2 Main steps of the proof of Theorem 1.1

2.1 Reformulation

The general idea of the construction is to build an operator whose fixed points will give a solution of the controllability problem. It is based on the resolution of controllability problems for suitable approximations of equation (1.1) near the trajectory $(\overline{\rho}, \overline{u})$.

In our fixed point argument, it will be convenient to work within a class of functions vanishing at time $t = 0$. Therefore, to take the initial data into account, we extend (ρ_0, u_0) into smooth functions on $\mathbb{R}$, still denoted the same, such that $(\rho_0 - \overline{\rho}, u_0 - \overline{u})$ vanish outside $(-1, L+1)$, in such a way that we still have

$$\|\rho_0 - \overline{\rho}\|_{H^3(\mathbb{R})} + \|u_0 - \overline{u}\|_{H^3(\mathbb{R})} < C\kappa, \tag{2.1}$$

for some constant $C > 0$ depending on L only.

We then define (ρ_{in}, u_{in}) as the solution of

$$\begin{cases} \partial_t \rho_{in} + \partial_x((\overline{\rho} + \rho_{in})(\overline{u} + u_{in})) = 0 \text{ in } [0,T] \times \mathbb{R}, \\ (\overline{\rho} + \rho_{in})(\partial_t u_{in} + (\overline{u} + u_{in})\partial_x u_{in}) - \nu \partial_{xx} u_{in} + p'(\overline{\rho} + \rho_{in})\partial_x \rho_{in} = 0 \text{ in } [0,T] \times \mathbb{R}, \end{cases} \tag{2.2}$$

with initial data

$$\rho_{in}(0) = \rho_0 - \overline{\rho} \text{ and } u_{in}(0) = u_0 - \overline{u} \quad \text{ on } \mathbb{R}. \tag{2.3}$$

The existence of (ρ_{in}, u_{in}) is given in the next proposition, which is a direct consequence of a paper by Matsumura and Nishida [18] (see also [13] for a related result).

Proposition 2.1. *Set $(\overline{\rho}, \overline{u}) \in \mathbb{R}_+^* \times \mathbb{R}$ and $T > 0$. There exists $\kappa, K > 0$ such that, for any $u_0 \in \overline{u} + H^3(\mathbb{R})$ and $\rho_0 \in \overline{\rho} + H^3(\mathbb{R})$ satisfying (2.1), there exists a solution (ρ_{in}, u_{in}) in $L^\infty(0,T; H^3(\mathbb{R})) \cap W^{1,\infty}(0,T; H^2(\mathbb{R})) \times L^\infty(0,T; H^3(\mathbb{R})) \cap W^{1,\infty}(0,T; H^1(\mathbb{R}))$ of (2.2)-(2.3), satisfying:*

$$\|\rho_{in}\|_{L^\infty(0,T;H^3(\mathbb{R})) \cap W^{1,\infty}(0,T;H^2(\mathbb{R}))} + \|u_{in}\|_{L^\infty(0,T;H^3(\mathbb{R})) \cap W^{1,\infty}(0,T;H^1(\mathbb{R}))} \leq K \left(\|\rho_0 - \overline{\rho}\|_{H^3(\mathbb{R})} + \|u_0 - \overline{u}\|_{H^3(\mathbb{R})} \right). \tag{2.4}$$

We give some explanations on Proposition 2.1 in Appendix 6.3.

As a consequence of Proposition 2.1, we will be able to suppose that ρ_{in} and u_{in} are suitably small by choosing initial data (ρ_0, u_0) sufficiently close to $(\overline{\rho}, \overline{u})$. To express this in a convenient manner, we introduce

$$R_{in} := \|\rho_{in}\|_{L^\infty(0,T;W^{2,\infty}(\mathbb{R})) \cap W^{1,\infty}(0,T;W^{1,\infty}(\mathbb{R}))} + \|u_{in}\|_{L^\infty(0,T;W^{2,\infty}(\mathbb{R})) \cap W^{1,\infty}(0,T;L^\infty(\mathbb{R}))}, \tag{2.5}$$

which we will be able to consider small when taking κ small enough in (1.5). In particular it will be systematically supposed to satisfy:

$$R_{in} \leq \min\left\{1, \frac{\overline{u}}{4}, \frac{\overline{\rho}}{4}\right\}. \tag{2.6}$$

We can now reformulate the problem as follows. First, recall that T has been chosen large enough so that (1.4) holds. We can thus introduce $T_0 > 0$ such that

$$T_0 \in \left(0, \frac{1}{4}\right) \quad \text{and} \quad 10T_0 < T - \frac{L}{\overline{u}}. \tag{2.7}$$

Now we choose a smooth cut-off function Λ such that

$$\Lambda : [0,T] \to [0,1], \quad \Lambda(t) = \begin{cases} 1 & \text{for } t \in [0, T_0], \\ 0 & \text{for } t \in [2T_0, T], \end{cases} \tag{2.8}$$

and set

$$\rho = \rho_{\mathcal{S}} - \overline{\rho} - \Lambda \rho_{in} \quad \text{and} \quad u = u_{\mathcal{S}} - \overline{u} - \Lambda u_{in}. \tag{2.9}$$

Then our goal is to show that there exists a solution (ρ, u) of

$$\partial_t \rho + (\overline{u} + u + \Lambda u_{in})\partial_x \rho + \overline{\rho}\partial_x u + \frac{\overline{\rho}}{\nu} p'(\overline{\rho})\rho = f(\rho, u) \text{ in } [0,T] \times (0,L), \tag{2.10}$$

$$(\overline{\rho} + \Lambda \rho_{in})(\partial_t u + \overline{u}\partial_x u) - \nu \partial_{xx} u = g(\rho, u) \text{ in } [0,T] \times (0,L), \tag{2.11}$$

where $f(\rho, u)$ and $g(\rho, u)$ are given as follows:

$$f(\rho, u) = -\Lambda' \rho_{in} + (\Lambda - \Lambda^2)\partial_x(\rho_{in} u_{in}) - \Lambda \partial_x(\rho_{in} u) - \Lambda \rho \partial_x u_{in} - \rho \partial_x u + \frac{\overline{\rho}}{\nu} p'(\overline{\rho})\rho \tag{2.12}$$

and

$$\begin{aligned} g(\rho, u) = &- (\overline{\rho} + \Lambda \rho_{in})\Lambda' u_{in} - (p'(\overline{\rho} + \Lambda \rho_{in}) - p'(\overline{\rho} + \rho_{in}))\Lambda \partial_x \rho_{in} \\ &+ \rho_{in}\partial_t u_{in}(\Lambda - \Lambda^2) + \rho_{in}\overline{u}\partial_x u_{in}(\Lambda - \Lambda^2) + \overline{\rho} u_{in}\partial_x u_{in}(\Lambda - \Lambda^2) + \rho_{in} u_{in} \partial_x u_{in}(\Lambda - \Lambda^3) \\ &- \Lambda(\overline{\rho} + \Lambda \rho_{in})\partial_x(u u_{in}) - (\overline{\rho} + \Lambda \rho_{in}) u \partial_x u \\ &- \rho(\partial_t(\Lambda u_{in} + u) + (\overline{u} + \Lambda u_{in} + u)\partial_x(\Lambda u_{in} + u)) \\ &- (p'(\overline{\rho} + \Lambda \rho_{in} + \rho) - p'(\overline{\rho} + \Lambda \rho_{in}))\partial_x(\Lambda \rho_{in} + \rho) - p'(\overline{\rho} + \Lambda \rho_{in})\partial_x \rho, \end{aligned} \tag{2.13}$$

satisfying

$$\rho(0, \cdot) = \rho(T, \cdot) = 0 \text{ and } u(0, \cdot) = u(T, \cdot) = 0. \tag{2.14}$$

The lengthy computations leading to the expressions of f and g are detailed in Appendix 6.1 and 6.2 respectively.

Now to obtain a solution of (2.10)-(2.14), the idea is to find a fixed point to the application

$$F(\hat{\rho}, \hat{u}) := (\rho, u), \tag{2.15}$$

where (ρ, u) is a suitable solution of

$$\partial_t \rho + (\overline{u} + u + \Lambda u_{in})\partial_x \rho + \overline{\rho}\partial_x u + \frac{\overline{\rho}}{\nu} p'(\overline{\rho})\rho = f(\hat{\rho}, \hat{u}) \text{ in } [0,T] \times (0,L), \tag{2.16}$$

$$(\overline{\rho} + \Lambda \rho_{in})(\partial_t u + \overline{u}\partial_x u) - \nu \partial_{xx} u = g(\hat{\rho}, \hat{u}) \text{ in } [0,T] \times (0,L), \tag{2.17}$$

satisfying

$$\rho(0) = \rho(T) = 0 \text{ and } u(0) = u(T) = 0. \tag{2.18}$$

Of course, for this map to be well-defined, we need to make precise in which spaces the map F is defined and how the solution (ρ, u) is constructed. Indeed, the existence of such (ρ, u) is not obvious since it is a solution of a control problem that involves a heat type equation for the equation of the velocity and a transport equation for the density. Details on the construction of F will be given afterwards.

Besides, to complete the proof of Theorem 1.1, we will have to construct a convex set which is stable by F. This will be the main difficulty of the proof.

To simplify notations, we shall denote $f(\hat{\rho}, \hat{u})$ and $g(\hat{\rho}, \hat{u})$ simply by $\hat{f}$ and $\hat{g}$, respectively.

2.2 Construction of the fixed point map

The map F is constructed in two steps that will be detailed in the sections afterwards:

Step 1. Controlling u. For this to be done, we shall use a global Carleman estimate involving a weight function that will "travel" at velocity $\overline{u}$. This is the object of Section 3. The idea is very close to the control of the classical heat equation, except that one should be cautious about the fact that the weight functions travel along the characteristics.

Step 2. Constructing ρ. The idea is to use a backward solution vanishing at time T and a forward solution vanishing at time 0 and to glue them along the characteristics of the flow. This construction is very naive and natural, but the main difficulty is then to estimate the obtained ρ in an appropriate space. Such an estimate is derived in Section 4.

We finally end this section by giving a description of the fixed point space.

2.3 Description of the fixed point space

The space where F is to be defined is a weighted space connected to the aforementioned Carleman estimate. Let us first describe the weight function that we use. Set $\psi \in C^\infty(\mathbb{R};\mathbb{R}_+)$ such that

$$3 \leq \min_{[-5\overline{u}T,L]} \psi \leq \max_{[-5\overline{u}T,L]} \psi \leq 4, \qquad \max_{[-3\overline{u}T,L]} \psi' < 0 \quad \text{and} \quad \min_{[-5\overline{u}T,-4\overline{u}T]} \psi' > 0. \tag{2.19}$$

Then, let $\theta = \theta(t) \in C^2([0,T];\mathbb{R}_+)$ defined by

$$\theta(t) = \begin{cases} t & \text{in } [0, 2T_0] \\ 1 & \text{in } [3T_0, T-3T_0] \\ T-t & \text{in } [T-2T_0, T], \end{cases} \tag{2.20}$$

and being such that θ is increasing on $[0,3T_0]$ and decreasing on $[T-3T_0,T]$.

We then define the weight function $\varphi(t,x)$, depending on a positive parameter λ as follows

$$\varphi(t,x) = \frac{1}{\theta(t)}\left(e^{5\lambda} - e^{\lambda\psi(x-\overline{u}t)}\right), \ (t,x) \in (0,T)\times\mathbb{R}. \tag{2.21}$$

To this weight we associate the time-dependent function

$$\check{\varphi}(t) := \min_{x\in[0,L]} \varphi(t,x) = \varphi(t,0), \ t \in (0,T). \tag{2.22}$$

We also denote

$$\xi(t,x) = \frac{1}{\theta(t)} e^{\lambda\psi(x-\overline{u}t)}, \ (t,x) \in (0,T)\times\mathbb{R}. \tag{2.23}$$

Note in particular that for all $(t,x) \in (0,T)\times\mathbb{R}$,

$$\xi \geq 1. \tag{2.24}$$

The parameter λ used in the above definition of φ in (2.21) will always be assumed to be positive and larger than one, as well as the second parameter, called s, of the Carleman estimates:

$$s \geq 1 \ \text{ and } \ \lambda \geq 1. \tag{2.25}$$

We can now define the set on which F is to be defined. It depends on two constants

$$R_\rho \in (0,1) \quad \text{and} \quad R_u \in (0,1). \tag{2.26}$$

Given s, λ, R_ρ and R_u, we define the spaces X_{s,λ,R_ρ} and Y_{s,λ,R_u} as follows:

$$X_{s,\lambda,R_\rho} = \{\rho \ \text{ such that}$$
$$\left.\begin{array}{ll} \xi^{-1}e^{s\varphi}\rho \in L^2((0,T)\times(0,L)) & \text{with } \|\xi^{-1}e^{s\varphi}\rho\|_{L^2((0,T)\times(0,L))} \leq R_\rho, \\ \xi^{-3/2}e^{s\varphi}\partial_x\rho \in L^2((0,T)\times(0,L)) & \text{with } \|\xi^{-3/2}e^{s\varphi}\partial_x\rho\|_{L^2((0,T)\times(0,L))} \leq R_\rho, \\ \partial_t\rho \in L^2((0,T)\times(0,L)) & \text{with } \|\partial_t\rho\|_{L^2((0,T)\times(0,L))} \leq R_\rho, \\ e^{s\check{\varphi}/2}\rho \in L^\infty((0,T)\times(0,L)) & \text{with } \|e^{s\check{\varphi}/2}\rho\|_{L^\infty((0,T)\times(0,L))} \leq R_\rho, \\ e^{s\check{\varphi}/2}\partial_x\rho \in L^\infty((0,T);L^2(0,L)) & \text{with } \|e^{s\check{\varphi}/2}\partial_x\rho\|_{L^\infty((0,T);L^2(0,L))} \leq R_\rho, \\ (\xi^{-3/2}e^{s\varphi}\rho)(\cdot,0) \in L^2(0,T) & \text{with } \|\lambda^{1/2}[\xi^{-3/2}e^{s\varphi}\rho](\cdot,0)\|_{L^2(0,T)} \leq R_\rho, \\ (\xi^{-3/2}e^{s\varphi}\rho)(\cdot,L) \in L^2(0,T) & \text{with } \|\lambda^{1/2}[\xi^{-3/2}e^{s\varphi}\rho](\cdot,L)\|_{L^2(0,T)} \leq R_\rho, \end{array}\right\} \tag{2.27}$$

$$
\begin{aligned}
Y_{s,\lambda,R_u} = \{&u \text{ such that } u(t,L)=0,\ t\in(0,T),\\
&\left.\begin{array}{ll}
e^{s\varphi}u \in L^2((0,T)\times(0,L)) & \text{with } \|s^{3/2}\lambda^2 e^{s\varphi}u\|_{L^2((0,T)\times(0,L))} \le R_u,\\
\xi^{-1}e^{s\varphi}\partial_x u \in L^2((0,T)\times(0,L)) & \text{with } \|s^{1/2}\lambda\xi^{-1}e^{s\varphi}\partial_x u\|_{L^2((0,T)\times(0,L))} \le R_u,\\
\xi^{-2}e^{s\varphi}\partial_{xx} u \in L^2((0,T)\times(0,L)) & \text{with } \|s^{-1/2}\xi^{-2}e^{s\varphi}\partial_{xx} u\|_{L^2((0,T)\times(0,L))} \le R_u,\\
\xi^{-2}e^{s\varphi}\partial_t u \in L^2((0,T)\times(0,L)) & \text{with } \|s^{-1/2}\xi^{-2}e^{s\varphi}\partial_t u\|_{L^2((0,T)\times(0,L))} \le R_u.
\end{array}\right\}.
\end{aligned}
\tag{2.28}
$$

Let us remark that both sets are convex and compact for the topology of $L^2((0,T)\times(0,L))$. Therefore, if one shows that the map F maps $X_{s,\lambda,R_\rho}\times Y_{s,\lambda,R_u}$ into itself for convenient choices of parameters $s,\lambda\ge 1$ and R_ρ, R_u small enough, we are in position to prove the existence of a fixed point by Schauder's fixed point theorem, provided the continuity of F on $X_{s,\lambda,R_\rho}\times Y_{s,\lambda,R_u}$ endowed with the $(L^2((0,T)\times(0,L)))^2$-topology is proved. This will be the object of Section 5.

3 Controlling the velocity

In this section, we study the controllability problem attached to the parabolic equation (2.17). The term $\hat{g} = g(\hat{\rho},\hat{u})$ is considered as a source term. We are then in a familiar framework which can be handled using Carleman estimates and duality arguments.

3.1 Construction of u

For sake of simplicity, let us introduce the following general heat equation:

$$
a\,\partial_t u + b\,\partial_x u - \nu\partial_{xx}u = g \text{ in } (0,T)\times(0,L), \quad u(t,L)=0, \text{ in } (0,T) \tag{3.1}
$$

where $a(t,x)\in W^{1,\infty}((0,T)\times(0,L))$, $b(t,x)\in L^\infty(0,T;W^{1,\infty}(0,L))$ and

$$
\inf_{(t,x)\in(0,T)\times(0,L)}\{a(t,x)\} > 0. \tag{3.2}
$$

The source term g is assumed to be given.

We also introduce the following control problem: find a trajectory u of (3.1) such that

$$
u(0,\cdot) = u(T,\cdot) = 0 \text{ in } (0,L). \tag{3.3}
$$

Here again, the control is hidden in the lack of boundary condition at $x=0$ in (3.1).

To be more precise, we shall look for conditions on the source term g that guarantee the existence of a controlled trajectory of (3.1) satisfying (3.3).

Of course, this corresponds to the construction of the u-part of $F(\hat{\rho},\hat{u})$ with

$$
a(t,x) := \overline{\rho} + \Lambda\rho_{in}(t,x), \quad b(t,x) := (\overline{\rho}+\Lambda\rho_{in}(t,x))\overline{u} \quad \text{and} \quad g := \hat{g}, \tag{3.4}
$$

provided that R_{in} is small enough to guarantee that $a(t,x) := \overline{\rho}+\Lambda\rho_{in}(t,x)$ satisfies (3.2).

To solve this control problem, we first extend (3.1) on a larger domain, for instance $(-4\overline{u}T, L)$ and extend a and b on $(0,T)\times(-4\overline{u}T,L)$ such that the extensions, still denoted by a and b, satisfy:

$$
\begin{gathered}
a\in W^{1,\infty}((0,T)\times(-4\overline{u}T,L)), \quad b\in L^\infty(0,T;W^{1,\infty}(-4\overline{u}T,L)),\\
\|a\|_{W^{1,\infty}((0,T)\times(-4\overline{u}T,L))} + \|b\|_{L^\infty(0,T;W^{1,\infty}(-4\overline{u}T,L))} \le \beta,
\end{gathered}
\tag{3.5}
$$

and

$$
\inf_{(t,x)\in(0,T)\times(-4\overline{u}T,L)}\{a(t,x)\} \ge \alpha > 0. \tag{3.6}
$$

Note that, when constructing the u-part of $F(\hat{\rho},\hat{u})$, the coefficients a and b given by (3.4) are naturally defined on $(0,T)\times\mathbb{R}$ and then this extension argument is not really needed.

We shall also consider the extension of g by 0 in $(0,T)\times(-4\overline{u}T,0)$, that we still denote the same for sake of simplicity.

We then consider the following control problem: find a control v so that the solution u of

$$\begin{cases} a\,\partial_t u + b\,\partial_x u - \nu \partial_{xx} u = g + v\mathbf{1}_{(0,T)\times(-4\overline{u}T,-\overline{u}T)} \text{ in } (0,T)\times(-4\overline{u}T,L), \\ u(t,-4\overline{u}T) = u(t,L) = 0, \text{ in } (0,T), \\ u(0,\cdot) = 0 \text{ in } (-4\overline{u}T,L), \end{cases} \tag{3.7}$$

satisfies

$$u(T,\cdot) = 0 \text{ in } (-4\overline{u}T,L). \tag{3.8}$$

By restriction, solving (3.7)–(3.8) for some v yields a controlled trajectory u of (3.1) satisfying (3.3).

As it is classical now from the work of Fursikov-Imanuvilov [8], this issue can be addressed by proving a Carleman estimate for the adjoint of the heat operator under consideration.

Hence, setting

$$P_{a,b} := a\,\partial_t + b\,\partial_x - \nu\partial_{xx} \text{ on } (0,T)\times(-4\overline{u}T,L),$$
$$\text{with Dirichlet boundary conditions at } x = -4\overline{u}T \text{ and } x = L, \tag{3.9}$$

we are going to derive a Carleman estimate for the operator

$$P^*_{a,b} = -\partial_t(a\,\cdot\,) - \partial_x(b\,\cdot\,) - \nu\partial_{xx} \text{ on } (0,T)\times(-4\overline{u}T,L),$$
$$\text{with Dirichlet boundary conditions at } x = -4\overline{u}T \text{ and } x = L, \tag{3.10}$$

with observation on $(0,T)\times(-4\overline{u}T,-\overline{u}T)$.

We are now in position to state the following Carleman estimate:

Theorem 3.1. *Assume that a and b satisfy conditions* (3.5) *and* (3.6).

