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Local Exponential H^2 Stabilization of a 2×2 Quasilinear Hyperbolic System using Backstepping

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***Abstract*— We consider the problem of boundary stabilization for a quasilinear 2×2 system of first-order hyperbolic PDEs. We design a full-state feedback control law, with actuation on only one end of the domain, and prove local H^2 exponential stability of the closed-loop system. The proof of stability is based on the construction of a strict Lyapunov function. The feedback law is found using the recently developed backstepping method for 2×2 system of first-order hyperbolic linear PDEs, developed by the authors in a previous work, which is briefly reviewed.**

I. INTRODUCTION

In this paper we are concerned with the problem of boundary stabilization for a 2×2 system of first-order hyperbolic *quasilinear* PDEs. We consider actuation in only one of the boundaries. The quasilinear case is of interest since several relevant physical systems are described by 2×2 systems of first-order hyperbolic quasilinear PDEs, such as open channels, transmission lines, gas flow pipelines or road traffic models.

This problem has been previously considered for 2×2 systems [8] and even $n \times n$ systems [13], using the explicit evolution of the Riemann invariants along the characteristics. More recently, an approach using control Lyapunov functions has been developed, for 2×2 systems [3] and $n \times n$ systems [4]. These results use only static output feedback (the output being the value of the state on the boundaries). However these results do not deal with the same class of systems considered in this work (which includes an extra term in the equations); with this term, it has been shown in [1] that there are examples (even in the linear case) for which there are no control Lyapunov functions of the form $\int_0^1 z^T Q(x) z dx$ (see the next section for notation) which would allow the computation of a static output feedback law to stabilize the system (even when feedback is allowed on both sides of the boundary).

Several other authors have also studied this problem. For instance, the linear case has been analyzed in [22] (using a Lyapunov approach) and in [14] (using a spectral approach). The nonlinear case has been considered by [6] and [9] using a Lyapunov approach, and in [15], [16], and [7] using a Riemann invariants approach.

The basis of the design used in this paper, which needs actuation on only one end, is the backstepping method [10]; initially developed for parabolic equations, it has been used for first-order hyperbolic equations [12], delay systems [11], second-order hyperbolic equations [17], fluid flows [19], nonlinear PDEs [20] and even PDE adaptive designs [18].

Our design builds upon the recently developed backstepping method for 2×2 system of first-order hyperbolic linear PDEs [21]. This design allows us to design a full-state feedback law (with actuation on only one end of the domain) that makes the closed-loop system locally exponentially stable in the H^2 sense. The gains of the feedback law are obtained as solutions of a well-posed 4 x 4 system of first-order hyperbolic linear PDEs. The proof of stability is based on [4]; we construct a strict Lyapunov function, locally equivalent to the H^2 norm, and written in coordinates defined by the (invertible) backstepping transformation.

The paper is organized as follows. In Section II we formulate the problem. In Section III we briefly review the linear backstepping design from [21]. In Section IV we present our main result, which is proven in Section V. We finish in Section VI with some concluding remarks. We also include an appendix with technical definitions and lemmas.

II. PROBLEM STATEMENT

Consider the system

$$z_t+\Lambda(z,x)z_x+f(z,x)=0, \quad x\in[0,1],\, t\in[0,+\infty), \tag{1}$$

where $z : [0,1]\times[0,\infty) \to \mathbb{R}^2$, $\Lambda : \mathbb{R}^2\times[0,1] \to \mathcal{M}_{2,2}(\mathbb{R})$, $f : \mathbb{R}^2\times[0,1] \to \mathbb{R}^2$, where $\mathcal{M}_{2,2}(\mathbb{R})$ denotes the set of 2×2 real matrices. We consider that $\Lambda(z,x)$ is twice continuously differentiable with respect to z and x, and that $\Lambda(0,x)$ is a diagonal matrix with nonzero eigenvalues $\Lambda_1(x)$ and $\Lambda_2(x)$ which are, respectively, positive and negative, i.e.,

$$\Lambda(0,x) = \mathrm{diag}(\Lambda_1(x),\Lambda_2(x)), \Lambda_1(x) > 0, \Lambda_2(x) < 0, \tag{2}$$

where $\mathrm{diag}(\Lambda_1,\Lambda_2)$ denotes the diagonal matrix with Λ_1 in the first position of the diagonal and Λ_2 in the second.

Also assume that $f(0,x) = 0$ (so that there is an equilibrium at the origin) and that f is twice continuously differentiable with respect to z. Denote

$$\frac{\partial f}{\partial z}(0,x) = \begin{bmatrix} f_{11}(x) & f_{12}(x) \\ f_{21}(x) & f_{22}(x) \end{bmatrix}, \tag{3}$$

and assume that $f_{ij} \in \mathcal{C}^1([0,1])$.

Denoting $z = [z_1\ z_2]^T$, we study classical solutions of the system under the following boundary conditions

$$z_1(0,t) = q z_2(0,t),\, z_2(1,t) = U_c(t), \tag{4}$$

for $q \neq 0$, and $U_c(t)$ the actuation. Our task is to find a feedback law for $U_c(t)$ to make the origin of system (1),(4) locally exponentially stable.

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Remark 1: The case with $f = 0$ in (1) was addressed in [3] and [4] by using control Lyapunov functions to design a static output feedback law; this approach has been shown to fail [1] for some cases with $f \neq 0$, at least for a Lypunov function of the form $\int_0^1 z^T Q(x) z dx$.

Remark 2: The case $q = 0$ in (4) can be considered by modifying the design as in [21]. Also, nonlinear boundary conditions can be addressed by slightly modifying the proof.

III. Stabilization of 2×2 hyperbolic linear systems

To stabilize system (1) and (4), we use the design presented in [21] for a linear 2×2 hyperbolic system.

Consider the following linear system

$$w_t = \Sigma(x) w_x + C(x) w, \tag{5}$$

where

$$\Sigma(x)=\begin{pmatrix} -\epsilon_1(x) & 0 \\ 0 & \epsilon_2(x) \end{pmatrix}, C(x)=\begin{pmatrix} 0 & c_1(x) \\ c_2(x) & 0 \end{pmatrix}, \tag{6}$$

where $\epsilon_1(x), \epsilon_2(x) > 0$ and c_1, c_2 are $\mathcal{C}^2([0,1])$ functions and with boundary conditions $u(0,t) = qv(0,t)$ and $v(1,t) = U_c(t)$, where the components of w are $w = [u \ v]^T$.

