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Local Existence of the Borel Transform in Euclidean Φ_4^4

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Abstract. We bound rigorously the large order behaviour of Φ_4^4 euclidean perturbative quantum field theory, as the simplest example of renormalizable, but non-super-renormalizable theory. The needed methods are developed to take into account the structure of renormalization, which plays a crucial role in the estimates. As a main theorem, it is shown that the Schwinger functions at order n are bounded by $K^n n!$, which implies a finite radius of convergence for the Borel transform of the perturbation series.

I. Introduction

In this work we give rigorous bounds on the Feynman amplitudes at large order for renormalized Φ_4^4 euclidean quantum field theory. We prove the "local existence" of the Borel transform of its perturbative series for the connected Green's functions (or Schwinger functions): the Borel transformed perturbative series is shown to have a finite radius of convergence. Such a theorem is reached by finding direct estimates of the renormalized amplitudes related to Feynman graphs. We combine the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) renormalization scheme with the use of the so-called Hepp sectors to give bounds which depend on the renormalization structure of the graphs. Then we count the number of graphs with a given renormalization structure, and we can bound the term of order n of any Schwinger function by $K^n n!$, which proves our main result.

The same property was already known for Φ_ν^4 in the super-renormalizable domain $\mathrm{Re}\,\nu<4$ [1]. Through Borel summability for integer dimensions $\nu=2,3$ [2], perturbative and nonperturbative methods of constructive quantum field theory are connected [3]. In this domain, each Feynman amplitude is uniformly bounded by K^n. Conversely for $\nu=4$, such a bound cannot exist and the situation remains unresolved. Despite all efforts there has been almost no rigorous result. It has been remarked that some graphs of order n grow like $n!$ [4]. (Of course we recover this particular result and we show how few graphs are dangerous.) Heuristic arguments have been given, to make plausible the appearance of singularities, called renormalons, on the real positive axis of the Borel plane [5].

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That would destroy the usual Borel summability. However it was believed that the Borel transformed series has a finite radius of convergence; this has been evaluated by semi-classical approximations [6].

Since we prove rigorously this last fact, we hope to provide a new incitement to the study of renormalizable (but non-super-renormalizable) field theories. Starting from this result, one could try to perform an analytic continuation of the function we define in the Borel plane, or to use our methods for discovering a new constructive approach.

This paper is organized as follows: Sect. II is devoted to the definitions and main results. In Sect. III we define one of our main tools, the classification of forests, and we give the proof of the bound on individual amplitudes. Sect. IV gives a brief discussion on our results and their possible generalizations. Appendices A–C solve the graphical and combinatoric problems, very different from those encountered in the super-renormalizable case.

II. Definitions and Results

II.1. The Φ_4^4 Model

We consider an euclidean field theory of massive scalar self-interacting bosons in 4-dimensional space-time. The interaction is defined by

$$\mathscr{L}_I(\Phi) = -\lambda\Phi^4 + \text{counter-terms}.$$

Up to now this is a field theory only in the perturbative sense. We define the truncated Schwinger function of N fields, $N \geqq 2$, via a formal expansion in the coupling constant:

$$\mathbf{S}_N^T(\not{p}, \lambda) = \delta\left(\sum_{a=1}^{N} \not{p}_a\right) \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \mathbf{a}_n^N(\not{p}), \tag{II.1}$$

where $\not{p} = (\not{p}_1, \ldots, \not{p}_N)$ is a set of external euclidean momenta, and $\mathbf{a}_n^N$ is a sum of renormalized Feynman amplitudes, associated to Feynman graphs with n internal vertices and N external legs. These graphs have exactly 4 lines attached at each vertex.

The unrenormalized Feynman amplitude I_G associated to the graph G is given in the α-parametric representation by the following formal integral:

$$I_G(\not{p}) = \int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{\ell} d\alpha_i \left[\exp\left(-\mu^2 \sum_{i=1}^{\ell} \alpha_i\right)\right] Z_G(\not{p}, \alpha), \tag{II.2}$$

where a parameter α_i has been attached to each internal line $i = 1, \ldots, \ell$ of the graph G, and μ is the mass of the bosons, which we fix at the value 1 in the rest of the paper.

$$Z_G(\not{p}, \alpha) = 1/[U_G(\alpha)]^2 \exp[-V_G(\not{p}, \alpha)/U_G(\alpha)], \tag{II.3}$$

$$U_G(\alpha) = \sum_S \prod_{i \notin S} \alpha_i, \tag{II.4}$$

$$V_G(\not{p}, \alpha) = \sum_T \left(\prod_{i \notin T} \alpha_i\right)\left(\sum_{a \in T_1} \not{p}_a\right)^2. \tag{II.5}$$

U_G and V_G are the standard Symanzik polynomials: S runs over the spanning trees (or "one-trees") of G, T over the 2-trees of G which separate the external lines of G into two non-empty sets, one of which is T_1. It is also convenient to define:

$$|p| = 1 + N \sum_{a=1}^{N} p_a^2 . \tag{II.6}$$

This is not a norm in the usual sense, but we have:

$$V_G(p, \alpha) \leqq |p| \sum_T \prod_{i \notin T} \alpha_i . \tag{II.7}$$

Before renormalizing, hence defining a finite part of the integral (II.2), we introduce our definitions and notations for the graphs, used everywhere in the rest of the paper.

II.2. Graph Theory

Throughout this paper, the word graph means a "labeled" graph, hence a set of distinguished vertices with internal and external lines. For any graph G we define:

$n(G)$ as the number of vertices of G,
$\ell(G)$ as the number of internal lines of G,
$N(G)$ as the number of external lines of G,
$L(G)$ as the number of independent loops of G,
$c(G)$ as the number of connected components of G.
We have the topological relation

$$L(G) = \ell(G) - n(G) + c(G) . \tag{II.8}$$

Moreover if G appears in the expansion (II.1) it has exactly 4 lines attached at each vertex. Hence $N(G)$ is even, and

$$\ell(G) = 2n(G) - N(G)/2 . \tag{II.9}$$

The superficial degree of convergence of G is defined by:

$$\omega(G) = \ell(G) - 2L(G) . \tag{II.10}$$

From (II.8) and (II.9) we have also, if G is a connected graph of Φ_4^4

$$\omega(G) = N(G)/2 - 2 . \tag{II.11}$$

Definition II.1. A subgraph F of G is a set of internal lines of G, together with the corresponding attached vertices. An external vertex of F is a vertex attached to at least one external line of F.

We extend also the definitions of ℓ, n, L, N, c, ω to subgraphs in the natural way. $F \subset F'$ means that the set of internal lines defining F is strictly included in the set of internal lines defining F'. For non-strict inclusion we always use the symbol $\subseteqq$.

Since we look at the expansion of truncated Schwinger functions with $N \geqq 2$, there will be no vacuum ($N=0$) graphs or subgraphs in our problem. Moreover the graphs containing tadpoles (subgraphs with only one external vertex) will vanish in expansion (II.1) after renormalization (see Subsect. II.3).

Definition II.2. A subgraph F of G is said to be divergent if $\omega(F) \leqq 0$. Using (II.11) we remark that the connected divergent subgraphs F have $N(F)=4$ or $N(F)=2$. They will be called quadrupeds or bipeds, respectively.

Definition II.3. Two subgraphs F, F' are disjoint if they have no line and no vertex in common. They overlap if they are not disjoint and do not satisfy an inclusion relation ($F \subsetneqq F'$ or $F' \subsetneqq F$). A forest $\mathscr{F}$ is a set of non-overlapping connected subgraphs. F is said to be compatible with $\mathscr{F}$ if $\mathscr{F} \cup \{F\}$ is a forest.

For any $\mathscr{F}$ and F compatible with $\mathscr{F}$, we define

$$\mathscr{A}_{\mathscr{F}}(F) = \{F' \mid F' \in \mathscr{F}; F' \subset F; \nexists F'' \in \mathscr{F}, F' \subset F'' \subset F\},$$
$$A_{\mathscr{F}}(F) = \bigcup_{F' \in \mathscr{A}_{\mathscr{F}}(F)} F' = \bigcup_{\substack{F' \in \mathscr{F} \\ F' \subset F}} F'.$$

$A_{\mathscr{F}}(F)$ may be empty and generally does not belong to $\mathscr{F}$, except if $\mathscr{A}_{\mathscr{F}}(F)$ has exactly one element. Conversely we denote $B_{\mathscr{F}}(F)$ the smallest subgraph in $\mathscr{F}$ which strictly includes F

$$B_{\mathscr{F}}(F) \in \mathscr{F}; \quad B_{\mathscr{F}}(F) \supset F; \quad \nexists F'' \in \mathscr{F}, \quad F \subset F'' \subset B_{\mathscr{F}}(F).$$

If there exists no such subgraph, we define $B_{\mathscr{F}}(F) = G$. If $F \subset F'$ we have the usual notion of reduced graph: F'/F is the graph obtained from F' when each connected component of F has been reduced to a single vertex. If F is compatible with $\mathscr{F}$ we define: $F/\mathscr{F} = F/A_{\mathscr{F}}(F)$. Finally for $F \in \mathscr{F}$ and any subgraph X, denote by $X \cap F/\mathscr{F}$ the subgraph $[(X \cup A_{\mathscr{F}}(F)) \cap F]/\mathscr{F}$.

Definition II.4. A subgraph F is said to be one line reducible (OLR) if there exists a line of F such that by its removal the number of connected components of F increases. A single line is OLR. A connected subgraph which is not OLR is called proper. A proper component of a subgraph F is a maximal proper subgraph $F' \subsetneqq F$.

Definition II.5. A quadruped Q is said to be open if it is proper, and if there exists a proper biped B such that $Q \subset B$, and that both external vertices of B are external vertices of Q. The biped B is then unique and is called the closure Q^* of Q. In other words an open quadruped Q is obtained from its closure $B = Q^*$ by removing a single line, or a chain of proper bipeds, as pictured in Fig. 1.

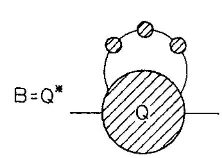

Fig. 1

A proper subgraph F is said to be closed if

$$\forall Q \text{ open quadruped}, \quad Q \subsetneqq F \Rightarrow Q^* \subsetneqq F. \tag{II.12}$$

Finally we define the closure F^* of any proper subgraph F as the smallest closed subgraph containing F. It is easy to verify that this extended definition coincides with the former one for open quadrupeds, that a proper biped is always closed and that the closure preserves the inclusion relations:

$$\forall F_1, F_2 \text{ proper}, \quad F_1 \subsetneqq F_2 \Rightarrow F_1^* \subsetneqq F_2^* . \tag{II.13}$$

Moreover if F_1^* and F_2^* overlap, then F_1 and F_2 also overlap. A connected divergent subgraph which is not closed is called unessential. Then either F is proper and is an open quadruped Q (Fig. 1), or it is not proper (Fig. 2 for bipeds, or Fig. 3 for quadrupeds).

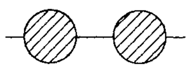

Fig. 2

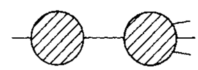

Fig. 3

A forest of proper (respectively closed, divergent closed, etc. ...) subgraphs is called in brief a proper (respectively closed, divergent closed, etc. ...) forest. We remark that in the reduction of a graph by a closed divergent forest no tadpole may appear.

Definition II.6. For any divergent forest $\mathscr{F}$ we define:

$$f(\mathscr{F}) = q(\mathscr{F}) + 2b(\mathscr{F}), \tag{II.14}$$

where $q(\mathscr{F})$ and $b(\mathscr{F})$ are respectively the numbers of quadrupeds and of bipeds in $\mathscr{F}$. Then we define

$$f_1(G) = \sup_{\mathscr{F} \text{ closed divergent forests of } G} [f(\mathscr{F})], \tag{II.15}$$

$$f_2(G) = \begin{cases} \sup\limits_{\mathscr{F} \text{ forests of quadrupeds of } G} [f(\mathscr{F})], & \text{if} \quad N(G) > 2, \\ 1 + \sup\limits_{\mathscr{F} \text{ forests of quadrupeds of } G} [f(\mathscr{F})], & \text{if} \quad N(G) = 2. \end{cases} \tag{II.16}$$

Lemma II.1.

$$f_1(G) \leqq f_2(G). \tag{II.17}$$

Proof. For any forest $\mathscr{F}$ we call $\mathscr{B}(\mathscr{F})$ the subforest of its bipeds. Let $\mathscr{F}$ be a closed divergent forest, with $\mathscr{B}(\mathscr{F}) = \{B_1, \ldots, B_j, \ldots, B_{b(\mathscr{F})}\}$. We choose a line i_1 in $B_1 - A_{\mathscr{F}}(B_1)$ (which is not empty) and an external line i'_1 of B_1, internal in G (if $B_1 \neq G$). Then we define

$$Q(B_1) = B_1 - \{i_1\}; \quad R(B_1) = B_1 \cup \{i'_1\}.$$

Any $F \in \mathscr{F}$ which overlaps with $R(B_1)$ is replaced by $F \cup R(B_1) = F^1$, and we define a new forest $\mathscr{F}_1$ by these replacements, by removing B_1 and by adding $Q(B_1)$ and $R(B_1)$ At this first step, since each F is proper, $F \cap R(B_1)$ is empty or is the vertex at the end of i'_1 which does not belong to B_1. In the second case $N(F \cup R(B_1)) = N(F)$ (see Figs. 2 or 3) and $\mathscr{F}_1$ satisfies the following conditions (H), which we take as induction hypotheses at the step j; let us assume we have built a divergent forest $\mathscr{F}_j$ such that

$$\text{(H)}\begin{cases} f(\mathscr{F}_j)=f(\mathscr{F}); \quad b(\mathscr{F}_j)=b(\mathscr{F})-j; & \text{(H1)}\\ \forall F\in\mathscr{F} \text{ and } \forall Q \text{ non-closed proper component of } F, & \\ Q^*=B_k \text{ with } k\leqq j; & \text{(H2)}\\ \mathscr{B}(\mathscr{F}_j)=\{B^j_{j+1},\ldots,B^j_{j'},\ldots,B^j_{b(\mathscr{F})}\}; \text{ and } F=B^j_{j'}\Leftrightarrow B_{j'} \text{ proper component of } F. & \text{(H3)}\end{cases}$$

Then defining i_{j+1}, i'_{j+1}, $Q(B^j_{j+1})$, $R(B^j_{j+1})$ in the same way as before, with $\mathscr{F}$ replaced by $\mathscr{F}_j$, it is easy to verify that

a) $i_{j+1}\in B_{j+1}$,

b) $Q(B^j_{j+1})\notin\mathscr{F}_j$ since it has a non-closed proper component whose closure is B_{j+1} [see (H2)],

c) $R(B^j_{j+1})\notin\mathscr{F}_j$ since it has B_{j+1} as a proper component [see (H3)],

d) If $F\in\mathscr{F}_j$ overlaps with $R(B^j_{j+1})$, $F\cap R(B^j_{j+1})$ is the vertex at the end of i'_{j+1} which does not belong to B^j_{j+1} [see again (H3)].