There exist $s_0 \geq 1$, $\lambda_0 \geq 1$ and $C > 0$, all depending on β and α, such that for all $s \geq s_0$ and $\lambda \geq \lambda_0$, any smooth function $z : [0,T]\times[-4\overline{u}T,L] \to \mathbb{R}$ satisfying $z(t,L) = 0$ and $z(t,-4\overline{u}T) = 0$ satisfies

$$\begin{aligned} s^3\lambda^4 \iint_{(0,T)\times(-4\overline{u}T,L)} \xi^3 e^{-2s\varphi}|z|^2 + s\lambda^2 \iint_{(0,T)\times(-4\overline{u}T,L)} \xi e^{-2s\varphi}|\partial_x z|^2 \\ + \frac{1}{s}\iint_{(0,T)\times(-4\overline{u}T,L)} \frac{1}{\xi} e^{-2s\varphi}\left(|\partial_{xx} z|^2 + |\partial_t z|^2\right) \\ \leq C \iint_{(0,T)\times(-4\overline{u}T,L)} e^{-2s\varphi}\left|P^*_{a,b} z\right|^2 + Cs^3\lambda^4 \iint_{(0,T)\times(-4\overline{u}T,-\overline{u}T)} \xi^3 e^{-2s\varphi}|z|^2. \end{aligned} \tag{3.11}$$

The proof of Theorem 3.1 is given in Subsection 3.2. It is mainly classical (see Fursikov and Imanuvilov [8]), except for what concerns the Carleman weight. Indeed, the classical Carleman weight usually takes the form

$$\tilde{\varphi}(t,x) = \frac{1}{t(T-t)}\left(e^{5\lambda} - e^{\lambda\psi(x)}\right).$$

The differences between the weight (2.21) and the classical one are then the following: the weight function θ (see (2.20)) is constant during a certain interval of time and the variable in the function ψ is $x - \overline{u}t$ instead of x. This latest point somehow reflects the hyperbolic nature of the equation of ρ and the fact that it is important to take into account the transport at velocity $\overline{u}$. See also [1] for a similar Carleman weight function.

As we shall see later, this particular form of the weight function will allow us to estimate the controlled density in weighted functional spaces, which is a crucial step to develop the fixed point argument.

Relying on this Carleman estimate, we develop a duality argument using Theorem 3.1 and the method developed by Fursikov and Imanuvilov [8]. Let us assume that $g : (0,T)\times(-4\overline{u}T,L) \to \mathbb{R}$ satisfies

$$\iint_{(0,T)\times(-4\overline{u}T,L)} \frac{1}{\xi^3} e^{2s\varphi}|g|^2 < \infty. \tag{3.12}$$

We then introduce the functional J defined by

$$J(z) = \frac{1}{2} \iint_{(0,T)\times(-4\overline{u}T,L)} e^{-2s\varphi} |P^*_{a,b} z|^2 + \frac{s^3\lambda^4}{2} \iint_{(0,T)\times(-4\overline{u}T,-\overline{u}T)} \xi^3 e^{-2s\varphi} |z|^2 - \iint_{(0,T)\times(-4\overline{u}T,L)} gz, \quad (3.13)$$

among all z belonging to the space $\overline{\mathcal{Y}}$ defined as the completion with respect to the norm

$$\|z\|^2_{obs} = \iint_{(0,T)\times(-4\overline{u}T,L)} e^{-2s\varphi} |P^*_{a,b} z|^2 + s^3\lambda^4 \iint_{(0,T)\times(-4\overline{u}T,-\overline{u}T)} \xi^3 e^{-2s\varphi} |z|^2$$

of the space of functions in $C^\infty([0,T]\times[-4\overline{u}T,L])$ vanishing at $x = L$ and $x = -4\overline{u}T$. Note that the fact that $\|\cdot\|_{obs}$ is a norm is a consequence of the Carleman estimate (3.11).

Observe that thanks to (3.11) and (3.12), the linear map

$$z \mapsto \iint_{(0,T)\times(-4\overline{u}T,L)} gz,$$

is well-defined and continuous on $\overline{\mathcal{Y}}$. Moreover, one easily checks that J is strictly convex and coercive on the space $\overline{\mathcal{Y}}$ endowed with the norm $\|\cdot\|_{obs}$.

Therefore, it has a unique minimizer Z, for which, due to the coercivity of J, we have

$$\|Z\|^2_{obs} \le C \frac{1}{s^3\lambda^4} \iint_{(0,T)\times(-4\overline{u}T,L)} \frac{1}{\xi^3} e^{2s\varphi} |g|^2. \quad (3.14)$$

Besides, as a minimizer of J, Z satisfies, for all $z \in \overline{\mathcal{Y}}$,

$$\iint_{(0,T)\times(-4\overline{u}T,L)} e^{-2s\varphi} P^*_{a,b} z P^*_{a,b} Z + s^3\lambda^4 \iint_{(0,T)\times(-4\overline{u}T,-\overline{u}T)} \xi^3 e^{-2s\varphi} zZ = \iint_{(0,T)\times(-4\overline{u}T,L)} gz. \quad (3.15)$$

Consequently, if we set

$$u := e^{-2s\varphi} P^*_{a,b} Z, \qquad v := -s^3\lambda^4\xi^3 e^{-2s\varphi} Z, \quad (3.16)$$

it is not difficult to see that u satisfies, in the transposition sense,

$$a\,\partial_t u + b\,\partial_x u - \nu\partial_{xx} u = g + v\mathbf{1}_{(-4\overline{u}T,-\overline{u}T)} \text{ in } (0,T)\times(-4\overline{u}T,L), \quad u(\cdot,-4\overline{u}T) = 0 = u(\cdot,L), \quad (3.17)$$

Besides, due to (3.14), we get:

$$\iint_{(0,T)\times(-4\overline{u}T,L)} e^{2s\varphi}|u|^2 + \frac{1}{s^3\lambda^4} \iint_{(0,T)\times(-4\overline{u}T,-\overline{u}T)} \frac{1}{\xi^3} e^{2s\varphi}|v|^2 \le C \frac{1}{s^3\lambda^4} \iint_{(0,T)\times(-4\overline{u}T,L)} \frac{1}{\xi^3} e^{2s\varphi}|g|^2. \quad (3.18)$$

Of course, thanks to the exponential blow up of the weight function φ as $t \to 0$ and as $t \to T$, (see (2.21)), this implies that $u(0,\cdot) = u(T,\cdot) = 0$ in $(-4\overline{u}T,L)$.

Moreover, by uniqueness of the solution in the transposition sense, since the source term belongs to $L^2((0,T)\times(-4\overline{u}T,L))$, u is a strong solution of (3.17).

With all these ingredients, we can obtain the following (the detailed proof is available in Section 3.3):

Theorem 3.2. *Given $g \in L^2((0,T)\times(-4\overline{u}T,L))$ satisfying* (3.12) *and a,b satisfying* (3.5)–(3.6)*, there exists a constant C depending only on β and α, such that for all $s \ge s_0$ and $\lambda \ge \lambda_0$, there exists a solution u of* (3.7)-(3.8) *and such that*

$$s^3\lambda^4 \iint_{(0,T)\times(-4\overline{u}T,L)} e^{2s\varphi}|u|^2 + s\lambda^2 \iint_{(0,T)\times(-4\overline{u}T,L)} \frac{1}{\xi^2} e^{2s\varphi}|\partial_x u|^2$$
$$+\frac{1}{s}\iint_{(0,T)\times(-4\overline{u}T,L)} \frac{1}{\xi^4} e^{2s\varphi}\left(|\partial_t u|^2 + |\partial_{xx} u|^2\right) \leq C \iint_{(0,T)\times(-4\overline{u}T,L)} \frac{1}{\xi^3} e^{2s\varphi}|g|^2. \quad (3.19)$$

Remark 3.3. *In this theorem and in the sequel, s_0 and λ_0 stand for two sufficiently large constants which may change from line to line.*

The u-part of $F(\hat{\rho},\hat{u})$ is given by this u for a, b, g as indicated above:

$$a(t,x) := \overline{\rho} + \Lambda\rho_{in}(t,x) \text{ in } (0,T)\times(-4\overline{u}T,L), \quad b(t,x) := (\overline{\rho} + \Lambda\rho_{in}(t,x))\overline{u} \text{ in } (0,T)\times(-4\overline{u}T,L) \quad (3.20)$$

with source term

$$g := \begin{cases} \hat{g} \text{ in } (0,T)\times(0,L), \\ 0 \text{ in } (0,T)\times(-4\overline{u}T,0). \end{cases} \quad (3.21)$$

Of course, it is easy to check that a and b satisfy (3.5)–(3.6) with $\beta = 3\overline{\rho u}$ and $\alpha = \overline{\rho}/2$ by taking $R_{in} \leq \overline{\rho}/2$. However, the fact that this g satisfies assumption (3.12) is not obvious. We will see later in Section 5 Lemma 5.1 that this can be proved using the fact that $(\hat{\rho},\hat{u}) \in X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$.

Remark 3.4. *Note that the u-part of the control constructed above is known at $x = L$ and corresponds to the following boundary conditions for $u_{\mathcal{S}}$:*

$$u_{\mathcal{S}}(t,L) = \overline{u} + \Lambda u_{in}.$$

This could very likely be reduced to $u_{\mathcal{S}}(t,L) = \overline{u}$ provided there is a regular solution (ρ_{in}, u_{in}) of (2.2) with $u_{in}(t,L) = 0$ for $t \in [0,T]$, which would of course entail strong compatibility conditions on (ρ_0, u_0) at $x = L$.

In Section 3.2 we prove Theorem 3.1 and we establish Theorem 3.2 in Section 3.3. For later use, in Section 3.4, we also prove interpolation estimates to get estimates on the boundary, i.e. at $x = 0$ and $x = L$, and in $L^\infty((0,T)\times(0,L))$ and $L^1((0,T);W^{1,\infty}(0,L))$ norms.

To simplify notations, in the following, we set

$$Q_T = (0,T)\times(-4\overline{u}T,L).$$

3.2 Proof of Theorem 3.1

Let us begin this section by giving some properties of the Carleman weights. Thanks to the structure of φ (see (2.21)–(2.23)) simple computations give

$$\partial_x\varphi(t,x) = -\lambda\psi'(x-\overline{u}t)\xi(t,x), \quad \partial_{xx}\varphi(t,x) = -\lambda^2(\psi'(x-\overline{u}t))^2\xi - \lambda\psi''(x-\overline{u}t)\xi(t,x).$$

Thus, due to (2.19) for some $\lambda_0 > 0$, there exists a constant $c_* > 0$ such that for $\lambda \geq \lambda_0$, ,

$$\begin{cases} -\partial_{xx}\varphi(t,x) \geq c_*\lambda^2\xi(t,x), \\ -\partial_x((\partial_x\varphi)^3)(t,x) \geq c_*\lambda^4\xi^3(t,x), \end{cases} \quad \forall (t,x) \in [0,T]\times[-2\overline{u}T,L], \quad (3.22)$$

whereas we obviously have for some constant C independent of λ

$$\begin{cases} |-\partial_{xx}\varphi(t,x)| \leq C\lambda^2\xi(t,x), \\ |-\partial_x((\partial_x\varphi)^3)(t,x)| \leq C\lambda^4\xi^3(t,x), \end{cases} \quad \forall (t,x) \in [0,T]\times[-4\overline{u}T,L]. \quad (3.23)$$

One also easily checks that

$$\begin{cases} \partial_x\varphi(t,-4\overline{u}T) \leq 0, \\ \partial_x\varphi(t,L) \geq 0, \end{cases} \quad \forall t \in [0,T]. \quad (3.24)$$

Besides

$$\partial_t \varphi = -\frac{\theta'}{\theta}\varphi + \lambda\, \overline{u}\, \psi'(x - \overline{u}t)\, \xi.$$

But $\varphi \le \theta\xi^2$ and $\lambda \le C\xi$ for some C independent of $\lambda > 0$ (recall that $\psi \ge 3$). We thus obtain the bound

$$|\partial_t \varphi| \le C\xi^2 \tag{3.25}$$

and, similarly,

$$|\partial_{tx}\varphi| \le C\lambda\xi^2, \quad |\partial_{tt}\varphi| \le C\xi^3. \tag{3.26}$$

for some constant C independent of λ. In the following, we shall always assume that $\lambda \ge \lambda_0$ so that formulas (3.22)–(3.26) hold.

Proof of Theorem 3.1. Let z be a smooth function on $[0,T] \times [-4\overline{u}T, L]$ satisfying $z(t, -4\overline{u}T) = z(t, L) = 0$ and set $h = a\,\partial_t z + \nu \partial_{xx} z$.

We then introduce the function $w = e^{-s\varphi} z$. Due to the blow up of the function φ as $t \to 0$ and $t \to T$, w satisfies

$$(\xi^2 w)(0, x) = (\xi^2 w)(T, x) = 0, \quad x \in (-4\overline{u}T, L),$$

still with the boundary conditions $w(t, -4\overline{u}T) = w(t, L) = 0$.

Then, setting

$$P_0 w = e^{-s\varphi}(a\,\partial_t + \nu\partial_{xx})(e^{s\varphi} w),$$

we have that $P_0 w = h e^{-s\varphi}$. We then compute the operator $P_0 w$:

$$P_0 w = P_1 w + P_2 w + R w,$$

where

$$\begin{cases} P_1 w = a\,\partial_t w + 2\nu\, s\,\partial_x\varphi\,\partial_x w, \\ P_2 w = \nu\,\partial_{xx} w + s\, a\,\partial_t\varphi\, w + \nu\, s^2\,(\partial_x\varphi)^2 w, \\ R w = \nu\, s\,\partial_{xx}\varphi\, w. \end{cases}$$

Let us now compute the mean value of $P_1 w\, P_2 w$. Integrations by parts in space and time yield

$$\nu \iint_{Q_T} a\,\partial_t w\,\partial_{xx} w = \frac{1}{2}\nu \iint_{Q_T} \partial_t a\,|\partial_x w|^2 - \nu \iint_{Q_T} \partial_x a\,\partial_t w\,\partial_x w$$

and

$$\iint_{Q_T} a\,\partial_t w (s\, a\,\partial_t\varphi\, w + \nu s^2\,(\partial_x\varphi)^2\, w) = -\frac{s}{2}\iint_{Q_T} \partial_t(a^2\,\partial_t\varphi)\,|w|^2 - \nu\frac{s^2}{2}\iint_{Q_T} \partial_t(a\,(\partial_x\varphi)^2)|w|^2.$$

Then, we integrate by parts in space and we obtain

$$2\,\nu^2\, s \iint_{Q_T} \partial_x\varphi\,\partial_x w\,\partial_{xx} w = -\nu^2\, s\iint_{Q_T} \partial_{xx}\varphi\,|\partial_x w|^2 + \nu^2\, s\int_0^T \partial_x\varphi(t,x)\,|\partial_x w(t,x)|^2\Big|_{x=-4\overline{u}T}^{x=L},$$

and

$$\begin{aligned} 2\,\nu\, s\iint_{Q_T} \partial_x\varphi\,\partial_x w\,(s\, a\,\partial_t\varphi\, w + \nu\, s^2 (\partial_x\varphi)^2\, w) = {} & -\nu\, s^2 \iint_{Q_T} \partial_x(a\,\partial_x\varphi\,\partial_t\varphi)|w|^2 \\ & - \nu^2\, s^3 \iint_{Q_T} \partial_x((\partial_x\varphi)^3)|w|^2. \end{aligned}$$

Combining all these computations, we get

$$\begin{aligned} \iint_{Q_T} P_1 w\, P_2 w \;=\; & \frac{1}{2}\nu\iint_{Q_T} \partial_t a\,|\partial_x w|^2 - \nu\iint_{Q_T} \partial_x a\,\partial_t w\,\partial_x w \\ & -\frac{s}{2}\iint_{Q_T} \partial_t(a^2\,\partial_t\varphi)|w|^2 - \nu\frac{s^2}{2}\iint_{Q_T} \partial_t(a\,(\partial_x\varphi)^2)|w|^2 \\ & -\nu^2\, s\iint_{Q_T} \partial_{xx}\varphi\,|\partial_x w|^2 - \nu\, s^2\iint_{Q_T} \partial_x(a\,\partial_x\varphi\,\partial_t\varphi)|w|^2 \\ & -\nu^2\, s^3\iint_{Q_T} \partial_x((\partial_x\varphi)^3)|w|^2 + \nu^2\, s\int_0^T \partial_x\varphi(t,x)\,|\partial_x w(t,x)|^2\Big|_{x=-4\overline{u}T}^{x=L}. \end{aligned}$$

Recalling the fact that $a \in W^{1,\infty}((0,T)\times(0,L))$ and the formulas (3.22)–(3.26), we obtain, for λ and s large enough,

$$\begin{aligned}
&\iint_{Q_T} P_1 w\, P_2 w \\
&\geq c_* s^3\lambda^4 \iint_{Q_T} \xi^3\,|w|^2 + c_* s\lambda^2 \iint_{Q_T} \xi|\partial_x w|^2 - C\iint_{Q_T} |\partial_t w||\partial_x w| \\
&\quad - C\left(s^3\lambda^4 \iint_{(0,T)\times(-4\overline{u}T,-2\overline{u}T)} \xi^3|w|^2 + s\lambda^2 \iint_{(0,T)\times(-4\overline{u}T,-2\overline{u}T)} \xi|\partial_x w|^2\right) \\
&\geq c_* s^3\lambda^4 \iint_{Q_T} \xi^3|w|^2 + \frac{c_*}{2} s\lambda^2 \iint_{Q_T} \xi|\partial_x w|^2 - \frac{C}{s\lambda^2}\iint_{Q_T} \frac{1}{\xi}|\partial_t w|^2 \\
&\quad - C\left(s^3\lambda^4 \iint_{(0,T)\times(-4\overline{u}T,-2\overline{u}T)} \xi^3|w|^2 + s\lambda^2 \iint_{(0,T)\times(-4\overline{u}T,-2\overline{u}T)} \xi|\partial_x w|^2\right),
\end{aligned} \tag{3.27}$$

for some $c_* > 0$ and $C > 0$, both independent of $s \geq s_1$ and $\lambda \geq \lambda_1$.

Now, we estimate the $L^2(L^2)$-norm of $\partial_t w$. In order to do that, we observe that

$$|\partial_t w| \leq C|P_1 w| + Cs\lambda\xi|\partial_x w|.$$

Therefore

$$\frac{1}{s}\iint_{Q_T} \frac{1}{\xi}|\partial_t w|^2 \leq C\iint_{Q_T} |P_1 w|^2 + Cs\lambda^2 \iint_{Q_T} \xi|\partial_x w|^2. \tag{3.28}$$

Similarly, from the definition of P_2 we get

$$\frac{1}{s}\iint_{Q_T} \frac{1}{\xi}|\partial_{xx} w|^2 \leq C\iint_{Q_T} |P_2 w|^2 + Cs^3\lambda^4 \iint_{Q_T} \xi^3|w|^2, \tag{3.29}$$

for s large enough.