Then, selecting $U_c(t)$ as

$$U_c = \int_0^1 K^{vu}(1,\xi) u(\xi,t) d\xi + \int_0^1 K^{vv}(1,\xi) v(\xi,t) d\xi, \tag{7}$$

it can be shown that the origin of system (5) is exponentially stable, where $K^{vu}(x,\xi)$ and $K^{vv}(x,\xi)$ are solution of:

$$\begin{aligned} \epsilon_1(x) K_x^{uu} + \epsilon_1(\xi) K_\xi^{uu} &= -\epsilon_1'(\xi) K^{uu} - c_2(\xi) K^{uv}, & (8)\\ \epsilon_1(x) K_x^{uv} - \epsilon_2(\xi) K_\xi^{uv} &= \epsilon_2'(\xi) K^{uv} - c_1(\xi) K^{uu}, & (9)\\ \epsilon_2(x) K_x^{vu} - \epsilon_1(\xi) K_\xi^{vu} &= \epsilon_1'(\xi) K^{vu} + c_2(\xi) K^{vv}, & (10)\\ \epsilon_2(x) K_x^{vv} + \epsilon_2(\xi) K_\xi^{vv} &= -\epsilon_2'(\xi) K^{vv} + c_1(\xi) K^{vu}, & (11) \end{aligned}$$

with boundary conditions

$$\begin{aligned} K^{uu}(x,0) &= \frac{\epsilon_2(0)}{q\epsilon_1(0)} K^{uv}(x,0), & (12)\\ K^{uv}(x,x) &= \frac{c_1(x)}{\epsilon_1(x)+\epsilon_2(x)}, & (13)\\ K^{vu}(x,x) &= -\frac{c_2(x)}{\epsilon_1(x)+\epsilon_2(x)}, & (14)\\ K^{vv}(x,0) &= \frac{q\epsilon_1(0)}{\epsilon_2(0)} K^{vu}(x,0). & (15) \end{aligned}$$

The equations evolve in the triangular domain $\mathcal{T} = \{(x,\xi) : 0 \le \xi \le x \le 1\}$. In [21] it is shown that if the coefficients ϵ_1, ϵ_2 are $\mathcal{C}^2([0,1])$, and if c_1 and c_2 are $\mathcal{C}^2([0,1])$, then the kernels belong to $\mathcal{C}^2(\mathcal{T})$. Define $\|w(\cdot,t)\|_{L^2} = \sqrt{\int_0^1 \left(u^2(\xi,t) + v^2(\xi,t)\right) d\xi}$. Then, it holds:

Theorem 1: Consider system (5) with control law (7) and initial condition $w_0 \in L^2([0,1])$. Then, there exists $\lambda > 0$ and $c > 0$ such that

$$\|w(\cdot,t)\|_{L^2} \le c\, \mathrm{e}^{-\lambda t} \|w_0\|_{L^2}. \tag{16}$$

Proof: The selection of $U_c(t)$ is based on the backstepping method. Denoting

$$K(x,\xi) = \begin{pmatrix} K^{uu}(x,\xi) & K^{uv}(x,\xi) \\ K^{vu}(x,\xi) & K^{vv}(x,\xi) \end{pmatrix}, \tag{17}$$

and defining a transformation

$$\gamma(x,t) = w(x,t) - \int_0^x K(x,\xi) w(\xi,t) d\xi, \tag{18}$$

where the "target" variable $\gamma(x,t) = [\alpha(x,t) \ \ \beta(x,t)]^T$, it can be proven that, if the kernels verify (8)–(15), then γ verifies the following equations:

$$\gamma(x,t)_t = \Sigma(x)\gamma_x(x,t), \alpha(0,t) = q\beta(0,t), \beta(1,t) = 0, \tag{19}$$

which has an exponentially stable equilibrium at the origin, as we show next. Define

$$D(x) = \begin{bmatrix} A\frac{\mathrm{e}^{-\mu x}}{\epsilon_1(x)} & 0 \\ 0 & B\frac{\mathrm{e}^{\mu x}}{\epsilon_2(x)} \end{bmatrix}, \tag{20}$$

where $A, B, \mu > 0$ will be computed later. Select:

$$U = \int_0^1 \gamma^T(x,t) D(x) \gamma(x,t) dx. \tag{21}$$

Notice that $\sqrt{U}$ defines a norm equivalent to $\|\gamma(\cdot,t)\|_{L^2}$. Computing the derivative of U and integrating by parts,

$$\begin{aligned} \dot{U} \ &= \ -\int_0^1 \gamma^T(x,t) \left(D(x)\Sigma(x)\right)_x \gamma(x,t) dx \\ &\quad + \left[\gamma^T(x,t) D(x)\Sigma(x)\gamma(x,t)\right]_0^1, \end{aligned} \tag{22}$$

where we have used that $\Sigma(x)$ and $D(x)$ commute. Since

$$\left(D(x)\Sigma(x)\right)_x = \mu \begin{bmatrix} A\mathrm{e}^{-\mu x} & 0 \\ 0 & B\mathrm{e}^{\mu x} \end{bmatrix} > 0, \tag{23}$$

and, on the other hand,

$$\begin{aligned} &\left[\gamma^T(x,t) D(x)\Sigma(x)\gamma_x(x,t)\right]_0^1 \\ &= \ -A\alpha^2(1,t)\mathrm{e}^{-\mu} - (B - q^2 A)\beta^2(0,t), \end{aligned} \tag{24}$$

choosing $B = q^2 A + \lambda_2$, $A = \lambda_2 \mathrm{e}^{\mu}$, and $\mu = \lambda_1 \bar{\epsilon}$, where $\bar{\epsilon} = \max_{x\in[0,1]} \left\{\frac{1}{\epsilon_1(x)}, \frac{1}{\epsilon_2(x)}\right\}$, we get that $\left(D(x)\Sigma(x)\right)_x \ge \lambda_1 D(x)$, therefore:

$$\dot{U} \le -\lambda_1 U - \lambda_2 \left(\alpha^2(1,t) + \beta^2(0,t)\right), \tag{25}$$

where $\lambda_1, \lambda_2 > 0$ can be chosen as large as desired. This shows exponential stability of the origin for the γ system. To extend the result to the original system, an inverse transformation to (18) is defined as follows

$$w(x,t) = \gamma(x,t) + \int_0^x L(x,\xi)\gamma(\xi,t) d\xi, \tag{26}$$

where the coefficients of the inverse kernel matrix L are solutions of a system of equations analogous to (8)–(15). The theorem then follows by using the inverse and direct transformations to relate the L^2 norms of w and γ. ■

IV. Main Result: Application of Backstepping to the Nonlinear System

We wish to show that the linear controller (7) designed using backstepping works locally for the nonlinear system.