This allows us to perform the step $j+1$, and to build $\mathscr{F}_{j+1}$ still satisfying the conditions (H). Finally $\mathscr{F}_{b(\mathscr{F})}$ is a forest of quadrupeds with $f(\mathscr{F})$ elements, or possibly $f(\mathscr{F})-1$ if G is a biped (in which case $R(G)$ cannot be defined). In any case, with the definintion (II.16), (II.17) is proved.

Remark. In fact, given a forest $\mathscr{F}$ of quadrupeds, by considering the forest $\mathscr{F}^*$ of the closures of the proper components of the subgraphs in $\mathscr{F}$, one could show $f_1(G)=f_2(G)$. Since we do not use this equality, but only (II.17), we leave its verification to the reader.

The numbers $f_1(G)$ and $f_2(G)$ will be useful to state precisely the results of the next sections. It is proved in Appendix C that $f_2(G)\leqq n(G)$. Hence by (II.17) we have also $f_1(G)\leqq n(G)$.

II.3. Renormalization

Following [7] we define the usual BPHZ renormalized Feynman amplitude with subtractions at zero external momenta by an operator acting directly on the integrand Z_G in the α-parameters representation (II.2). By this subtraction procedure, the graphs with tadpoles subgraphs vanish.

For every connected subgraph F of G we define a Taylor operator $\mathscr{T}_F$ acting on Z_G in the following way: let T^r be the usual Taylor operator of order r; if $f(x)$ is $\mathscr{C}^\infty$ at $x=0$ we have

$$T^r f=0, \quad \text{if} \quad r<0,$$

$$T^r f=\sum_{s=0}^{r}\frac{x^s}{s!}f^{(s)}(0), \quad \text{if} \quad r\geqq 0.$$

Now if f is such that $x^m f$ is $\mathscr{C}^\infty$ at $x=0$ for some integer m, we define

$$\mathscr{T}^r_x f=x^{-m}T^{m+r}(x^m f).$$

This definition is independent of m. Let $Z_G(\varrho_F, \alpha)$ be obtained from $Z_G(\alpha)$ by scaling α_i into $\varrho_F \alpha_i$ for $i \in F$; then $\mathscr{T}_F$ is finally defined by

$$\mathscr{T}_F Z_G(\alpha) = [\mathscr{T}_{\varrho_F}^{-\ell(F)} Z_G(\varrho_F, \alpha)]_{\varrho_F = 1}. \tag{II.18}$$

The renormalized amplitude associated to G is then expressed by the following absolutely convergent integral representation

$$I_G^R(p) = \int_0^\infty \dots \int_0^\infty \prod_{i=1}^{\ell} d\alpha_i \left[\exp\left(-\sum_{i=1}^{\ell} \alpha_i\right)\right] Z_G^R(p, \alpha), \tag{II.19}$$

$$Z_G^R(p, \alpha) = \mathscr{R}\{1/[U_G(\alpha)]^2 \exp[-V_G(p, \alpha)/U_G(\alpha)]\}. \tag{II.20}$$

The operator $\mathscr{R}$ may be defined in many equivalent ways, among which we select the following one [7]

$$\mathscr{R} = \sum_{\mathscr{F}} \prod_{F \in \mathscr{F}} (-\mathscr{T}_F), \tag{II.21}$$

where the sum is performed over all proper divergent forests of G, including the empty one. We note that Taylor operators of non-overlapping subgraphs commute, hence the products in (II.21) are taken in an arbitrary order.

It is however possible to simplify formulae (II.18) and (II.21) to obtain a more practical form of the renormalization, adapted to our specific Φ_4^4 model. We introduce the following notations, for F a subgraph of G: S_F (respectively T_F) is the set of spanning trees (respectively two-trees) whose restriction to F is a spanning tree of F; $S_{\bar{F}}$ (respectively $T_{\bar{F}}$) is the set of spanning trees (respectively two-trees) whose restriction to F is not a spanning tree of F.

U_F, $U_{\bar{F}}$, V_F, $V_{\bar{F}}$ are defined by formulae (II.4) and (II.5) with the sums respectively restricted to S_F, $S_{\bar{F}}$, T_F, $T_{\bar{F}}$. These notations help us to give explicitly the action of the Taylor operators:

i) if F is convergent,

$$\mathscr{T}_F Z_G = 0, \tag{II.22}$$

ii) if F is a quadruped, we have

$$U_G(\varrho_F, \alpha) = \varrho_F^{L(F)}[U_F(\alpha) + \varrho_F \tilde{U}_{\bar{F}}(\varrho_F, \alpha)],$$

$$V_G(\varrho_F, p, \alpha) = \varrho_F^{L(F)}[V_F(p, \alpha) + \varrho_F \tilde{V}_{\bar{F}}(\varrho_F, p, \alpha)],$$

where $\tilde{U}_{\bar{F}} = U_{\bar{F}}/\varrho_F$ and $\tilde{V}_{\bar{F}} = V_{\bar{F}}/\varrho_F$ are regular at $\varrho_F = 0$. Hence

$$\mathscr{T}_F Z_G = 1/[U_F(\alpha)]^2 \exp(-V_F(p, \alpha)/U_F(\alpha)) = \mathscr{T}_{\chi_F}^0 Z_G(\chi_F, p, \alpha), \tag{II.23}$$

with

$$Z_G(\chi_F, p, \alpha) = \frac{1}{[U_F + \chi_F U_{\bar{F}}]^2} \exp\left(-\frac{V_F + \chi_F V_{\bar{F}}}{U_F + \chi_F U_{\bar{F}}}\right). \tag{II.24}$$

By Taylor's integral formula

$$\begin{aligned}(1 - \mathscr{T}_F) Z_G &= \int_0^1 d\chi_F \frac{d}{d\chi_F} Z_G(\chi_F, p, \alpha) \\ &= \int_0^1 d\chi_F [U_F + \chi_F U_{\bar{F}}]^{-3} \left[-2U_{\bar{F}} + \frac{U_{\bar{F}} V_F - U_F V_{\bar{F}}}{U_F + \chi_F U_{\bar{F}}}\right] \exp\left(-\frac{V_F + \chi_F V_{\bar{F}}}{U_F + \chi_F U_{\bar{F}}}\right),\end{aligned} \tag{II.25}$$

iii) if F is a biped, we remark that a spanning tree or a two-tree which are not spanning trees in F have at most two connected components in F, each one containing one of the external vertices of F.

Hence

$$U_G(\varrho_F,\alpha)=\varrho_F^{L(F)}[U_F(\alpha)+\varrho_F U_{\bar{F}}(\alpha)],$$
$$V_G(\varrho_F,p\!\!\!/,\alpha)=\varrho_F^{L(F)}[V_F(p\!\!\!/,\alpha)+\varrho_F V_{\bar{F}}(p\!\!\!/,\alpha)].$$

It follows that

$$\mathscr{T}_F Z_G=\mathscr{T}^1_{\chi_F}Z_G(\chi_F,p\!\!\!/,\alpha)=\frac{1}{U_F^2}\left[1-2\frac{U_{\bar{F}}}{U_F}+\frac{U_{\bar{F}}V_F-U_F V_{\bar{F}}}{U_F^2}\right]\exp\left(-\frac{V_F}{U_F}\right), \quad \text{(II.26)}$$

and finally

$$\begin{aligned}(1-\mathscr{T}_F)Z_G&=\int_0^1 d\chi_F(1-\chi_F)\frac{d^2}{d\chi_F^2}Z_G(\chi_F,p\!\!\!/,\alpha)\\&=\int_0^1 d\chi_F(1-\chi_F)[U_F+\chi_F U_{\bar{F}}]^{-4}\exp\left(-\frac{V_F+\chi_F V_{\bar{F}}}{U_F+\chi_F U_{\bar{F}}}\right)\\&\quad\cdot\left[6U_{\bar{F}}^2-6U_{\bar{F}}\frac{U_{\bar{F}}V_F-U_F V_{\bar{F}}}{U_F+\chi_F U_{\bar{F}}}+\left(\frac{U_{\bar{F}}V_F-U_F V_{\bar{F}}}{U_F+\chi_F U_{\bar{F}}}\right)^2\right]. \quad \text{(II.27)}\end{aligned}$$

In the whole $\mathscr{R}$ operation, further simplifications appear since in fact Taylor operators for unessential subgraphs are unessential: the connected divergent subgraphs which are not proper already do not appear in (II.21). But we will show also that open quadrupeds do not contribute to the $\mathscr{R}$ operation, which is not a result quoted in [7]. Before proving it, we state for non-closed subgraphs a more general lemma which is useful in Sect. III:

Lemma II.2. *Let F be a proper subgraph and F^* its closure. Then*

$$S_{\bar{F}^*}\subsetneqq S_{\bar{F}}; \qquad T_{\bar{F}^*}\subsetneqq T_{\bar{F}}. \quad \text{(II.28)}$$

Proof. If F is an open quadruped, by definition (II.5) every external vertex of F^* is an external vertex of F. By induction, this remains true for any proper non-closed subgraph F, since F^* is obtained from F by adding some lines or chains of bipeds which have both ends in F.

Now let S be a one-tree (respectively T a two-tree) which is not a one-tree in F^*. Then $S\cap F^*$ (respectively $T\cap F^*$) has at least two connected components γ_1, γ_2, some of them possibly reduced to a single vertex, and each of them must contain at least one external vertex of F^*, say v_1, v_2. There is no path in $S\cap F^*$ (respectively in $T\cap F^*$) from v_1 to v_2, therefore no path in $S\cap F$ (respectively in $T\cap F$) from v_1 to v_2, which proves that S (respectively T) is not a one-tree in F.

Now from Lemma II.2 and formulae (II.23) and (II.27) it is easy to conclude that if F is an open quadruped,

$$\mathscr{T}_F(1-\mathscr{T}_{F^*})Z_G=0. \quad \text{(II.29)}$$

A definition of the $\mathscr{R}$ operator follows which is more convenient for the next sections:

Lemma II.3.

$$\mathscr{R} = \sum_{\mathscr{F}} \prod_{F \in \mathscr{F}} (-\mathscr{T}_F), \tag{II.30}$$

where the sum is performed over all (possibly empty) closed divergent forests of G.

Proof. Let us consider all forests $\mathscr{F}$ in which a given open quadruped F is maximal among the open quadrupeds of $\mathscr{F}$. Then F^* is a proper biped, which cannot overlap with any closed biped or quadruped (see Appendix B). Hence F^* cannot overlap with any subgraph of any such $\mathscr{F}$. This means that in the sum (II.21) one can factorize $\mathscr{T}_F(1-\mathscr{T}_{F^*})$ and the sum vanishes by (II.29). In the same way we eliminate inductively all forests containing any open quadruped, which proves Lemma II.3.

II.4. Borel Transform and Results

The coefficients in the expansion (II.1) are now precisely defined by

$$\mathbf{a}_n^N(\not{p}) = \sum_G I_G^R(\not{p}), \tag{II.31}$$

where the sum is performed over all possible sets of Wick contractions which give connected graphs G with $n(G)=n$ and $N(G)=N$, without tadpoles. The expansion (II.1) probably has a vanishing radius of convergence, and we will give a meaning only to its Borel transform in a small disk. We define the Borel transformed Schwinger functions $\mathbf{B}_N^T$ by

$$\mathbf{B}_N^T(\not{p}, t) = \delta\left(\sum_{a=1}^N \not{p}_a\right) \sum_{n=0}^\infty \frac{(-t)^n}{[n!]^2} \mathbf{a}_n^N(\not{p}). \tag{II.32}$$

Our main results are:

Theorem I. *For any graph G in euclidean Φ_4^4 theory, and any set of external momenta $\not{p}$, we have*

$$|I_G^R(\not{p})| \leqq \Gamma(\omega^R(G))(K_1|\not{p}|)^{n(G)}[f_1(G)]!, \tag{II.33}$$

where K_1 is a constant and

$$\omega^R(G) = \mathrm{Sup}\{1, N(G)/2-2\}. \tag{II.34}$$

The proof is in Sect. III and Appendices A and B.

Remark. The extra factor $f_1!$ in (II.33) does not appear in Φ_ν^4 theory, $\mathrm{Re}\,\nu<4$ [1]. It is characteristic of a renormalizable (and non-super-renormalizable) theory. f_1 can be seen as the maximal number of subtractions due to the renormalization. Indeed we have seen that, in the sense of (II.23) and (II.29), the Taylor operators subtract once for quadrupeds, and twice for bipeds.

Theorem II. *There exist $C(N)$ and a constant K_2 such that if $\gamma(N,n,f)$ is the number of (labeled) graphs G with $N(G)=N$, $n(G)=n$ and $f_2(G)=f$, then*

$$\gamma(N,n,f) \leqq K_2^n \frac{(n!)^2}{f!} n^{N/2} C(N). \tag{II.35}$$

The proof is in Appendix C. As a simple corollary of Theorems I, II, Lemma II.1 and formulae (II.31) and (II.32) we obtain:

Theorem III. *The expansion* (II.32) *converges and defines an analytic function of t, at least in a disk of radius* $1/|\not{p}|K_1K_2$.

Remark. Inequality (II.33) is only an upper bound on I_G^R. We will come back to this point in Sect. IV, and give some examples of classes of graphs which in fact increase more slowly with $n(G)$.