But, using the fact that $P_1 w + P_2 w = he^{-s\varphi} - Rw$,

$$\iint_{Q_T} |P_1 w|^2 + \iint_{Q_T} |P_2 w|^2 + \iint_{Q_T} P_1 w\, P_2 w \leq 2\iint_{Q_T} |h|^2 e^{-2s\varphi} + 2\iint_{Q_T} |Rw|^2,$$

and therefore estimates (3.27)–(3.28)–(3.29) yield, for $s \geq s_2$ and $\lambda \geq \lambda_2$ and for some constant $C > 0$ independent of s and λ

$$\begin{aligned}
s^3\lambda^4 \iint_{Q_T} \xi^3|w|^2 + s\lambda^2 \iint_{Q_T} \xi|\partial_x w|^2 + \frac{1}{s}\iint_{Q_T} \frac{1}{\xi}\left(|\partial_{xx} w|^2 + |\partial_t w|^2\right) + \iint_{Q_T} (|P_1 w|^2 + |P_2 w|^2) \\
\leq C\iint_{Q_T} |h|^2 e^{-2s\varphi} + C\iint_{Q_T} |Rw|^2 + \frac{C}{s\lambda^2}\iint_{Q_T} \frac{1}{\xi}|\partial_t w|^2 \\
+ C\left(s^3\lambda^4 \iint_{(0,T)\times(-4\overline{u}T,-2\overline{u}T)} \xi^3|w|^2 + s\lambda^2 \iint_{(0,T)\times(-4\overline{u}T,-2\overline{u}T)} \xi|\partial_x w|^2\right).
\end{aligned}$$

Of course, $|Rw| \leq Cs\lambda^2\xi|w|$ and thus this term can be easily absorbed by the left hand side: for some constant C independent of s and λ, for $s \geq s_3$ and $\lambda \geq \lambda_3$,

$$\begin{aligned}
s^3\lambda^4 \iint_{Q_T} \xi^3|w|^2 + s\lambda^2 \iint_{Q_T} \xi|\partial_x w|^2 + \frac{1}{s}\iint_{Q_T} \frac{1}{\xi}\left(|\partial_{xx} w|^2 + |\partial_t w|^2\right) \\
+ \iint_{Q_T} (|P_1 w|^2 + |P_2 w|^2) \leq C\iint_{Q_T} |h|^2 e^{-2s\varphi} \\
+ C\left(s^3\lambda^4 \iint_{(0,T)\times(-4\overline{u}T,-2\overline{u}T)} \xi^3|w|^2 + s\lambda^2 \iint_{(0,T)\times(-4\overline{u}T,-2\overline{u}T)} \xi|\partial_x w|^2\right).
\end{aligned} \tag{3.30}$$

Now, we introduce a nonnegative smooth function χ that vanishes identically on $(-\overline{u}T, L)$ and that takes value one on $(-4\overline{u}T, -2\overline{u}T)$ and we compute $P_2 w\, \xi\, \chi^2\, w$:

$$\iint_{Q_T} P_2 w\, \xi\, \chi^2\, w = \nu \iint_{Q_T} \partial_{xx} w\, \xi\, \chi^2\, w + \iint_{Q_T} s\, a\, \partial_t \varphi\, \xi\, \chi^2 |w|^2 + \nu s^2 \iint_{Q_T} (\partial_x \varphi)^2 \xi\, \chi^2 |w|^2.$$

But

$$\nu \iint_{Q_T} \partial_{xx} w\, \xi\, \chi^2 w \quad = \quad -\nu \iint_{Q_T} |\partial_x w|^2 \xi\, \chi^2 + \frac{\nu}{2} \iint_{Q_T} |w|^2 \partial_{xx}(\xi \chi^2),$$

and therefore,

$$\nu \iint_{Q_T} |\partial_x w|^2 \xi\, \chi^2 = - \iint_{Q_T} P_2 w\, \xi\, \chi^2 w + \frac{\nu}{2} \iint_{Q_T} |w|^2 \partial_{xx}(\xi \chi^2) + \iint_{Q_T} s\, a\, \partial_t \varphi\, \xi\, \chi^2 |w|^2 + \nu s^2 \iint_{Q_T} (\partial_x \varphi)^2 \xi\, \chi^2 |w|^2. \quad (3.31)$$

Using

$$\left| \iint_{Q_T} P_2 w\, \xi\, \chi^2 w \right| \leq \frac{C}{s^{3/2} \lambda^2} \left(\iint_{Q_T} |P_2 w|^2 + s^3 \lambda^4 \iint_{Q_T} \xi^2 \chi^4 |w|^2 \right),$$

we thus obtain

$$\begin{aligned} &\nu \iint_{(-4\overline{u}T, -2\overline{u}T)} |\partial_x w|^2 \xi\, \chi^2 \\ &\leq \frac{C}{s^{3/2} \lambda^2} \iint_{Q_T} |P_2 w|^2 + \left(C s^{3/2} \lambda^2 + C \lambda^2 + C s + C s^2 \lambda^2 \right) \iint_{(0,T) \times (-4\overline{u}T, -\overline{u}T)} \xi^3 |w|^2 \\ &\leq \frac{C}{s^{3/2} \lambda^2} \iint_{Q_T} |P_2 w|^2 + C s^2 \lambda^2 \iint_{(0,T) \times (-4\overline{u}T, -\overline{u}T)} \xi^3 |w|^2, \end{aligned}$$

for $s, \lambda \geq 1$.

From (3.30), we then obtain

$$s^3 \lambda^4 \iint_{Q_T} \xi^3 |w|^2 + s \lambda^2 \iint_{Q_T} \xi |\partial_x w|^2 + \frac{1}{s} \iint_{Q_T} \frac{1}{\xi} \left(|\partial_{xx} w|^2 + |\partial_t w|^2 \right) + \iint_{Q_T} \left(|P_1 w|^2 + |P_2 w|^2 \right) \leq C \iint_{Q_T} |h|^2 e^{-2s\varphi} + C s^3 \lambda^4 \iint_{(0,T) \times (-4\overline{u}T, -\overline{u}T)} \xi^3 |w|^2. \quad (3.32)$$

We now recall that $z = w e^{s\varphi}$, and thus

$$\begin{aligned} |z| e^{-s\varphi} &\leq |w|, & |\partial_x z| e^{-s\varphi} &\leq C(|\partial_x w| + s \lambda \xi |w|), \\ |\partial_t z| e^{-s\varphi} &\leq C(|\partial_t w| + s \xi^2 |w|), & |\partial_{xx} z| e^{-s\varphi} &\leq C(|\partial_{xx} w| + s \lambda \xi |\partial_x w| + s^2 \lambda^2 \xi^2 |w|). \end{aligned}$$

Of course, this immediately yields

$$s^3 \lambda^4 \iint_{Q_T} \xi^3 e^{-2s\varphi} |z|^2 + s \lambda^2 \iint_{Q_T} \xi\, e^{-2s\varphi} |\partial_x z|^2 + \frac{1}{s} \iint_{Q_T} \frac{1}{\xi} e^{-2s\varphi} \left(|\partial_{xx} z|^2 + |\partial_t z|^2 \right) \leq C \iint_{Q_T} |a\, \partial_t z + \nu \partial_{xx} z|^2 e^{-2s\varphi} + C s^3 \lambda^4 \iint_{(0,T) \times (-4\overline{u}T, -\overline{u}T)} \xi^3 |z|^2 e^{-2s\varphi}.$$

Taking s large enough, the lower order terms $(\partial_t a) z + \partial_x (b z)$ can be absorbed by the left hand side due to conditions (3.5) on a, b, thus yielding (3.11). □

3.3 Proof of Theorem 3.2.

Proof of Theorem 3.2. Let us multiply the equation (3.17) by $u\,\xi^{-2}\,e^{2s\varphi}$:

$$\iint_{Q_T} (a\,\partial_t u + b\,\partial_x u - \nu \partial_{xx} u)\, u\, e^{2s\varphi} \frac{1}{\xi^2} = \iint_{Q_T} (g + v\mathbf{1}_{(-4\overline{u}T, -\overline{u}T)})\, u\, e^{2s\varphi} \frac{1}{\xi^2}. \tag{3.33}$$

Note that this computation and the ones afterwards are mainly formal since the weight function $\theta(t)$ vanishes at time $t = 0$ and $t = T$. To make these computations rigorous, one could introduce, for $\varepsilon > 0$,

$$\theta_\varepsilon(t) = \begin{cases} \theta(t+\varepsilon) \text{ for } t \in (0, 3T_0), \\ 1 \text{ for } t \in (3T_0, T - 3T_0) \\ \theta(t-\varepsilon) \text{ for } t \in (T - 3T_0, T), \end{cases} \quad \text{and} \quad \varphi_\varepsilon(t,x) = \frac{1}{\theta_\varepsilon(t)} \left(e^{5\lambda} - e^{\lambda \psi(x - \overline{u}t)} \right). \tag{3.34}$$

Then, all the computations below can be done with φ_ε instead of φ and passing to the limit $\varepsilon \to 0$, we recover the desired estimates. We will not detail this passage to the limit below, which is left to the readers.

Let us now come back to identity (3.34) and estimate each term in it:

$$\left| \iint_{Q_T} a\,\partial_t u\, u\, e^{2s\varphi} \frac{1}{\xi^2} \right| = \left| -\frac{1}{2} \iint_{Q_T} |u|^2 \partial_t \left(a\, e^{2s\varphi} \frac{1}{\xi^2} \right) \right| \leq Cs \iint_{Q_T} |u|^2 e^{2s\varphi},$$

$$\left| \iint_{Q_T} b\,\partial_x u\, u\, e^{2s\varphi} \frac{1}{\xi^2} \right| = \left| -\frac{1}{2} \iint_{Q_T} |u|^2 \partial_x \left(b\, e^{2s\varphi} \frac{1}{\xi^2} \right) \right| \leq Cs\lambda \iint_{Q_T} \frac{1}{\xi} |u|^2 e^{2s\varphi},$$

and

$$\begin{aligned} &\left| \iint_{Q_T} (g + v\mathbf{1}_{(-4\overline{u}T, -\overline{u}T)}) u e^{2s\varphi} \frac{1}{\xi^2} \right| \\ &\leq \frac{C}{s^{3/2}\lambda^2} \iint_{Q_T} (|g|^2 + |v\mathbf{1}_{(-4\overline{u}T, -\overline{u}T)}|^2) e^{2s\varphi} \frac{1}{\xi^3} + Cs^{3/2}\lambda^2 \iint_{Q_T} \frac{1}{\xi} |u|^2 e^{2s\varphi}, \end{aligned}$$

for $s, \lambda \geq 1$. Therefore we focus on the term

$$-\nu \iint_{Q_T} \partial_{xx} u\, u e^{2s\varphi} \frac{1}{\xi^2} = \nu \iint_{Q_T} |\partial_x u|^2 e^{2s\varphi} \frac{1}{\xi^2} - \frac{\nu}{2} \iint_{Q_T} |u|^2 \partial_{xx} \left(e^{2s\varphi} \frac{1}{\xi^2} \right),$$

which yields

$$\nu \iint_{Q_T} |\partial_x u|^2 e^{2s\varphi} \frac{1}{\xi^2} \leq \left| \nu \iint_{Q_T} \partial_{xx} u\, u e^{2s\varphi} \frac{1}{\xi^2} \right| + Cs^2\lambda^2 \iint_{Q_T} |u|^2 e^{2s\varphi},$$

for $s, \lambda \geq 1$. Combining the above estimates and the identity (3.33), we obtain

$$\iint_{Q_T} |\partial_x u|^2 e^{2s\varphi} \frac{1}{\xi^2} \leq Cs^2\lambda^2 \iint_{Q_T} |u|^2 e^{2s\varphi} + \frac{C}{s^{3/2}\lambda^2} \iint_{Q_T} (|g|^2 + |v\mathbf{1}_{(-4\overline{u}T, -\overline{u}T)}|^2) e^{2s\varphi} \frac{1}{\xi^3}, \tag{3.35}$$

and, according to (3.18),

$$s\lambda^2 \iint_{Q_T} |\partial_x u|^2 e^{2s\varphi} \frac{1}{\xi^2} \leq C \iint_{Q_T} |g|^2 e^{2s\varphi} \frac{1}{\xi^3}, \tag{3.36}$$

Now, multiply (3.17) by $\partial_t u\, e^{2s\varphi}/\xi^4$:

$$\begin{aligned} &\iint_{Q_T} a |\partial_t u|^2 e^{2s\varphi} \frac{1}{\xi^4} + \iint_{Q_T} b\,\partial_x u\, \partial_t u\, e^{2s\varphi} \frac{1}{\xi^4} - \nu \iint_{Q_T} \partial_{xx} u\, \partial_t u\, e^{2s\varphi} \frac{1}{\xi^4} \\ &= \iint_{Q_T} (g + v\mathbf{1}_{(-4\overline{u}T, -\overline{u}T)}) \partial_t u\, e^{2s\varphi} \frac{1}{\xi^4}. \end{aligned} \tag{3.37}$$

We then have

$$\inf_{(t,x)} \{a\} \iint_{Q_T} |\partial_t u|^2 e^{2s\varphi} \frac{1}{\xi^4} \leq \iint_{Q_T} a |\partial_t u|^2 e^{2s\varphi} \frac{1}{\xi^4},$$

whereas the second and the last terms in (3.37) can be handled as follows:

$$\left|\iint_{Q_T} b\,\partial_x u\,\partial_t u\,e^{2s\varphi}\frac{1}{\xi^4}\right| \leq C\left(s\iint_{Q_T}\frac{1}{\xi^3}|\partial_x u|^2 e^{2s\varphi} + \frac{1}{s}\iint_{Q_T}\frac{1}{\xi^5}|\partial_t u|^2 e^{2s\varphi}\right),$$

$$\begin{aligned}&\left|\iint_{Q_T}(g+v\mathbf{1}_{(-4\overline{u}T,-\overline{u}T)})\,\partial_t u\,e^{2s\varphi}\frac{1}{\xi^4}\right|\\ &\leq Cs\iint_{Q_T}(|g|^2+|v\mathbf{1}_{(-4\overline{u}T,-\overline{u}T)}|^2)e^{2s\varphi}\frac{1}{\xi^3}+\frac{C}{s}\iint_{Q_T}|\partial_t u|^2 e^{2s\varphi}\frac{1}{\xi^5}.\end{aligned}$$

We then focus on the cross term:

$$-\nu\iint_{Q_T}\partial_{xx}u\,\partial_t u\,e^{2s\varphi}\frac{1}{\xi^4} = -\frac{\nu}{2}\iint_{Q_T}|\partial_x u|^2\partial_t\left(\frac{e^{2s\varphi}}{\xi^4}\right)+\nu\iint_{Q_T}\partial_x u\partial_t u\partial_x\left(\frac{e^{2s\varphi}}{\xi^4}\right),$$

which implies that

$$\begin{aligned}\left|-\nu\iint_{Q_T}\partial_{xx}u\,\partial_t u\,e^{2s\varphi}\frac{1}{\xi^4}\right| &\leq Cs\iint_{Q_T}|\partial_x u|^2\frac{e^{2s\varphi}}{\xi^2}+Cs\lambda\iint_{Q_T}|\partial_x u||\partial_t u|\frac{e^{2s\varphi}}{\xi^3}\\ &\leq C(s+s^2\lambda^2)\iint_{Q_T}|\partial_x u|^2\frac{e^{2s\varphi}}{\xi^2}+\frac{\inf_{(t,x)}\{a\}}{2}\iint_{Q_T}|\partial_t u|^2\frac{e^{2s\varphi}}{\xi^4}.\end{aligned} \tag{3.38}$$

Putting the above estimates in (3.37) and choosing s large enough, we obtain

$$\inf_{(t,x)}\{a\}\iint_{Q_T}|\partial_t u|^2 e^{2s\varphi}\frac{1}{\xi^4}\leq Cs\iint_{Q_T}(|g|^2+|v\mathbf{1}_{(-4\overline{u}T,-\overline{u}T)}|^2)e^{2s\varphi}\frac{1}{\xi^3}+Cs^2\lambda^2\iint_{Q_T}|\partial_x u|^2\frac{e^{2s\varphi}}{\xi^2},$$

which, due to (3.36), implies

$$\frac{1}{s}\iint_{Q_T}|\partial_t u|^2 e^{2s\varphi}\frac{1}{\xi^4}\leq C\iint_{Q_T}|g|^2 e^{2s\varphi}\frac{1}{\xi^3}. \tag{3.39}$$

Finally, to obtain an estimate on $\partial_{xx}u$, we use the equation (3.17):

$$\frac{1}{s}|\partial_{xx}u|^2 e^{2s\varphi}\frac{1}{\xi^4}\leq\frac{C}{s}|\partial_t u|^2 e^{2s\varphi}\frac{1}{\xi^4}+\frac{C}{s}(|g|^2+|v\mathbf{1}_{(-4\overline{u}T,-\overline{u}T)}|^2)e^{2s\varphi}\frac{1}{\xi^4}.$$

Integrating this estimate and using (3.18) and (3.39), we easily obtain

$$\frac{1}{s}\iint_{Q_T}|\partial_{xx}u|^2 e^{2s\varphi}\frac{1}{\xi^4}\leq C\iint_{Q_T}|g|^2 e^{2s\varphi}\frac{1}{\xi^3}. \tag{3.40}$$

This concludes the proof of Theorem 3.2. □

3.4 Interpolation estimates

In the sequel, it will be important to have estimates on the value of u and $\partial_x u$ at $x=0$ and $x=L$. In order to do this, we will use the following result:

Proposition 3.5. *There exists a constant C independent of $s,\lambda\geq 1$ and R_u, such that for all $w\in Y_{s,\lambda,R_u}$,*

$$s^2\lambda^3\int_0^T|w(t,0)|^2 e^{2s\varphi(t,0)}\frac{1}{\xi(t,0)}\,dt\leq CR_u^2. \tag{3.41}$$

and

$$\lambda\left(\int_0^T|\partial_x w(t,0)|^2 e^{2s\varphi(t,0)}\frac{1}{\xi^3(t,0)}\,dt+\int_0^T|\partial_x w(t,L)|^2 e^{2s\varphi(t,L)}\frac{1}{\xi^3(t,L)}\,dt\right)\leq CR_u^2. \tag{3.42}$$

Proof of Proposition 3.5. We focus on the estimate of w at $x = 0$, the other ones being completely similar.

Let $\eta = \eta(x)$ be a smooth positive function on $(0, L)$ that takes value 1 close to $x = 0$ and vanishing at $x = 1$. Then

$$\begin{aligned} s^2\lambda^3 \int_0^T |w(t,0)|^2 e^{2s\varphi(t,0)} \frac{1}{\xi(t,0)} &= -s^2\lambda^3 \iint_{(0,T)\times(0,L)} \partial_x \left(\eta |w|^2 e^{2s\varphi} \frac{1}{\xi}\right) \\ = -s^2\lambda^3 \iint_{(0,T)\times(0,L)} |w|^2 \partial_x \left(\eta e^{2s\varphi}\frac{1}{\xi}\right) &- 2s^2\lambda^3 \iint_{(0,T)\times(0,L)} w\,\partial_x w\,\eta e^{2s\varphi}\frac{1}{\xi} \\ \le C s^3\lambda^4 \iint_{(0,T)\times(0,L)} |w|^2 e^{2s\varphi} &+ C s^2\lambda^3 \iint_{(0,T)\times(0,L)} |w||\partial_x w| e^{2s\varphi}\frac{1}{\xi} \\ \le C s^3\lambda^4 \iint_{(0,T)\times(0,L)} |w|^2 e^{2s\varphi} &+ C s\lambda^2 \iint_{(0,T)\times(0,L)} |\partial_x w|^2 e^{2s\varphi}\frac{1}{\xi^2}, \end{aligned}$$

for $s, \lambda \ge 1$.

The proof of (3.42) follows the same lines and is left to the reader. ☐

We will also need estimates on some norms of the elements of Y_{s,λ,R_u}.

Lemma 3.6. *There exists a constant C independent of $s, \lambda \ge 1$ and R_u such that for any $w \in Y_{s,\lambda,R_u}$,*

$$\|w\|_{L^\infty((0,T)\times(0,L))} \le C R_u \exp\left(-\frac{s}{2}(e^{5\lambda} - e^{4\lambda})\right) \le C R_u, \tag{3.43}$$

$$\|w\|_{L^1((0,T);W^{1,\infty}(0,L))} \le C R_u \exp\left(-\frac{s}{2}(e^{5\lambda} - e^{4\lambda})\right) \le C R_u, \tag{3.44}$$

Proof of Lemma 3.6. Estimate (3.43) follows from the fact that $w \in Y_{s,\lambda,R_u}$ implies $s^{-1/2}\xi^{-2}e^{s\varphi}w$ lies in the ball of $H^1(0,T;L^2(0,L)) \cap L^2(0,T;H^2(0,L))$ of radius R_u. Hence it belongs to the ball of $L^\infty((0,T)\times(0,L))$ with radius CR_u, where the constant comes from the injection

$$H^1(0,T;L^2(0,L)) \cap L^2(0,T;H^2(0,L)) \to L^\infty((0,T)\times(0,L)).$$

We then remark that there exists a constant C such that for all $s, \lambda \ge 1$,

$$s^{-1/2}\xi^{-2}e^{s\varphi} \ge C \exp\left(\frac{s}{2}(e^{5\lambda} - e^{4\lambda})\right).$$

This concludes the proof of (3.43).

The proof of (3.44) follows the same line, by using the continuous injection of $L^2(0,T;H^2(0,L))$ into $L^1(0,T;W^{1,\infty}(0,L))$. ☐

4 Controlling ρ

In this section, we construct a solution of the controllability problem attached to the ρ-part of the map F defined in (2.16). We assume that u has been constructed as in Section 3 and belongs to some Y_{s,λ,R_u}.

4.1 Constructing ρ

As we will see below, the construction of the controlled density ρ is very natural. Indeed, the main remark consists in the fact that the density is transported among the flow of velocity $\overline{u} + u + \Lambda u_{in}$, which is close to $\overline{u}$. Hence, we will construct a forward solution ρ_f of (2.16), a backward solution ρ_b of (2.16) and glue these two solutions according to the characteristics of the flow. To be more precise, we introduce ρ_f defined by

$$\begin{cases} \partial_t \rho_f + (\overline{u} + u + \Lambda u_{in})\partial_x \rho_f + \overline{\rho}\partial_x u + \frac{\overline{\rho}}{\nu} p'(\overline{\rho})\rho_f = \hat{f} \text{ in } [0,T]\times(0,L), \\ \rho_f(0,x) = 0 \text{ in } (0,L), \\ \rho_f(t,0) = 0 \text{ in } (0,T), \end{cases} \tag{4.1}$$

and ρ_b defined by

$$\begin{cases} \partial_t \rho_b + (\overline{u} + u + \Lambda u_{in})\partial_x \rho_b + \overline{\rho}\partial_x u + \frac{\overline{\rho}}{\nu} p'(\overline{\rho})\rho_b = \hat{f} \text{ in } [0,T]\times(0,L), \\ \rho_b(T,x) = 0 \text{ in } (0,L), \\ \rho_b(t,L) = 0 \text{ in } (0,T). \end{cases} \tag{4.2}$$

For equations (4.1) and (4.2) to be well-posed, first remark that $\overline{u} + u + \Lambda u_{in}$ is in $L^1(0,T;W^{1,\infty}(0,L))$ so the transport equation is easily solvable by characteristics. But one should also guarantee that $\overline{u} + u + \Lambda u_{in}$ is positive on the space boundaries $(0,T)\times\{0,L\}$. Actually, we will need an even more restrictive condition on that quantity.