For that, we write our quasilinear system in a form equivalent (up to linear terms) to (5). Define

$$\varphi_1(x) = \exp\left(\int_0^x \frac{f_{11}(s)}{\Lambda_1(s)} ds\right), \quad (27)$$

$$\varphi_2(x) = \exp\left(-\int_0^x \frac{f_{22}(s)}{\Lambda_2(s)} ds\right). \quad (28)$$

We obtain a new state variable w from z using the following invertible transformation:

$$w = \begin{bmatrix} \varphi_1(x) & 0 \\ 0 & \varphi_2(x) \end{bmatrix} z = \Phi(x) z. \quad (29)$$

It follows that w verifies the following equation:

$$w_t + \bar{\Lambda}(w,x) w_x + \bar{f}(w,x) = 0, \quad (30)$$

where

$$\bar{\Lambda}(w,x) = \Phi(x)\Lambda(\Phi^{-1}(x)w,x)\Phi^{-1}(x), \quad (31)$$

$$\bar{f}(w,x) = \Phi(x) f(\Phi^{-1}(x)w,x) + \bar{\Lambda}(w,x)\begin{bmatrix} -\frac{f_{11}(x)}{\Lambda_1(x)} & 0 \\ 0 & \frac{f_{22}(x)}{\Lambda_2(x)} \end{bmatrix} w. \quad (32)$$

It is evident that $\bar{\Lambda}(0,x) = \Phi(x)\Lambda(0,x)\Phi^{-1}(x) = \Lambda(0,x)$ and that $\bar{f}(0,x) = 0$. Also,

$$C(x) = -\left.\frac{\partial \bar{f}(w,x)}{\partial w}\right|_{w=0} = \begin{bmatrix} 0 & -f_{12} \\ -f_{21} & 0 \end{bmatrix}. \quad (33)$$

Thus, it is possible to write (30) as a linear system plus nonlinear terms:

$$w_t - \Sigma(x) w_x - C(x) w + \Lambda_{NL}(w,x) w_x + f_{NL}(w,x) = 0, \quad (34)$$

where $\Sigma(x) = -\Lambda(0,x)$ and

$$\Lambda_{NL}(w,x) = \bar{\Lambda}(w,x) + \Sigma(x), \quad (35)$$

$$f_{NL}(w,x) = \bar{f}(w,x) + C(x) w. \quad (36)$$

It is clear that the nonlinear terms verify $\Lambda_{NL}(0,x) = 0$, $f_{NL}(0,x) = \frac{\partial f_{NL}}{\partial w}(0,x) = 0$.

Computing the boundary conditions of (34) by combining (4) with the transformation (29), one obtains

$$u(0,t) = qv(0,t),\ v(1,t) = \bar{U}_c(t) = U_c(t)/\varphi_2(1). \quad (37)$$

Notice that the linear part of (34) is identical to (5), the boundary conditions are the same, and the coefficients $C(x)$ and $\Sigma(x)$ verify the assumptions of Section III. Thus we consider using the feedback law:

$$\bar{U}_c = \int_0^1 K^{vu}(1,\xi) u(\xi,t) d\xi + \int_0^1 K^{vv}(1,\xi) v(\xi,t) d\xi, \quad (38)$$

which implies:

$$U_c = \varphi_2(1)\int_0^1 K^{vu}(1,\xi)\frac{z_1(\xi,t)}{\varphi_1(\xi)} d\xi + \varphi_2(1)\int_0^1 K^{vv}(1,\xi)\frac{z_2(\xi,t)}{\varphi_2(\xi)} d\xi, \quad (39)$$

where the kernels are computed from (8)–(15)) using the coefficients $C(x)$ and $\Sigma(x)$ as defined before.

Denoting:

$$q_0 = \begin{bmatrix} 1 \\ -q \end{bmatrix}, q_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, k(x) = \begin{bmatrix} \frac{\varphi_2(1)K^{vu}(1,x)}{\varphi_1(x)} \\ \frac{\varphi_2(1)K^{vv}(1,x)}{\varphi_2(x)} \end{bmatrix}, \quad (40)$$

the boundary conditions of the closed loop system can be written as:

$$q_0^T z(0,t) = 0,\ q_1^T z(1,t) = \int_0^1 k^T(\xi) z(\xi,t) d\xi. \quad (41)$$

Define the norms $\|z(\cdot,t)\|_{H_1} = \|z(\cdot,t)\|_{L^2} + \|z_x(\cdot,t)\|_{L^2}$ and $\|z(\cdot,t)\|_{H_2} = \|z(\cdot,t)\|_{H^1} + \|z_{xx}(\cdot,t)\|_{L^2}$. We now state the main result of the paper.

Theorem 2: Consider system (1) with boundary conditions (41) and initial conditions $z_0 = [z_{0_1}\ z_{0_2}]^T \in H^2([0,1])$, where the kernels K^{vu} and K^{vv} are obtained from (8)–(15). Then, under the assumptions of smoothness for the coefficients stated in Section II, and if the following compatibility conditions are verified

$$0 = q_0^T z_0(0), \quad (42)$$

$$0 = \int_0^1 k^T(\xi) z_0(\xi) d\xi - q_1^T z_0(1), \quad (43)$$

$$0 = q_0^T \left(\Lambda(z_0(0),0) z_{0_x}(0) + f(z_0(0),0)\right), \quad (44)$$

$$0 = \int_0^1 k^T(\xi)\left(\Lambda(z_0(\xi),\xi) z_{0_x}(\xi) + f(z_0(\xi),\xi)\right) d\xi - q_1^T\left(\Lambda(z_0(1),1) z_{0_x}(1) + f(z_0(1),1)\right), \quad (45)$$

there exists $\delta > 0$, $\lambda > 0$ and $c > 0$ such that if $\|z_0\|_{H^2} \leq \delta$

$$\|z(\cdot,t)\|_{H^2} \leq c\, \mathrm{e}^{-\lambda t} \|z_0\|_{H^2}. \quad (46)$$

Remark 3: The compatibility conditions (43) and (45) depend on our feedback laws and therefore are not natural. They can be omitted by considering a dynamical extension in the spirit of [2] (compare Theorem 2.4 and Theorem 2.8 of [2]). We shall give more details in a forthcoming paper.