III. Proof of Theorem I

III.1. Hepp's Sectors and Classification of Forests

We call $\mathfrak{S}_G$ the group of permutations of the internal lines of G. For any $\sigma\in\mathfrak{S}_G$, we call G_i^σ the subgraph of G made from the lines $\sigma(1), \ldots, \sigma(i)$. The Hepp's sector h_σ associated to σ is defined by

$$h_\sigma=\{\alpha|0\leqq\alpha_{\sigma(1)}\leqq \ldots \leqq\alpha_{\sigma(\ell)}\}. \tag{III.1}$$

We split the integral (II.19) into corresponding contributions:

$$I_G^R=\sum_{\sigma\in\mathfrak{S}_G} I_{G,\sigma}^R, \tag{III.2}$$

$$I_{G,\sigma}^R=\int_{h_\sigma}\prod_{i=1}^{\ell} d\alpha_i\left[\exp\left(-\sum_{i=1}^{\ell}\alpha_i\right)\right]Z_G^R. \tag{III.3}$$

For a given sector, we group the divergent closed forests appearing in the $\mathscr{R}$ operation according to appropriate classes. This classification is inspired by the Ω-construction of [7], which classifies the nests. The presentation here will be quite different because we want to avoid difficulties related to the unessential divergent subgraphs.

To any closed divergent forest $\mathscr{F}$ we associate a larger proper forest $\Omega^\sigma(\mathscr{F})$ in the following way: for

$$\forall F\in\mathscr{F}\cup\{G\} \text{ we consider } F\cap[G_i^\sigma\cup A_{\mathscr{F}}(F)], \tag{III.4}$$

and we take the proper components of this subgraph. By varying F in $\mathscr{F}\cup\{G\}$ and i from 1 to ℓ we thus obtain a set $\Omega^\sigma(\mathscr{F})$ of proper subgraphs.

Since

$$F=F\cap G_\ell^\sigma\cup A_{\mathscr{F}}(F),$$

$$F\subset F'\Rightarrow F\cap G_i^\sigma\cup A_{\mathscr{F}}(F)\subseteqq F\subseteqq A_{\mathscr{F}}(F')\subseteqq F'\cap G_j^\sigma\cup A_{\mathscr{F}}(F'), \forall i,j, \tag{III.5}$$

$$F\cap F'=\emptyset\Rightarrow[F\cap G_i^\sigma\cup A_{\mathscr{F}}(F)]\cap[F'\cap G_j^\sigma\cup A_{\mathscr{F}}(F')]=\emptyset, \forall i,j, \tag{III.6}$$

$$G_i^\sigma\subset G_j^\sigma\Rightarrow A_{\mathscr{F}}(F)\subseteqq F\cap G_i^\sigma\cup A_{\mathscr{F}}(F)\subseteqq F\cap G_j^\sigma\cup A_{\mathscr{F}}(F)\subseteqq F, \tag{III.7}$$

it is easy to verify that $\Omega^\sigma(\mathscr{F})$ is a forest which contains $\mathscr{F}$.

Next for any $F\in\mathscr{F}$ we define an index $x(F,\mathscr{F})$, and a subgraph $X(F,\mathscr{F})$ in $\Omega^\sigma(\mathscr{F})$ by

a) $X(F,\mathscr{F})$ is a proper component of $F\cap G_{x(F,\mathscr{F})}^\sigma\cup A_{\mathscr{F}}(F)$,

b) $X^*(F,\mathscr{F})=F$,

c) $\forall i<x(F,\mathscr{F})$ and $\forall Y$ a proper component of $F\cap G_i^\sigma\cup A_{\mathscr{F}}(F)$, $Y^*\neq F$.

We also define inductively, starting from the maximal elements of $\mathscr{F}$, an index $y(F,\mathscr{F})$ for any $F\in\mathscr{F}$, and a subforest $\mathscr{S}(\mathscr{F})$ by

d) $\mathscr{S}(\mathscr{F})=\{F|F\in\mathscr{F}\,;\,x(F,\mathscr{F})>y(F,\mathscr{F})\}$,

e) if $G\in\mathscr{F}$, $y(G,\mathscr{F})=l+1$ by convention,

f) if $F\in\mathscr{F}$, $F\neq G$, is a quadruped, $y(F,\mathscr{F})=\inf_{\sigma(j)\in E} j$,

g) if $F\in\mathscr{F}$ is a biped, $y(F,\mathscr{F})=\sup_{\sigma(j)\in E} j$, except when one external line of F is an external line of G, in which case we put $y(F,\mathscr{F})=\ell+1$.

In f) and g), E is the set of external lines of F, internal in $B_{\mathscr{S}(\mathscr{F})}(F)$ and $B_{\mathscr{S}(\mathscr{F})}(F)$ is defined in Sect. II as the smallest subgraph in $\mathscr{S}(\mathscr{F})$ containing F [if there is no such subgraph $B_{\mathscr{S}(\mathscr{F})}(F)=G$]. It is clear from these definitions that $G\notin\mathscr{S}(\mathscr{F})$ and that

$$x(F,\mathscr{F})\neq y(F,\mathscr{F}),\qquad \forall F\in\mathscr{F}. \tag{III.8}$$

Lemma III.1. *Let F,H be two distinct subgraphs in a closed divergent forest $\mathscr{F}$. Then if $H\notin\mathscr{S}(\mathscr{F})$ we have*

$$x(F,\mathscr{F})=x(F,\mathscr{F}-\{H\}), \tag{III.9}$$

$$y(F,\mathscr{F})=y(F,\mathscr{F}-\{H\}), \tag{III.10}$$

$$\mathscr{S}(\mathscr{F})=\mathscr{S}(\mathscr{F}-\{H\}), \tag{III.11}$$

$$\forall i, L(G_i^\sigma\cap F/\mathscr{F})+L(G_i^\sigma\cap H/\mathscr{F})=L(G_i^\sigma\cap F/\mathscr{F}-\{H\}). \tag{III.12}$$

Proof. (III.11) is a simple corollary of (III.9) and (III.10). (III.10) is trivial from the definitions of y and $\mathscr{S}$. (III.9) and (III.12) are trivial except when $H\in\mathscr{A}_{\mathscr{F}}(F)$. In this last case (III.12) could only be violated if the reduction by the subgraph H would add supplementary loops to $G_i^\sigma\cap F$. But this is not possible since

i) either $i>x(H,\mathscr{F})$ and $G_i^\sigma\cap H\supseteq X(H,\mathscr{F})$,

ii) or $i\leq x(H,\mathscr{F})<y(H,\mathscr{F})$ which means that H is disjoint from $(G_i^\sigma\cap F)-(G_i^\sigma\cap H)$ (if H is a quadruped), or joined by at most one line to it (if H is a biped).

To prove (III.9) when $H\in\mathscr{A}_{\mathscr{F}}(F)$, we remark that $F\cap G_i^\sigma\cup A_{\mathscr{F}-\{H\}}(F)$ is included in $F\cap G_i^\sigma\cup A_{\mathscr{F}}(F)$, hence

$$x(F,\mathscr{F})\leq x(F,\mathscr{F}-\{H\}).$$

We show the converse inequality

$$x(F,\mathscr{F})\geq x(F,\mathscr{F}-\{H\}) \tag{III.13}$$

by distinguishing two cases:

i) either $X(F,\mathscr{F})$ is disjoint from H, then $X(F,\mathscr{F})$ is a proper component of

$$F\cap G_{x(F,\mathscr{F})}^\sigma\cup A_{\mathscr{F}-\{H\}}(F)\quad\text{and}\quad X^*(F,\mathscr{F})=F,$$

which proves (III.13),

ii) or $X(F,\mathscr{F})\supset H$, then $X(F,\mathscr{F})$ contains at least two external legs of H, internal in F, and

$$x(F,\mathscr{F})\geq y(H,\mathscr{F})>x(H,\mathscr{F}). \tag{III.14}$$

Now $X(F,\mathscr{F})\supset H$ and $X^*(H,\mathscr{F})=H$ imply that the subgraph $Y=X(F,\mathscr{F})-[H-X(H,\mathscr{F})]$ is proper. Then $Y^*=F$ and by (III.14) Y is contained in a proper component of

$$F\cap G_{x(F,\mathscr{F})}^\sigma\cup A_{\mathscr{F}-\{H\}}(F).$$

This proves (III.13) and achieves the proof of the lemma.

Definition III.1. A closed divergent forest is called a skeleton forest if $\mathscr{S}(\mathscr{F})=\mathscr{F}$.

By simple induction one sees from Lemma III.1 that for any closed divergent forest $\mathscr{F}$, $\mathscr{S}(\mathscr{F})$ is a skeleton forest itself: $\mathscr{S}(\mathscr{S}(\mathscr{F}))=\mathscr{S}(\mathscr{F})$.

Definition III.2. For a skeleton forest $\mathscr{F}$ we define:

$$\mathscr{H}(\mathscr{F})=\left\{H\,\middle|\begin{array}{l}H \text{ is a closed divergent subgraph, compatible with } \mathscr{F}\\ \text{and } \mathscr{S}(\mathscr{F}\cup\{H\})=\mathscr{F}\,.\end{array}\right\}.$$

Lemma III.2. *$\mathscr{F}\cup\mathscr{H}(\mathscr{F})$ is a forest.*

Proof. We only have to verify that two different subgraphs H_1 and H_2 in $\mathscr{H}$ cannot overlap. But we have seen in Sect. II that the closures of two non-overlapping subgraphs cannot overlap. Thus it is sufficient to prove that $X(H_1,\mathscr{F}\cup\{H_1\})$ and $X(H_2,\mathscr{F}\cup\{H_2\})$ cannot overlap. This is true since both subgraphs belong to $\Omega^\sigma(\mathscr{F})$, being proper components of

$$B_{\mathscr{F}}(H_j)\cap G^\sigma_{x(H_j,\mathscr{F}\cup\{H_j\})}\cup A_{\mathscr{F}}(B_{\mathscr{F}}(H_j))$$

for $j=1,2$ respectively.

By Lemmas III.1 and III.2 we can conclude:

Lemma III.3. *Given a skeleton forest $\mathscr{F}$ and an arbitrary closed divergent forest $\mathscr{F}'$ we have*

$$\mathscr{S}(\mathscr{F}')=\mathscr{F}\Leftrightarrow\mathscr{F}\subseteq\mathscr{F}'\subseteq\mathscr{F}\cup\mathscr{H}(\mathscr{F})\,. \tag{III.15}$$

We may now use Lemma III.3 to write the $\mathscr{R}$ operation and the renormalized amplitude (III.2) in another form, by grouping all the closed divergent forests according to their skeleton forests

$$I_G^R=\sum_{\mathscr{F}}\sum_{\sigma\in\mathfrak{S}_G^{\mathscr{F}}}I^{\mathscr{F}}_{G,\sigma}, \tag{III.16}$$

where the first sum runs over all closed divergent forests of G, $\mathfrak{S}_G^{\mathscr{F}}$ is the set of $\sigma\in\mathfrak{S}_G$ such that $\mathscr{F}$ is a skeleton forest for σ and,

$$I^{\mathscr{F}}_{G,\sigma}=\int_{h_\sigma}\prod_{i=1}^{\ell}d\alpha_i\left[\exp\left(-\sum_{i=1}^{\ell}\alpha_i\right)\right]\prod_{F\in\mathscr{F}}(-\mathscr{T}_F)\prod_{H\in\mathscr{H}(\mathscr{F})}(1-\mathscr{T}_H)Z_G\,. \tag{III.17}$$

Although it is not obvious, the main interest of this classification lies in the fact that each integral (III.17) is absolutely convergent and may be explicitly estimated, as we will show now. Such a classification seems to realize, in a given sector h_σ, the smallest and most practical partial sums of terms in the $\mathscr{R}$ operation, which give a finite result when applied to Z_G and integrated over h_σ.

III.2. The Subtraction Process

In this paragraph we write explicitly the result of the subtractions in the integrand of (III.17). For $\mathscr{F}$ and σ fixed, $\mathscr{F}$ being a skeleton forest for σ, denote by $\mathscr{B}$ (respectively $\mathscr{Q}$) the forest of bipeds (respectively quadrupeds) in $\mathscr{F}$, with b (respectively q) elements; $\mathscr{H}$ the forest $\mathscr{H}(\mathscr{F})$; $\mathscr{B}'$ (respectively $\mathscr{Q}'$) the forest of bipeds (respectively quadrupeds) in $\mathscr{H}$, with b' (respectively q') elements; $s=|\mathscr{F}|=b+q$.