In this section, we will assume that u belongs to Y_{s,λ,R_u} for some parameter s, λ, R_u to be determined. And we will also assume that R_u and R_{in} are small enough so that the $L^\infty((0,T)\times(0,L))$-bound of u given by Lemma 3.6 and the smallness of u_{in} (coming from (2.5)–(2.6)) imply

$$\overline{u} + u + \Lambda u_{in} \geq \frac{L}{T - 8T_0} \text{ in } [0,T]\times[0,L], \tag{4.3}$$

where T_0 is defined in (2.7). Note that this choice can be done independently of s and λ thanks to Lemma 3.6.

Then we introduce the flow associated to the transport equation of ρ, given by

$$\partial_t X(t,\tau,a) = \overline{u} + u(t, X(t,\tau,a)) + \Lambda u_{in}(t, X(t,\tau,a)), \quad X(\tau,\tau,a) = a. \tag{4.4}$$

For later use, it is convenient to introduce extensions of u and Λu_{in} to $(t,x)\in[0,T]\times\mathbb{R}$ (with comparable norms), so that we can consider the flow $X(t,\tau,a)$ to be defined on $[0,T]\times[0,T]\times\mathbb{R}$.

Due to (4.3), it is easy to check that there exists

$$[a_0, b_0] \subset (-\infty, 0),$$

such that

$$X(T,0,a_0) > L, \ X(\cdot,0,a_0)^{-1}(L) \leq T - 3T_0 \ \text{ and } \ X(\cdot,0,b_0)^{-1}(0) \geq 3T_0,$$

see Figure 1.

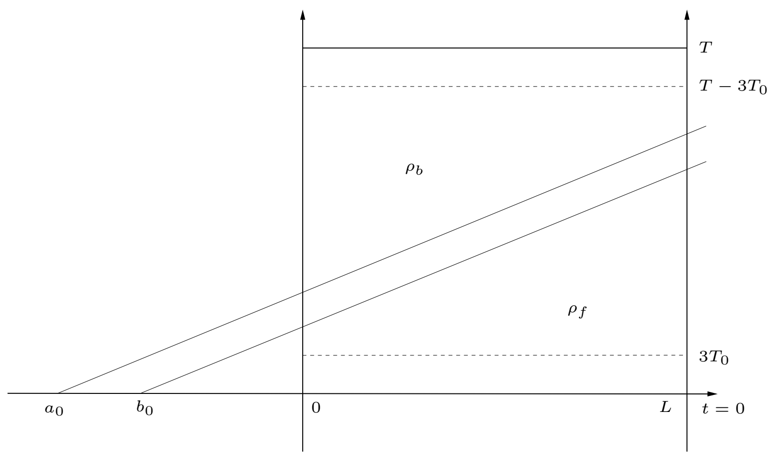

Figure 1: Geometric setting on a_0, b_0. The straight lines represent the lines $t \mapsto (t, a_0 + \overline{u}t)$ and $t \mapsto (t, b_0 + \overline{u}t)$, which approximate the flow X.

We take $\eta \in C^\infty(\mathbb{R};\mathbb{R})$ such that

$$\eta(a) = 1 \text{ for } a < a_0 \ \text{ and } \ \eta(a) = 0 \text{ for } a > b_0. \tag{4.5}$$

Remark that for R_u small enough, a_0, b_0 and η can be taken to be independent of u. We set

$$\rho(t,x) = \rho_f(t,x)(1 - \eta(X(0,t,x))) + \rho_b(t,x)\eta(X(0,t,x)). \tag{4.6}$$

Easy computations then show that ρ solves the equation of conservation of mass (2.16), and that

$$\rho(0,x) = \rho(T,x) = 0 \text{ in } [0,L],$$

due to the time boundary conditions on ρ_f and ρ_b and (4.5).

Since this ρ is admissible for the control problem corresponding to the ρ-part of F, we choose this ρ.

4.2 A new unknown μ

An important argument concerning the control of ρ consists in introducing a new quantity which we will denote by μ. This quantity will be easier to handle in the estimates.

To explain why this new unknown is relevant, let us consider for a few lines the following simplified form of (2.10)–(2.11):

$$\partial_t\tilde{\rho} + \overline{u}\partial_x\tilde{\rho} + \partial_x\tilde{u} = 0 \text{ in } [0,T]\times(0,L), \tag{4.7}$$

$$\partial_t\tilde{u} + \overline{u}\partial_x\tilde{u} - \partial_{xx}\tilde{u} + \partial_x\tilde{\rho} = g \text{ in } [0,T]\times(0,L), \tag{4.8}$$

where g belongs to $C_c^\infty((0,T)\times(0,L))$, our goal being to find a trajectory $(\tilde{\rho},\tilde{u})$ such that $(\tilde{\rho}(0),\tilde{u}(0)) = (\tilde{\rho}(T),\tilde{u}(T)) = 0$.

Of course, a natural strategy would be to use a Carleman estimate directly on the parabolic part to estimate $\tilde{u}$ in terms of $\tilde{\rho}$ and a corresponding weighted estimate for $\tilde{\rho}$ in terms of $\tilde{u}$, but we did not manage to find a suitable set for a fixed point argument. However, if one introduces $\tilde{\mu} = \tilde{u} + \partial_x\tilde{\rho}$, one easily checks that (4.7)–(4.8) could be reduced to

$$\partial_t\tilde{\mu} + \overline{u}\partial_x\tilde{\mu} = 0 \text{ in } [0,T]\times(0,L), \tag{4.9}$$

$$\partial_t\tilde{u} + \overline{u}\partial_x\tilde{u} - \partial_{xx}\tilde{u} + (\tilde{\mu}-\tilde{u}) = g \text{ in } [0,T]\times(0,L). \tag{4.10}$$

Here, the coupling between the two equations is somewhat weaker and the freedom on the choices of the parameters s and λ in the Carleman estimates of Section 3 will hence allow us to set up a fixed point strategy.

Differentiating (2.16) with respect to x and multiplying it by the constant $\nu/\overline{\rho}$, we have

$$\overline{\rho}\left(\partial_t\left(\frac{\nu}{\overline{\rho}^2}\partial_x\rho\right) + (\overline{u}+u+\Lambda u_{in})\partial_x\left(\frac{\nu}{\overline{\rho}^2}\partial_x\rho\right)\right) + \overline{\rho}\left(p'(\overline{\rho})\frac{\overline{\rho}}{\nu} + \partial_x(u+\Lambda u_{in})\right)\left(\frac{\nu}{\overline{\rho}^2}\partial_x\rho\right) + \nu\partial_{xx}u = \frac{\nu}{\overline{\rho}}\partial_x\hat{f}. \tag{4.11}$$

Of course, since both ρ_f and ρ_b satisfy (2.16), they also satisfy equation (4.11).

Besides, adding it to the equation of u (see (2.17)), one easily obtains that

$$\mu_f(t,x) = u + \frac{\nu}{\overline{\rho}^2}\partial_x\rho_f, \quad \mu_b(t,x) = u + \frac{\nu}{\overline{\rho}^2}\partial_x\rho_b. \tag{4.12}$$

both solve the equation

$$\overline{\rho}(\partial_t\mu + (\overline{u}+u+\Lambda u_{in})\partial_x\mu) + \overline{\rho}\left(p'(\overline{\rho})\frac{\overline{\rho}}{\nu} + \partial_x(u+\Lambda u_{in})\right)\mu = \frac{\nu}{\overline{\rho}}\partial_x\hat{f} + \hat{g} - \Lambda\rho_{in}(\partial_t u + \overline{u}\partial_x u) + \overline{\rho}\partial_x\left[u(u+\Lambda u_{in})\right] + p'(\overline{\rho})\frac{\overline{\rho}^2}{\nu}u. \tag{4.13}$$

or, equivalently

$$\partial_t\mu + (\overline{u}+u+\Lambda u_{in})\partial_x\mu + k\mu = h, \tag{4.14}$$

where the source term h is defined by

$$\overline{\rho}h := \frac{\nu}{\overline{\rho}}\partial_x\hat{f} + \hat{g} - \Lambda\rho_{in}(\partial_t u + \overline{u}\partial_x u) + \overline{\rho}\partial_x\left[u(u+\Lambda u_{in})\right] + p'(\overline{\rho})\frac{\overline{\rho}^2}{\nu}u,$$

and the potential term k is

$$k := p'(\overline{\rho})\frac{\overline{\rho}}{\nu} + \partial_x(u + \Lambda u_{in}). \tag{4.15}$$

Note that, to complete the equations (4.14), one should further introduce boundary conditions in space and time. From the definition of μ_f and μ_b in (4.12), one easily checks that the boundary conditions in time simply are

$$\mu_f(0,x) = 0 \text{ for } x \in (0,L), \qquad \mu_b(T,x) = 0 \text{ for } x \in (0,L), \tag{4.16}$$

whereas the boundary conditions in space are given by the equations (4.1)–(4.2) satisfied by ρ_f and ρ_b respectively:

$$\begin{aligned} \mu_f(t,0) &= m_f(t) := u(t,0) + \frac{\nu}{\overline{\rho}^2}\left(\frac{1}{\overline{u} + u(t,0) + \Lambda u_{in}(t,0)}\right)\left(\hat{f}(t,0) - \overline{\rho}\,\partial_x u(t,0)\right), & (4.17)\\ \mu_b(t,L) &= m_b(t) := \frac{\nu}{\overline{\rho}^2}\left(\frac{1}{\overline{u} + \Lambda u_{in}(t,L)}\right)\left(\hat{f}(t,L) - \overline{\rho}\,\partial_x u(t,L)\right), & (4.18)\end{aligned}$$

where we have used in (4.18) that the function u constructed in Section 3 vanishes at $x = L$.

Note that, due to the fact that $\rho_f(t,0) = \rho_b(t,L) = 0$, we have the following identities

$$\rho_f(t,x) := \frac{\overline{\rho}^2}{\nu}\int_0^x (\mu_f - u)(t,y)\,dy, \quad \rho_b(t,x) := -\frac{\overline{\rho}^2}{\nu}\int_x^L (\mu_b - u)(t,y)\,dy, \tag{4.19}$$

which will be used in the sequel.

Remark 4.1. *Note that ρ_f and ρ_b correspond to primitives of μ_f and μ_b respectively according to the formula* (4.12). *However, $\mu = u + \nu\partial_x\rho/\overline{\rho}^2$ is a priori different from $\mu_f(t,x)(1 - \eta(X(0,t,x))) + \mu_b(t,x)\eta(X(0,t,x))$.*

Our goal in the next subsections is to obtain suitable estimates on the functions μ_f, μ_b, ρ_f, ρ_b that we constructed.

4.3 Preliminaries: estimates on the flow

In order to estimate ρ, we will first need estimates on the flow X. In particular, the estimates measure how close X is to $(t,x) \mapsto x + t\overline{u}$ when R_{in} and R_u are small, and give consequences on the weight functions of Section 2.3 (since the Carleman weight is calibrated with respect to the straight flow $(t,x) \mapsto x + t\overline{u}$).

Lemma 4.2. *For all $(t,x) \in (0,T) \times (0,L)$ and $\tau \in (0,T)$ such that $X(\tau,t,x) \in (0,L)$,*

$$|(X(\tau,t,x) - \tau\overline{u}) - (x - t\overline{u})| \le C|\tau - t|\|u + \Lambda u_{in}\|_{L^\infty((0,T)\times(0,L))}. \tag{4.20}$$

Proof of Lemma 4.2. Let us define

$$\Gamma(\tau,t,x) = (X(\tau,t,x) - \tau\overline{u}) - (x - t\overline{u}).$$

As one immediately checks, $\Gamma(t,t,x) = 0$. Besides, $\Gamma(\tau,t,x)$ satisfies the equation

$$\begin{aligned} \frac{d\Gamma(\tau,t,x)}{d\tau} &= (\overline{u} + u(\tau, X(\tau,t,x)) + \Lambda u_{in}(\tau, X(\tau,t,,x))) - \overline{u}\\ &= u(\tau, X(\tau,t,x)) + \Lambda u_{in}(\tau, X(\tau,t,,x))\\ \Gamma(t,t,x) &= 0. \end{aligned}$$

Therefore,

$$\left|\frac{d\Gamma(\tau,t,x)}{d\tau}\right| \le \|u + \Lambda u_{in}\|_{L^\infty((0,T)\times(0,L))},$$

and estimate (4.20) immediately follows. □

In the following, we shall use the following simple identity on the Carleman weight, which comes from the design of the weight function in (2.21):

$$\varphi(\tau, x-(t-\tau)\overline{u}) \begin{cases} \geq \varphi(t,x) \text{ for all } (t,\tau) \text{ satisfying } 0<\tau\leq t\leq T-3T_0, \\ = \varphi(t,x) \text{ for all } (t,\tau) \text{ satisfying } 3T_0<\tau\leq t\leq T-3T_0. \end{cases} \tag{4.21}$$

Of course, when following the characteristic flow associated to $\overline{u}+u+\Lambda u_{in}$, these formula are not true anymore but we still obtain the following approximation lemma:

Lemma 4.3. *There exist constants $C_0>0$, $\lambda_0>0$, $s_0>0$ such that for all $p\in[-4,-2]$, for all $(t,x)\in(0,T-3T_0)\times(0,L)$, for all $\tau\leq t$ such that $X(\tau,t,x)\in(0,L)$, for all $\lambda\geq\lambda_0$ and $s\geq s_0$,*

$$p\log\left(\xi(\tau,X(\tau,t,x))\right)-2s\varphi(\tau,X(\tau,t,x))\leq p\log\left(\xi(t,x)\right)-2s\varphi(t,x) + C_0 s\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty((0,T)\times(0,L))}. \tag{4.22}$$

Proof of Lemma 4.3. This follows from an explicit computation of the difference and we shall prove the following equivalent form of (4.22):

$$2s(\varphi(\tau,X(\tau,t,x))-\varphi(t,x))+p\log\left(\frac{\xi(t,x)}{\xi(\tau,X(\tau,t,x))}\right)\geq -Cs\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty((0,T)\times(0,L))}. \tag{4.23}$$

First, for all $\tau\in(0,t)$ and $t\leq T-3T_0$,

$$\begin{aligned}\varphi(\tau,X(\tau,t,x))-\varphi(t,x) &= \frac{1}{\theta(\tau)}\left(e^{5\lambda}-e^{\lambda\psi(X(\tau,t,x)-\overline{u}\tau)}\right)-\frac{1}{\theta(t)}\left(e^{5\lambda}-e^{\lambda\psi(x-\overline{u}t)}\right)\\ &= \left(\frac{1}{\theta(\tau)}-\frac{1}{\theta(t)}\right)\left(e^{5\lambda}-e^{\lambda\psi(X(\tau,t,x)-\overline{u}\tau)}\right)\\ &\quad+\frac{1}{\theta(t)}\left(e^{\lambda\psi(x-\overline{u}t)}-e^{\lambda\psi(X(\tau,t,x)-\overline{u}\tau)}\right)\\ &= \left(\frac{\theta(t)}{\theta(\tau)}-1\right)\frac{1}{\theta(t)}\left(e^{5\lambda}-e^{\lambda\psi(X(\tau,t,x)-\overline{u}\tau)}\right)\\ &\quad+\frac{1}{\theta(t)}e^{\lambda\psi(X(\tau,t,x)-\overline{u}\tau)}(e^{\lambda(\psi(x-\overline{u}t)-\psi(X(\tau,t,x)-\overline{u}\tau))}-1).\end{aligned}$$

Using (2.19), Lemma 4.2, $\tau\leq t$ and $\exp(y)-1\geq y$, we thus obtain

$$\begin{aligned}\varphi(\tau,X(\tau,t,x))-\varphi(t,x) &\geq \left(\frac{\theta(t)}{\theta(\tau)}-1\right)\frac{1}{\theta(t)}\left(e^{5\lambda}-e^{\lambda\psi(X(\tau,t,x)-\overline{u}\tau)}\right)\\ &\quad-\frac{C}{\theta(t)}e^{4\lambda}\lambda t\|u+\Lambda u_{in}\|_{L^\infty((0,T)\times(0,L))}.\end{aligned}$$

Since $t\leq T-3T_0$, $t/\theta(t)$ is bounded:

$$\begin{aligned}\varphi(\tau,X(\tau,t,x))-\varphi(t,x) &\geq \left(\frac{\theta(t)}{\theta(\tau)}-1\right)\frac{1}{\theta(t)}\left(e^{5\lambda}-e^{\lambda\psi(X(\tau,t,x)-\overline{u}\tau)}\right)\\ &\quad-Ce^{4\lambda}\lambda\|u+\Lambda u_{in}\|_{L^\infty((0,T)\times(0,L))}.\end{aligned} \tag{4.24}$$

Let us emphasize that the first term in the right-hand side is positive for $\tau<t$.

We now focus on the estimate of $\log(\xi(t,x)/\xi(\tau,X(\tau,t,x)))$. According to the definition of ξ in (2.23),

$$\log\left(\frac{\xi(t,x)}{\xi(\tau,X(\tau,t,x))}\right)=\log\left(\frac{\theta(\tau)}{\theta(t)}\right)+\lambda\left(\psi(x-\overline{u}t)-\psi(X(\tau,t,x)-\overline{u}\tau)\right).$$

Of course, from (4.20), we immediately deduce that, for $p\in[-4,-2]$ and $\tau\leq t$,

$$\left|p\log\left(\frac{\xi(t,x)}{\xi(\tau,X(\tau,t,x))}\right)\right|\leq C\left(\frac{\theta(t)}{\theta(\tau)}-1\right)+C\lambda t\|u+\Lambda u_{in}\|_{L^\infty((0,T)\times(0,L))}, \tag{4.25}$$

where we used

$$\left|\log\left(\frac{\theta(\tau)}{\theta(t)}\right)\right|=\log\left(\frac{\theta(t)}{\theta(\tau)}\right)\leq\frac{\theta(t)}{\theta(\tau)}-1.$$

We then deduce (4.23) from (4.24) and (4.25) for s and λ large enough. □

Lemma 4.4. *There exist constants $C_0 > 0$, $\lambda_0 \geq 1$, $s_0 \geq 1$ such that for all $p \in [-4,-2]$, for all $(t,x) \in (0, T-3T_0) \times (0,L)$, for all $\lambda \geq \lambda_0$ and $s \geq s_0$,*

$$\int_{t^*(t,x)}^{t} \xi^p(\tau, X(\tau,t,x)) e^{-2s\varphi(\tau,X(\tau,t,x))} d\tau \leq t\xi^p(t,x) e^{-2s\varphi(t,x)} e^{C_0 s\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}}, \tag{4.26}$$

where $t^*(t,x)$ is defined as follows:

$$t^*(t,x) := \inf\left\{\tau_0 \in (0,t) \text{ such that } \forall \tau \in (\tau_0, t),\ X(\tau,t,x) \in (0,L)\right\}. \tag{4.27}$$

We also have

$$\xi^p(t^*(t,x), X(t^*(t,x),t,x)) e^{-2s\varphi(t^*(t,x),X(t^*(t,x),t,x))} \leq \xi^p(t,x) e^{-2s\varphi(t,x)} e^{C_0 s\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}}, \tag{4.28}$$

The time $t^*(t,x)$ corresponds to the entrance time in $(0,T)\times(0,L)$ of the line of the characteristic through (t,x). Accordingly,

$$t^*(t,x) = \begin{cases} 0 & \text{if } x \geq X(t,0,0), \\ X(\cdot,t,x)^{-1}(0) & \text{if } x \leq X(t,0,0). \end{cases}$$

Proof of Lemma 4.4. Taking the exponential of (4.22), we obtain, for all $\tau \leq t$,

$$\xi^p(\tau, X(\tau,t,x)) e^{-2s\varphi(\tau,X(\tau,t,x))} \leq \xi^p(t,x) e^{-2s\varphi(t,x)} \exp(C_0 s\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}).$$

This immediately yields (4.28) by taking $\tau = t^*(t,x)$ and (4.26) by integration between $t^*(t,x)$ and t. □

We now prove that, for t fixed, the map $x \in [0, \min\{X(t,0,0), L\}] \mapsto t^*(t,x)$ is a C^1-diffeomorphism:

Lemma 4.5. *For $t \in (0,T)$, the map*

$$t^*_t : x \in [0, \min\{X(t,0,0), L\}] \mapsto t^*(t,x)$$

is a C^1 diffeormorphism of bounded jacobian for

$$\|u + \Lambda u_{in}\|_{L^\infty} \leq \frac{\overline{u}}{2} \quad \text{and} \quad \partial_x(\Lambda u_{in} + u) \in L^1((0,T); L^\infty(0,L)).$$

We then have the estimate:

$$\frac{2}{3\overline{u}} \exp(-\|\partial_x(\Lambda u_{in}+u)\|_{L^1((0,T);L^\infty(0,L))}) \leq |\partial_x t^*_t(x)| \leq \frac{2}{\overline{u}} \exp(\|\partial_x(\Lambda u_{in}+u)\|_{L^1((0,T);L^\infty(0,L))}). \tag{4.29}$$

Proof. We rather study the inverse of t^*_t, which is deduced easily by the formula

$$X(t, t^*(t,x), 0) = x.$$

Hence we define

$$\tilde{x}_t : \tau \in [0,t] \mapsto X(t,\tau,0).$$

One easily checks that X satisfies

$$\frac{d}{dt}(\partial_\tau X(t,\tau,0)) = \partial_x(\Lambda u_{in} + u)(t, X(t,\tau,0)) \partial_\tau X(t,\tau,0), \tag{4.30}$$

whereas

$$\partial_\tau X(t,\tau,0)_{|t=\tau} = \frac{d}{d\tau}(X(\tau,\tau,0)) - \partial_t X(t,\tau,0)_{|t=\tau} = -(\overline{u} + \Lambda u_{in} + u)(\tau, 0).$$

Hence

$$\partial_\tau \tilde{x}_t(\tau) = -(\overline{u} + \Lambda u_{in} + u)(\tau, x) \exp\left(\int_\tau^t \partial_x(\Lambda u_{in}+u)(\tau', X(\tau',\tau,0)) \partial_\tau X(\tau',\tau,0)\, d\tau'\right),$$

from which we easily deduce Lemma 4.5. □

4.4 Estimates on μ

In this section, we focus on getting estimates for μ_f and μ_b.