V. Proof of Theorem 2

We first establish some notation. For an $\mathbb{R}^2$ vector $\gamma(x) = [\alpha(x)\ \beta(x)]^T$ denote $|\gamma(x)| = |\alpha(x)| + |\beta(x)|$, and

$$\|\gamma\|_\infty = \sup_{x\in[0,1]} |\gamma(x)|,\ \|\gamma\|_{L^1} = \int_0^1 |\gamma(\xi)| d\xi. \quad (47)$$

For a 2×2 matrix M, denote:

$$|M| = \max\{|Mv|; \gamma \in \mathbb{R}^2, |\gamma| = 1\}. \quad (48)$$

For the kernel matrices $K(x,\xi)$ and $L(x,\xi)$ denote

$$\|K\|_\infty = \sup_{(x,\xi)\in\mathcal{T}} |K(x,\xi)|. \quad (49)$$

Denote $|\gamma| = |\gamma(x,t)|$ and $\|\gamma\| = \|\gamma(\cdot,t)\|$. For $\gamma \in H^2([0,1])$, recall the following well-known inequalities:

$$\|\gamma\|_{L^1} \leq C_1\|\gamma\|_{L^2} \leq C_2\|\gamma\|_\infty, \quad (50)$$
$$\|\gamma\|_\infty \leq C_3\left[\|\gamma\|_{L^2} + \|\gamma_x\|_{L^2}\right] \leq C_4\|\gamma\|_{H^1}, \quad (51)$$
$$\|\gamma_x\|_\infty \leq C_5\left[\|\gamma_x\|_{L^2} + \|\gamma_{xx}\|_{L^2}\right] \leq C_6\|\gamma\|_{H^2}. \quad (52)$$

To prove Theorem 2, we notice that if we apply the (invertible) backstepping transformation (18) to the nonlinear system (34) we obtain the following transformed system:

$$0 = \gamma_t - \Sigma(x)\gamma_x + \Lambda_{NL}(w,x)w_x + f_{NL}(w,x) + \int_0^x K(x,\xi)\left(\Lambda_{NL}(w,\xi)w_x(\xi) - f_{NL}(w,\xi)\right)d\xi, \quad (53)$$

and using the inverse transformation (26) the equation can be expressed fully in terms of γ as:

$$\gamma_t - \Sigma(x)\gamma_x + F_3[\gamma,\gamma_x] + F_4[\gamma] = 0, \quad (54)$$

where the functionals F_3 and F_4 are defined in the Appendix. The boundary conditions are $\alpha(0,t) = q\beta(0,t)$ and $\beta(1,t) = 0$. Differentiating twice with respect to x in the transformations (29), (18) and (26), H^2 norm of γ is equivalent to the H^2 norm of z. Thus, from H^2 local stability of the origin for (54), the same holds for z.

We proceed by analyzing (using a Lyapunov function) the growth of $\|\gamma\|_{L^2}$, $\|\gamma_t\|_{L^2}$ and $\|\gamma_{tt}\|_{L^2}$. Relating these norms with $\|\gamma\|_{H^2}$, we then prove H^2 local stability for γ.

A. Analyzing the growth of $\|\gamma\|_{L^2}$

Define

$$U = \int_0^1 \gamma^T(x,t)D(x)\gamma(x,t)dx, \quad (55)$$

for $D(x)$ as in (20) with the coefficients found in the proof of Theorem 1. Proceeding analogously to (22)–(25), we get:

$$\dot{U} \leq -\lambda_1 U - \lambda_2\left(\alpha^2(1,t) + \beta^2(0,t)\right) - 2\int_0^1 \gamma^T(x,t)D(x)\left(F_3[\gamma,\gamma_x] + F_4[\gamma]\right)dx. \quad (56)$$

Let us analyze the last term:

$$2\left|\int_0^1 \gamma^T(x,t)D(x)\left(F_3[\gamma,\gamma_x] + F_4[\gamma]\right)dx\right| \leq K_1\int_0^1 |\gamma|\left(|F_3[\gamma,\gamma_x]| + |F_4[\gamma]|\right)dx. \quad (57)$$

Applying Lemma 2 (see the Appendix), we obtain that there exists a δ_1, such that for $\|\gamma\|_\infty < \delta_1$,

$$\int_0^1 |\gamma||F_3[\gamma,\gamma_x]|dx \leq K_2\|\gamma_x\|_\infty\|\gamma\|^2_{L^2}, \quad (58)$$
$$\int_0^1 |\gamma||F_4[\gamma]|dx \leq K_3\|\gamma\|_\infty\|\gamma\|^2_{L^2}, \quad (59)$$

and using inequality (51) and noting $\|\gamma\|_{L^2} \leq K_4 U^{1/2}$, we obtain the following theorem:

Theorem 3: There exists δ_1 such that if $\|\gamma\|_\infty < \delta_1$ then

$$\dot{U} \leq -\lambda_1 U - \lambda_2\left(\alpha^2(1,t) + \beta^2(0,t)\right) + C_1 U^{3/2} + C_2\|\gamma_x\|_\infty U, \quad (60)$$

where λ_1, λ_2, C_1 and C_2 are positive constants.

B. Analyzing the growth of $\|\gamma_t\|_{L^2}$

Define $\eta = \gamma_t$. Notice that the norms of η and γ_x are related (see Lemma 5 in the Appendix). Taking a partial derivative in t in (53) we obtain an equation for η as follows:

$$\eta_t + \left(F_1[\gamma] - \Sigma(x)\right)\eta_x + F_5[\gamma,\gamma_x,\eta] + F_6[\gamma,\eta] = 0, \quad (61)$$

where F_1, F_5 and F_6 are defined in the Appendix. The boundary conditions for $\eta = [\eta_1\ \eta_2]^T$ are $\eta_1(0,t) = q\eta_2(0,t)$ and $\eta_2(1,t) = 0$.