In order to apply repeatedly formulae (II.23) and (II.25)–(II.27) it is convenient to introduce the following definitions:

Definition III.3. $\mathscr{I}$ and $\mathscr{J}$ being two disjoint subforests of $\mathscr{F}\cup\mathscr{H}$, we call $S_{\mathscr{I}\bar{\mathscr{J}}}$ (respectively $T_{\mathscr{I}\bar{\mathscr{J}}}$) the set of spanning trees (respectively two-trees) of G whose restriction to F is a spanning tree of F for any $F\in\mathscr{I}$, and is not a spanning tree of F for any $F\in\mathscr{J}$. Then $U_{\mathscr{I}\bar{\mathscr{J}}}$ (respectively $V_{\mathscr{I}\bar{\mathscr{J}}}$) is defined by formula (II.4) [respectively (II.5)] in which the sum is restricted to $S_{\mathscr{I}\bar{\mathscr{J}}}$ (respectively $T_{\mathscr{I}\bar{\mathscr{J}}}$). Finally we attach a variable χ_F to every $F\in\mathscr{F}\cup\mathscr{H}$, and if $\mathscr{I}$, $\mathscr{J}$, $\mathscr{K}$ are three disjoint subforests of $\mathscr{F}\cup\mathscr{H}$, we put

$$U^{\mathscr{K}}_{\mathscr{I}\bar{\mathscr{J}}}=\sum_{\mathscr{K}'\subseteq\mathscr{K}}\Big(\prod_{F\in\mathscr{K}'}\chi_F\Big)U_{\mathscr{I}\cup(\mathscr{K}-\mathscr{K}'),\,\overline{\mathscr{J}\cup\mathscr{K}'}}\,, \tag{III.18}$$

$$V^{\mathscr{K}}_{\mathscr{I}\bar{\mathscr{J}}}=\sum_{\mathscr{K}'\subseteq\mathscr{K}}\Big(\prod_{F\in\mathscr{K}'}\chi_F\Big)V_{\mathscr{I}\cup(\mathscr{K}-\mathscr{K}'),\,\overline{\mathscr{J}\cup\mathscr{K}'}}\,. \tag{III.19}$$

With these notations we find:

Lemma III.4.

$$\prod_{F\in\mathscr{F}}(-\mathscr{T}_F)\prod_{H\in\mathscr{H}}(1-\mathscr{T}_H)Z_G=\sum_{\delta}Z^{\delta}_G(p,\alpha)\,, \tag{III.20}$$

where

a) *the index δ runs over a set of at most* $4^{f(\mathscr{F}\cup\mathscr{H})}\dfrac{[f(\mathscr{F}\cup\mathscr{H})]!}{s!}$ *elements.*

b) $$Z^{\delta}_G=\varepsilon\int_0^1\prod_{F\in\mathscr{K}}d\chi_F\prod_{F\in\mathscr{L}}(1-\chi_F)[U^{\mathscr{K}}_{\mathscr{F}}]^{-2-k}\left[\prod_{m=1}^{k}P_m\right]\exp[-W(\chi,\alpha,p)]\,, \tag{III.21}$$

(ε, $\mathscr{K}$, $\mathscr{L}$, k, P_m, *and* W *depend on the index* δ).

c) $$\varepsilon=\pm1\,;\qquad W=V^{\mathscr{K}}_{\mathscr{F}}/U^{\mathscr{K}}_{\mathscr{F}}\geqq0\,;\qquad P_m\geqq0\,, \tag{III.22}$$

$$\mathscr{L}\subseteq\mathscr{K}\subseteq\mathscr{H}\,, \tag{III.23}$$

$$k\leqq2f(\mathscr{F}\cup\mathscr{H})\,. \tag{III.24}$$

d) $$P_m=U^{\mathscr{K}_m}_{\mathscr{I}_m\bar{\mathscr{J}}_m}\ \text{or}\ V^{\mathscr{K}_m}_{\mathscr{I}_m\bar{\mathscr{J}}_m}\ \textit{(in both cases denote it } P^{\mathscr{K}_m}_{\mathscr{I}_m\bar{\mathscr{J}}_m}\textit{)}\,. \tag{III.25}$$

Moreover $P_m=V^{\mathscr{K}_m}_{\mathscr{I}_m\bar{\mathscr{J}}_m}$ *for at most* $f(\mathscr{F}\cup\mathscr{H})$ *values of m.*

e) $$\left.\begin{aligned}&F\in\mathscr{Q}\Rightarrow F\in\mathscr{I}_m\forall m\,,\\&F\in\mathscr{Q}'\Rightarrow\exists!m\,,\quad F\in\mathscr{J}_m\,,\\&F\in\mathscr{B}\Rightarrow \text{ either } F\in\mathscr{I}_m\forall m,\text{ or }\exists!m\,,\quad F\in\mathscr{J}_m\,,\\&F\in\mathscr{B}'\Rightarrow\exists m_1,m_2\,,\quad F\in\mathscr{J}_m\Leftrightarrow m=m_1\text{ or }m=m_2\,.\end{aligned}\right\} \tag{III.26}$$

f) $$\forall m,\mathscr{F}\subseteq\mathscr{I}_m\cup\mathscr{J}_m\quad\text{and}\quad\mathscr{F}\cup\mathscr{K}\subseteq\mathscr{I}_m\cup\mathscr{J}_m\cup\mathscr{K}_m\,. \tag{III.27}$$

g) *If* $F\in\mathscr{J}_m$ *and* $F'\in\mathscr{A}_{\mathscr{F}\cup\mathscr{H}}(F)$, *then* $F'\in\mathscr{I}_m\cup\mathscr{J}_m$, *except perhaps if* $F\in\mathscr{B}'$ *and* $F'\in\mathscr{B}\cup\mathscr{Q}'$; *in this last case,* m_1 *and* m_2 *being the indices defined in* (III.26), $F\in\mathscr{J}_{m_1}\Rightarrow F'\in\mathscr{I}_{m_2}\cup\mathscr{J}_{m_2}$ *and* $F\in\mathscr{J}_{m_2}\Rightarrow F'\in\mathscr{I}_{m_1}\cup\mathscr{J}_{m_1}$.

Proof. The proof is by induction on b, q, b', q'. We add new subgraphs one by one, starting from the maximal elements of the final forest. Assuming Lemma III.4 at a given step, we add a new subgraph F and perform the corresponding $\mathcal{T}_F$ or $1-\mathcal{T}_F$ operation. The dependence on the corresponding χ_F is obtained by replacing $\mathcal{K}$ by $\mathcal{K}\cup\{F\}$. Then we use (II.23), (II.25), (II.26) or (II.27) and verify that Lemma III.4 remains true. Among all possible cases, we write explicitly the most complicated one, i.e.: $F\in\mathcal{B}'$ (hence we perform $1-\mathcal{T}_F$) and $B_{\mathcal{F}}(F)\in\mathcal{B}'$ [hence $B_{\mathcal{F}}(F)\in\mathcal{I}_{m_1}$ and $B_{\mathcal{F}}(F)\in\mathcal{I}_{m_2}$, which is the most difficult case for the verification of item g)]. We simplify the notations by numbering the indices in such a way that $m_1=1$, $m_2=2$. Then the $1-\mathcal{T}_F$ operator subtracts 0, 1, or 2 times and we obtain

$$(1-\mathcal{T}_F)Z_G^\delta=Z_0+Z_1+Z_2\,, \tag{III.28}$$

$$Z_0=\varepsilon\int_0^1\prod_{F'\in\mathcal{K}}d\chi_{F'}\prod_{F'\in\mathcal{L}}(1-\chi_{F'})[U_{\mathcal{F}}^{\mathcal{K}}]^{-2-k}\exp\left(-\frac{V_{\mathcal{F}}^{\mathcal{K}}}{U_{\mathcal{F}}^{\mathcal{K}}}\right)$$
$$\cdot\sum_{1\leqq t<t'\leqq k}\left[\prod_{\substack{m=1\\m\neq t}}^{t'-1}P^{\mathcal{K}_m}_{\mathcal{I}_m\cup\{F\},\overline{\mathcal{I}_m}}\right]\left[\prod_{m=t'+1}^{k}P^{\mathcal{K}_m}_{\mathcal{I}_m,\overline{\mathcal{I}_m}}\right]P^{\mathcal{K}_t}_{\mathcal{I}_t,\overline{\mathcal{I}_t\cup\{F\}}}P^{\mathcal{K}_{t'}}_{\mathcal{I}_{t'},\overline{\mathcal{I}_{t'}\cup\{F\}}}\,, \tag{III.29}$$

$$Z_1=\varepsilon\int_0^1\prod_{F'\in\mathcal{K}\cup\{F\}}d\chi_{F'}\prod_{F'\in\mathcal{L}}(1-\chi_{F'})\exp\left(-\frac{V_{\mathcal{F}}^{\mathcal{K}\cup\{F\}}}{U_{\mathcal{F}}^{\mathcal{K}\cup\{F\}}}\right)\sum_{t=1}^{k}\prod_{\substack{m=1\\m\neq t}}^{k}P^{\mathcal{K}_m}_{\mathcal{I}_m\cup\{F\},\overline{\mathcal{I}_m}}$$
$$\cdot P^{\mathcal{K}_t}_{\mathcal{I}_t,\overline{\mathcal{I}_t\cup\{F\}}}\left\{-\frac{(2+k)U^{\mathcal{K}}_{\mathcal{F},\overline{\{F\}}}}{[U_{\mathcal{F}}^{\mathcal{K}\cup\{F\}}]^{2+k+1}}+\frac{U^{\mathcal{K}}_{\mathcal{F},\overline{\{F\}}}V^{\mathcal{K}}_{\mathcal{F}\cup\{F\}}-U^{\mathcal{K}}_{\mathcal{F}\cup\{F\}}V^{\mathcal{K}}_{\mathcal{F},\overline{\{F\}}}}{[U_{\mathcal{F}}^{\mathcal{K}\cup\{F\}}]^{2+k+2}}\right\}, \tag{III.30}$$

$$Z_2=\varepsilon\int_0^1\prod_{F'\in\mathcal{K}\cup\{F\}}d\chi_{F'}\prod_{F'\in\mathcal{L}\cup\{F\}}(1-\chi_{F'})\exp\left(-\frac{V_{\mathcal{F}}^{\mathcal{K}\cup\{F\}}}{U_{\mathcal{F}}^{\mathcal{K}\cup\{F\}}}\right)\prod_{m=1}^{k}P^{\mathcal{K}_m}_{\mathcal{I}_m\cup\{F\},\bar{\mathcal{I}}_m}$$
$$\cdot\left\{(2+k)(3+k)\frac{[U^{\mathcal{K}}_{\bar{\mathcal{F}},\{\bar{F}\}}]^2}{[U_{\mathcal{F}}^{\mathcal{K}\cup\{F\}}]^{2+k+2}}\right.$$
$$-2(2+k+1)U^{\mathcal{K}}_{\mathcal{F},\overline{\{F\}}}\frac{U^{\mathcal{K}}_{\mathcal{F},\overline{\{F\}}}V^{\mathcal{K}}_{\mathcal{F}\cup\{F\}}-U^{\mathcal{K}}_{\mathcal{F}\cup\{F\}}V^{\mathcal{K}}_{\mathcal{F},\overline{\{F\}}}}{[U_{\mathcal{F}}^{\mathcal{K}\cup\{F\}}]^{2+k+3}}$$
$$\left.+\frac{[U^{\mathcal{K}}_{\mathcal{F},\overline{\{F\}}}V^{\mathcal{K}}_{\mathcal{F}\cup\{F\}}-U^{\mathcal{K}}_{\mathcal{F}\cup\{F\}}V^{\mathcal{K}}_{\mathcal{F},\overline{\{F\}}}]^2}{[U_{\mathcal{F}}^{\mathcal{K}\cup\{F\}}]^{2+k+4}}\right\}. \tag{III.31}$$

We can verify Lemma III.4 on these formulae. The other cases, following F and $B_{\mathcal{F}}(F)$ belong to $\mathcal{B}$, $\mathcal{Q}$, $\mathcal{B}'$, $\mathcal{Q}'$, give formulae which are shorter. The verification of items a)–g) is easier, and is left to the reader.

Corollaries of Lemma III.4. *From* (III.21) *and* (III.22) *we conclude*

$$|Z_G^\delta(\not p,\alpha)|\leqq Y_G^\delta(\not p,\alpha)=\int_0^1\prod_{F\in\mathcal{K}}d\chi_F\left[\prod_{m=1}^{k}P_m\right][U_{\mathcal{F}}^{\mathcal{K}}]^{-2-k}. \tag{III.32}$$

From g) and (III.26), it is easy to verify

$$\forall\delta,\ \sum_{m=1}^{k}\sum_{F\in\mathcal{I}_m}\ell(F/\mathcal{I}_m\cup\mathcal{J}_m)\leqq 2\sum_{F\in\mathcal{F}\cup\mathcal{H}}\ell(F/\mathcal{F}\cup\mathcal{H})\leqq 2\ell(G)\,. \tag{III.33}$$

III.3. Change of Variables and Estimates

In the integrand of (III.17) expressed by means of Lemma III.4, we perform the usual change of variables

$$\alpha_k = \prod_{G_i^\sigma \ni k} \beta_i . \tag{III.34}$$

Each variable β_i varies from 0 to 1, except β_ℓ which varies from 0 to ∞. Since the various polynomials U, V, P_m are homogeneous with respect to β_ℓ, we perform first the β_ℓ integration, [using (III.26) if G is divergent] and we obtain from (III.17), (III.20), and (III.32)

$$|I_{G,\sigma}^{\mathscr{F}}| \leqq \int_0^1 \dots \int_0^1 \prod_{i=1}^{\ell-1} \beta_i^{i-1} d\beta_i \left[\sum_\delta Y_G^\delta(p, \beta_i)\right] \cdot \Gamma(\omega^R(G)) . \tag{III.35}$$

Since the global homogeneity in β_ℓ has been taken out, $Y_G^\delta(p, \beta_i)$ as defined in (III.32) is only a function of $\beta_1, \dots, \beta_{\ell-1}$, and $\omega^R(G)$ is defined in (II.34).

Lemma III.5.

$$\forall \delta, \int_0^1 \dots \int_0^1 \prod_{i=1}^{\ell-1} \beta_i^{i-1} d\beta_i Y_G^\delta(p, \beta_i) \leqq (K_3 |p|)^{n(G)} \prod_{i=1}^{\ell-1} \frac{1}{\tilde{N}_i(\sigma)}, \tag{III.36}$$

where

$$\tilde{N}_i(\sigma) = \sum_{\substack{F \in \mathscr{F} \cup \{G\} \\ G_i^\sigma \cap F/\mathscr{F} \neq F/\mathscr{F}}} N(G_i^\sigma \cap F/\mathscr{F}) . \tag{III.37}$$

The proof of Lemma III.5 is rahter lengthy and we give it in the next subsection. As a consequence of the definition of the skeleton forests, $\tilde{N}_i(\sigma)$ never vanish. Then we are left with the sum over Hepp's sectors. This problem is treated in Appendix B, where the following result is proved:

Lemma III.6.

$$\forall \mathscr{F}, \sum_{\sigma \in \mathfrak{S}_G^{\mathscr{F}}} \frac{1}{s!} \prod_{i=1}^{\ell-1} \frac{1}{\tilde{N}_i(\sigma)} \leqq K_4^{n(G)} . \tag{III.38}$$

From Lemma C.1 and Definition II.6, the index δ runs over a set of at most $4^n f_1!/s!$ elements. Finally we can bound the sum in (III.16), using Appendix A to bound the possible choices of $\mathscr{F}$, and (III.35), (III.36), (III.38) to bound $\sum_{\sigma \in \mathfrak{S}_G^{\mathscr{F}}} I_{G,\sigma}^{\mathscr{F}}$. Hence Theorem I is proved.

III.4. Proof of Lemma III.5

To prove Lemma III.5 we will define a natural correspondence between the subsets $S_{\mathscr{I}_m \bar{\mathscr{J}}_m}$ or $T_{\mathscr{I}_m \bar{\mathscr{J}}_m}$ which index the sums P_m in the numerator of (III.32), and the subset $S_{\mathscr{F}}$ which indexes the denominator $U_{\mathscr{F}}^{\mathscr{K}}$ of (III.32).