In order to do that, we will assume that h writes

$$h = h_1 + h_2, \tag{4.31}$$

with

$$\iint_{(0,T)\times(0,L)} \frac{1}{\xi^4} e^{2s\varphi}|h_1|^2 + \iint_{(0,T)\times(0,L)} \frac{1}{\xi^3} e^{2s\varphi}|h_2|^2 < \infty. \tag{4.32}$$

In other words, h_1 has a bit less "integrability" near $t = 0$ and $t = T$ than h_2.

To be more precise on the decomposition (4.31), we will introduce $\tilde{f}$ and $\tilde{g}$ defined by:

$$\tilde{f} := \hat{f} - \frac{\overline{\rho}}{\nu} p'(\overline{\rho})\hat{\rho} + \Lambda\rho_{in}\partial_x\hat{u}, \tag{4.33}$$

$$\tilde{g} := \hat{g} + p'(\overline{\rho} + \Lambda\rho_{in})\partial_x\hat{\rho}, \tag{4.34}$$

(recall that $\hat{f}$ and $\hat{g}$ were introduced in (2.16)-(2.17)) and h_1 and h_2 as follows:

$$\overline{\rho}h_1 := -\frac{\nu}{\overline{\rho}}\Lambda\rho_{in}\partial_{xx}\hat{u} - \Lambda\rho_{in}\partial_t u, \tag{4.35}$$

$$\begin{aligned}\overline{\rho}h_2 := {} & (p'(\overline{\rho}) - p'(\overline{\rho} + \Lambda\rho_{in}))\,\partial_x\hat{\rho} + \frac{\nu}{\overline{\rho}}\partial_x\tilde{f} + \tilde{g} - \frac{\nu}{\overline{\rho}}\Lambda\partial_x\rho_{in}\partial_x\hat{u} \\ & + \overline{\rho}\partial_x\left[u(u + \Lambda u_{in})\right] + p'(\overline{\rho})\frac{\overline{\rho}^2}{\nu}u - \Lambda\rho_{in}\overline{u}\partial_x u.\end{aligned} \tag{4.36}$$

In particular, we have (see (2.8))

$$h_1(t,x) = 0 \quad \forall (t,x) \in (3T_0, T) \times (0,L). \tag{4.37}$$

Of course, we shall check later, see Section 5, that these choices for h_1 and h_2 indeed satisfy condition (4.32).

We shall also assume that $\Lambda u_{in} + u \in L^\infty((0,T)\times(0,L)) \cap L^1((0,T); W^{1,\infty}(0,L))$ with

$$\|\Lambda u_{in} + u\|_{L^\infty((0,T)\times(0,L))\cap L^1((0,T);W^{1,\infty}(0,L))} \leq 1. \tag{4.38}$$

Let us emphasize that this can be done for $R_{in} \leq 1/2$ and $u \in Y_{s,\lambda,R_u}$ with R_u small enough independent of s, λ according to Lemma 3.6.

Note that this also imposes $k \in L^1(0,T; L^\infty(0,L))$ and

$$\|k\|_{L^1((0,T);L^\infty(0,L))} \leq T\left(p'(\overline{\rho})\frac{\overline{\rho}}{\nu} + 1\right) \tag{4.39}$$

(see (4.15)).

Finally, we will also assume that the boundary conditions μ_f and μ_b satisfy;

$$[\xi^{-3/2}e^{s\varphi}](t,0)m_f(t) \in L^2(0, T - 3T_0), \qquad [\xi^{-3/2}e^{s\varphi}](t,L)m_b(t) \in L^2(3T_0, T). \tag{4.40}$$

We now explain how to estimate μ_f and μ_b.

We first focus on μ_f, solution of (4.14), (4.16), (4.17) and in that section only, we remove the subscript f (we will explain in Lemma 4.7 that our estimates also apply to μ_b):

$$\begin{cases} \partial_t\mu + (\overline{u} + u + \Lambda u_{in})\partial_x\mu + k\mu = h \text{ in } (0,T)\times(0,L), \\ \mu(t,0) = m(t), \quad \mu(0,\cdot) = 0. \end{cases} \tag{4.41}$$

Using the characteristics $X(t,\tau,a)$ defined in (4.4), one easily checks that, for $(t,\tau,a) \in [0,T] \times [0,T] \times [0,L]$, such that $X(t,\tau,a) \in [0,L]$,

$$\mu(t, X(t,\tau,a)) = \mu(\tau,a)e^{-\int_\tau^t k(\tau', X(\tau',\tau,a))\,d\tau'} + \int_\tau^t h(\tilde{\tau}, X(\tilde{\tau},\tau,a))e^{-\int_\tau^{\tilde{\tau}} k(\tau',X(\tau',\tau,a))\,d\tau'}d\tilde{\tau}.$$

Of course, due to the fact that the characteristics go from left to right, see (4.3), for $x \in [0,L]$ and $t \in [0,T]$, we have two cases, depending on the position of x with respect to the characteristic $X(t,0,0)$:

- $x \geq X(t,0,0)$: in this case, we use the above formula to get:

$$\mu(t,x) = \int_0^t h(\tilde{\tau}, X(\tilde{\tau},t,x)) e^{-\int_0^{\tilde{\tau}} k(\tau, X(\tau,t,x))\, d\tau} d\tilde{\tau}. \tag{4.42}$$

- $x \leq X(t,0,0)$: in this case, the characteristic through (t,x) lies outside $(0,L)$ at time $t=0$. We shall therefore take $\tau = t^*(t,x)$ and $a = 0$ in the above formula:

$$\mu(t,x) \;=\; m(t^*(t,x)) e^{-\int_{t^*(t,x)}^t k(\tau, X(\tau,t,x))\, d\tau} + \int_{t^*(t,x)}^t h(\tilde{\tau}, X(\tilde{\tau},t,x)) e^{-\int_{t^*(t,x)}^{\tilde{\tau}} k(\tau, X(\tau,t,x))\, d\tau} d\tilde{\tau}. \tag{4.43}$$

Recall that k is supposed to be in $L^1(0,T;L^\infty(0,L))$ (see (4.39)), so that in particular

$$\left| e^{-\int_\tau^t k(\tau', X(\tau',t,x))\, d\tau'} \right| \leq C, \qquad \forall (t,\tau) \in [0,T]^2. \tag{4.44}$$

Let us begin with the estimates in the zone "below the diagonal", that is for (t,x) satisfying $x > X(t,0,0)$. Using (4.26) for $p=-3$ and $p=-4$, for $(t,x) \in (0,T-3T_0) \times (0,L)$ with $x > X(t,0,0)$,

$$\begin{aligned}
|\mu(t,x)|^2 \leq & C \left(\int_0^t |h(\tau, X(\tau,t,x))| d\tau \right)^2 \\
\leq & C \left(\int_0^t |h_1(\tau, X(\tau,t,x))|^2 \frac{e^{2s\varphi(\tau,X(\tau,t,x))}}{\xi^4(\tau, X(\tau,t,x))} d\tau \right) \left(\int_0^t \xi^4(\tau, X(\tau,t,x)) e^{-2s\varphi(\tau,X(\tau,t,x))} d\tau \right) \\
& + C \left(\int_0^t |h_2(\tau, X(\tau,t,x))|^2 \frac{e^{2s\varphi(\tau,X(\tau,t,x))}}{\xi^3(\tau, X(\tau,t,x))} d\tau \right) \left(\int_0^t \xi^3(\tau, X(\tau,t,x)) e^{-2s\varphi(\tau,X(\tau,t,x))} d\tau \right) \\
\leq & C \left(\int_0^t |h_1(\tau, X(\tau,t,x))|^2 \frac{e^{2s\varphi(\tau,X(\tau,t,x))}}{\xi^4(\tau, X(\tau,t,x))} d\tau \right) t\xi^4(t,x) e^{-2s\varphi(t,x)} e^{C_0 s\lambda e^{4\lambda} \|u+\Lambda u_{in}\|_{L^\infty}} \\
& + C \left(\int_0^t |h_2(\tau, X(\tau,t,x))|^2 \frac{e^{2s\varphi(\tau,X(\tau,t,x))}}{\xi^3(\tau, X(\tau,t,x))} d\tau \right) \xi^3(t,x) e^{-2s\varphi(t,x)} e^{C_0 s\lambda e^{4\lambda} \|u+\Lambda u_{in}\|_{L^\infty}},
\end{aligned}$$

In particular, this implies that, for all $t \leq T - 3T_0$ such that $X(t,0,0) \leq L$,

$$\begin{aligned}
\int_{X(t,0,0)}^L |\mu(t,x)|^2 \frac{e^{2s\varphi(t,x)}}{\xi^3(t,x)} dx \leq & C e^{4\lambda} e^{C_0 s\lambda e^{4\lambda} \|u+\Lambda u_{in}\|_{L^\infty}} \int_0^t \int_0^L |h_1(\tau,y)|^2 \frac{e^{2s\varphi(\tau,y)}}{\xi^4(\tau,y)} d\tau dy \\
& + C e^{C_0 s\lambda e^{4\lambda} \|u+\Lambda u_{in}\|_{L^\infty}} \int_0^t \int_0^L |h_2(\tau,y)|^2 \frac{e^{2s\varphi(\tau,y)}}{\xi^3(\tau,y)} d\tau dy.
\end{aligned} \tag{4.45}$$

Here, we have used Lemma 4.5. Of course, similar estimates can be done in the zone "above the diagonal", that is for $(t,x) \in (0,T-3T_0) \times (0,L)$ with $x < X(t,0,0)$, except for what concerns the boundary term. This term can be handled using (4.28) with $p=-3$ as follows:

$$\begin{aligned}
& \left| m(t^*(t,x)) e^{-\int_{t^*(t,x)}^t k(\tau, X(\tau,t,x))\, d\tau} \right|^2 \\
& \qquad \leq |m(t^*(t,x))|^2 \left(\frac{e^{2s\varphi(t^*(t,x),0)}}{\xi^3(t^*(t,x),0)} \right) \left(\xi^3(t,x) e^{-2s\varphi(t,x)} \right) e^{C_0 s\lambda e^{4\lambda} \|u+\Lambda u_{in}\|_{L^\infty}}.
\end{aligned}$$

In particular, this implies that, for all $t \leq T - 3T_0$,

$$\begin{aligned}
\int_0^{\min\{X(t,0,0),L\}} |\mu(t,x)|^2 \frac{e^{2s\varphi(t,x)}}{\xi^3(t,x)} dx \leq & C e^{C_0 s\lambda e^{4\lambda} \|u+\Lambda u_{in}\|_{L^\infty}} \int_0^t |m(\tau)|^2 \frac{e^{2s\varphi(\tau,0)}}{\xi^3(\tau,0)} d\tau \\
& + C e^{4\lambda} e^{C_0 s\lambda e^{4\lambda} \|u+\Lambda u_{in}\|_{L^\infty}} \int_0^t \int_0^L |h_1(\tau,y)|^2 \frac{e^{2s\varphi(\tau,y)}}{\xi^4(\tau,y)} d\tau dy \\
& + C e^{C_0 s\lambda e^{4\lambda} \|u+\Lambda u_{in}\|_{L^\infty}} \int_0^t \int_0^L |h_2(\tau,y)|^2 \frac{e^{2s\varphi(\tau,y)}}{\xi^3(\tau,y)} d\tau dy,
\end{aligned} \tag{4.46}$$

where we have used that the map $x \mapsto t^*(t,x)$ defines a change of variable of bounded jacobian, see Lemma 4.5.

Therefore, combining (4.45)–(4.46), for all $t \in [0, T-3T_0]$, we have

$$\begin{aligned}\int_0^L |\mu(t,x)|^2 \frac{e^{2s\varphi(t,x)}}{\xi^3(t,x)}\,dx \leq & Ce^{C_0 s\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}} \int_0^{T-3T_0} |m(\tau)|^2 \,\frac{e^{2s\varphi}(\tau,0)}{\xi^3(\tau,0)}\,d\tau \\ & + Ce^{4\lambda} e^{C_0 s\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}} \int_0^{T-3T_0}\int_0^L |h_1(\tau,y)|^2 \frac{e^{2s\varphi(\tau,y)}}{\xi^4(\tau,y)} d\tau dy \\ & + Ce^{C_0 s\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}} \int_0^{T-3T_0}\int_0^L |h_2(\tau,y)|^2 \frac{e^{2s\varphi(\tau,y)}}{\xi^3(\tau,y)} d\tau dy.\end{aligned} \tag{4.47}$$

We can now estimate μ_f.

Lemma 4.6 (Estimates on μ_f)**.** *Assume that*

- h_1 *and* h_2 *given by* (4.35)–(4.36) *satisfy* (4.32);
- $\Lambda u_{in} + u$ *belongs to* $L^\infty((0,T)\times(0,L)) \cap L^1((0,T); W^{1,\infty}(0,L))$ *and satisfies* (4.38);
- $[\xi^{-3/2} e^{s\varphi} \mu_f](t,0)$ *belongs to* $L^2(0,T)$.

Then there exist constants C, s_0 *and* λ_0 *such that for* $s \geq s_0$ *and* $\lambda \geq \lambda_0$,

$$\begin{aligned}&\sup_{[0,T-3T_0]} \int_0^L |\mu_f(t,x)|^2 \frac{e^{2s\varphi(t,x)}}{\xi^3(t,x)}\,dx + \int_0^{T-3T_0}\int_0^L |\mu_f(t,x)|^2 \frac{e^{2s\varphi(t,x)}}{\xi^3(t,x)}\,dtdx \\ &\leq Ce^{C_0 s\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}} \int_0^{T-3T_0} |m_f(\tau)|^2 \,\frac{e^{2s\varphi}(\tau,0)}{\xi^3(\tau,0)}\,d\tau \\ &+ Ce^{4\lambda} e^{C_0 s\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}} \int_0^{T-3T_0}\int_0^L |h_1(\tau,y)|^2 \frac{e^{2s\varphi(\tau,y)}}{\xi^4(\tau,y)} d\tau dy \\ &+ Ce^{C_0 s\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}} \int_0^{T-3T_0}\int_0^L |h_2(\tau,y)|^2 \frac{e^{2s\varphi(\tau,y)}}{\xi^3(\tau,y)} d\tau dy.\end{aligned} \tag{4.48}$$

Proof of Lemma 4.6. The proof follows directly from (4.47) and the fact that

$$\int_0^{T-3T_0}\int_0^L |\mu_f(t,x)|^2 \frac{e^{2s\varphi(t,x)}}{\xi^3(t,x)}\,dtdx \leq C \sup_{[0,T-3T_0]} \int_0^L |\mu_f(t,x)|^2 \frac{e^{2s\varphi(t,x)}}{\xi^3(t,x)}\,dx.$$

□

Similarly, one can derive estimates on μ_b:

Lemma 4.7 (Estimates on μ_b)**.** *Assume that*

- h_1 *and* h_2 *given by* (4.35)–(4.36) *satisfy* (4.32) *and* (4.37);
- $\Lambda u_{in} + u$ *belongs to* $L^\infty((0,T)\times(0,L)) \cap L^1((0,T); W^{1,\infty}(0,L))$ *and satisfies* (4.38);
- $[\xi^{-3/2} e^{s\varphi} \mu_b](t,L)$ *belongs to* $L^2(0,T)$.

Then there exist constants C, s_0 *and* λ_0 *such that for* $s \geq s_0$ *and* $\lambda \geq \lambda_0$,

$$\begin{aligned}&\sup_{[3T_0,T]} \int_0^L |\mu_b(t,x)|^2 \frac{e^{2s\varphi(t,x)}}{\xi^3(t,x)}\,dtdx + \int_{3T_0}^{T}\int_0^L |\mu_b(t,x)|^2 \frac{e^{2s\varphi(t,x)}}{\xi^3(t,x)}\,dtdx \\ &\leq Ce^{C_0 s\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}} \int_{3T_0}^{T} |m_b(\tau)|^2 \,\frac{e^{2s\varphi}(\tau,L)}{\xi^3(\tau,L)}\,d\tau \\ &+ Ce^{C_0 s\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}} \int_{3T_0}^{T}\int_0^L |h_2(\tau,y)|^2 \frac{e^{2s\varphi(\tau,y)}}{\xi^3(\tau,y)} d\tau dy.\end{aligned} \tag{4.49}$$

Proof of Lemma 4.7. Set $\mu(t,x) = \mu_b(T-t, L-x)$. Then μ solves an equation of the form (4.14) ($k(t,x)$ replaced by $-k(T-t, L-x)$), where h_1 can be taken to be 0 since it vanishes outside $(0, 3T_0) \times (0, L)$, and thus estimate (4.47) applies since we never use the sign of the derivative of ψ (which has changed doing this transform), but only the direction of monotonicity of θ. Undoing the change of variables, we obtain (4.49). $\square$

In the following, we explain how to deduce estimates on $\partial_x \rho$ and ρ from (4.48)–(4.49) for μ_f and μ_b.

4.5 Estimates on $\partial_x \rho$

Having obtained estimates on μ_f and μ_b, we can deduce estimates on $\partial_x \rho_f$ and $\partial_x \rho_b$.

By construction, we have

$$\partial_x \rho_f = \frac{\overline{\rho}^2}{\nu}(\mu_f - u).$$

Thus estimates on $\partial_x \rho_f$ can be immediately deduced from the ones on μ_f and u:

$$\|\xi^{-3/2} e^{s\varphi} \partial_x \rho_f\|_{L^2((0,T-3T_0)\times(0,L))} \leq \|\xi^{-3/2} e^{s\varphi} \mu_f\|_{L^2((0,T-3T_0)\times(0,L))} + \|\xi^{-3/2} e^{s\varphi} u\|_{L^2((0,T)\times(0,L))}. \tag{4.50}$$

Similarly, estimates on $\partial_x \rho_b$ follows from the ones on μ_b and u:

$$\|\xi^{-3/2} e^{s\varphi} \partial_x \rho_b\|_{L^2((3T_0,T)\times(0,L))} \leq \|\xi^{-3/2} e^{s\varphi} \mu_b\|_{L^2((3T_0,T)\times(0,L))} + \|\xi^{-3/2} e^{s\varphi} u\|_{L^2((0,T)\times(0,L))}. \tag{4.51}$$

Remark that, since we assume that $u \in Y_{s,\lambda,R_u}$ for some R_u, $\xi^{-2} e^{s\varphi} u \in H^1(0,T; L^2(0,L))$, hence it is $L^\infty(0,T; L^2(0,L))$. Therefore, using the $L^\infty(0, T-3T_0; L^2(0,L))$ estimates on $\xi^{-3/2} e^{s\varphi} \mu_f$ in (4.48), we deduce that $\xi^{-2} e^{s\varphi} \partial_x \rho_f \in L^\infty(0, T-3T_0; L^2(0,L))$. Similarly, $\xi^{-2} e^{s\varphi} \partial_x \rho_b \in L^\infty(3T_0, T; L^2(0,L))$ and we have the estimates:

$$\|\xi^{-2} e^{s\varphi} \partial_x \rho_f\|_{L^\infty(0,T-3T_0;L^2(0,L))} \leq \|\xi^{-3/2} e^{s\varphi} \mu_f\|_{L^\infty(0,T-3T_0;L^2(0,L))} + \|\xi^{-2} e^{s\varphi} u\|_{H^1(0,T;L^2(0,L))}, \tag{4.52}$$

$$\|\xi^{-2} e^{s\varphi} \partial_x \rho_b\|_{L^\infty(3T_0,T;L^2(0,L))} \leq \|\xi^{-3/2} e^{s\varphi} \mu_b\|_{L^\infty(3T_0,T;L^2(0,L))} + \|\xi^{-2} e^{s\varphi} u\|_{H^1(0,T;L^2(0,L))}. \tag{4.53}$$

In the following, we assume that we have estimates on the $L^2((0, T-3T_0) \times (0,L))$ and $L^2((3T_0, T) \times (0,L))$ norms of $\xi^{-3/2} e^{s\varphi} \partial_x \rho_f$ and $\xi^{-3/2} e^{s\varphi} \partial_x \rho_b$, respectively, and also on the $L^\infty(0, T-3T_0; L^2(0,L))$ and $L^\infty(3T_0, T; L^2(0,L))$ norms of $\xi^{-2} e^{s\varphi} \partial_x \rho_f$ and $\xi^{-2} e^{s\varphi} \partial_x \rho_b$.