To find a Lyapunov function for η, we use the next lemma:

Lemma 1: There exists $\delta > 0$ such that if $\|\gamma\|_\infty < \delta$, there exists a symmetric matrix $R[\gamma] > 0$ verifying

$$R[\gamma]\left(\Sigma(x) - F_1[\gamma]\right) - \left(\Sigma(x) - F_1[\gamma]\right)^T R[\gamma] = 0, \quad (62)$$

and the following bounds:

$$R[\gamma](x) \leq c_1 + c_2\|\gamma\|_\infty, \quad (63)$$
$$\left|\left(\left(R[\gamma] - D(x)\right)\Sigma(x)\right)_x\right| \leq c_2\|\gamma\|_\infty(1 + \|\gamma_x\|_\infty), \quad (64)$$
$$\left|\left(R[\gamma]\right)_t\right| \leq c_3\left(|\eta| + \|\eta\|_{L^1}\right), \quad (65)$$

where c_1, c_2, c_3 are positive constants.
The proof is skipped due to page limitation.

Define:

$$V = \int_0^1 \eta^T(x,t)R[\gamma](x)\eta(x,t)dx. \quad (66)$$

Computing the time derivative of the Lyapunov function, applying Lemma 1, and integrating by parts, we find

$$\begin{aligned}\dot{V} = &-\int_0^1 \eta^T(x,t)\left(R[\gamma]\left(\Sigma(x) - F_1[\gamma]\right)\right)_x\eta(x,t)dx \\ &+ \left[\eta^T(x,t)R[\gamma](x)\left(\Sigma(x) - F_1[\gamma](x)\right)\eta(x,t)\right]_{x=0}^{x=1} \\ &+ \int_0^1 \eta^T(x,t)\left(R[\gamma]\right)_t\eta(x,t)dx \\ &-2\int_0^1 \eta^T(x,t)R[\gamma]F_5[\gamma,\gamma_x,\eta,\eta_x,]dx \\ &-2\int_0^1 \eta^T(x,t)R[\gamma]F_6[\gamma,\eta]dx. \end{aligned} \quad (67)$$

The first three terms of (67) are analyzed using Lemma 1. Thus, there exists δ_1 such that, for $\|\gamma\|_\infty < \delta$, we find, for the first term:

$$-\int_0^1 \eta^T(x,t)\left(R[\gamma]\left(\Sigma(x) - F_1[\gamma]\right)\right)_x\eta(x,t)dx \leq -\lambda_1 V + K_1\|\eta\|^2_{L^2}\left(\|\gamma\|_\infty + \|\gamma_x\|_\infty\right). \quad (68)$$

The second term of (67) is bounded using the boundary conditions and Lemma 1, as:

$$\left[\eta^T(x,t)R[\gamma](x)\left(\Sigma(x) - F_1[\gamma](x)\right)\eta(x,t)\right]_{x=0}^{x=1} \leq -\lambda_2\left(\eta_1^2(1,t) + \eta_2^2(0,t)\right) + K_2\|\gamma\|_\infty\left(\eta_2^2(0,t) + \eta_1^2(1,t)\right). \quad (69)$$

Finally, we bound the third term of (67) applying Lemma 1 as follows:

$$\int_0^1 \eta^T(x,t)\left(R[\gamma]\right)_t\eta(x,t)dx \leq K_3\int_0^1 |\eta|^2\left(|\eta| + \|\eta\|_{L^1}\right)dx \leq K_4\|\eta\|^2_{L^2}\|\eta\|_\infty. \quad (70)$$

Applying Lemmas 1 and 3 to the last two terms of (67), we obtain that, for $\|\gamma\|_\infty < \delta$,

$$\begin{aligned}&2\left|\int_0^1 \eta^T(x,t)R[\gamma]F_5[\gamma,\gamma_x,\eta,\eta_x]dx\right| \\ \leq\ & K_5\int_0^1 |\eta||F_5[\gamma,\eta]|dx \leq K_6\|\eta\|^2_{L^2}\left(\|\gamma\|_\infty+\|\gamma_x\|_\infty\right) \\ &+K_7\|\eta\|_{L^2}|\eta(0,t)||\gamma(0,t)|, \qquad (71)\end{aligned}$$

and

$$\begin{aligned}&2\left|\int_0^1 \eta^T(x,t)R[\gamma]F_6[\gamma,\eta]dx\right| \\ \leq\ & K_8\int_0^1 |\eta||F_6[\gamma,\eta]|dx \leq K_9\|\eta\|^2_{L^2}\|\gamma\|_\infty. \qquad (72)\end{aligned}$$

Thus, for $\|\gamma\|_\infty$ small enough, using Lemma 5 to bound $\|\gamma_x\|_\infty$ by $\|\eta\|_\infty$, and noting $\|\eta\|_{L^2} \leq K_{10}V^{1/2}$, we obtain

Theorem 4: There exists δ_2 such that if $\|\gamma\|_\infty < \delta_2$

$$\dot{V} \leq -\lambda_3 V - \lambda_4\left(\eta_1^2(1,t)+\eta_2^2(0,t)\right) + K_1 V\|\eta\|_\infty, \qquad (73)$$

for $\lambda_2, \lambda_3, K_1$ positive constants.