To fix the language we only consider the case $P_m = U_{\mathscr{I}_m \bar{\mathscr{J}}_m}^{\mathscr{K}_m}$. The case $P_m = V_{\mathscr{I}_m \bar{\mathscr{J}}_m}^{\mathscr{K}_m}$ would be treated along the same lines, with minor changes which we indicate later in a remark. In this case, we bound at the beginning the dependence on the external momenta by (II.7).

Let us now describe a first elementary mapping φ_F in the set of spanning trees. We put $\mathscr{I}_m \cup \mathscr{J}_m = \mathscr{E}_m$, and $\forall F \in \mathscr{J}_m$ we define $\mathscr{E}_{mF} = \{F' | F' \in \mathscr{E}_m;\ F' \subset F\}$. Let S be a spanning tree of G whose restriction to each $F' \in \mathscr{E}_{mF}$ is a spanning tree of F'. Then we define a mapping φ_F from $S_{\mathscr{E}_{mF}}$ into $S_{\mathscr{E}_{mF} \cup \{F\}}$ in the following way:

1) If S is a one-tree in $F/\mathscr{E}_m$ (hence a one-tree in F) we put

$$\varphi_F(S) = S. \tag{III.39}$$

2) If S is not a one-tree in $F/\mathscr{E}_m$, by Lemma II.2 S is not a one-tree in $X_F = X(F, \mathscr{E}_m)/\mathscr{E}_m$; hence $S \cap X_F$ has several connected components (some of them eventually reduced to a single vertex); since each of these connected components must contain an external vertex of F, their number $c_F(S)$ satisfies:

$$2 \leqq c_F(S) \leqq N(F) \leqq 4. \tag{III.40}$$

We choose an arbitrary connected component γ of $S \cap X_F$. Since X_F is connected, and all its vertices belong to S, there exists another connected component γ' of $S \cap X_F$ linked to γ by one line $\lambda' \notin S$. Then if i' is the rank of λ' in the sector, $\lambda' = \sigma(i')$, we have by (III.9) and (III.27):

$$i' \leqq x(F, \mathscr{E}_m) = x(F, \mathscr{F}). \tag{III.41}$$

Next we define an external line λ of F by distinguishing three cases:

a) If $B_{\mathscr{F}}(F) \in \mathscr{Q}$, then S is a one-tree in $B_{\mathscr{F}}(F)$ by (III.26). Then there is a path in $S \cap B_{\mathscr{F}}(F)$ from γ to γ', which contains at least two external lines of F, one attached to γ, the other one to γ'. Among these two lines, we call λ the one which has the highest rank in the sector σ.

b) If $B_{\mathscr{F}}(F) \in \mathscr{B}$ and F is a biped, there is a path in S from γ to γ' which contains both external lines of F, one attached to γ and the other one to γ'. Moreover these two lines are internal in $B_{\mathscr{F}}(F)$ since $B_{\mathscr{F}}(F)$ is proper. We define λ in the same way as in case a).

c) If $B_{\mathscr{F}}(F) \in \mathscr{B}$ and F is a quadruped, there is a path in S from γ to γ' which contains at least two external lines of F, one attached to γ, the other one to γ'. These two lines cannot be both external lines of $B_{\mathscr{F}}(F)$ since F is closed, and we define λ as one of these two lines which is internal in $B_{\mathscr{F}}(F)$ (any one of them if both are).

In the three cases a)–c), we conclude that if $\lambda = \sigma(i)$ then $i \geqq y(F, \mathscr{E}_m)$, hence by (III.27) and (III.10)

$$\lambda = \sigma(i) \Rightarrow i \geqq y(F, \mathscr{F}). \tag{III.42}$$

By construction $S \cup \{\lambda'\} - \{\lambda\}$ is a one-tree in G and has $c_F(S) - 1$ connected components in $F/\mathscr{E}_m$. Iterating this operation $c_F(S) - 1$ times, we build

$$\varphi_F(S) = S \cup \{\lambda'_1, \ldots, \lambda'_{c_F(S)-1}\} - \{\lambda_1, \ldots, \lambda_{c_F(S)-1}\}, \tag{III.43}$$

which is a one-tree in $F/\mathscr{E}_m$, hence in F.

We note that by this construction, there are at most $2^4 \cdot 2^{\ell(F/\mathscr{E}_m)}$ one-trees S such that $\varphi_F(S)$ is equal to some given one-tree.

Remark. When $P_m = V^{\mathscr{K}_m}_{\mathscr{I}_m \bar{\mathscr{I}}_m}$, we associate a one-tree $\varphi_F(T)$ to a two-tree T by an almost identical algorithm: if γ and γ' cannot be joined by a path in T, λ cannot be defined and we simply add λ' without removing any line, which gives a one-tree. If λ can be defined at each step, we simply add an arbitrary line at the end, to get a one-tree.

Now we define a second mapping φ_m for any $S \in S_{\mathscr{I}_m \bar{\mathscr{I}}_m}$ by

$$\varphi_m(S) = \left[\prod_{F \in \mathscr{I}_m} \varphi_F\right](S), \tag{III.44}$$

the product (in the sense of the composition of mappings) being taken in an order compatible with the inclusion relations of the subgraphs F: if $\mathscr{D}_{mF}$ is the set of $F' \in \mathscr{E}_m$ such that $\varphi_{F'}$ is applied before φ_F, we have $F' \in \mathscr{D}_{mF} \Rightarrow F \nsubseteq F'$. By this definition, $\varphi_m(S)$ is a one-tree; $\forall F'$ compatible with $\mathscr{E}_m$,

$$S \in S_{\mathscr{E}_{mF} \cup \{F'\}} \Rightarrow \forall F \in \mathscr{E}_m, \varphi_F(S) \quad \text{is a one-tree in } F', \tag{III.45}$$

since the number of lines of $S \cap F'$ can only increase by applying φ_F. The mapping φ_m maps $S_{\mathscr{I}_m, \bar{\mathscr{I}}_m}$ into $S_{\mathscr{E}_m} \subseteq S_{\mathscr{F}}$ since $\mathscr{F} \subseteq \mathscr{E}_m$ by (III.27). By construction, the number of one-trees which become a given one-tree S_1 by this mapping is bounded by

$$|\{S \mid \varphi_m(S) = S_1\}| \leqq 2^{4|\mathscr{I}_m|} \cdot 2^{\sum_{F \in \bar{\mathscr{I}}_m} \ell(F/\mathscr{E}_m)}. \tag{III.46}$$

By (III.18) we have

$$P_m = \sum_{S \in S_{\mathscr{I}_m, \bar{\mathscr{I}}_m}} \left[\prod_{\substack{F \in \mathscr{K}_m \\ S\{F\} \not\ni S}} \chi_F\right] \cdot u(S), \tag{III.47}$$

$$U^{\mathscr{K}}_{\mathscr{F}} = \sum_{S \in S_{\mathscr{F}}} \left[\prod_{\substack{F \in \mathscr{K} \\ S\{F\} \not\ni S}} \chi_F\right] \cdot u(S), \tag{III.48}$$

where

$$u(S) = \prod_{i' \notin S} \prod_{G_i^{\mathscr{F}} \ni i'} \beta_i. \tag{III.49}$$

By (III.27) $\mathscr{K} \subseteq \mathscr{K}_m \cup \mathscr{E}_m$. Therefore by (III.45) and (III.46)

$$\frac{P_m}{U^{\mathscr{K}}_{\mathscr{F}}} \leqq 2^{4|\mathscr{I}_m|} \cdot 2^{\sum_{F \ni \bar{\mathscr{I}}_m} \ell(F/\mathscr{E}_m)} \cdot \operatorname*{Sup}_{\substack{S \in S_{\mathscr{I}_m \bar{\mathscr{I}}_m} \\ S' = \varphi_m(S)}} \frac{u(S)}{u(S')}. \tag{III.50}$$

Lemma III.7.

$$\operatorname*{Sup}_{\substack{S \in S_{\mathscr{I}_m \bar{\mathscr{I}}_m} \\ S' = \varphi_m(S)}} \frac{u(S)}{u(S')} \leqq \frac{\prod_{F \in \mathscr{I}_m \cap \mathscr{H}} \prod_{i \in M_F} \beta_i}{\prod_{F \in \mathscr{I}_m \cap \mathscr{B}} \prod_{i \in M_F} \beta_i}, \tag{III.51}$$

where

$$M_F = \{i \mid x(F, \mathscr{F}) \leqq i < y(F, \mathscr{F}) \quad \text{if} \quad F \in \mathscr{H}\,; y(F, \mathscr{F}) \leqq i < x(F, \mathscr{F}) \ \text{if} \ F \in \mathscr{F}\}. \tag{III.52}$$

Proof. We analyse the action of φ_m by looking at the composition of mappings: for $S \in S_{\mathcal{J}_m \bar{\mathcal{J}}_m}$ and $F \in \mathcal{J}_m$ we examine how φ_F acts on $S' = \left[\prod_{F' \in \mathcal{D}_{mF}} \varphi_{F'}\right](S)$.

a) If $F \in \mathcal{B}$, then by (III.40) φ_F acts on S' by changing at most one line λ of S' into another line λ', and by (III.41) and (III.42) this implies

$$\frac{u(\varphi_F(S'))}{u(S')} \leqq \left[\prod_{y(F,\mathcal{F}) \leqq i < x(F,\mathcal{F})} \beta_i\right]^{-1}. \tag{III.53}$$

b) If $F \in \mathcal{H}$ and S' is not a one-tree in F, then $\varphi_F(S')$ is given by (III.43), and we have directly from (III.41) and (III.42)

$$\frac{u(\varphi_F(S'))}{u(S')} \leqq \prod_{x(F,\mathcal{F}) \leqq i < y(F,\mathcal{F})} \beta_i. \tag{III.54}$$

c) If $F \in \mathcal{H}$ and S' is a one-tree in F, then $\varphi_F(S') = S'$.

Since $S \in S_{\mathcal{J}_m \bar{\mathcal{J}}_m}$, S is not a one-tree in F, and there must exist some $F_1 \subset F$ with

$$\begin{cases} S_1 = \left[\prod_{F' \in \mathcal{D}_{mF_1}} \varphi_{F'}\right](S) & \text{not a one-tree in } F, \\ \varphi_{F_1}(S_1) & \text{one tree in } F. \end{cases}$$

In other words, φ_{F_1} removes from S_1 at least one line λ which is external in F and F_1, and internal in $B_{\mathcal{F}}(F_1)$. This implies

$$F_1 \subset F \subset B_{\mathcal{F}}(F_1).$$

More generally the whole set $\{F_1, F_2, \ldots, F_r\}$ of subgraphs F_j, such that $F_1 \subsetneqq F_j \subset B_{\mathcal{F}}(F_1)$ and λ is external in F_j, is totally ordered by inclusion: $F_1 \subset F_2 \subset \ldots \subset F_r \subset B_{\mathcal{F}}(F_1)$. Applying inductively the argument given in the proof of Lemma III.1, we have as a generalization of (III.14)

$$x(F_1,\mathcal{F}) < y(F_1,\mathcal{F}) \leqq x(F_2,\mathcal{F}) < y(F_2,\mathcal{F}) \leqq \ldots \leqq x(F_r,\mathcal{F}) < y(F_r,\mathcal{F}) \leqq k,$$

where k is the rank of λ in the sector: $\lambda = \sigma(k)$. From (III.41) and (III.42) again, we have

$$\frac{u(\varphi_{F_1}(S_1))}{u(S_1)} \leqq \prod_{x(F_1,\mathcal{F}) \leqq i < k} \beta_i \leqq \prod_{j=1}^{r} \prod_{x(F_j,\mathcal{F}) \leqq i < y(F_j,\mathcal{F})} \beta_i. \tag{III.55}$$

In particular, like (III.54), (III.55) contains the factor

$$\prod_{x(F,\mathcal{F}) \leqq i < y(F,\mathcal{F})} \beta_i.$$

Gathering the results a)–c) with the definition (III.44) achieves the proof of Lemma III.7.

By Lemma III.7, (III.25), (III.26), (III.32), (III.33), and (III.50), we can perform trivially the χ_F integrations and use $U_{\mathcal{F}}^{\mathcal{H}}(\beta_i, \chi_F) \geqq U_{\mathcal{F} \cup \mathcal{H}}(\beta_i)$ to get

$$Y_G^\delta(\not{p}, \beta_i) \leqq |\not{p}|^{n(G)} \frac{2^{2\ell(G)} \cdot 2^{4n(G)}}{[U_{\mathcal{F} \cup \mathcal{H}}(\beta_i)]^2} \cdot \frac{\left[\prod_{F \in \mathcal{Q}'} \prod_{i \in M_F} \beta_i\right]\left[\prod_{F \in \mathcal{B}'} \prod_{i \in M_F} \beta_i\right]^2}{\prod_{F \in \mathcal{B}} \prod_{i \in M_F} \beta_i} \tag{III.56}$$

$\left[\text{we have used (III.26) to obtain: } \sum_{m=1}^{k} |\mathscr{I}_m| \leqq f(\mathscr{H}) + b(\mathscr{F}) \leqq f_1(G) \leqq n(G)\right]$. From (III.56) we deduce

$$Y_G^\delta(\not{p}, \beta_i) \leqq |\not{p}|^{n(G)} \cdot 2^{2\ell(G)} \cdot 2^{4n(G)} \prod_{i=1}^{\ell-1} \beta_i^{\eta_i}, \tag{III.57}$$

with

$$\eta_i = -2 \sum_{F \in \mathscr{F} \cup \mathscr{H} \cup \{G\}} L(G_i^\sigma \cap F/\mathscr{F} \cup \mathscr{H}) + \sum_{\substack{F \in \mathscr{B}' \\ M_F \ni i}} 2 + \sum_{\substack{F \in \mathscr{Q}' \\ M_F \ni i}} 1 - \sum_{\substack{F \in \mathscr{B} \\ M_F \ni i}} 1. \tag{III.58}$$

Therefore (III.36) and Lemma III.5 will be proved by the following:

Lemma III.8.

$$\forall i, \quad i + \eta_i \geqq \frac{\tilde{N}_i(\sigma)}{6}. \tag{III.59}$$

Proof. Using inductively (III.12) we can change (III.58) into

$$i + \eta_i = \sum_{F \in \mathscr{F} \cup \{G\}} \omega(G_i^\sigma \cap F/\mathscr{F}) + \sum_{\substack{F \in \mathscr{B}' \\ M_F \ni i}} 2 + \sum_{\substack{F \in \mathscr{Q}' \\ M_F \ni i}} 1 - \sum_{\substack{F \in \mathscr{B} \\ M_F \ni i}} 1. \tag{III.60}$$

In this formula, any connected component C of $G_i^\sigma \cap F/\mathscr{F}$ can contain vertices with only two lines, by reduction of elements of $\mathscr{B}$. Hence if $b(C)$ is the number of such vertices, (II.11) has to be modified into

$$\omega(C) = \frac{N(C)}{2} - 2 + b(C). \tag{III.61}$$

But for $F \in \mathscr{B}$, $i \in M_F$, we have $y(F, \mathscr{F}) \leqq i < x(F, \mathscr{F})$; therefore by definition of $y(F, \mathscr{F})$ there exists in $B_{\mathscr{F}}(F)/\mathscr{F}$ a connected component C of $G_i^\sigma \cap B_{\mathscr{F}}(F)/\mathscr{F}$ which contains the reduction vertex of F.