4.6 Estimates on ρ

We can now deduce estimates on ρ.

- *Step 1. Estimates on $\rho_f(t, L)$.*

Note that ρ_f solves equation (2.16) with $\rho_f(0,x) = 0$ and $\rho_f(t,0) = 0$ by construction. Therefore, for t such that $X(t,0,0) \leq L$, $\rho_f(t,L)$ is given by

$$\rho_f(t,L) = \int_0^t (\hat{f} - \overline{\rho} \partial_x u)(\tau, X(\tau, t, L)) \exp\left(-\frac{\overline{\rho}}{\nu} p'(\overline{\rho})(t-\tau)\right) d\tau,$$

whereas, for t such that $X(t,0,0) \geq L$, $\rho_f(t,L)$ is given by

$$\rho_f(t,L) = \int_{t^*(t,L)}^t (\hat{f} - \overline{\rho} \partial_x u)(\tau, X(\tau, t, L)) \exp\left(-\frac{\overline{\rho}}{\nu} p'(\overline{\rho})(t-\tau)\right) d\tau.$$

Therefore, following the proof of Lemma 4.6, we get

Lemma 4.8. *There exist constants C, s and λ_0 such that for $s \geq s_0$ and $\lambda \geq \lambda_0$,*

$$\int_0^{T-3T_0} |\rho_f(t,L)|^2 \frac{e^{2s\varphi(t,L)}}{\xi^2(t,L)} dt \leq C e^{Cs\lambda e^{4\lambda} \|u + \Lambda u_{in}\|_{L^\infty}} \int_0^{T-3T_0} \int_0^L |\hat{f}(\tau,y)|^2 \frac{e^{2s\varphi(\tau,y)}}{\xi^2(\tau,y)} d\tau dy$$
$$+ C e^{Cs\lambda e^{4\lambda} \|u + \Lambda u_{in}\|_{L^\infty}} \int_0^{T-3T_0} \int_0^L |\partial_x u(\tau,y)|^2 \frac{e^{2s\varphi(\tau,y)}}{\xi^2(\tau,y)} d\tau dy. \tag{4.54}$$

Proof of Lemma 4.8. The proof follows line to line the one of Lemma 4.6 and is left to the reader. $\square$

• *Step 2. Global estimates on ρ.*

Here is a key lemma that will allow us to obtain global estimates on ρ directly from the ones on $\partial_x \rho_f$, $\partial_x \rho_b$ and the one of $\rho_f(t, L)$ above:

Lemma 4.9. *There exists a constant $C > 0$ independent of s and λ such that for all $a \in H^1(0, L)$, for all $t \in (0, T)$, for all s, $\lambda \geq 1$,*

$$|a(0)|^2 \frac{e^{2s\varphi(t,0)}}{\xi^2(t,0)} + s\lambda \int_0^L |a(x)|^2 \frac{e^{2s\varphi(t,x)}}{\xi(t,x)}\, dx \leq \frac{C}{s\lambda} \int_0^L |a'(x)|^2 \frac{e^{2s\varphi(t,x)}}{\xi^3(t,x)}\, dx + |a(L)|^2 \frac{e^{2s\varphi(t,L)}}{\xi^2(t,L)}. \tag{4.55}$$

Proof of Lemma 4.9. The proof is based on the following identity:

$$\begin{aligned} |a(L)|^2 \frac{e^{2s\varphi(t,L)}}{\xi^2(t,L)} - |a(0)|^2 \frac{e^{2s\varphi(t,0)}}{\xi^2(t,0)} &= \int_0^L \partial_x \left(|a(x)|^2 \frac{e^{2s\varphi(t,x)}}{\xi^2(t,x)} \right) dx \\ &= 2 \int_0^L a(x) a'(x) \frac{e^{2s\varphi(t,x)}}{\xi^2(t,x)}\, dx - 2s\lambda \int_0^L |a(x)|^2 \psi'(x - \overline{u}t) \frac{e^{2s\varphi(t,x)}}{\xi(t,x)}\, dx. \end{aligned}$$

Since $\psi'(x - \overline{u}t)$ is negative on $(0, L)$ for $t \in (0, T)$ by construction (see (2.19)), there exists $c_* > 0$ such that

$$\psi'(x - \overline{u}t) \leq -c_*, \quad (t, x) \in (0, L) \times (0, T).$$

But we also have

$$\left| 2 \int_0^L a(x) a'(x) \frac{e^{2s\varphi(t,x)}}{\xi^2(t,x)}\, dx \right| \leq c_* s\lambda \int_0^L |a(x)|^2 \frac{e^{2s\varphi(t,x)}}{\xi(t,x)}\, dx + \frac{1}{c_* s\lambda} \int_0^L |a'(x)|^2 \frac{e^{2s\varphi(t,x)}}{\xi^3(t,x)}\, dx,$$

which yields the result. $\square$

Using Lemma 4.9, we immediately obtain:

Lemma 4.10. *For $s \geq s_0$ and $\lambda \geq \lambda_0$,*

$$\begin{aligned} s\lambda \int_0^{T-3T_0} \int_0^L |\rho_f(t,x)|^2 \frac{e^{2s\varphi(t,x)}}{\xi(t,x)}\, dtdx \leq \frac{C}{s\lambda} \int_0^{T-3T_0} \int_0^L |\partial_x \rho_f(t,x)|^2 \frac{e^{2s\varphi(t,x)}}{\xi^3(t,x)}\, dtdx \\ + C \int_0^{T-3T_0} |\rho_f(t,L)|^2 \frac{e^{2s\varphi(t,L)}}{\xi^2(t,L)}\, dt \end{aligned} \tag{4.56}$$

and

$$\begin{aligned} \int_{3T_0}^T |\rho_b(t,0)|^2 \frac{e^{2s\varphi(t,0)}}{\xi^2(t,0)}\, dt + s\lambda \int_{3T_0}^T \int_0^L |\rho_b(t,x)|^2 \frac{e^{2s\varphi(t,x)}}{\xi(t,x)}\, dtdx \\ \leq \frac{C}{s\lambda} \int_{3T_0}^T \int_0^L |\partial_x \rho_b(t,x)|^2 \frac{e^{2s\varphi(t,x)}}{\xi^3(t,x)}\, dtdx. \end{aligned} \tag{4.57}$$

Using Lemma 4.10 and the definition of ρ, we obtain the following estimates on ρ:

$$\begin{aligned} \int_0^T \int_0^L |\rho|^2 \frac{e^{2s\varphi}}{\xi}\, dtdx \leq \frac{C}{s^2\lambda^2} \int_{3T_0}^T \int_0^L |\partial_x \rho_b|^2 \frac{e^{2s\varphi}}{\xi^3}\, dtdx \\ + \frac{C}{s^2\lambda^2} \int_0^{T-3T_0} \int_0^L |\partial_x \rho_f|^2 \frac{e^{2s\varphi}}{\xi^3}\, dtdx + \frac{C}{s\lambda} \int_0^{T-3T_0} |\rho_f(t,L)|^2 \frac{e^{2s\varphi(t,L)}}{\xi^2(t,L)}\, dt. \end{aligned} \tag{4.58}$$

Using (4.57) and since $\rho_f(t, 0) = 0$ by construction, we deduce

$$\int_0^T |\rho(t,0)|^2 \frac{e^{2s\varphi(t,0)}}{\xi^2(t,0)}\, dt \leq \frac{C}{s\lambda} \int_{3T_0}^T \int_0^L |\partial_x \rho_b|^2 \frac{e^{2s\varphi}}{\xi^3}\, dtdx. \tag{4.59}$$

Similarly, $\rho_b(t, L) = 0$, and then

$$\int_0^T |\rho(t,L)|^2 \frac{e^{2s\varphi(t,L)}}{\xi^2(t,L)}\, dt = \int_0^T |\rho_f(t,L)|^2 \frac{e^{2s\varphi(t,L)}}{\xi^2(t,L)}\, dt, \tag{4.60}$$

which is estimated by Lemma 4.8.

Finally, let us explain how to obtain $L^\infty((0,T)\times(0,L))$ bounds on ρ. We do it independently for ρ_f and ρ_b. Using that $\rho_f(t,0) = 0$ and (4.52), we immediately get by Sobolev embedding that $\rho_f e^{s\check{\varphi}/2} \in L^\infty((0, T-3T_0)\times(0,L))$. Similarly, $\rho_b e^{s\check{\varphi}/2} \in L^\infty((3T_0, T)\times(0,L))$. Thus, $\rho e^{s\check{\varphi}/2} \in L^\infty((0,T)\times(0,L))$.

To get an estimate on $\partial_t \rho$ in $L^2((0,T)\times(0,L))$, we then use the equation of ρ (see (2.16)).

5 The fixed point argument

In this section we prove that the operator described in Section 2 admits a fixed point provided that the initial data is chosen suitably small and that the parameters s, λ, R_ρ and R_u are suitably chosen. This fixed point is obtained via Schauder's fixed point theorem. Hence we are going to focus on the two following issues:

- the operator $F : (\hat\rho, \hat u) \mapsto (\rho, u)$ maps the set $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$ into itself for conveniently chosen parameters s, λ, R_ρ and R_u;
- F is continuous on $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$ equipped with the $L^2((0,T)\times(0,L))^2$- topology.

We first focus on the first item in Sections 5.1–5.2 and then develop the fixed point argument in Section 5.3.

5.1 Estimates on u

To get estimates on the function u constructed in Section 3, we shall use Theorem 3.2 and Proposition 3.5. Therefore, we shall first derive an estimate on the $L^2((0,T)\times(0,L))$-norm of $e^{s\varphi}\, \hat g\, \xi^{-3/2}$:

Lemma 5.1. *There exists a constant C independent of $s, \lambda \geq 1$ and $R_\rho, R_u, R_{in} \leq 1$ such that for all $(\hat\rho, \hat u) \in X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$,*

$$\|\hat g\, e^{s\varphi}\, \xi^{-3/2}\|_{L^2((0,T)\times(0,L))} \leq C \left(R_\rho + \mathcal{O}_{s,\lambda}(R_{in}) + R_u^2, \right). \tag{5.1}$$

$$\|\tilde g\, e^{s\varphi}\, \xi^{-3/2}\|_{L^2((0,T)\times(0,L))} \leq C \left(\mathcal{O}_{s,\lambda}(R_{in}) + R_\rho^2 + R_u^2. \right), \tag{5.2}$$

where $\hat g = g(\hat\rho, \hat u)$ is defined in (2.13) *and $\tilde g$ is defined in* (4.34).

Proof of Lemma 5.1. It is a matter of estimating the different terms in $\hat g$ and $\tilde g$ by using the estimates on $\hat\rho$ and $\hat u$ in X_{s,λ,R_ρ} and Y_{s,λ,R_u}. We regroup the terms that are treated likewise. The various constants C below are independent of s, λ and R_ρ, R_u and R_{in}.

• First using the uniform bound on ρ_{in} and the definition of X_{s,λ,R_ρ} one has immediately

$$\|p'(\overline{\rho} + \Lambda\rho_{in})\partial_x \hat\rho\, e^{s\varphi} \xi^{-3/2}\|_{L^2((0,T)\times(0,L))} \leq C R_\rho.$$

It is the only term estimated by R_ρ; it appears in $\hat g$ but not in $\tilde g$.

• Now, using that the following terms are compactly supported in time in $(T_0, 2T_0)$, we also have

$$\Big\| \Big(-(\overline{\rho} + \Lambda\rho_{in})\Lambda' u_{in} - (p'(\overline{\rho} + \Lambda\rho_{in}) - p'(\overline{\rho} + \rho_{in}))\Lambda\partial_x\rho_{in} + \rho_{in}\partial_t u_{in}(\Lambda - \Lambda^2) + \rho_{in}\overline{u}\partial_x u_{in}(\Lambda - \Lambda^2) + \overline{\rho} u_{in}\partial_x u_{in}(\Lambda - \Lambda^2) + \rho_{in} u_{in}\partial_x u_{in}(\Lambda - \Lambda^3) \Big) e^{s\varphi}\xi^{-3/2} \Big\|_{L^2((0,T)\times(0,L))} \leq \mathcal{O}_{s,\lambda}(R_{in})$$

(see (2.5)).

• Next, by using the definition of Y_{s,λ,R_u} and (2.24),

$$\left\| \big(\Lambda(\overline{\rho} + \Lambda \rho_{in}) \partial_x (\hat{u} u_{in})\big) e^{s\varphi} \xi^{-3/2} \right\|_{L^2((0,T)\times(0,L))} \leq \mathcal{O}_{s,\lambda}(R_{in}) R_u.$$

• We obtain the following estimate by using the definition of Y_{s,λ,R_u}, (2.24) and (3.43) in Lemma 3.6:

$$\left\| \big((\overline{\rho} + \Lambda \rho_{in}) \hat{u}\, \partial_x \hat{u}\big) e^{s\varphi} \xi^{-3/2} \right\|_{L^2((0,T)\times(0,L))} \leq C R_u^2.$$

• Next, again using Lemma 3.6, one obtains

$$\left\| \hat{\rho} \Big(\partial_t (\Lambda u_{in}) + (\overline{u} + \Lambda u_{in} + \hat{u}) \partial_x (\Lambda u_{in}) \Big) e^{s\varphi} \xi^{-3/2} \right\|_{L^2((0,T)\times(0,L))} \leq C R_\rho R_{in}.$$

• Using that for some constant c independent of $s, \lambda \geq 1$ one has $\sup_{(t,x)} \{ s^{1/2} \xi^{1/2} e^{-s\check{\varphi}/2} \} \leq c$, one obtains:

$$\left\| \hat{\rho} \Big(\partial_t \hat{u} + (\overline{u} + \Lambda u_{in} + \hat{u}) \partial_x \hat{u} \Big) e^{s\varphi} \xi^{-3/2} \right\|_{L^2((0,T)\times(0,L))} \leq C R_\rho R_u.$$

• Using the regularity of p and the boundedness of ρ and $\hat{\rho}$, we get that pointwise

$$\left| p'(\overline{\rho} + \Lambda \rho_{in} + \hat{\rho}) - p'(\overline{\rho} + \Lambda \rho_{in}) \right| \leq C |\hat{\rho}|,$$

and similarly as above,

$$\left\| \Big(\big[p'(\overline{\rho} + \Lambda \rho_{in} + \hat{\rho}) - p'(\overline{\rho} + \Lambda \rho_{in})\big] \partial_x (\Lambda \rho_{in} + \hat{\rho}) \Big) e^{s\varphi} \xi^{-3/2} \right\|_{L^2((0,T)\times(0,L))} \leq C R_\rho (R_{in} + R_\rho).$$

Gathering all the estimates above, we reach the conclusion. □

Using the estimates of Lemma 5.1, according to Theorem 3.2, we obtain

$$\begin{aligned} s^{3/2} \lambda^2 \| u\, e^{s\varphi} \|_{L^2((0,T)\times(0,L))} &+ s^{1/2} \lambda \| \partial_x u\, e^{s\varphi}\, \xi^{-1} \|_{L^2((0,T)\times(0,L))} \\ &+ s^{-1/2} \| \partial_{xx} u\, e^{s\varphi}\, \xi^{-2} \|_{L^2((0,T)\times(0,L))} + s^{-1/2} \| \partial_t u\, e^{s\varphi}\, \xi^{-2} \|_{L^2((0,T)\times(0,L))} \\ &\leq C_1 \left(R_\rho + \mathcal{O}_{s,\lambda}(R_{in}) + R_u^2 \right). \end{aligned} \tag{5.3}$$

Hence we get to the following statement.

Corollary 5.2. *There exist $c_1 > 0$, $R_1 > 0$ independent of s, λ such that, if*

$$R_u \leq R_1, \tag{5.4}$$

and

$$R_\rho \leq c_1 R_u, \tag{5.5}$$

then for any $s \geq s_1$, $\lambda \geq \lambda_1$, there exists $K_1(s,\lambda,R_u) > 0$ such that if

$$R_{in} \leq K_1(s,\lambda,R_u), \tag{5.6}$$

then the u-part of $F(\hat{\rho},\hat{u})$ belongs to Y_{s,λ,R_u} for any $(\hat{\rho},\hat{u})$ in $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$ and conditions (4.3) *and* (4.38) *are satisfied.*

Proof. The fact that one can choose c_1 and R_1 such that the u-part of $F(\hat{\rho},\hat{u})$ belongs to Y_{s,λ,R_u} follows from (5.3). Indeed, take $\tilde{R}_1$ small enough such that $C_1 \tilde{R}_1^2 \leq \tilde{R}_1/3$. Then, set $c_1 = 1/(3C_1)$ and take R_{in} small enough so that $\mathcal{O}_{s,\lambda}(R_{in}) \leq \tilde{R}_u/(3C_1)$.

Conditions (4.3) and (4.38) need to be proved. Applying Lemma 3.6, there exists a constant $\hat{R}_1$ independent of $s, \lambda \geq 1$ such that, taking R_u and R_{in} smaller than $\hat{R}_1$, we can furthermore guarantee that conditions (4.3) and (4.38) hold.

We thus set $R_1 = \min\{\tilde{R}_1, \hat{R}_1\}$. □

In the sequel, we choose R_u, R_ρ and R_{in} so that (5.4), (5.5), (5.6) are satisfied. In particular, $u \in Y_{s,\lambda,R_u}$ and conditions (4.3) and (4.38) are satisfied.

5.2 Estimates on ρ

To get estimates on ρ, we shall use the estimates given in Section 4. They will be based on estimates on μ_f, μ_b. Of course, these first require to get estimates on the source terms h_1, h_2 given in (4.35)-(4.36), and the boundary terms m_f, m_b given by (4.17)–(4.18).

Lemma 5.3. *There exists a constant C independent of s, λ and R_ρ, R_u, R_{in} such that for all $\hat\rho \in X_{s,\lambda,R_\rho}$ and $\hat u, u \in Y_{s,\lambda,R_u}$,*

$$\exp(C_0 s\lambda e^{4\lambda}\|u + \Lambda u_{in}\|_{L^\infty((0,T)\times(0,L))}) \le C(1 + \mathcal{O}_{s,\lambda}(R_{in})), \tag{5.7}$$

$$\|\tilde f\, e^{s\varphi}\xi^{-1}\|_{L^2((0,T)\times(0,L))} \le C\left(\mathcal{O}_{s,\lambda}(R_{in}) + R_u^2 + R_\rho^2\right), \tag{5.8}$$

$$\|\hat f\, e^{s\varphi}\xi^{-1}\|_{L^2((0,T)\times(0,L))} \le C\left(\mathcal{O}_{s,\lambda}(R_{in}) + R_\rho + R_u^2\right), \tag{5.9}$$

$$\|h_1\, e^{s\varphi}\xi^{-2}\|_{L^2((0,T)\times(0,L))} \le C e^{4\lambda} s R_{in}^2 + e^{-4\lambda} R_u^2, \tag{5.10}$$

$$\|h_2\, e^{s\varphi}\xi^{-3/2}\|_{L^2((0,T)\times(0,L))} \le C\left(\mathcal{O}_{s,\lambda}(R_{in}) + \frac{1}{s^{3/2}\lambda^2} R_u + R_\rho^2 + R_u^2\right), \tag{5.11}$$

$$\|m_f(\cdot) e^{s\varphi}\xi^{-3/2}\|_{L^2(0,T)} \le C\left(\frac{1}{\sqrt{\lambda}}(R_u + R_\rho) + \mathcal{O}_{s,\lambda}(R_{in})\right), \tag{5.12}$$

$$\|m_b(\cdot) e^{s\varphi}\xi^{-3/2}\|_{L^2(0,T)} \le C\left(\frac{1}{\sqrt{\lambda}}(R_u + R_\rho) + \mathcal{O}_{s,\lambda}(R_{in})\right), \tag{5.13}$$

where $\hat f = f(\hat\rho, \hat u)$ and $\tilde f$ are given by (2.12) *and* (4.33).

Proof of Lemma 5.3. All these estimates are obtained independently and we prove them one by one.

• *Proof of* (5.7). Using Lemma 3.6,

$$\begin{aligned}\exp(C_0 s\lambda e^{4\lambda}\|u\|_{L^\infty((0,T)\times(0,L))}) &\le \exp(C_0 s\lambda e^{4\lambda}\exp(-s\check\varphi(t)/2)R_u)\\ &\le \exp(C_0 s\lambda e^{4\lambda}\exp(-s(e^{5\lambda} - e^{4\lambda})/2)R_u) \le C,\end{aligned}$$

since $\lambda \ge 1$. On the other hand,

$$\exp(C_0 s\lambda e^{4\lambda}\|u_{in}\|_{L^\infty((0,T)\times(0,L))}) = (1 + \mathcal{O}_{s,\lambda}(R_{in})).$$

These estimates yield (5.7).