C. Analyzing the growth of $\|\gamma_{tt}\|_{L^2}$

Define $\theta = \eta_t$. Notice that the norms of θ and η_x are related (see Lemma 5 in the Appendix). Taking a partial derivative in t in (61) we obtain an equation for θ:

$$\begin{aligned}&\theta_t + \left(F_1[\gamma]-\Sigma(x)\right)\theta_x \\ &+F_7[\gamma,\gamma_x,\eta,\eta_x,\theta]+F_8[\gamma,\eta,\theta]=0, \qquad (74)\end{aligned}$$

where F_7 and F_8 are defined in the Appendix. The boundary conditions for $\theta = [\theta_1\ \theta_2]^T$ are $\theta_1(0,t) = q\theta_2(0,t)$ and $\theta_2(1,t) = 0$. Since (74) is similar to (61), we define:

$$W = \int_0^1 \theta^T(x,t)R[\gamma](x)\theta(x,t)dx. \qquad (75)$$

Computing the time derivative of the Lyapunov function, and proceeding exactly as in (67), we reach:

$$\begin{aligned}\dot{W} \leq\ & -\lambda_1 W + K_1\|\theta\|^2_{L^2}\left(\|\gamma\|_\infty+\|\gamma_x\|_\infty+\|\eta\|_\infty\right) \\ &+\left(K_2\|\gamma\|_\infty-\lambda_2\right)\left(\theta_1^2(1,t)+\theta_2^2(0,t)\right) \\ &+2\left|\int_0^1 \theta^T(x,t)R[\gamma]F_7[\gamma,\gamma_x,\eta,\eta_x,\theta]dx\right| \\ &+2\left|\int_0^1 \theta^T(x,t)R[\gamma]F_6[\gamma,\eta,\theta]dx\right|. \qquad (76)\end{aligned}$$

Finally, applying Lemmas 1 and 4 in the last two terms of (76), there exists a δ, such that for $\|\gamma\|_\infty < \delta$,

$$\begin{aligned}&2\left|\int_0^1 \theta^T(x,t)R[\gamma]F_7[\gamma,\gamma_x,\eta,\eta_x,\theta]dx\right| \\ &\leq K_5\|\theta\|^2_{L^2}\left(\|\gamma\|_\infty+\|\gamma_x\|_\infty\right)+K_6\|\theta\|_{L^2}\|\eta\|^2_{L^2} \\ &\quad+K_7\|\theta\|_{L^2}\left(\|\eta\|_{L^2}\|\eta\|^2_\infty+|\eta(0,t)|^2+|\gamma(0,t)||\theta(0,t)|\right) \\ &\quad+K_8\|\theta\|_{L^2}\|\eta_x\|_{L^2}\|\eta\|_\infty+K_9\|\theta\|_{L^2}, \qquad (77)\end{aligned}$$

and

$$\begin{aligned}&2\left|\int_0^1 \theta^T(x,t)R[\gamma]F_8[\gamma,\eta,\theta]dx\right| \\ &\leq K_{11}\|\theta\|^2_{L^2}\|\gamma\|_\infty+K_{12}\|\eta\|_{L^2}\|\theta\|_{L^2}\|\eta\|_\infty \\ &\quad+K_{13}\|\eta\|_{L^2}\|\theta\|^2_{L^2}+K_{14}\|\eta\|^2_{L^2}\|\theta\|_{L^2}. \qquad (78)\end{aligned}$$

Thus, we finally obtain the following theorem:

Theorem 5: There exists δ_3 such that if $\|\gamma\|_\infty + \|\eta\|_\infty < \delta_3$ then

$$\begin{aligned}\dot{W} \leq\ & -\lambda_5 W - \lambda_6\left(\theta_1^2(1,t)+\theta_2^2(0,t)\right)+K_1 W V^{1/2} \\ &+K_2 V W^{1/2}+K_3 W^{3/2}, \qquad (79)\end{aligned}$$

where $\lambda_5, \lambda_6, K_1, K_2, K_3$ are positive constants.

D. Proof of H^2 stability of γ

Defining $S = U + V + W$, and combining Theorems 3, 4, and 5, there exists δ such that if $\|\gamma\|_\infty + \|\eta\|_\infty < \delta$

$$\dot{S} \leq -\lambda S + CS^{3/2}, \qquad (80)$$

for $\lambda, C > 0$. Following [3] and noting $\|\gamma\|_\infty + \|\eta\|_\infty \leq C_2 S$, then for sufficiently small $S(0)$, it follows that $S(t) \to 0$ exponentially.

Given that S (by Lemma 5) is equivalent to the H^2 norm of γ when $\|\gamma\|_\infty + \|\eta\|_\infty$ is sufficiently small, we obtain that there exists $\delta > 0$ and $c > 0$ such that if $\|\gamma_0\|_{H^2} \leq \delta$, then:

$$\|\gamma\|_{H^2} \leq c\,\mathrm{e}^{-\lambda t}\|\gamma_0\|_{H^2}. \qquad (81)$$

This proves Theorem 2.

VI. CONCLUDING REMARKS

We have solved the problem of boundary stabilization for a 2×2 system of first-order hyperbolic quasilinear PDEs with actuation on only one end of the boundary. We have shown, using a strict Lyapunov function, H^2 local exponential stability of the state.

It would be of interest to extend the method to $n \times n$ systems. For instance, a 3×3 first-order hyperbolic system of interest is the Saint-Venant-Exner system, which models open channels with a moving sediment bed [5]. While extending the Lyapunov analysis to $n \times n$ systems has been done [4], it remains an open problem to extend backstepping to such systems, even in the linear case. In general, the method would need n^2 kernels resulting in a $n^2 \times n^2$ system of coupled first-order hyperbolic equations, whose well-posedness depends critically on the exact choice of the transformation and target system; these are still to be defined.

REFERENCES

[1] G. Bastin and J.-M. Coron, "Further Results on boundary feedback stabilisation of 2×2 hyperbolic systems over a bounded interval," *Proceedings of 8th NOLCOS*, pp. 1081–1085, 2010.

[2] J.-M. Coron, "On the null asymptotic stabilization of the two-dimensional incompressible Euler equations in a simply connected domain," *SIAM J. Control Optim.*, Vol. 37, pp. 1874–1896, 1999.

[3] J.-M. Coron, B. d'Andrea-Novel and G. Bastin, "A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws," *IEEE Trans. on Automatic Control*, vol. 52, pp. 2–11, 2006.

[4] J.-M. Coron, G. Bastin and B. d'Andrea-Novel, "Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems," *SIAM Journal of Control and Optimization*, vol. 47, pp. 1460–1498, 2008.

[5] A. Diagne, G. Bastin and J.-M. Coron, "Lyapunov exponential stability of linear hyperbolic systems of balance laws," to appear in *Proceedings of the 18th IFAC World Congress*, 2011.

[6] M. Dick, M. Gugat and G. Leugering, "Classical solutions and feedback stabilisation for the gas flow in a sequence of pipes," *Networks and heterogeneous media*, vol. 5, pp. 691–709, 2010.