Thus we have

$$\sum_{F \in \mathscr{F}} \left[\sum_{\substack{C \text{ conn. comp.} \\ \text{of } G_i^\sigma \cap B_{\mathscr{F}}(F)/\mathscr{F}}} b(C) \right] - \sum_{\substack{F \in \mathscr{B} \\ M_F \ni i}} 1 \geqq \sum_{\substack{F \in \mathscr{B} \\ x(F, \mathscr{F}) \leqq i}} 1, \tag{III.62}$$

and (III.60) becomes

$$i + \eta_i \geqq \sum_{F \in \mathscr{F} \cup \{G\}} \left[\sum_{\substack{C \text{ conn. comp.} \\ \text{of } G_i^\sigma \cap F/\mathscr{F}}} \left(\frac{N(C)}{2} - 2 \right) \right] + \sum_{\substack{F \in \mathscr{B}' \\ M_F \ni i}} 2 + \sum_{\substack{F \in \mathscr{Q}' \\ M_F \ni i}} 1 + \sum_{\substack{F \in \mathscr{B} \\ x(F, \mathscr{F}) \leqq i}} 1. \tag{III.63}$$

We can now evaluate $i + \eta_i$, by distinguishing, given $F \in \mathscr{F} \cup \{G\}$, the possible cases for connected components C of $G_i^\sigma \cap F/\mathscr{F}$:

a) If $\frac{N(C)}{2} - 2 > 0$, then $\frac{N(C)}{2} - 2 \geqq \frac{N(C)}{6}$. (III.64)

b) If $\frac{N(C)}{2} - 2 \leqq 0$ and C is not proper, then C is represented by Figs 2 or 3 and contains a proper biped $B(C)$ which satisfies $B(C) = B'/\mathscr{F}$, $B' \in \mathscr{B}'$ and $M_{B'} \ni i$ from the definitions of Subsect. III.1. Hence we have

$$\left(\frac{N(C)}{2}-2\right)+2=\frac{N(C)}{2}>\frac{N(C)}{6}. \tag{III.65}$$

c) If $\frac{N(C)}{2}-2\leqq 0$ and C is proper but not closed, then C is a quadruped, $C^* = B'/\mathscr{F}$ and $B'\in\mathscr{B}\cup\mathscr{B}'$.

Either $B'\in\mathscr{B}'$ and $M_{B'}\ni i$. Hence

$$\left(\frac{N(C)}{2}-2\right)+2=2>\frac{N(C)}{6}. \tag{III.66}$$

Or else $B'\in\mathscr{B}$; then $X(B',\mathscr{F})/\mathscr{F}\subset C$, that is $x(B',\mathscr{F})\leqq i$.
Hence

$$\left(\frac{N(C)}{2}-2\right)+1=1>\frac{N(C)}{6}. \tag{III.67}$$

d) If $\frac{N(C)}{2}-2\leqq 0$ and C is proper and closed, then $C=F'/\mathscr{F}$ with the following possibilities:

$$F'\in\mathscr{Q}', \quad M_{F'}\ni i, \quad \text{hence} \quad \left(\frac{N(C)}{2}-2\right)+1=1>\frac{N(C)}{6}, \tag{III.68}$$

$$F'\in\mathscr{B}', \quad M_{F'}\ni i, \quad \text{hence} \quad \left(\frac{N(C)}{2}-2\right)+2=1>\frac{N(C)}{6}, \tag{III.69}$$

$$F'\in\mathscr{Q}, \quad \text{hence} \quad F'=F \quad \text{and} \frac{N(C)}{2}-2=0, \tag{III.70}$$

$$F'\in\mathscr{B}, \quad \text{hence} \quad F'=F, x(F,\mathscr{F})\leqq i \quad \text{and} \quad \left(\frac{N(C)}{2}-2\right)+1=0. \tag{III.71}$$

All these cases are mutually exclusive and allow us to bound the sum in the right-hand side of (III.63), proving Lemma III.8. The cases (III.70) and (III.71) correspond to the restriction $G_i^q\cap F/\mathscr{F}\neq F/\mathscr{F}$ in (III.37).

IV. Discussion

IV.1. Discussion of Theorem 1

We mainly want to emphasize that Theorem I gives only an upper bound on the renormalized Feynman amplitudes, but that their true behaviour may be quite different. First we note that (III.33) does not describe the true behaviour of these amplitudes at large momenta, because we used in (III.32) the very crude bound $\exp(-W)\leqq 1$. The asymptotic behaviour of a given amplitude, when any set of external momenta is scaled to infinity, can be computed in a systematic way, for instance by the multiple or complete Mellin techniques [8]. The result is an expansion in powers and powers of logarithms, which generalizes the Weinberg theorem. In particular the leading term, when all external momenta are scaled by ϱ and ϱ goes to infinity, behaves like

$$\varrho^{\sup_{F \subseteq G}(-2\omega(F))} \ln^{\zeta(G)} \varrho, \tag{IV.1}$$

where

$$\zeta(G) = \sup_{\mathcal{N}} |\mathcal{N}| - \delta, \quad \delta = 0 \quad \text{or} \quad 1,$$

$$\mathcal{N} = \left\{ F_1 \subset \ldots \subset F_r \ldots \subset F_{|\mathcal{N}|}, \quad \text{with} \quad \omega(F_r) = \inf_{F \subseteq G} (\omega(F)) \forall r \right\}. \tag{IV.2}$$

We believe that it is possible to take into account the decrease of $\exp(-W)$ in (III.21), to improve Theorem I and obtain bounds at large order with the right behaviour (IV.1) at large momenta. More generally it would be interesting to estimate jointly the large order and the various large momenta behaviours of Feynman amplitudes. Both problems are linked, in fact, the large momenta behaviour of the chain in Fig. 4 led to the discovery of the factorial behaviour of some Feynman amplitudes [4].

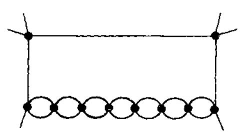

Fig. 4

Even at fixed external momenta, Theorem I does not give the right large order behaviour (up to exponentials of the order). As a first trivial example, when the graph is one-vertex reducible, one can bound its amplitude by the product of the bounds on its one-vertex irreducible components, which generally improves the bound of Theorem I.

We also remark that a biped does not necessarily have to be counted twice in the true factorial behaviour at large order. As a simple example of this fact, one can study the graph of Fig. 5 and prove easily:

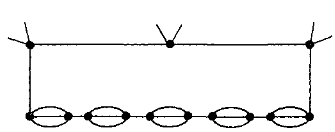

Fig. 5

Proposition IV.1. *Let G_m be the graph of Fig. 5, with n vertices and $m = n/2 - 3/2$ proper bipeds. Then there exists two positive constants K_6 and K_7 such that*

$$K_6^n m! \leqq I_{G_m}^R(0) \leqq K_7^n m! \tag{IV.3}$$

Therefore the large order behaviour of $I_{G_m}^R$ is given by $(n!)^{1/2}$ rather than $n!$. This discrepancy could perhaps be corrected by identifying all the reduction lines in a chain of proper bipeds. In the properly modified α-parametric representation, there would be fewer Hepp's sectors; our definition of skeleton forests in Subsect. III.1 would become more natural by treating bipeds and quadrupeds in the same way.

Finally we must mention that we did not try to find the smallest possible constants in Theorems I and II. By looking at many places in Sect. III and in the appendices, one could lower these constants.

IV.2. Analytic Continuation in the Borel Plane and Renormalons

Although the expansion in the coupling constant is Borel summable for Φ_2^4 or Φ_3^4 field theory [2], this is not believed to be the case for Φ_4^4. The analytic continuation of the function $\mathbf{B}_N^T$ depends in a crucial way on the signs of the coefficients $\mathbf{a}_n^N$, and the renormalization is supposed to destroy the alternation of signs in the perturbative expansion. More precisely it has been argued [5] that the partial series retaining only the graphs of Fig. 4 (since they behave like $K^n n!$ and present no alternation of signs) is responsible for the appearance of a singularity ("renormalon") of $\mathbf{B}_N^T(t)$, at the positive real value $t=1/K$. Of course this is not a rigorous argument: there could be a conspiracy between the graphs of Fig. 4 and the other ones, which cancels the singularity. Such a cancellation would certainly be spectacular, but we are not sure it can be ruled out a priori. For example, if we separate the expansion of the Schwinger functions of Φ_2^4 into two parts, respectively given by the amplitudes with even and odd number of vertices, each of these parts presents singularities on the positive real axis of the Borel plane. Constructive field theory proves that these singularities disappear when we add them, a property which is not obvious at all at the perturbative level.

On this open problem of the existence of renormalons, we may note that our study generalizes the argument based on the graphs of Fig. 4. For example the amplitudes of the G_m in Fig. 5 also define a partial series with no alternation of signs. Since their number is about $n!$, and they grow as $(n!)^{1/2}$, the corresponding partial series in the Borel plane has an infinite radius of convergence. However, it defines a function which incrases like $\exp(t^2)$ at infinity on the real positive axis, destroying Borel summability. Such a feature can be thought of as a "renormalon at infinity". Since some factorial behaviour is a very common property of the renormalized amplitudes, there could be many such "renormalons at infinity". Again, we cannot say whether they cancel or not in the analytic continuation of the whole function $\mathbf{B}_N^T$.

Finally let us mention that our methods can be applied to other renormalizable theories, in spite of the tedious complications due to spinor fields. Our opinion is that the Borel transform exists locally again, for the Yukawa model or quantum electrodynamics in 4 dimensions for instance. Yet the problem of the analytic continuation may present very different features, depending on the model.

Appendix A

In this appendix we bound the number of closed divergent forests in the graphs with n vertices.

A.1. Overlapping Divergent Subgraphs

If two subgraphs F_1 and F_2 overlap, we have evidently

$$N(F_1)+N(F_2)=N(F_1\cup F_2)+N(F_1\cap F_2), \tag{A.1}$$

where $N(F)$ is the number of external lines of F. If F_1 and F_2 are divergent, $N(F_1)$ and $N(F_2)$ may be equal to 2 or 4. Moreover if F_1 and F_2 are proper, $F_1\cap F_2$ must be linked to $F_1-(F_1\cap F_2)$ by at least two lines, and to $F_2-(F_1\cap F_2)$ by at least two different lines. Thus

$$N(F_1\cap F_2)\geqq 4. \tag{A.2}$$

Hence two proper bipeds cannot overlap if there is no vacuum graph, and we are left with the following cases:

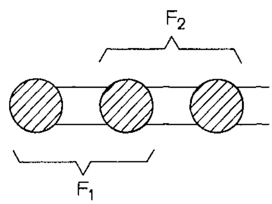

Fig. 6

$$N(F_1)=2;\quad N(F_2)=4;\quad N(F_1\cup F_2)=2;\quad N(F_1\cap F_2)=4 \text{ (Fig. 6)}$$

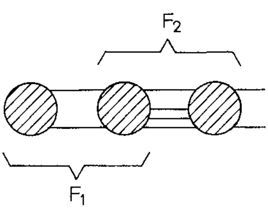

Fig. 7

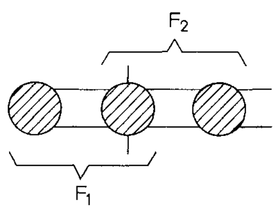

Fig. 8

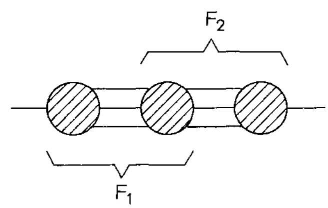

Fig. 9

$$N(F_1)=N(F_2)=4;\quad N(F_1\cup F_2)=2;\quad N(F_1\cap F_2)=6 \text{ (Figs. 7–9)}$$

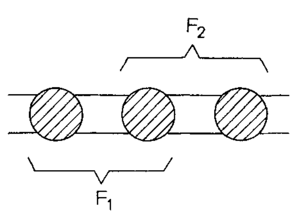

Fig. 10

$$N(F_1)=N(F_2)=N(F_1\cup F_2)=N(F_1\cap F_2)=4. \text{ (Fig. 10)}$$

If F_1 and F_2 are closed, Figs. 6–8 are forbidden. We see in particular that a proper biped cannot overlap with any closed divergent subgraph.

A.2. Closed Divergent Forests

A closed divergent subgraph F of G is said to be maximal if $F\subset G$, and there exists no other closed divergent subgraph F' satisfying $F\subset F'\subset G$. A closed divergent forest is maximal if there is no other closed divergent forest containing it. Given a maximal closed divergent forest $\mathscr{F}$:

i) either the biggest subgraphs in $\mathscr{F}$ (apart from G itself if $G\in\mathscr{F}$) are the maximal closed divergent subgraphs of G.

ii) or there exists a maximal subgraph of G, say F_1, with $F_1\notin\mathscr{F}$. $\mathscr{F}$ being maximal, $\mathscr{F}\cup\{F_1\}$ is not a forest, that is F_1 must overlap with at least one biggest subgraph in $\mathscr{F}$, say F_2. From (A.1) and (A.2) $F_1\cup F_2$ is divergent, since F_1 is maximal, $F_1\cup F_2=G$ and G has the structure of Fig. 9 (respectively Fig. 10) if G is a biped (respectively a quadruped). By writing the complete 3- (respectively 2-)particles reducibility of G, the possible biggest subgraphs in $\mathscr{F}$ are obtained in cutting some set of 3 (respectively 2) lines as in Fig. 11 (respectively Fig. 12).