• *Proof of* (5.8). The function $\tilde f$ is defined by (4.33): using the definition of $\hat f = f(\hat\rho, \hat u)$ in (2.12), we get:

$$\tilde f = -\Lambda'\rho_{in} + (\Lambda - \Lambda^2)\partial_x(\rho_{in}u_{in}) - \Lambda(\partial_x\rho_{in})u - \Lambda\,\hat\rho\,\partial_x u_{in} - \hat\rho\,\partial_x\hat u, \tag{5.14}$$

The first two terms $-\Lambda'\rho_{in} + (\Lambda - \Lambda^2)\partial_x(\rho_{in}u_{in})$ are compactly supported in time away from $t = 0$ and $t = T$ (in $(T_0, 2T_0)$) and depend only on $\rho_{in}u_{in}$, so

$$\|\left(-\Lambda'\rho_{in} + (\Lambda - \Lambda^2)\partial_x(\rho_{in}u_{in})\right) e^{2s\varphi}\xi^{-1}\|_{L^2((0,T)\times(0,L))} \le \mathcal{O}_{s,\lambda}(R_{in}).$$

Next, using the $L^\infty((0,T)\times(0,L))$ norm of $\partial_x\rho_{in}$, we infer

$$\| - \Lambda(\partial_x\rho_{in})u\, e^{s\varphi}\xi^{-1}\|_{L^2((0,T)\times(0,L))} \le C R_{in}R_u \le C R_{in}^2 + C R_u^2.$$

Similarly,

$$\| - \Lambda\,\hat\rho\,\partial_x u_{in} e^{s\varphi}\xi^{-1}\|_{L^2((0,T)\times(0,L))} \le C R_\rho R_{in} \le C R_\rho^2 + C R_{in}^2.$$

Finally, the term $\rho\,\partial_x u$ is quadratic:

$$\|\hat\rho\,\partial_x\hat u\, e^{s\varphi}\xi^{-1}\|_{L^2((0,T)\times(0,L))} \le C R_\rho R_u \le C R_\rho^2 + C R_u^2.$$

This concludes the estimate (5.8) on $\tilde f$.

• *Proof of* (5.9). Of course, we already have the estimate (5.8), so we only need to estimate

$$\hat f - \tilde f = \frac{\overline{\rho}}{\nu} p'(\overline{\rho})\hat\rho - \Lambda \rho_{in} \partial_x \hat u.$$

By definition,

$$\|\hat\rho\, e^{s\varphi}\xi^{-1}\|_{L^2((0,T)\times(0,L))} \leq R_\rho.$$

The last term satisfies

$$\|\Lambda\rho_{in}\partial_x \hat u\ e^{s\varphi}\xi^{-1}\|_{L^2((0,T)\times(0,L))} \leq C R_{in} R_u \leq C R_{in}^2 + C R_u^2.$$

This concludes the proof of (5.9) since, due to (2.26), $R_\rho^2 \leq R_\rho$.

• *Proof of* (5.10). Using the definition of Y_{s,λ,R_u},

$$\|\Lambda\rho_{in}\,\partial_{xx}\hat u\, e^{s\varphi}\xi^{-2}\|_{L^2((0,T)\times(0,L))} \leq C\sqrt{s} R_{in} R_u \leq C s e^{4\lambda} R_{in}^2 + e^{-4\lambda} R_u^2.$$

and using Corollary 5.2, that

$$\|\Lambda\rho_{in}\partial_t u\, e^{s\varphi}\xi^{-2}\|_{L^2((0,T)\times(0,L))} \leq C\sqrt{s} R_{in} R_u \leq C s e^{4\lambda} R_{in}^2 + e^{-4\lambda} R_u^2. \tag{5.15}$$

According to the definition of h_1 in (4.35), we thus obtain (5.10).

• *Proof of* (5.11). Recall the definition of h_2 in (4.36):

$$\overline{\rho} h_2 = \left(p'(\overline{\rho}) - p'(\overline{\rho} + \Lambda\rho_{in})\right)\partial_x\hat\rho + \frac{\nu}{\overline{\rho}}\partial_x \tilde f + \tilde g - \frac{\nu}{\overline{\rho}}\Lambda\partial_x\rho_{in}\partial_x\hat u \\ + \overline{\rho}\partial_x\left[u(u+\Lambda u_{in})\right] + p'(\overline{\rho})\frac{\overline{\rho}^2}{\nu}u - \Lambda\rho_{in}\overline{u}\partial_x u. \tag{5.16}$$

We shall estimate each term separately.

⋆ Using the fact that p' is Lipschitz (in a neighborhood of $\overline{\rho}$), we deduce

$$\|\left(p'(\overline{\rho}) - p'(\overline{\rho} + \Lambda\rho_{in})\right)\partial_x\hat\rho\, e^{s\varphi}\xi^{-3/2}\|_{L^2((0,T)\times(0,L))} \leq C R_{in} R_\rho \leq C R_{in}^2 + R_\rho^2.$$

⋆ *Estimates on* $\partial_x \tilde f$. To estimate the second term $\nu\partial_x \tilde f/\overline{\rho}$, we develop it. Differentiating $\tilde f$, we have

$$\partial_x \tilde f = -\Lambda'\partial_x\rho_{in} + (\Lambda - \Lambda^2)\partial_{xx}(\rho_{in}u_{in}) - \Lambda\partial_x((\partial_x\rho_{in})u) - \Lambda\partial_x(\hat\rho\,\partial_x u_{in}) - \partial_x\hat\rho\,\partial_x\hat u - \hat\rho\,\partial_{xx}\hat u. \tag{5.17}$$

The first two terms are compactly supported in time away from $t=0$ and $t=T$ and depend only on (ρ_{in}, u_{in})

$$\|\left(-\Lambda'\partial_x\rho_{in} + (\Lambda-\Lambda^2)\partial_{xx}(\rho_{in}u_{in})\right)e^{s\varphi}\xi^{-3/2}\|_{L^2((0,T)\times(0,L))} \leq \mathcal{O}_{s,\lambda}(R_{in}).$$

The third one is estimated as follows

$$\|\Lambda\partial_x((\partial_x\rho_{in})u)e^{s\varphi}\xi^{-3/2}\|_{L^2((0,T)\times(0,L))} \leq C R_{in} R_u \leq C R_{in}^2 + C R_u^2.$$

Similarly,

$$\|\Lambda\partial_x(\hat\rho\,\partial_x u_{in})e^{s\varphi}\xi^{-3/2}\|_{L^2((0,T)\times(0,L))} \leq C R_{in} R_\rho \leq C R_{in}^2 + C R_\rho^2.$$

Finally, the last terms are quadratic:

$$\|\partial_x\hat\rho\,\partial_x\hat u\, e^{s\varphi}\xi^{-3/2}\|_{L^2((0,T)\times(0,L))} \leq \|s^{1/2}\xi^{1/2}\partial_x\hat\rho\|_{L^\infty((0,T);L^2(0,L))}\|s^{-1/2}\partial_x\hat u\, e^{s\varphi}\xi^{-2}\|_{L^2(0,T;L^\infty(0,L))} \\ \leq C R_\rho R_u \leq C R_\rho^2 + C R_u^2,$$

where we used the Sobolev embedding $L^2(0,T;H^1(0,L)) \to L^2(0,T;L^\infty(0,L))$ on $s^{-1/2}\partial_x u\, e^{s\varphi}\xi^{-2}$ and the fact that $s^{1/2}\xi^{1/2}e^{-s\check\varphi/2}$ is uniformly bounded on $(0,T)\times(0,L)$.

Similarly

$$\|\hat\rho\, \partial_{xx}\hat u\, e^{s\varphi}\xi^{-3/2}\|_{L^2((0,T)\times(0,L))} \leq \|s^{1/2}\hat\rho\xi^{1/2}\|_{L^\infty((0,T)\times(0,L))}\|s^{-1/2}\partial_{xx}\hat u e^{s\varphi}\xi^{-2}\|_{L^2((0,T)\times(0,L))} \leq CR_\rho R_u \leq CR_\rho^2 + CR_u^2.$$

To sum up, we have obtained the following estimate on $\partial_x \tilde f$

$$\|\partial_x\tilde f\, e^{s\varphi}\xi^{-3/2}\|_{L^2((0,T)\times(0,L))} \leq C\left(\mathcal{O}_{s,\lambda}(R_{in}) + R_\rho^2 + R_u^2\right).$$

$\star$ Let us now come back to the estimates of the terms of h_2. We already have an estimate on $\tilde g$, which is the one given by Lemma 5.1. Going on,

$$\|\Lambda\partial_x\rho_{in}\partial_x\hat u\, e^{s\varphi}\xi^{-3/2}\|_{L^2((0,T)\times(0,L))} \leq CR_{in}R_u \leq CR_{in}^2 + R_u^2.$$

Similarly,

$$\|\Lambda\rho_{in}\partial_x u\, e^{s\varphi}\xi^{-3/2}\|_{L^2((0,T)\times(0,L))} \leq CR_{in}R_u \leq CR_{in}^2 + R_u^2.$$

Then we have again a quadratic term

$$\|\partial_x\left[u(u+\Lambda u_{in})\right] e^{s\varphi}\xi^{-3/2}\|_{L^2((0,T)\times(0,L))} \leq \|\partial_x u\, e^{s\varphi}\xi^{-1}\|_{L^2((0,T)\times(0,L))}\|u\|_{L^\infty((0,T)\times(0,L))} + CR_{in}\left(\|\partial_x u\, e^{s\varphi}\xi^{-1}\|_{L^2((0,T)\times(0,L))} + \|ue^{s\varphi}\|_{L^2((0,T)\times(0,L))}\right) \leq CR_u^2 + CR_{in}^2.$$

Finally, there is a linear term in u:

$$\|u\, e^{s\varphi}\xi^{-3/2}\|_{L^2((0,T)\times(0,L))} \leq \frac{C}{s^{3/2}\lambda^2}R_u.$$

These estimates all together yield (5.11).

• *Proof of* (5.12). Thanks to (4.3), we have

$$\left\|\frac{1}{\overline{u} + u(t,0) + \Lambda u_{in}(t,0)}\right\|_{L^\infty(0,T)} \leq \frac{2}{\overline{u}} \leq C.$$

It follows that

$$|m_f(t)| \leq |u(t,0)| + C|\hat f(t,0)| + C|\partial_x u(t,0)|.$$

The difficult part consists in the estimate of $\hat f(t,0)$: for all $t\in(0,T)$,

$$\hat f(t,0) = -\Lambda'\rho_{in}(t,0) + (\Lambda-\Lambda^2)\partial_x(\rho_{in}u_{in})(t,0) - \Lambda\partial_x(\rho_{in}\hat u)(t,0) - \Lambda\hat\rho(t,0)\partial_x u_{in}(t,0) - \hat\rho(t,0)\partial_x\hat u(t,0) + \frac{\overline{\rho}}{\nu}p'(\overline{\rho})\hat\rho(t,0).$$

Hence

$$|m_f(t)| \leq |u(t,0)| + C|\partial_x u(t,0)| + C|-\Lambda'\rho_{in}(t,0) + (\Lambda-\Lambda^2)\partial_x(\rho_{in}u_{in})(t,0)| + C|\hat\rho(t,0)|.$$

Using the interpolation results of Proposition 3.5, we have that

$$\|(e^{s\varphi}\xi^{-3/2}u)(t,0)\|_{L^2(0,T)} + \|(e^{s\varphi}\xi^{-3/2}\partial_x u)(t,0)\|_{L^2(0,T)} \leq \frac{C}{s\lambda^{3/2}}R_u + \frac{C}{\lambda^{1/2}}R_u \leq \frac{C}{\sqrt\lambda}R_u.$$

Then, since $\hat\rho\in X_{s,\lambda,R_\rho}$,

$$\|(e^{s\varphi}\xi^{-3/2}\hat\rho)(t,0)\|_{L^2(0,T)} \leq \frac{C}{\sqrt\lambda}R_\rho.$$

Finally, since $\rho_{in}(t,0), \partial_x\rho_{in}(t,0), \partial_x u_{in}(t,0)$ all are in $L^\infty(0,T)$, we have

$$\|\left(-\Lambda'\rho_{in}(t,0) + (\Lambda-\Lambda^2)\partial_x(\rho_{in}u_{in}\right) e^{s\varphi}\xi^{-3/2})(t,0)\|_{L^2(0,T)} \leq \mathcal{O}_{s,\lambda}(R_{in}),$$

which proves (5.12).

• *Proof of* (5.13). This is the same proof as the one of (5.12). Actually, it is even easier to get (5.13) since $u(t,L)=0$.

The proof of Lemma 5.3 is complete. □

We can now turn to the proof that the ρ-part of F is sent into X_{s,λ,R_ρ} for a proper choice of the parameters.

• All the assumptions of Lemmas 4.6 and 4.7 are satisfied due to Corollary 5.2 and Lemma 5.3. We therefore obtain, for $s \geq s_0$ and $\lambda \geq \lambda_0$,

$$\begin{aligned}
&\|\mu_f e^{s\varphi}\xi^{-3/2}\|_{L^\infty((0,T-3T_0);L^2(0,L))} + \|\mu_f e^{s\varphi}\xi^{-3/2}\|_{L^2((0,T-3T_0);L^2(0,L))} \\
&\quad \leq C(1+\mathcal{O}_{s,\lambda}(R_{in}))\left[\left(\frac{1}{\sqrt{\lambda}}(R_u+R_\rho)+\mathcal{O}_{s,\lambda}(R_{in})\right)\right. \\
&\qquad \left. + \left(e^{8\lambda}sR_{in}^2+R_u^2\right)+\left(\mathcal{O}_{s,\lambda}(R_{in})+\frac{1}{s^{3/2}\lambda^2}R_u+R_\rho^2+R_u^2\right)\right] \\
&\quad \leq C\left(\frac{1}{\sqrt{\lambda}}(R_u+R_\rho)+\mathcal{O}_{s,\lambda}(R_{in})+R_\rho^2+R_u^2\right),
\end{aligned}$$

provided that R_{in} is sufficiently small depending on s and λ. Here, we have used (5.10)-(5.12). Similarly,

$$\begin{aligned}
&\|\mu_b e^{s\varphi}\xi^{-3/2}\|_{L^\infty((3T_0,T);L^2(0,L))} + \|\mu_b e^{s\varphi}\xi^{-3/2}\|_{L^2((3T_0,T);L^2(0,L))} \\
&\quad \leq C\left(\frac{1}{\sqrt{\lambda}}(R_u+R_\rho)+\mathcal{O}_{s,\lambda}(R_{in})+R_\rho^2+R_u^2\right).
\end{aligned}$$

From estimates (4.50)-(4.51), we deduce

$$\begin{aligned}
\|\partial_x\rho_f e^{s\varphi}\xi^{-3/2}\|_{L^2((0,T-3T_0);L^2(0,L))} &\leq C\left(\frac{1}{\sqrt{\lambda}}(R_u+R_\rho)+\mathcal{O}_{s,\lambda}(R_{in})+R_\rho^2+R_u^2\right)+\frac{C}{s^{3/2}\lambda^2}R_u \\
&\leq C\left(\frac{1}{\sqrt{\lambda}}(R_u+R_\rho)+\mathcal{O}_{s,\lambda}(R_{in})+R_\rho^2+R_u^2\right), \qquad (5.18)
\end{aligned}$$

and, similarly,

$$\|\partial_x\rho_b e^{s\varphi}\xi^{-3/2}\|_{L^2((3T_0,T);L^2(0,L))} \leq C\left(\frac{1}{\sqrt{\lambda}}(R_u+R_\rho)+\mathcal{O}_{s,\lambda}(R_{in})+R_\rho^2+R_u^2\right). \qquad (5.19)$$

Then, using estimates (4.52),

$$\begin{aligned}
&\|\partial_x\rho_f e^{s\varphi}\xi^{-2}\|_{L^\infty((0,T-3T_0);L^2(0,L))} + \|\partial_x\rho_b e^{s\varphi}\xi^{-2}\|_{L^\infty((3T_0,T);L^2(0,L))} \\
&\quad \leq C\left(\frac{1}{\sqrt{\lambda}}(R_u+R_\rho)+\mathcal{O}_{s,\lambda}(R_{in})+R_\rho^2+R_u^2\right)+\sqrt{s}R_u.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\|\partial_x\rho_f e^{s\check{\varphi}/2}\|_{L^\infty((0,T-3T_0);L^2(0,L))} + \|\partial_x\rho_b e^{s\check{\varphi}/2}\|_{L^\infty((3T_0,T);L^2(0,L))} \\
&\quad \leq Ce^{-s\varphi(T/2,0)/4}\left(\frac{1}{\sqrt{\lambda}}(R_u+R_\rho)+\sqrt{s}R_u+\mathcal{O}_{s,\lambda}(R_{in})+R_\rho^2+R_u^2\right) \\
&\quad \leq \frac{1}{\sqrt{\lambda}}(R_u+R_\rho)+\frac{1}{\sqrt{s}}R_u+\mathcal{O}_{s,\lambda}(R_{in})+R_\rho^2+R_u^2, \qquad (5.20)
\end{aligned}$$

since $\varphi(T/2,0)$ is the minimum of φ on $(0,T)\times(0,L)$ and $s\exp(-s\check{\varphi}(T/2,0)/4)$ is bounded uniformly in $s \geq 1$.

According to Corollary 5.2 and to estimate (5.9), Lemma 4.8 then yields, for all $s \geq s_0$ and $\lambda \geq \lambda_0$,

$$\|(e^{s\varphi}\xi^{-1}\rho_f)(t,L)\|_{L^2(0,T-3T_0)} \leq C(1+\mathcal{O}_{s,\lambda}(R_{in}))\left(\mathcal{O}_{s,\lambda}(R_{in}) + R_\rho + R_u^2\right) + C(1+\mathcal{O}_{s,\lambda}(R_{in}))\frac{R_u}{s^{1/2}\lambda}$$
$$\leq C\left(\mathcal{O}_{s,\lambda}(R_{in}) + R_\rho + R_u^2 + \frac{R_u}{s^{1/2}\lambda}\right). \tag{5.21}$$

Note that this last estimate and the fact that $\rho_b(t,L) = 0$ by construction imply in particular that

$$\sqrt{\lambda}\|(e^{s\varphi}\xi^{-3/2}\rho)(t,L)\|_{L^2(0,T)} \leq \sqrt{\lambda}\|(e^{s\varphi}\xi^{-3/2}\rho_f)(t,L)\|_{L^2(0,T-3T_0)}$$
$$\leq \frac{C}{\lambda}\left(\mathcal{O}_{s,\lambda}(R_{in}) + R_\rho + R_u^2 + \frac{R_u}{s^{1/2}\lambda}\right), \tag{5.22}$$

where we used that $\lambda^{3/2}\xi^{-1/2}$ is uniformly bounded in $s, \lambda \geq 1$, $(t,x) \in [0,T]\times[0,L]$. According to Lemma 4.10, using (5.18) and (5.21), we thus have

$$\|\rho_f e^{s\varphi}\xi^{-1/2}\|_{L^2((0,T-3T_0)\times(0,L))} \leq \frac{C}{s\lambda}\left(\frac{1}{\sqrt{\lambda}}(R_u+R_\rho) + \mathcal{O}_{s,\lambda}(R_{in}) + R_\rho^2 + R_u^2\right)$$
$$+ \frac{C}{\sqrt{s\lambda}}\left(\mathcal{O}_{s,\lambda}(R_{in}) + R_\rho + R_u^2 + \frac{R_u}{s^{1/2}\lambda}\right)$$
$$\leq C\left(\frac{1}{\sqrt{s\lambda}}(R_u+R_\rho) + \mathcal{O}_{s,\lambda}(R_{in}) + R_\rho^2 + R_u^2\right). \tag{5.23}$$

Using Lemma 4.10 estimate (4.57), the fact that $\rho_f(t,0) = 0$ by construction and $\lambda^{3/2}\xi^{-1/2}$ bounded uniformly in s, λ and (t,x), we also have

$$\sqrt{\lambda}\|\rho(t,0)e^{s\varphi}\xi^{-3/2}\|_{L^2(0,T)} \leq \sqrt{\lambda}\|\rho_b(t,0)e^{s\varphi}\xi^{-3/2}\|_{L^2(3T_0,T)} \leq \frac{1}{\lambda}\|\rho_b(t,0)e^{s\varphi}\xi^{-1}\|_{L^2(3T_0,T)}$$
$$\leq C\frac{1}{\sqrt{s\lambda}\,\lambda}\left(\frac{1}{\sqrt{\lambda}}(R_u+R_\rho) + \mathcal{O}_{s,\lambda}(R_{in}) + R_\rho^2 + R_u^2\right). \tag{5.24}$$

and

$$\|\rho_b e^{s\varphi}\xi^{-1/2}\|_{L^2((3T_0,T)\times(0,L))} \leq \frac{C}{s\lambda}\left(\frac{1}{\sqrt{\lambda}}(R_u+R_\rho) + \mathcal{O}_{s,\lambda}(R_{in}) + R_\rho^2 + R_u^2\right). \tag{5.25}$$

Combining (5.23) and (5.25), we obtain

$$\|\rho\, e^{s\varphi}\xi^{-1}\|_{L^2((0,T)\times(0,L))} \leq \|\rho\, e^{s\varphi}\xi^{-1/2}\|_{L^2((0,T)\times(0,L))} \leq C\left(\frac{1}{\sqrt{s\lambda}}(R_u+R_\rho) + \mathcal{O}_{s,\lambda}(R_{in}) + R_\rho^2 + R_u^2\right). \tag{5.26}$$

Similarly, combining (5.23), (5.25) and estimates (5.18), (5.19), we obtain

$$\|\partial_x\rho\, e^{s\varphi}\,\xi^{-3/2}\|_{L^2((0,T)\times(0,L))} \leq C\left(\frac{1}{\sqrt{\lambda}}(R_u+R_\rho) + \mathcal{O}_{s,\lambda}(R_{in}) + R_\rho^2 + R_u^2\right). \tag{5.27}$$

Finally, to get an $L^\infty((0,T)\times(0,L))$-bound on ρ, we first obtain $L^\infty((0,T)\times(0,L))$-bounds on ρ_f, ρ_b, using the fact that $\rho_f(t,0) = 0$ and $\rho_b(t,L) = 0$. Therefore, since $\partial_x\rho_f e^{s\check{\varphi}/2} \in L^\infty((0,T-3T_0);L^2(0,L))$ and $\partial_x\rho_b e^{s\check{\varphi}/2} \in L^\infty((3T_0,T);L^2(0,L))$, we can use Poincaré estimate:

$$\|\rho_f e^{s\check{\varphi}/2}\|_{L^\infty((0,T-3T_0)\times(0,L))} + \|\rho_b e^{s\check{\varphi}/2}\|_{L^\infty((0,T-3T_0)\times(0,L))}$$
$$\leq \frac{1}{\sqrt{\lambda}}(R_u+R_\rho) + \frac{1}{\sqrt{s}}R_u + \mathcal{O}_{s,\lambda}(R_{in}) + R_\rho^2 + R_u^2, \tag{5.28}$$

according to the estimates (5.20).