[7] V. Dos Santos and C. Prieur, "Boundary control of open channels with numerical and experimental validations," *IEEE Trans. Control Syst. Tech.*, vol. 16, pp. 1252–1264, 2008.

[8] J-M. Greenberg and T-t. Li, "The effect of boundary damping for the quasilinear wave equations," *Journal of Differential Equations*, vol. 52, pp. 66–75, 1984.

[9] M. Gugat and M. Herty, "Existence of classical solutions and feedback stabilisation for the flow in gas networks," *ESAIM Control Optimisation and Calculus of Variations*, vol. 17, pp. 28–51, 2011.

[10] M. Krstic and A. Smyshlyaev, *Boundary Control of PDEs*, SIAM, 2008

[11] M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*, Birkhauser, 2009.

[12] M. Krstic and A. Smyshlyaev, "Backstepping boundary control for first order hyperbolic PDEs and application to systems with actuator and sensor delays," *Syst. Contr. Lett.*, vol. 57, pp. 750–758, 2008.

[13] T-t. Li, *Global Classical Solutions for Quasilinear Hyperbolic Systems*, Wiley, 1994.

[14] X. Litrico and V. Fromion, "Boundary control of hyperbolic conservation laws using a frequency domain approach," *Automatica*, vol. 45, pp. 647–656, 2009.

[15] C. Prieur, "Control of systems of conservation laws with boundary errors," *Networks and Heterogeneous Media*, vol. 4, pp. 393–407, 2009.

[16] C. Prieur, J. Winkin and G. Bastin, "Robust boundary control of systems of conservation laws," *Mathematics of Control, Signals, and Systems*, vol. 20, pp. 173–197, 2008.

[17] A. Smyshlyaev, E. Cerpa and M. Krstic, "Boundary stabilization of a 1-D wave equation with in-domain antidamping," *SIAM Journal of Control and Optimization*, vol. 48, pp. 4014–4031, 2010.

[18] A. Smyshlyaev and M. Krstic, *Adaptive Control of Parabolic PDEs*, Princeton University Press, 2010.

[19] R. Vazquez and M. Krstic, *Control of Turbulent and Magnetohydrodynamic Channel Flow*, Birkhauser, 2008.

[20] R. Vazquez and M. Krstic, "Control of 1-D parabolic PDEs with Volterra nonlinearities — Part I: Design," *Automatica*, vol. 44, pp. 2778–2790, 2008.

[21] R. Vazquez, M. Krstic and J.-M. Coron, "Backstepping Boundary Stabilization and State Estimation of a 2×2 Linear Hyperbolic System," *50th IEEE CDC and ECC*, 2011.

[22] C.Z. Xu and G. Sallet, "Exponential stability and transfer functions of processes governed by symmetric hyperbolic systems," *ESAIM Control Optimisation and Calculus of Variations*, vol. 7, pp. 421–442, 2002.

APPENDIX

Define first the following linear functionals:

$$\mathcal{K}[\gamma](x) = \gamma(x,t) - \int_0^x K(x,\xi)\gamma(\xi,t)d\xi, \quad (82)$$

$$\mathcal{L}[\gamma](x) = \gamma(x,t) + \int_0^x L(x,\xi)\gamma(\xi,t)d\xi, \quad (83)$$

$$\mathcal{K}_\xi[\gamma](x) = -K(x,x)\gamma(x,t) + \int_0^x K_\xi(x,\xi)\gamma(\xi,t)d\xi, \quad (84)$$

$$\mathcal{L}_x[\gamma](x) = L(x,x)\gamma(x,t) + \int_0^x L_x(x,\xi)\gamma(\xi,t)d\xi. \quad (85)$$

Next we drop the x and t dependence. It clearly holds that $|\mathcal{K}[\gamma]|, |\mathcal{L}[\gamma]|, |\mathcal{K}_\xi[\gamma]|, |\mathcal{L}_x[\gamma]| \leq C_1 (|\gamma| + \|\gamma\|_{L^1})$.

Using (82) and (83), we define $F_1[\gamma]$ and $F_2[\gamma]$ as:

$$F_1 = \Lambda_{NL}(\mathcal{L}[\gamma], x), \ F_2 = f_{NL}(\mathcal{L}[\gamma], x). \quad (86)$$

From (86), $F_3[\gamma,\gamma_x](x)$ and $F_4[\gamma](x)$ are defined as

$$F_3 = \mathcal{K}[F_1[\gamma]\gamma_x], \ F_4 = \mathcal{K}[F_1[\gamma]\mathcal{L}_x[\gamma] + F_2[\gamma]]. \quad (87)$$

Similarly, $F_5[\gamma,\gamma_x,\eta](x)$ and $F_6[\gamma,\eta](x)$ are defined as

$$\begin{aligned} F_5 &= \mathcal{K}_\xi[F_1[\gamma]\eta] + K(x,0)\Lambda_{NL}(\gamma(0),0)\eta(0) \\ &\quad + \int_0^x K(x,\xi)F_{12}[\gamma,\gamma_x]\eta(\xi)d\xi + \mathcal{K}[F_{11}[\gamma,\eta]\gamma_x], \quad (88) \\ F_6 &= \mathcal{K}[F_{11}[\gamma,\eta]\mathcal{L}_x[\gamma] + F_1[\gamma]\mathcal{L}_x[\eta] + F_{21}[\gamma,\eta]], \quad (89) \end{aligned}$$

where $F_{11} = \frac{\partial F_1}{\partial \gamma}$, $F_{12} = \frac{\partial F_1}{\partial x}$ and $F_{21} = \frac{\partial F_2}{\partial \gamma}$. The functionals $F_7[\gamma,\gamma_x,\eta,\eta_x,\theta](x)$ and $F_8[\gamma,\eta,\theta](x)$ are defined as