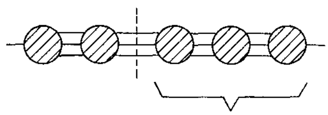

Fig. 11

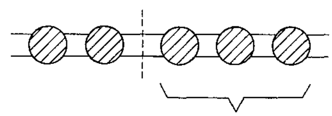

Fig. 12

Let $d(G)$ be the number of maximal closed divergent forests of G, and d_n the maximal value of $d(G)$ over all graphs G with n vertices. Since there is at least one vertex in each maximal divergent closed subgraph and in each bubble of Figs. 11 and 12, we may conclude in both cases i) and ii) that

$$d_n\leqq\sum_{n'=1}^{n-1} d_{n'}d_{n-n'}\,. \tag{A.3}$$

Lemma A.1.

$$d_n\leqq 4^n\,. \tag{A.4}$$

Proof. Putting $e_n=\sum_{n'=1}^{n-1} e_{n'}e_{n-n'}$ with $e_1=1$, and defining

$$e(z)=\sum_{n=1}^{\infty} e_n z^n\,,$$

we have $e^2(z)=e(z)-z$, or $e(z)=1/2[1-(1-4z)^{1/2}]$, which is analytic for $|z|<1/4$. It is easy to verify, by computing the n^{th} derivative of $e(z)$, that $e_n\leqq 4^n$. Now from (A.3), $d_n\leqq e_n$, which proves the lemma.

Finally any closed divergent forest is a subset of a maximal closed divergent forest, which contains at most n subgraphs by Lemma C.1. Using Lemma A.1 we may conclude:

Lemma A.2. *Let $v(G)$ be the number of closed divergent forests of a graph G of Φ_4^4, with n vertices. Then*

$$v(G)\leqq 8^n\,. \tag{A.5}$$

Appendix B

This appendix is devoted to the proof of Lemma III.6. We fix a given closed divergent forest $\mathscr{F}$, with s elements. We number in an arbitrary way the elements

F_j of $\mathscr{F}\cup\{G\}$ by an index $j\in J=\{1,\ldots,s+1\}$. A spanning tree S of G is called an $\mathscr{F}$-tree of G if $S\cap F_j$ is a spanning tree of F_j for every $j\in J$. Any $\mathscr{F}$-tree of G is defined as well by the family $\{S_j\}$, $j\in J$, where $S_j=S\cap F_j/\mathscr{F}$ is a spanning tree of $F_j/\mathscr{F}$.

Then to every permutation $\sigma\in\mathfrak{S}_G^{\mathscr{F}}$ we associate an $\mathscr{F}$-tree $S(\sigma)$ by defining the corresponding family $\{S_j(\sigma)\}$

$$S_j(\sigma)=\{\sigma(i)|\sigma(i)\in F_j/\mathscr{F}\,;L(G_i^\sigma\cap F_j/\mathscr{F})=L(G_{i-1}^\sigma\cap F_j/\mathscr{F})\}\,, \tag{B.1}$$

and we define

$$k_j(\sigma)=\sup_{\sigma(i)\in S_j(\sigma)} i\,, \tag{B.2}$$

$$\bar{\mathfrak{S}}_G^{\mathscr{F}}=\{\sigma|\sigma\in\mathfrak{S}_G^{\mathscr{F}},k_1(\sigma)<k_2(\sigma)<\ldots<k_{s+1}(\sigma)\}\,. \tag{B.3}$$

Since there are $(s+1)!$ ways to choose the arbitrary numbering of the subgraphs F_j, and since

$$(s+1)!/s!=s+1\leqq n(G)+1\leqq 2^{n(G)}\,, \tag{B.4}$$

we see that Lemma III.6 is a direct consequence of the existence of a constant K_4 such that

$$\sum_{\sigma\in\bar{\mathfrak{S}}_G^{\mathscr{F}}}\left[\prod_{\substack{i\\ \sigma(i)\in S(\sigma)}}\frac{1}{\tilde{N}_i(\sigma)}\right]\cdot\left[\prod_{\substack{i\\ \sigma(i)\notin S(\sigma)}}\frac{1}{\tilde{N}_i(\sigma)}\right]\leqq(\mathrm{K}_4/2)^{n(G)}\,. \tag{B.5}$$

In order to prove (B.5), we introduce some new definitions: Given two subgraphs F and F', with $F'\subsetneqq F$ denote by $N_F(F')$ the number of internal lines of F which are external lines of F'. Given an $\mathscr{F}$-tree S (defined by the family $\{S_j\}$, $j\in J$) and a permutation $\sigma\in\mathfrak{S}_S$, we define $k_j(\sigma)$ by (B.2) and we put

$$\bar{\mathfrak{S}}_S=\{\sigma|\sigma\in\mathfrak{S}_S,k_1(\sigma)<k_2(\sigma)<\ldots<k_{s+1}(\sigma)\}\,, \tag{B.6}$$

$$M(S_i^\sigma)=1+\sum_{\substack{j\in J\\ k_j(\sigma)>i}}[N_{S_j}(S_j\cap S_i^\sigma)+1]\,, \tag{B.7}$$

where S_i^σ is defined for $\sigma\in\mathfrak{S}_S$, just like G_i^σ for $\sigma\in\mathfrak{S}_G$.

$$M_S(\sigma)=\prod_{i=1}^{\ell(S)}1/M(S_i^\sigma)\,. \tag{B.8}$$

Finally to any permutation $\sigma\in\bar{\mathfrak{S}}_G^{\mathscr{F}}$ we associate the permutation $\psi(\sigma)=\sigma'\in\bar{\mathfrak{S}}_{S(\sigma)}$ naturally induced by σ: $\sigma'(i')=\sigma(i)\Leftrightarrow\sigma(i)$ is the i'th line belonging to $S(\sigma)$. That is

$$\sigma'(i')=\sigma(i)\Leftrightarrow\sigma(i)\in S(\sigma)\,;|\{i''<i|\sigma(i'')\in S(\sigma)\}|=i'-1\,. \tag{B.9}$$

Then we have:

Lemma B.1. *The left-hand side of* (B.5) *is bounded by*

$$3^{n(G)}\sum_{\substack{S\\ S\,\mathscr{F}\text{-tree of }G}}\;\sum_{\sigma'\in\bar{\mathfrak{S}}_S}M_S(\sigma')\sum_{\substack{\sigma\in\bar{\mathfrak{S}}_G^{\mathscr{F}}\\ S(\sigma)=S\\ \psi(\sigma)=\sigma'}}\;\prod_{\substack{i\\ \sigma(i)\notin S(\sigma)}}1/\tilde{N}_i(\sigma)\,. \tag{B.10}$$

Proof. We have $\tilde{N}_i(\sigma) \geqq 1$ since $\sigma \in \mathfrak{S}_G^{\mathscr{F}}$. Moreover if $\sigma'(i') = \sigma(i) \in S$ and $k_j(\sigma') > i'$, we have

$$G_i^\sigma \cap F_j/\mathscr{F} \neq F_j/\mathscr{F}$$

and

$$N(G_i^\sigma \cap F_j/\mathscr{F}) \geqq \mathrm{Sup}\{1, N_{S_j}(S_j \cap S_{i'}^{\sigma'})\}.$$

Thus

$$2N(G_i^\sigma \cap F_j/\mathscr{F}) \geqq N_{S_j}(S_j \cap S_{i'}^{\sigma'}) + 1,$$

and

$$\sigma'(i') = \sigma(i) \in S \Rightarrow 3\tilde{N}_i(\sigma) \geqq M(S_{i'}^{\sigma'}). \tag{B.11}$$

Therefore

$$\prod_{\substack{i \\ \sigma(i) \in S(\sigma)}} 1/\tilde{N}_i(\sigma) \leqq 3^{\ell(S(\sigma))} M_{S(\sigma)}(\psi(\sigma)). \tag{B.12}$$

Since $S(\sigma)$ is a spanning tree of G, $\ell(S(\sigma)) = n(G) - 1$, which achieves the proof of Lemma B.1.

Lemma B.2.

$$\forall S, \sigma', \sum_{\substack{\sigma \in \mathfrak{S}_G^{\mathscr{F}} \\ S(\sigma) = S \\ \psi(\sigma) = \sigma'}} \prod_{\substack{i \\ \sigma(i) \notin S(\sigma)}} \frac{1}{\tilde{N}_i(\sigma)} \leqq 2^{\ell(G) - 1}. \tag{B.13}$$

Proof. By induction, let us assume that we have chosen $\sigma(1), \ldots, \sigma(i-1)$. We must then choose a line $\sigma(i)$ such that $S(\sigma) = S$ and $\psi(\sigma) = \sigma'$. We get a factor 2 by deciding whether $\sigma(i)$ belongs to S or not.

i) If $\sigma(i) \in S$, $\sigma(i)$ is determined by σ'.

ii) If $\sigma(i) \notin S$, $\sigma(i)$ belongs to one $F_j/\mathscr{F}$ and adds a loop to $G_{i-1}^\sigma \cap F_j/\mathscr{F}$. In this case there is at most $\tilde{N}_{i-1}(\sigma)/2$ possible choices for $\sigma(i)$, and we have $\tilde{N}_i(\sigma) = \tilde{N}_{i-1}(\sigma) - 2$, hence $\tilde{N}_{i-1}(\sigma) \geqq 4$ (since there is no vacuum graph) and $\tilde{N}_i(\sigma) \geqq \tilde{N}_{i-1}(\sigma)/2$. Therefore the number of possible choices is bounded by $\tilde{N}_i(\sigma)$.

At the last step, $\sigma(\ell(G))$ is completely determined, which achieves the proof of Lemma B.2.

Now we want to evaluate $\sum_{\sigma' \in \mathfrak{S}_S} M_S(\sigma')$. Before doing it we construct a one-to-one correspondence ξ_S between the lines of a given connected tree S, and the lines of the chain-like tree D, with $n(D) = n(S)$ (Fig. 13): drawing S on a plane, we "turn around it", starting from an arbitrary extremal line, and we number the lines in the order we meet them the first time. The lines of D are numbered in the same way, and ξ_S associates the lines with the same number. Then we have:

Lemma B.3. *For any (connected or disconnected) subgraph R of S*

$$N_S(R) + 1 \geqq c(\xi_S(R)), \tag{B.14}$$

where $c(\xi_S(R))$ is the number of connected components of $\xi_S(R)$.

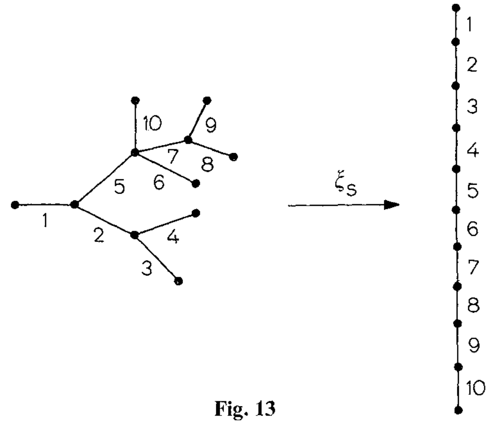

Fig. 13

Proof. Lemma B.3 is trivial for $n(S)=2$ and 3. Inductively, we assume it for $n(S)\leqq n$, and we consider a given S with $n(S)=n+1$ [thus $\ell(S)=n$]. Then the line numbered as n in the preceding numbering is certainly an extremal line of S. Let R be a given subgraph of S.

a) $n\notin R$. Applying the induction hypothesis to R as a subgraph of $S'=S-\{n\}$ proves (B.14).

b) $n\in R$ and $n-1\in R$. Applying the induction hypothesis to $R'=R-\{n\}$ and $S'=S-\{n\}$ gives (B.14) since $N_S(R)=N_{S'}(R')$; $c(\xi_S(R))=c(\xi_{S'}(R'))$.

c) $n\in R$, $n-1\notin R$ and $n-1$ connects with R. Applying the induction hypothesis to $R'=R-\{n\}$ and $S'=S-\{n,n-1\}$ gives (B.14) since

$$N_S(R)\geqq N_{S'}(R')+1 \quad \text{and} \quad c(\xi_S(R))=c(\xi_{S'}(R'))+1\,.$$

d) $n\in R$, $n-1\notin R$ and $n-1$ does not connect with R. Then $n-1$ is extremal in S and $E\cap R=\emptyset$, if E is the set of lines of S which connect with $n-1$. Moreover $E\neq\emptyset$.

Either all the lines in E connect with R. Since there exists one line $j\in E$ with

$$j+1\in E\cup\{n-1\}\,,$$

we can contract this line and renumber the tree $S'=S/\{j\}$ by the same procedure. Applying the induction hypothesis to S' and $R'=R\cup\{j\}/\{j\}$ gives (B.14) since $N_S(R)=N_{S'}(R')$ and

$$c(\xi_S(R))=c(\xi_{S'}(R'))\,.$$

Or else there exists a line k in E which does not connect with R. Then $k+1$ connects with k (trivially if k is not extremal, because k connects with $n-1$ if k is extremal), hence $k+1\notin R$. Applying the induction hypothesis to $S'=S/\{k\}$ and $R'=R\cup\{k\}/\{k\}$ gives (B.14) since again $N_S(R)=N_{S'}(R')$ and $c(\xi_S(R))=c(\xi_{S'}(R'))$.