Thus, gluing these estimates, we obtain

$$\|\rho\, e^{s\check{\varphi}/2}\|_{L^\infty((0,T)\times(0,L))} \leq \frac{1}{\sqrt{\lambda}}(R_u+R_\rho) + \frac{1}{\sqrt{s}}R_u + \mathcal{O}_{s,\lambda}(R_{in}) + R_\rho^2 + R_u^2. \tag{5.29}$$

We have obtained the following.

Proposition 5.4. *There exist $R_2 > 0$, $s_2 \geq s_1$ and $\lambda_2 \geq \lambda_1$ (s_1 and λ_1 are the ones given by Corollary 5.2) such that the following holds. If*

$$R_u \leq R_2 \text{ and } R_\rho = c_1 R_u, \tag{5.30}$$

where c_1 is given by Corollary 5.2, there exists $K_2(s_2, \lambda_2, R_u) \leq K_1(s_2, \lambda_2, R_u)$ (K_1 is the one given by Corollary 5.2) such that if

$$R_{in} \leq K_2(s_2, \lambda_2, R_u), \tag{5.31}$$

then the ρ-part of $F(\hat{\rho}, \hat{u})$ belongs to X_{s_2,λ_2,R_ρ} for any $(\hat{\rho}, \hat{u})$ in $X_{s_2,\lambda_2,R_\rho} \times Y_{s_2,\lambda_2,R_u}$.

Moreover, using Corollary 5.2, the map F maps $X_{s_2,\lambda_2,R_\rho} \times Y_{s_2,\lambda_2,R_u}$ into itself.

Proof of Proposition 5.4. Estimates (5.20), (5.22), (5.24), (5.26), (5.27), (5.28) and (5.29) show that ρ satisfies

$$\begin{aligned} &\|\xi^{-1} e^{s\varphi} \rho\|_{L^2((0,T)\times(0,L))} + \|\xi^{-3/2} e^{s\varphi} \partial_x \rho\|_{L^2((0,T)\times(0,L))} + \|e^{s\check{\varphi}/2} \rho\|_{L^\infty((0,T)\times(0,L))} \\ &\quad + \|e^{s\check{\varphi}/2} \partial_x \rho\|_{L^\infty((0,T);L^2(0,L))} + \|\lambda^{1/2} [\xi^{-3/2} e^{s\varphi} \rho](\cdot, 0)\|_{L^2(0,T)} + \|\lambda^{1/2} [\xi^{-3/2} e^{s\varphi} \rho](\cdot, L)\|_{L^2(0,T)} \leq R \end{aligned}$$

where, for some C_2 independent of s, λ and R_ρ, R_u, R_{in},

$$R = C_2 \left(\left(\frac{1}{\sqrt{s}} + \frac{1}{\sqrt{\lambda}} \right) (R_\rho + R_u) + R_\rho^2 + R_u^2 + \mathcal{O}_{s,\lambda}(R_{in}) \right).$$

Using Corollary 5.2, with the choices proposed in the Proposition 5.4, we already know that the u-part of $F(\hat{\rho}, \hat{u})$ belongs to Y_{s_2,λ_2,R_u} for any $(\hat{\rho}, \hat{u})$ in $X_{s_2,\lambda_2,R_\rho} \times Y_{s_2,\lambda_2,R_u}$.

Furthermore, using the constants $c_1 > 0$ and $R_1 > 0$ of Corollary 5.2, taking $R_\rho = c_1 R_u$, we can choose $R_2 \leq R_1$ such that for $R_u \leq R_2$, and $R_\rho = c_1 R_u$

$$C_2 R_\rho^2 \leq R_\rho/4 \quad \text{and} \quad C_2 R_u^2 \leq R_\rho/4.$$

We then can choose $\tilde{s}_2 \geq s_1$ and $\lambda_2 \geq \lambda_1$ so that for $s \geq \tilde{s}_2$,

$$C_2 \left(\frac{1}{\sqrt{s}} + \frac{1}{\sqrt{\lambda_2}} \right) (R_\rho + R_u) \leq \frac{R_\rho}{4}.$$

We then finally choose $R_{in} \leq K_1(s, \lambda_2, R_u)$ small enough so that $C_2 \mathcal{O}_{s,\lambda_2}(R_{in}) \leq R_\rho/4$. We thus obtain $R \leq R_\rho$ provided $R_u \leq R_2$, $R_\rho = c_1 R_u$, $\lambda = \lambda_2$, $s \geq \tilde{s}_2$ and $R_{in} \leq K_2(s, \lambda_2, R_u)$.

Of course, we shall furthermore estimate $\partial_t \rho$ in $L^2((0,T) \times (0,L))$: Using equation (2.16), we have

$$\begin{aligned} \|\partial_t \rho\|_{L^2((0,T)\times(0,L))} \leq C \|\hat{f}\|_{L^2((0,T)\times(0,L))} + C \|\partial_x u\|_{L^2((0,T)\times(0,L))} \\ + C \|\partial_x \rho\|_{L^2((0,T)\times(0,L))} + C \|\rho\|_{L^2((0,T)\times(0,L))}. \end{aligned}$$

But all the terms in the right hand side can be bounded by

$$\exp(-s\varphi(T/2, 0)/4)(R_\rho + R_u),$$

hence we can choose $s_2 \geq \tilde{s}_2$ large enough such that

$$\|\partial_t \rho\|_{L^2((0,T)\times(0,L))} \leq R_\rho.$$

This completes the proof of Proposition 5.4. □

Remark 5.5. *We emphasize that the possibility of choosing the second parameter λ (besides s) is required in our proof in order to suitably estimate $m_f e^{s\varphi} \xi^{-3/2}$ and $m_b e^{s\varphi} \xi^{-3/2}$ in $L^2(0,T)$, see estimates* (5.12) *and* (5.13) *respectively. More precisely, this comes from the fact that m_f and m_b involve the terms $\partial_x u(t, 0)$ and $\partial_x u(t, L)$ respectively.*

5.3 Conclusion

Proof of Theorem 1.1. We begin with the topological aspects. We equip $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$ with the $L^2((0,T)\times(0,L))^2$ topology.

Let us first check that $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$ is compact. It is closed under the $L^2((0,T)\times(0,L))^2$ convergence, because clearly the uniform inequalities defining it are stable under a passage to the limit in the sense of distributions. Now that it is relatively compact is a consequence of the uniform estimate defining $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$. Let (ρ_n, u_n) a sequence in $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$. Then (u_n) is bounded in $L^2(0,T;H^2(0,L))$ and in $H^1(0,T;L^2(0,L))$, hence is it relatively compact in $L^2((0,T)\times(0,L))$ by interpolation and Rellich's theorem. All the same, (ρ_n) is bounded in $L^2(0,T;H^1(0,L))$ and in $H^1(0,T;L^2(0,L))$, so the compactness follows easily.

Now, we choose the parameters R_ρ, R_u, R_{in}, $s = s_2$ and $\lambda = \lambda_2$ as to satisfy the assumptions of Proposition 5.4. Hence the map F maps $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$ into itself.

Let us now turn to the continuity of the operator F described above under the L^2 topology. Consider (ρ_n, u_n) a sequence in $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$, converging to (ρ, u) in $L^2((0,T)\times(0,L))^2$, and consequently in any topology stronger than $L^2((0,T)\times(0,L))^2$ for which $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$ is still relatively compact: (u_n) also converges in the sense of the weak $L^2(0,T;H^2(0,L))$ and $H^1(0,T;L^2(0,L))$ topologies and the strong $L^\infty((0,T)\times(0,L))$ and $L^2((0,T);W^{1,\infty}(0,L))$ ones; (ρ_n) also converges in the sense of the weak $L^2(0,T;H^1(0,L))$ and $H^1(0,T;L^2(0,L))$ topologies and the strong $L^\infty((0,T);L^2(0,L))$ and $L^2((0,T);L^\infty(0,L))$ ones.

Let us prove that the images under F converge correspondingly. By the compactness of $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$, we only have to prove that $F(\rho,u)$ is the unique limit point of the sequence $(F(\rho_n,u_n))$. Hence we suppose (relabeling the subsequence) that $F(\rho_n,u_n)$ converges to (ρ_∞, u_∞) and have to prove that $(\rho_\infty,u_\infty) = F(\rho,u)$. Then it is clear using the convergences above that each term in $g(\rho_n,u_n)$ converges in the sense of distributions to its counterpart in $g(\rho_\infty,u_\infty)$. Due to the uniform estimates of $(g(\rho_n,u_n))_n)$ in $L^2((0,T)\times(0,L); e^{s\varphi}\xi^{-3/2}\,dx\,dt)$ (see Subsection 5.1), one has the weak $L^2((0,T)\times(0,L))$ convergence of $e^{s\varphi}\xi^{-3/2}g(\rho_n,u_n)$ towards $e^{s\varphi}\xi^{-3/2}g(\rho,u)$. Hence one sees that we can pass to the limit in the variational formulation (3.15), so by uniqueness in Lax-Milgram's theorem, the u-part of $F(\rho,u)$ coincides with u_∞. Reasoning in the same way, using the uniqueness of the solution of the transport equations (4.1)-(4.2), we obtain $F(\rho,u) = (\rho_\infty, u_\infty)$.

In that case, all the assumptions of Schauder's fixed point theorem are fulfilled. Consequently, F admits a fixed point (ρ,u) in $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$. That it satisfies the equation comes from the construction. That $(\rho,u)(T) = 0$ comes from the definition of the space $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$ and of the weight function φ. The regularity of the controlled trajectory also follows easily.

This concludes the proof of Theorem 1.1. □

6 Appendix

6.1 Computation of f

To compute f in (2.12), we use that

$$\partial_t \rho_{\mathcal{S}} + \partial_x(\rho_{\mathcal{S}} u_{\mathcal{S}}) = 0 \text{ in } (0,T)\times(0,L).$$

Thus, setting $\rho = \rho_{\mathcal{S}} - \overline{\rho} - \Lambda\rho_{in}$ and $u = u_S - \overline{u} - \Lambda u_{in}$, we have

$$\begin{aligned}
0 =& \partial_t(\rho + \Lambda\rho_{in}) + \partial_x\left((\overline{\rho} + \rho + \Lambda\rho_{in})(\overline{u} + u + \Lambda u_{in})\right) \\
=& \partial_t\rho + \Lambda'\rho_{in} + \Lambda\partial_t\rho_{in} + \partial_x\left((\overline{\rho} + \Lambda\rho_{in})(\overline{u} + \Lambda u_{in})\right) + \partial_x\left(\rho(\overline{u} + u + \Lambda u_{in})\right) + \partial_x\left((\overline{\rho} + \Lambda\rho_{in})u\right) \\
=& \partial_t\rho + (\overline{u} + u + \Lambda u_{in}))\,\partial_x\rho + \overline{\rho}\partial_x u \\
& + \Lambda'\rho_{in} - \Lambda\partial_x\left((\overline{\rho} + \rho_{in})(\overline{u} + u_{in})\right) + \partial_x\left((\overline{\rho} + \Lambda\rho_{in})(\overline{u} + \Lambda u_{in})\right) + \rho\partial_x(u + \Lambda u_{in}) + \Lambda\partial_x(\rho_{in}u),
\end{aligned}$$

where we used (2.2) in the last identity.

This yields to f as in (2.12) once we have remarked that:

$$-\Lambda\partial_x\left((\overline{\rho} + \rho_{in})(\overline{u} + u_{in})\right) + \partial_x\left((\overline{\rho} + \Lambda\rho_{in})(\overline{u} + \Lambda u_{in})\right) = \partial_x(\rho_{in}u_{in})\left(\Lambda^2 - \Lambda\right).$$

6.2 Computation of g

We start by using the equation of $u_{\mathcal{S}}$ (see the second equation in (1.1)) as well as the expressions of $u_{\mathcal{S}}$ and $\rho_{\mathcal{S}}$ (see (2.9)) :

$$\begin{aligned} 0 =&(\rho + \overline{\rho} + \Lambda\rho_{in})[\partial_t u + \partial_t(\Lambda u_{in}) + (u + \overline{u} + \Lambda u_{in})(\partial_x u + \Lambda \partial_x u_{in})] \\ &- \nu\partial_{xx}u - \nu\Lambda\partial_{xx}u_{in} + p'(\rho + \overline{\rho} + \Lambda\rho_{in})(\partial_x\rho + \Lambda\partial_x\rho_{in}). \end{aligned}$$

Since we look for the equation of u written in (2.11), we regroup the previous expression in the following way:

$$\begin{aligned} 0 =&(\overline{\rho} + \Lambda\rho_{in})(\partial_t u + \overline{u}\partial_x u) + \Lambda(\overline{\rho} + \Lambda\rho_{in})(\partial_t u_{in} + (\overline{u} + \Lambda u_{in})\partial_x u_{in}) \\ &+ (\overline{\rho} + \Lambda\rho_{in})(\Lambda' u_{in} + \Lambda\partial_x(u u_{in}) + u\partial_x u) \\ &+ \rho[\partial_t u + \partial_t(\Lambda u_{in}) + (u + \overline{u} + \Lambda u_{in})(\partial_x u + \Lambda\partial_x u_{in})] \\ &- \nu\partial_{xx}u - \nu\Lambda\partial_{xx}u_{in} + p'(\overline{\rho} + \Lambda\rho_{in})\partial_x\rho + \Lambda p'(\overline{\rho} + \Lambda\rho_{in})\partial_x\rho_{in} \\ &+ (p'(\rho + \overline{\rho} + \Lambda\rho_{in}) - p'(\overline{\rho} + \Lambda\rho_{in}))(\partial_x\rho + \Lambda\partial_x\rho_{in}). \end{aligned}$$

Next, we replace $(\overline{\rho} + \Lambda\rho_{in})(\partial_t u + \overline{u}\partial_x u) - \nu\partial_{xx}u$ by g. This yields

$$\begin{aligned} g(\rho, u) =& -\Lambda((\overline{\rho} + \Lambda\rho_{in})(\partial_t u_{in} + (\overline{u} + \Lambda u_{in})\partial_x u_{in}) - \nu\partial_{xx}u_{in} + p'(\overline{\rho} + \Lambda\rho_{in})\partial_x\rho_{in}) \\ &- (\overline{\rho} + \Lambda\rho_{in})(\Lambda' u_{in} + \Lambda\partial_x(u u_{in}) + u\partial_x u) \\ &- \rho[\partial_t u + \partial_t(\Lambda u_{in}) + (u + \overline{u} + \Lambda u_{in})(\partial_x u + \Lambda\partial_x u_{in})] \\ &- (p'(\rho + \overline{\rho} + \Lambda\rho_{in}) - p'(\overline{\rho} + \Lambda\rho_{in}))(\partial_x\rho + \Lambda\partial_x\rho_{in}) - p'(\overline{\rho} + \Lambda\rho_{in})\partial_x\rho. \end{aligned} \tag{6.1}$$

The last two lines in this expression are exactly the two last lines in (2.13). In the second line of (6.1), the first term is the first one in the first line of (2.13) while the second and third terms correspond to the third line of (2.13).

We still have to work with the first line of (6.1). For this, we make the difference between the first line of (6.1) and the equation of u_{in} (see (2.2)) :

$$(\overline{\rho} + \rho_{in})(\partial_t u_{in} + (\overline{u} + u_{in})\partial_x u_{in}) - \nu\partial_{xx}u_{in} + p'(\overline{\rho} + \rho_{in})\partial_x\rho_{in} = 0.$$

We obtain

$$\begin{aligned} &-\Lambda[(\overline{\rho} + \Lambda\rho_{in})(\partial_t u_{in} + (\overline{u} + \Lambda u_{in})\partial_x u_{in}) - \nu\partial_{xx}u_{in} + p'(\overline{\rho} + \Lambda\rho_{in})\partial_x\rho_{in}] \\ &+\Lambda[(\overline{\rho} + \rho_{in})(\partial_t u_{in} + (\overline{u} + u_{in})\partial_x u_{in}) - \nu\partial_{xx}u_{in} + p'(\overline{\rho} + \rho_{in})\partial_x\rho_{in}] \\ &= -\Lambda[(\overline{\rho} + \Lambda\rho_{in})(\partial_t u_{in} + (\overline{u} + \Lambda u_{in})\partial_x u_{in}) - (\overline{\rho} + \rho_{in})(\partial_t u_{in} + (\overline{u} + u_{in})\partial_x u_{in})] \\ &-\Lambda\partial_x\rho_{in}(p'(\overline{\rho} + \Lambda\rho_{in}) - p'(\overline{\rho} + \rho_{in})). \end{aligned}$$

In this last identity, the last term is the second term in the first line of (2.13) while, by a simple computation, the first term equals

$$\rho_{in}\partial_t u_{in}(\Lambda - \Lambda^2) + \rho_{in}\overline{u}\partial_x u_{in}(\Lambda - \Lambda^2) + \overline{\rho}u_{in}\partial_x u_{in}(\Lambda - \Lambda^2) + \rho_{in}u_{in}\partial_x u_{in}(\Lambda - \Lambda^3),$$

which constitutes exactly the second line of (2.13).

6.3 Remarks of Proposition 2.1

Actually, Matsumura and Nishida [18, Theorem 7.1] prove a much stronger result than the one stated in Proposition 2.1 (see also [6]):

Theorem 6.1. *Let $\overline{\rho}$ be such that $p'(\overline{\rho}) > 0$. Then there exists a constant $c > 0$ such that, if $(\rho_0 - \overline{\rho}) \in H^3(\mathbb{R}^3)$, $u_0 \in H^3(\mathbb{R}^3)$ and*

$$\|\rho_0 - \overline{\rho}\|_{H^3(\mathbb{R})} + \|u_0\|_{H^3(\mathbb{R})} \leq c,$$

then the three-dimensional isentropic compressible Navier-Stokes equation:

$$\begin{cases} \partial_t \rho + div(\rho u) = 0, \\ \partial_t(\rho u) + div(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu) \nabla div u + \nabla P(\rho) = 0, \\ (\rho, u)_{|t=0} = (\rho_0, u_0), \end{cases}$$

has a unique global solution (ρ, u) *such that the density* $\rho - \overline{\rho} \in C(\mathbb{R}^+; H^3(\mathbb{R})) \cap C^1(\mathbb{R}^+; H^2(\mathbb{R}))$ *and the velocity* $u \in C(\mathbb{R}^+; H^3(\mathbb{R}^3)) \cap C^1(\mathbb{R}^+; H^1(\mathbb{R}^3))$*. Moreover for some* $C > 0$*:*

$$\|(\rho - \overline{\rho}, u)\|_{L^\infty(\mathbb{R}^+; H^3(\mathbb{R})^2) \cap W^{1,\infty}(\mathbb{R}^+; H^2(\mathbb{R}) \times H^1(\mathbb{R}^3))} \leq C \|(\rho_0 - \overline{\rho}, u)\|_{H^3(\mathbb{R})}.$$

Let us add several comments on this result.

- Mastumura and Nishida's result give global in time solutions. We merely need the local result.
- In fact Mastumura and Nishida consider even the more general system, non isentropic, with the equation of temperature. The isentropic case is actually simpler and still contained in their analysis (see the end of [18, Section 1]).
- Mastumura and Nishida's result is three-dimensional, but their analysis (relying only on energy estimates and characteristics for the density equation) applies in the one dimensional setting. Actually, the one dimensional case would be much simpler, since the Morrey-Sobolev injections are better, and the energy estimates way simplify.
- In the above result, the reference velocity $\overline{u}$ is not taken into account as in Proposition 2.1. But it is just a matter of taking the Galilean invariance of the equation into account to deduce this statement.

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