$$\begin{aligned} F_7 &= \mathcal{K}_\xi[F_{11}[\gamma,\eta]\eta] + \int_0^x K(x,\xi)F_{12}[\gamma,\gamma_x]\theta(\xi)d\xi \\ &\quad + \mathcal{K}_\xi[F_1[\gamma]\theta] + \int_0^x K(x,\xi)F_{14}[\gamma,\gamma_x,\eta,\eta_x]\eta(\xi)d\xi \\ &\quad + K(x,0)\frac{\partial \Lambda_{NL}}{\partial \gamma}(\gamma(0),0)\eta(0)\eta(0) \\ &\quad + K(x,0)\Lambda_{NL}(\gamma(0),0)\theta(0) \\ &\quad + \mathcal{K}[F_{11}[\gamma,\eta]\eta_x] + \mathcal{K}[F_{13}[\gamma,\eta,\theta]\gamma_x], \quad (90) \\ F_8 &= 2\mathcal{K}[F_{11}[\gamma,\eta]\mathcal{L}_x[\eta]] + \mathcal{K}[F_1[\gamma]\mathcal{L}_x[\theta]] \\ &\quad + \mathcal{K}[F_{13}[\gamma,\eta,\theta]\mathcal{L}_x[\gamma]] + \mathcal{K}[F_{22}[\gamma,\eta,\theta]], \quad (91) \end{aligned}$$

where $F_{13} = \frac{\partial F_{11}}{\partial \gamma}$, $F_{14} = \frac{\partial F_{12}}{\partial \gamma}$ and $F_{22} = \frac{\partial F_{21}}{\partial \gamma}$.

Next we give without proof some technical lemmas.

Lemma 2: There exists $\delta > 0$ such that for $\|\gamma\|_\infty < \delta$,

$$\begin{aligned} |F_1| &\leq C_5(|\gamma| + \|\gamma\|_{L^1}), \quad (92) \\ |F_2| &\leq C_6(|\gamma|^2 + \|\gamma\|_{L^1}^2), \quad (93) \\ |F_3| &\leq C_7(\|\gamma\|_{L^2}\| + |\gamma|)(\|\gamma_x\|_{L^2} + |\gamma_x(x)|), \quad (94) \\ |F_4| &\leq C_8(|\gamma|^2 + \|\gamma\|_{L^2}^2). \quad (95) \end{aligned}$$

Lemma 3: There exists $\delta > 0$ such that for $\|\gamma\|_\infty < \delta$,

$$\begin{aligned} |F_{11}| &\leq C_9(|\eta| + \|\eta\|_{L^1}), \quad (96) \\ |F_{12}| &\leq C_{10}(|\gamma_x| + |\gamma| + \|\gamma\|_{L^1}), \quad (97) \\ |F_{21}| &\leq C_{11}(|\gamma| + \|\gamma\|_{L^1})(|\eta| + \|\eta\|_{L^1}), \quad (98) \\ |F_5| &\leq C_{12}(|\eta| + \|\eta\|_{L^2})(|\gamma| + \|\gamma\|_{L^2}) \\ &\quad + C_{14}(|\eta| + \|\eta\|_{L^2})(|\gamma_x| + \|\gamma_x\|_{L^2}) \\ &\quad + C_{15}|\gamma(0)||\eta(0)|, \quad (99) \\ |F_6| &\leq C_{16}(|\eta| + \|\eta\|_{L^2})(|\gamma| + \|\gamma\|_{L^2}). \quad (100) \end{aligned}$$

Lemma 4: There exists $\delta > 0$ such that for $\|\gamma\|_\infty < \delta$,

$$\begin{aligned} |F_{13}| &\leq C_{17}(|\eta|^2 + \|\eta\|_{L^1}^2) + C_{18}(|\theta| + \|\theta\|_{L^1}), \quad (101) \\ |F_{14}| &\leq C_{19}(|\eta| + \|\eta\|_{L^1})(1 + |\gamma_x| + |\gamma| + \|\gamma\|_{L^1}) \\ &\quad + C_{20}(|\eta_x| + |\eta| + \|\eta\|_{L^1}), \quad (102) \\ |F_{22}| &\leq C_{21}(|\gamma| + \|\gamma\|_{L^1})(|\theta| + \|\theta\|_{L^1}) \\ &\quad + C_{22}(|\eta|^2 + \|\eta\|_{L^1}^2), \quad (103) \\ |F_7| &\leq C_{23}(|\eta|^2 + \|\eta\|_{L^2}^2)(1 + \|\gamma\|_\infty + \|\gamma_x\|_\infty) \\ &\quad + C_{24}(|\eta| + \|\eta\|_{L^2})(|\eta_x| + \|\eta\|_{L^2}) \\ &\quad + C_{25}(|\gamma| + \|\gamma\|_{L^2} + \|\gamma_x\|_\infty)(|\theta| + \|\theta\|_{L^2}) \\ &\quad + C_{26}(|\eta(0)|^2 + |\gamma(0)||\theta(0)|), \quad (104) \\ |F_8| &\leq C_{27}(|\eta|^2 + \|\eta\|_{L^2}^2)(1 + \|\gamma\|_\infty) \\ &\quad + C_{28}(|\gamma| + \|\gamma\|_{L^2})(|\theta| + \|\theta\|_{L^2}). \quad (105) \end{aligned}$$

Lemma 5: There exists δ such that if $\|\gamma\|_\infty < \delta$ then

$$\begin{aligned} \|\eta\|_\infty &\leq c_1(\|\gamma_x\|_\infty + \|\gamma\|_\infty), \quad (106) \\ \|\eta\|_{L^2} &\leq c_2(\|\gamma_x\|_{L^2} + \|\gamma\|_{L^2}), \quad (107) \\ \|\gamma_x\|_\infty &\leq c_3(\|\eta\|_\infty + \|\gamma\|_\infty), \quad (108) \\ \|\gamma_x\|_{L^2} &\leq c_4(\|\eta\|_{L^2} + \|\gamma\|_{L^2}), \quad (109) \\ \|\theta\|_\infty &\leq c_1(\|\eta_x\|_\infty + \|\eta\|_\infty + \|\gamma\|_\infty), \quad (110) \\ \|\theta\|_{L^2} &\leq c_2(\|\eta_x\|_{L^2} + \|\eta\|_{L^2} + \|\gamma\|_{L^2}), \quad (111) \\ \|\eta_x\|_\infty &\leq c_3(\|\theta\|_\infty + \|\eta\|_\infty + \|\gamma\|_\infty), \quad (112) \\ \|\eta_x\|_{L^2} &\leq c_4(\|\theta\|_{L^2} + \|\eta\|_{L^2} + \|\gamma\|_{L^2}). \quad (113) \end{aligned}$$