Lemma B.4. *Let D be the chain-like tree with n vertices. Then if we put*

$$\Delta_n=\sum_{\sigma\in\mathfrak{S}_D}\prod_{i=1}^{n-1}1/c(D_i^\sigma)\,, \tag{B.15}$$

we have

$$\Delta_n\leqq 4^n\,. \tag{B.16}$$

Proof. Let $G_1, \ldots, G_q$ be q disjoint graphs (each of them being connected or disconnected), with $\ell_1, \ldots, \ell_q$ internal lines. Let G be their union, with $\ell = \sum_{j=1}^{q} \ell_j$ internal lines. For any $\sigma \in \mathfrak{S}_G$, let $\psi_j(\sigma)$ be the permutation naturally induced on the lines of G_j by (B.9). Then, given $\sigma_1 \in \mathfrak{S}_{G_1}, \ldots, \sigma_q \in \mathfrak{S}_{G_q}$, we have

$$\sum_{\substack{\sigma\in\mathfrak{S}_G \\ \psi_j(\sigma)=\sigma_j \forall j}} \prod_{i=1}^{\ell} 1/c(G_i^\sigma) = \prod_{j=1}^{q} \prod_{i=1}^{\ell_j} 1/c[(G_j)_i^{\sigma_j}]. \tag{B.17}$$

This can be proved by induction on ℓ: it is true for $\ell = 1$ and we have

$$\sum_{\substack{\sigma\in\mathfrak{S}_G \\ \psi_j(\sigma)=\sigma_j}} \prod_{i=1}^{\ell} 1/c(G_i^\sigma) = \frac{1}{c(G)} \sum_{j=1}^{q} \sum_{\substack{\sigma\in\mathfrak{S}_G \\ \psi_j(\sigma)=\sigma_j \\ \sigma(\ell)=\sigma_j(\ell_j)}} \prod_{i=1}^{\ell-1} 1/c(G_i^\sigma).$$

If we assume (B.17) for $G - \{\sigma(\ell)\} = \left[\bigcup_{\substack{k=1 \\ k\neq j}}^{q} G_k\right] \cup [G_j - \sigma_j(\ell_j)]$, we get

$$\sum_{\substack{\sigma\in\mathfrak{S}_G \\ \psi_j(\sigma)=\sigma_j}} \prod_{i=1}^{\ell} 1/c(G_i^\sigma) = \frac{1}{c(G)} \sum_{j=1}^{q} c(G_j) \prod_{k=1}^{q} \prod_{i=1}^{\ell_k} 1/c[(G_k)_i^{\sigma_k}],$$

which proves (B.17) for G since $c(G) = \sum_{j=1}^{q} c(G_j)$. Now since $c(D_{n-1}^\sigma) = c(D) = 1$, we have

$$\Delta_n = \sum_{i=1}^{n-1} \sum_{\substack{\sigma_1\in\mathfrak{S}_{D_1(i)} \\ \sigma_2\in\mathfrak{S}_{D_2(i)}}} \sum_{\substack{\sigma\in\mathfrak{S}_D \\ \psi_1(\sigma)=\sigma_1 \\ \psi_2(\sigma)=\sigma_2 \\ \sigma(n-1)=i}} \prod_{i'=1}^{n-2} 1/c(D_{i'}^\sigma), \tag{B.18}$$

where $D_1(i)$ is the chain $\{1, \ldots, i-1\}$ and $D_2(i)$ the chain $\{i+1, \ldots, n-1\}$. By (B.17), with the convention $\Delta_1 = 1$, (B.18) may be written

$$\Delta_n = \sum_{i=1}^{n-1} \Delta_i \Delta_{n-i}. \tag{B.19}$$

Therefore, by the proof of Lemma A.1, $\Delta_n = e_n$ and Lemma B.4 is proved.

Lemma B.5.

$$\sum_{\sigma\in\mathfrak{S}_S} M_S(\sigma) \leqq 4^{n(G)}. \tag{B.20}$$

Proof. Given an $\mathscr{F}$-tree S defined by the family $\{S_j\}$, $j \in J$, we take the correspondence ξ_{S_j} between the lines of each S_j and the lines of the related chain D_j with $\ell(D_j) = \ell(S_j)$. Then we renumber the lines of $D_j = \{1, \ldots, \ell(D_j)\}$ as $\{1 + t_j, \ldots, \ell(D_j) + t_j\}$, where $t_j = \sum_{j'<j} \ell(D_{j'})$. By connecting the chains $D_1, \ldots, D_{s+1}$, we build the chain D with $\ell(D) = \ell(S) = n(G) - 1$ and we get a one-to-one correspondence τ_S between the lines of S and the lines of D.

For $\sigma \in \bar{\mathfrak{S}}_S$, we have from (B.6) $k_j(\sigma) \leqq i \Leftrightarrow j \leqq j_{\max}(i)$. Therefore

$$\bigcup_{\substack{j \in J \\ k_j(\sigma) \leqq i}} S_j \subseteqq S_i^\sigma$$

and

$$c\left[\tau_S\left(\bigcup_{\substack{j \in J \\ k_j(\sigma) \leqq i}} S_j\right)\right] = c\left(\bigcup_{j=1}^{j_{\max}(i)} D_j\right) = 1 . \tag{B.21}$$

For $k_j(\sigma) > i$ we have from Lemma B.3

$$1 + N_{S_j}(S_j \cap S_i^\sigma) \geqq c[\tau(S_j \cap S_i^\sigma)] . \tag{B.22}$$

Using (B.21) and (B.22) in the definition (B.7), we have

$$\forall \sigma \in \bar{\mathfrak{S}}_S, \qquad M(S_i^\sigma) \geqq c(\tau_S(S_i^\sigma)) . \tag{B.23}$$

Since τ_S is a one-to-one correspondence between the lines of S and the lines of D, (B.23) and Lemma B.4 achieve the proof of (B.20).

Finally Lemmas B.1, B.2, and B.5 achieve the proof of formula (B.5) and Lemma III.6

Appendix C

In this appendix we prove Theorem II. Let us remark first:

Lemma C.1.

$$\forall G, f_2(G) \leqq n(G) . \tag{C.1}$$

Proof. For any forest $\mathscr{F}$ of quadrupeds of G, and any $F \in \mathscr{F}$, we have

$$n(F) = n(F/\mathscr{F}) + \sum_{F' \in \mathscr{A}_{\mathscr{F}}(F)} [n(F') - 1] ,$$

and $\forall F' \in \mathscr{F}$, $n(F'/\mathscr{F}) \geqq 2$. A simple induction gives

$$f(\mathscr{F}) = q(\mathscr{F}) \leqq n(G) - 1 . \tag{C.2}$$

From the definition (II.16) of $f_2(G)$, (C.2) proves (C.1).

Now let $\gamma(N, n, f)$ be the number of graphs G with $N(G) = N$, $n(G) = n$ and $f_2(G) = f$. To any such graph we associate a given arbitrary forest $\mathscr{F}(G)$ of quadrupeds of G, such that $G \notin \mathscr{F}(G)$ and $f(\mathscr{F}(G))$ is maximal with this restriction, that is

$$f(\mathscr{F}(G)) = f' = f - 1 , \quad \text{if} \quad N(G) \leqq 4 ,$$
$$f(\mathscr{F}(G)) = f' = f , \quad \text{if} \quad N(G) > 4 ,$$

and in any case, by (C.2)

$$f \leqq f' + 1 \leqq n . \tag{C.3}$$

We call $\lambda(G)$ the number of elements in a maximal nest of $\mathscr{F}(G)$, i.e. in a maximal subset of $\mathscr{F}(G)$ totally ordered by inclusion. Then we separate $\mathscr{F}(G)$ into layers $\mathscr{L}_i$, $1 \leqq i \leqq \lambda(G)$, which realize a partition of $\mathscr{F}(G)$

$$\begin{aligned} \mathscr{L}_{\lambda(G)} &= \{F | F \in \mathscr{F}(G) ; B_{\mathscr{F}(G)}(F) = G\} \\ \mathscr{L}_i &= \{F | F \in \mathscr{F}(G) ; B_{\mathscr{F}(G)}(F) \in \mathscr{L}_{i+1}\} \quad \text{for} \quad 1 \leqq i \leqq \lambda(G) - 1 . \end{aligned} \tag{C.4}$$

We also define

$$h_i(G)=|\mathscr{L}_i|, \tag{C.5}$$

$$k_i(G)=\sum_{i'=1}^{i} h_i(G). \tag{C.6}$$

Therefore

$$h_i(G)>0 \quad \text{for} \quad 1\leqq i\leqq \lambda(G), \tag{C.7}$$

$$0<k_1(G)<k_2(G)< \dots <k_{\lambda(G)}=f'. \tag{C.8}$$

Next we order arbitrarily the disjoint subgraphs in each layer, and we label them in the following way

$$\mathscr{L}_i=\{F_j(G)|j=k_{i-1}+1,\dots,k_i\}, \tag{C.9}$$

with the conventions

$$k_0=0, \quad h_0=0, \quad \text{and} \quad G=F_{f'+1} \tag{C.10}$$

(There are of course $\prod_{i=1}^{\lambda(G)} [h_i(G)!]$ possible different labelings.) We define

$$n_j(G)=n[F_j(G)/\mathscr{F}(G)], \tag{C.11}$$

$$s_j(G)=|\mathscr{A}_{\mathscr{F}(G)}(F_j(G))|, \tag{C.12}$$

for $1\leqq j\leqq f'+1$. With all these definitions we get

$$\gamma(N,n,f)=\sum_{\lambda}\sum_{\{k_i|i=1,\dots,\lambda\}}\left[\prod_{i=1}^{\lambda} 1/h_i!\right] \cdot \sum_{\{n_j,s_j|j=1,\dots,f'+1\}} \gamma(N,n,f,\lambda,\{k_i\},\{n_j\},\{s_j\}), \tag{C.13}$$

where $\gamma(N,n,f,\lambda,\{k_i\},\{n_j\},\{s_j\})$ is the number of graphs G with $N(G)=N$, $n(G)=n, \dots, s_j(G)=s_j$. If this number does not vanish, the following relations must be satisfied:

$$\forall j, 1\leqq j\leqq f'+1, \quad 2\leqq n_j\leqq n, \tag{C.14}$$

$$\sum_{j=1}^{f'+1} n_j=n+f', \tag{C.15}$$

$$\sum_{j=k_{i-1}+1}^{k_i} s_j=h_{i-1}; \quad \sum_{j=1}^{f'+1} s_j=f'. \tag{C.16}$$

Now the number of sequences $\{k_i\}$ satisfying (C.8) is equal to $\binom{f'-1}{\lambda-1}$, and

$$\sum_{\lambda=1}^{f'}\binom{f'-1}{\lambda-1}=2^{f'-1}\leqq 2^n. \tag{C.17}$$

By similar standard combinatoric arguments, the number of sets $\{n_j\}$ (respectively $\{s_j\}$) satisfying (C.14), (C.15) [respectively (C.16)] is bounded by 2^n (respectively $2^{2f'}$). Therefore the proof of Theorem II is achieved by the following lemma:

Lemma C.2. *$\forall\lambda$, $\{k_i\}$, $\{n_j\}$, $\{s_j\}$ satisfying the preceding relations, we have*

$$\left[\prod_{i=1}^{\lambda}\frac{1}{h_i!}\right]\gamma(N,n,f,\lambda,\{k_i\},\{n_j\},\{s_j\}\leqq K_8^n\frac{(n!)^2}{f!}n^{N/2}C(N), \tag{C.18}$$

K_8 being a positive constant.

Proof. The number of graphs with N external legs and n internal vertices is bounded by $(4n+N-1)!!$ [1]; therefore it is bounded by $K_9^n(n!)^2n^{N/2}C(N)$ for any N, and by $K_9^n(n!)^2$ for $N=4$, K_9 being a positive constant.

Hence it is possible to bound the left hand side of (C.18) by

$$\prod_{i=1}^{\lambda}\left\{\frac{1}{h_i!}\prod_{j=k_{i-1}+1}^{k_i}\left[\binom{n-\sum_{j'=1}^{j-1}r_{j'}}{r_j}\binom{h_{i-1}-\sum_{j'=k_{i-1}+1}^{j-1}s_{j'}}{s_j}K_9^{n_j}(n_j!)^2(4!)^{s_j}\right]\right\}$$
$$\cdot K_9^{n_{f'+1}}(n_{f'+1}!)^2n_{f'+1}^{N/2}C(N)(4!)^{s_{f'+1}}, \tag{C.19}$$

where $r_j=n_j-s_j$. Indeed we estimate the left-hand side of (C.18) by building successively

$$F_1/\mathscr{F},\ldots,F_j/\mathscr{F},\ldots,F_{f'+1}/\mathscr{F}.$$

The binomial coefficients

$$\binom{n-\sum_{j'=1}^{j-1}r_{j'}}{r_j}\quad\text{and}\quad\binom{h_{i-1}-\sum_{j'=k_{i-1}+1}^{j-1}s_{j'}}{s_j}$$

choose respectively the ordinary vertices and the reduction vertices which belong to $F_j/\mathscr{F}$; the terms $K_9^{n_j}(n_j!)^2$ or $K_9^{n_{f'+1}}(n_{f'+1}!)^2n_{f'+1}^{N/2}C(N)$ bound the number of graphs made with these vertices. The factors $(4!)^{s_j}$ take into account the possible identifications between the 4 external legs of F_j, and the 4 lines attached to the reduction vertex corresponding to F_j in $B_{\mathscr{F}(G)}(F_j)/\mathscr{F}$.

But, using (C.15) and (C.16), (C.19) is reduced to

$$K_9^{n+f'}[4!]^{f'}n_{f'+1}^{N/2}C(N)n!\prod_{j=1}^{f'+1}n_j!\binom{n_j}{s_j}. \tag{C.20}$$

Therefore by (C.14), (C.15), and (C.3), since $\binom{n_j}{s_j}\leqq 2^{n_j}$, (C.20) is bounded by

$$[4(4!)K_9^2]^n n^{N/2}C(N)n!\prod_{j=1}^{f'+1}n_j! \tag{C.21}$$

By (C.14) and (C.15) again, $\prod_{j=1}^{f'+1}n_j!$ is a product of $n+f'$ integers smaller than n, among which at least $f'+1$ are 1 and $f'+1$ are 2. Therefore by Stirling's formula, and (C.3)

$$\prod_{j=1}^{f'+1}n_j!\leqq 2^{f'+1}n^{n-f'-2}\leqq(2e)^n\frac{n!}{n^{f'+1}}\leqq(2e)^n\frac{n!}{f!}, \tag{C.22}$$

which achieves the proof of (C.18) and Theorem II.

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